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A variational problem involving a polyconvex integrand

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Abstract.

Existence of solutions to systems of parabolic equations obtained from polyconvex functions, remains a challenge in PDEs. In the current notes, we keep our focus on a variational problem which originates from a discretization of such a system. We state a duality result for a functional whose integrand is polyconvex and fails to satisfy growth conditions imposed in the standard theory of the calculus of variations.

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§1. Introduction

A theory, now called direct methods of the calculus of variations, was developed and pioneered by Morrey [15] in the middle of the last century where he introduced various fundamental concepts of convexity. The theory has wide application to many fields, including Ball's fundamental work in elasticity theory [4], and problems involving given Dirichlet boundary conditions. In these notes, we keep our focus on a class of problems which we think are ground breaking, very educational and help understanding some major open problems of the calculus of variations. We refer those seeking further background on the subject to the short introduction written by J. Ball [6]. There, a more serious biography and accountability of major mathematical contribution to the field of the calculus of variations can be found.

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Let

$$\Omega, \Lambda \subset \mathbb{R}^d \text{ be two open bounded convex sets.}$$

For convenience, rescaling and translating the sets if necessary, we assume without loss of generality that

$$|\Omega| = 1 \quad \text{and} \quad \vec{0} \in \Lambda. \tag{1.1}$$

Let W_1 be a real valued Borel map defined on the set $\mathbb{R}^{d\times d}$ of $d \times d$ matrices and let $W_2 : \bar{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be a Borel map. Let

$$p \in (1, \infty), \quad q = \frac{p}{p-1}, \quad 0 < C_0 \leq C_1.$$

In order to compare our main results to the state–of–the–art in the calculus of variations, let us first make standard assumptions on W_1 and W_2. Suppose

$$C_0(|\xi|^p - 1) \leq W_1(\xi) \leq C_1\big(|\xi|^p + 1\big) \tag{1.2}$$

or

$$-C_0 \leq W_1(\xi) \leq C_1\big(|\xi|^p + 1\big). \tag{1.3}$$

Suppose

$$|W_1(\xi_2) - W_1(\xi_1)| \leq C_1\big(|\xi_1|^{p-1} + |\xi_2|^{p-1} + 1\big)|\xi_1 - \xi_2| \tag{1.4}$$

for all $\xi \in \mathbb{R}^{d\times d}$.

Similarly, we suppose that

$$-C_0 \leq W_2(x, u) \leq C_1\big(|x|^p + |u|^p + 1\big) \tag{1.5}$$

and

$$|W_2(x, u_2) - W_2(x, u_1)| \leq C_1\big(|u_1|^{p-1} + |u_2|^{p-1} + 1\big)|u_1 - u_2| \tag{1.6}$$

for all $x \in \Omega$ and all $u_1, u_2 \in \mathbb{R}^d$.

Note that (1.5) and (1.6) yield existence of $a \in L^\infty(\Omega)$ and $b \in L^p(\Omega)$ such that

$$|W_2(x, u)| \leq a(x)|u|^p + b(x) \tag{1.7}$$

and

$$|\nabla_u W_2(x, u)| \leq 2C_1\big(|u|^{p-1} + 1\big) \tag{1.8}$$

for all $x \in \Omega$ and all $u \in \mathbb{R}^d$

We assume existence of a monotone nondecreasing function $\eta \in C(\mathbb{R})$ such that $\eta(0) = 0$ and

$$|W_2(x_2, u) - W_2(x_1, u)| \leq \eta(|x_2 - x_1|)\big(|u|^p + 1\big) \tag{1.9}$$

for all $x_1, x_2 \in \Omega$ and all $u \in \mathbb{R}^d$.

We define

$$I[\mathbf{u}] = \int_\Omega W\big(x, \mathbf{u}(x), \nabla \mathbf{u}(x)\big)\,dx, \tag{1.10}$$

where

$$W(x, u, \xi) = W_1(\xi) + W_2(x, u).$$

When W_1 is convex, bounded below and W_2 satisfies the growth condition in (1.6) and (1.7), then (cf. Lemma 2.1) the functional I is weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^d)$ and so, for any nonempty weakly closed subset $\mathcal{A}$ of $W^{1,p}(\Omega, \mathbb{R}^d)$, if (1.2) holds, then the variational problem

$$\inf_{\mathbf{u} \in \mathcal{A}} I[\mathbf{u}], \tag{1.11}$$

admits a minimizer.

Since the fundamental work of Morrey [15], it has been well-understood that when (1.2), (1.4), (1.5), (1.6), (1.9) hold, then I is is weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^d)$ if and only if W_1 is quasiconvex. Let us recall that W_1 is quasiconvex means that

$$\int_{(0,1)^d} W_1\big(\xi + \nabla \mathbf{u}(x)\big)\,dx \geq W_1(\xi) \tag{1.12}$$

for all $\xi \in \mathbb{R}^{d\times d}$ and all $u \in C_c^\infty\big((0,1)^d, \mathbb{R}^d\big)$. A sufficient condition for property (1.12) to hold is that W_1 be a polyconvex function, meaning $W_1(\xi)$ is a convex function of the minors of ξ (cf. e.g. [11]). When $d = 2$, the minors of $\xi \in \mathbb{R}^{2\times 2}$ are

$$M(\xi) = (\xi, \det \xi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$$

and so, W_1 polyconvex means there exists $\omega_1 : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ convex such that

$$W_1(\xi) = \omega_1\big(M(\xi)\big).$$

Under the above quasiconvexity and growth conditions on W_1 and W_2, it is apparent that if $\mathcal{A}$ is a weakly closed subset of $W^{1,p}(\Omega, \mathbb{R}^d)$, then (1.11) admits a minimizer. However, unless W_1 is convex, characterizing minimizers of (1.11) in terms of their Euler–Lagrange equations or developping a theory which ensures uniqueness of a minimizer, continue to defy the current theory of the calculus of variations. The situation becomes much more complicated if in addition, one imposes the following condition, satisfied by stored energy functionals appearing in elasticity theory:

$$\lim_{|\det \xi| \to 0^+} W_1(\xi) = \infty. \tag{1.13}$$

Indeed, Equation (1.13) is a variance with the upper bound condition on W_1 imposed in (1.2).

Our goal in [3] has been to develop a duality theory which helps address issues such as stabilities and complete characterization of optima in some nonconvex setting. We are concerned with a study which encompass a class of functionals W_1 satisfying Equation (1.13).

Motivated by the study of stored energy functionals which appear in elasticity theory in the study of Ogden materials (cf. [16]), in these notes, we restrict our attention to the following functionals: let ω be a strictly convex differentiable function defined on the set of $d \times d$ matrices and satisfying

(1.14)
$$c|\xi|^p \le \omega(\xi) \le c^{-1}(|\xi|^p + 1) \quad |\nabla\omega(\xi)| \le c^{-1}(|\xi|^{p-1} + 1) \quad \forall \xi \in \mathbb{R}^{d\times d}.$$

Let $h \in C^1(0,\infty)$ be a convex functional such that

$$\lim_{t\to 0^+} h(t) = \lim_{t\to\infty} \frac{h(t)}{t} = \infty.$$

Define

$$W_1(A) = \begin{cases} \omega(A) + h(\det A) & \text{if} \quad \det A > 0 \\ \infty & \text{if} \quad \det A \le 0. \end{cases}$$

The goal of these notes is to help understand in which sense we can solve the extremely difficult initial value problem

$$\partial_t \mathbf{u} = \operatorname{div}\Big(DW_1(\nabla \mathbf{u})\Big), \quad \det \nabla\mathbf{u} > 0, \quad \mathbf{u}(0,\cdot) = \mathbf{u}_0. \tag{1.15}$$

Here, $\mathbf{u}_0 \in W^{1,p}(\Omega,\Lambda)$ is a prescribed orientation preserving diffeomorphism of Ω onto Λ, and the unknown $\mathbf{u}$ is to be found in $L^1\big(0,T; W^{1,p}(\Omega,\Lambda)\big)$.

It seems more realistic to look for $\mathbf{u}(t,\cdot)$ is a set larger than the set of orientation preserving diffeomorphism of Ω onto Λ. Let $\mathcal{U}$ be the set of Borel maps $\mathbf{u} \in W^{1,p}(\Omega,\Lambda)$ such that the symmetric difference of $\mathbf{u}(\Omega)$ and Λ is a set of zero measure. Let $\mathcal{V}$ be the set of pairs $(\beta, \mathbf{u})$

such that $\beta : \Omega \to \mathbb{R} \cup \{\infty\}$ and $\mathbf{u} : \Omega \to \Lambda$ are Borel maps such that $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$ and

$$\int_\Omega l(\mathbf{u}(x))\beta(x)dx = \int_\Lambda l(y)dy$$

for all $l \in C_c(\mathbb{R}^d)$.

For each $\tau > 0$, the discrete energy functional associated to the system of equations in (1.15) is

$$I_\tau(\mathbf{u}) = \int_\Omega \Big(W_1(\nabla \mathbf{u}(x)) + \frac{|\mathbf{u}_0(x) - \mathbf{u}(x)|^2}{2\tau}\Big)dx.$$

We also consider a functional which depends on τ (but we don't display the dependence in τ)

$$E(\beta, \mathbf{u}) = \int_\Omega \Big(\omega(\nabla \mathbf{u}(x)) + h(\beta) + \frac{|\mathbf{u}_0(x) - \mathbf{u}(x)|^2}{2\tau}\Big)dx.$$

In order to study the evolutive system in (1.15), one needs to understand the following variational problem:

Problem 1.1. *Does*

$$\inf_{\mathbf{u} \in \mathcal{U}} I_\tau(\mathbf{u}) \tag{1.16}$$

has a minimizer? What are the Euler–Lagrange equations of (1.16)*?*

Problem 1.1 remains a great challenge in the calculus of variations. The techniques developed in [3] allow to thoroughly study Problems 1.2 and 1.3, which in some sense, are relaxations of Problem (1.16).

Problem 1.2. *We note that existence of a minimizer of*

$$\inf_{(\beta,\mathbf{u}) \in \mathcal{V}} E(\beta, \mathbf{u}) \tag{1.17}$$

is not an issue. The challenge is to know under what condition on the data, is the minimizer in (1.17) *unique? What are the Euler–Lagrange equations of* (1.17)*? Can we characterize the minimizers of* (1.17) *in terms of a system of PDEs?*

Finally, we introduce another problem which we hope will shade some light on Problem (1.17). In Section 3, we will comment on the connections between these problems.

Consider the set of Radon measures γ supported by

$$C = \mathbb{R}^d \times (0, \infty) \times \bar{\Lambda} \times \mathbb{R}^{d \times d}$$

and that satisfy the conditions

$$\int_C b(x)\gamma(dx, dt, du, d\xi) = \int_\Omega b(x)dx \quad \forall\, b \in C_b(\mathbb{R}^d) \tag{1.18}$$

and

$$\int_C \omega(\xi)\gamma(dx, dt, du, d\xi) < \infty. \tag{1.19}$$

We define Γ to be the set of Radon measures on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ supported by C such that not only (1.18) and (1.19) hold, but the conditions

$$\int_C tl(u)\gamma(dx, dt, du, d\xi) = \int_\Lambda l(y)dy \quad \forall\, l \in C_b(\mathbb{R}^d) \tag{1.20}$$

and

$$\int_C \langle \xi, \psi(x)\rangle\gamma(dx, dt, du, d\xi) = -\int_C u \cdot \operatorname{div} \psi(x)\gamma(dx, dt, du, d\xi) \quad \forall\, \psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d}) \tag{1.21}$$

are satisfied. We set

$$\bar{E}(\gamma) = \int_C \Big(\omega(\xi) + h(t) + \frac{|\mathbf{u}_0(x) - u|^2}{2\tau}\Big)\gamma(dt, dx, du, d\xi)$$

Problem 1.3. *Can we characterize the measures γ which minimize $\bar{E}$ over Γ? What are the Euler–Lagrange equations of*

$$\inf_{\gamma\in\Gamma} \bar{E}(\gamma)? \tag{1.22}$$

Under what conditions is the minimizer of (1.22) *unique? What is the dual of* (1.22)*?*

We identify a problem dual to Problem (1.22) and give a geometric characterization of the minimizer of Problem (1.22) is provided. Suppose that $\mathbf{u}_0$ is non degenerate in the sense that it pushes forward the Lebesgue measure restricted to Ω to an absolutely continuous measure. The theory developed in [3] ensures that if the infima in Problems (1.17) and (1.22) coincide, then Problem (1.17) has a unique minimizer.

§2. Review of the classical theory for an existence of a minimizer

The goal of this section is to make the unfamiliar reader learn about what makes things work when dealing with quasiconvex integrands satisfying the standard growth condition in (1.2). To keep the arguments as simple as possible, we consider integrands W in (1.10) that assume a special form

$$W(x, u, \xi) = W_1(\xi) + W_2(x, u). \tag{2.1}$$

Throughout this section, $W_2 : \bar{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ is a Borel map such that $W_2(x, \cdot)$ is continuous for every $x \in \Omega$. We further assume that (1.7) holds and will later make various assumptions on W_1.

Let $C_P > 0$ be a Sobolev constant, in the sense that

$$||\varphi||_{L^p(\Omega)} \leq C_P ||\nabla\varphi||_{L^p(\Omega)} \tag{2.2}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

Let $\bar{\mathbf{u}} \in W_0^{1,p}(\Omega, \mathbb{R}^d)$ and set

$$\mathcal{A} = \bar{\mathbf{u}} + W_0^{1,p}(\Omega, \mathbb{R}^d).$$

Given $\mathbf{u} \in \mathcal{A}$ we have

$$||\mathbf{u} - \bar{\mathbf{u}}||_{L^p(\Omega)} \leq C_P ||\nabla\mathbf{u} - \nabla\bar{\mathbf{u}}||_{L^p(\Omega)}$$

and so, there exists a constant β_p depending on p and $\bar{\mathbf{u}}$, but independent of $\mathbf{u}$ such that

$$||\mathbf{u}||^p_{L^p(\Omega)} \leq C_P^p ||\nabla\mathbf{u}||^p_{L^p(\Omega)} + \beta_p \tag{2.3}$$

2.1. The convex integrand case

In this subsection, we assume that W_1 is a real valued convex function defined on $\mathbb{R}^{d\times d}$ and so, W_1 is locally Lipschitz. Given $\xi^0 \in \mathbb{R}^{d\times d}$, the subdifferential of W_1 at ξ^0, denoted by $\partial W_1(\xi^0)$, is non empty and consists of the set of $p^0 \in \mathbb{R}^{d\times d}$ such that

$$W_1(\xi) \geq W_1(\xi^0) + \langle p^0, \xi - \xi^0\rangle \tag{2.4}$$

for all $\xi \in \mathbb{R}^{d\times d}$. The set $\partial W_1(\xi^0)$ is a convex set which reduces to a single element if and only if W_1 is differentiable at ξ^0. In that case, $\partial W_1(\xi^0) = \{DW_1(\xi^0)\}$.

One checks that $\partial W_1(\xi^0)$ is a compact set and so, the standard theory of convex analysis theory (cf. e.g. [8]) ensures existence of a Borel map $D.W_1 : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ such that

$$D.W_1(\xi^0) \subset \partial W_1(\xi^0)$$

for all $\xi^0 \in \mathbb{R}^{d\times d}$. The fact that $\partial W_1(K)$ is a compact set for every compact set $K \subset \mathbb{R}^{d\times d}$ ensures that $D.W_1 \in L^\infty_{loc}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d})$.

Lemma 2.1. *Suppose W_2 satisfies* (1.6) *and* (1.7)*, $W_1 : \mathbb{R}^{d\times d} \to \mathbb{R}$ is convex and there exists a constant C_0 such that $W_1 \geq -C_0$. Then,*

(i) *the functional I is weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^d)$.*
(ii) *Further assume that* (1.2) *holds and*

$$||a||_{L^\infty(\Omega)} C_P^p < C_0. \tag{2.5}$$

If I is not indentically ∞ on $\mathcal{A}$, then it admits a minimizer on $\mathcal{A}$.

Proof. (i) Weak lower semicontinuity. Let $\{\mathbf{u}_n\}_n \subset W^{1,p}(\Omega, \mathbb{R}^d)$ be a sequence converging weakly to some $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$. We are to prove that

$$\liminf_{n\to\infty} I(\mathbf{u}_n) \geq I(\mathbf{u}). \tag{2.6}$$

Extracting from $\{\mathbf{u}_n\}_n$ a subsequence if necessary, thanks to the Sobolev embedding theorem, we may assume without loss of generality that $\{\mathbf{u}_n\}_n$ converges to $\mathbf{u}$ strongly in $L^p(\Omega, \mathbb{R}^d)$ and pointwise almost everywhere. Observe that $\{|\mathbf{u}_n|^p\}_n$ converges to $|\mathbf{u}|^p$ strongly in $L^1(\Omega, \mathbb{R}^d)$ and so, extracting from $\{\mathbf{u}_n\}_n$ a subsequence, we use the Lebesgue dominated convergence to infer the existence of $v \in L^p(\Omega)$ such that

$$|\mathbf{u}_n| \leq v. \tag{2.7}$$

The continuity property of $W_2(x, \cdot)$ ensures that $\{W_2(x, \mathbf{u}_n(x))\}_n$ converge pointwise almost everywhere to $W_2(x, \mathbf{u}(x))$. By (1.7) and (2.7),

$$|W_2(x, \mathbf{u}_n(x))| \leq a(x)v(x)^p + b(x).$$

Since $av^p + b \in L^1(\Omega)$, we apply the Lebesgue dominated convergence theorem to conclude that

$$\lim_{n\to\infty} \int_\Omega W_2(x, \mathbf{u}_n(x))dx = \int_\Omega W_2(x, \mathbf{u}(x))dx. \tag{2.8}$$

Set

$$\Omega_\epsilon = \Big\{ x \in \Omega \mid \big|DW_1(\nabla \mathbf{u}(x))\big| \leq \epsilon^{-1} \Big\}.$$

Since $W_1 + C_0 \geq 0$ and W_1 is convex, we have

$$
\begin{aligned}
(2.9) \qquad & \int_\Omega \big(W_1(\nabla \mathbf{u}_n) + C_0\big)dx \\
& \geq \int_{\Omega_\epsilon} \big(W_1(\nabla \mathbf{u}_n) + C_0\big)dx \\
& \geq \int_{\Omega_\epsilon} \big(W_1(\nabla \mathbf{u}) + C_0 + \langle D.W_1(\nabla \mathbf{u}), \nabla \mathbf{u}_n - \nabla \mathbf{u}\rangle\big)dx
\end{aligned}
$$

Since

$$\chi_{\Omega_\epsilon} D.W_1(\nabla \mathbf{u}) \in L^\infty(\Omega, \mathbb{R}^{d\times d}) \subset L^q(\Omega, \mathbb{R}^{d\times d}),$$

we exploit in (2.9) the fact that $\{\nabla \mathbf{u}_n\}_n$ converges weakly to $\nabla \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ to conclude that

(2.10)

$$\liminf_{n\to\infty} \int_\Omega W_1(\nabla \mathbf{u}_n)dx + C_0\mathcal{L}^d(\Omega) \geq \int_{\Omega_\epsilon} W_1(\nabla \mathbf{u})dx + C_0\mathcal{L}^d(\Omega_\epsilon).$$

We let ϵ tend to 0 in (2.10) and combine the subsequent inequality with (2.8) to obtain (2.6).

(ii) Existence of a minimizer. Let $\epsilon > 0$ be such that $C_P^p|a| \leq C_0 - \epsilon$ on Ω and let $\mathcal{A}$ be a weakly closed subset of $W^{1,p}(\Omega, \mathbb{R}^d)$ such that I is not indentically ∞ on $\mathcal{A}$. Let $\{\mathbf{u}_n\}_n \subset \mathcal{A}$ be such that

$$\lim_{n\to\infty} I(\mathbf{u}_n) = \inf_{\mathcal{A}} I \quad \text{and} \quad I(\mathbf{u}_n) \leq \inf_{\mathcal{A}} I + 1.$$

Thanks to (1.2), (1.7) and (2.5)

$$\int_\Omega \Big(C_0|\nabla \mathbf{u}_n|^p - C_0 - ||a||_{L^\infty(\Omega)}|\mathbf{u}_n|^p - |b(x)|\Big) \leq \inf_{\mathcal{A}} I + 1$$

and so,

$$\int_\Omega \Big(\big(C_0 - ||a||_{L^\infty(\Omega)}C_P\big)|\nabla \mathbf{u}_n|^p - \big(C_0 + ||a||_{L^\infty(\Omega)}\beta_p + |b(x)|\big)\Big) \leq \inf_{\mathcal{A}} I + 1.$$

Thus,

$$\sup ||\nabla \mathbf{u}_n||_{L^p(\Omega)} < \infty. \qquad (2.11)$$

We combine (2.3) and (2.11) to conclude that $\{\mathbf{u}_n\}_n$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^d)$ and so, it is precompact for the weak topology there. We may assume without loss of generality that $\{\mathbf{u}_n\}_n$ converges weakly to some $\mathbf{u}$ in $W^{1,p}(\Omega, \mathbb{R}^d)$. We have $\mathbf{u} \in \mathcal{A}$ and by (i),

$$I(\mathbf{u}) \leq \liminf_{n\to\infty} I(\mathbf{u}_n) = \inf_{\mathcal{A}} I.$$

Q.E.D.

Lemma 2.2. *Suppose $W_1 \in C^1(\mathbb{R}^{d\times d})$ is convex and* (1.2), (1.4) *hold. Suppose W_2 is a Borel function satisfying* (1.7), (1.8) *and $W_2(x,\cdot)$ is convex. If $\mathbf{u}_0 \in \mathcal{A}$, then the following are equivalent:*

(i) $I(\mathbf{u}_0) \leq I(\mathbf{u})$ *for all* $\mathbf{u} \in \mathcal{A}$.

(ii)

$$-\mathrm{div}\Big(DW_1(\nabla \mathbf{u}_0)\Big) = \nabla_u W_2(\cdot, \mathbf{u}_0) \qquad \mathcal{D}'(\Omega, \mathbb{R}^d). \tag{2.12}$$

Proof. Suppose (i) holds and let $\psi \in C_c^\infty(\Omega, \mathbb{R}^d)$. For $\epsilon \in \mathbb{R}$, setting $\mathbf{u}_\epsilon = \mathbf{u}_0 + \epsilon\psi$, we have that $\mathbf{u}_\epsilon \in \mathcal{A}$ and so, $\epsilon \to f(\epsilon) = I(\mathbf{u}_\epsilon)$ achieves its minimum at 0. We have

$$f(\epsilon) - f(0)$$
$$= \epsilon \int_\Omega \Big(\int_0^1 \Big[\langle DW_1(\nabla \mathbf{u}_0 + t\epsilon \nabla\psi), \nabla\psi\rangle + \nabla_u W_2(x, \mathbf{u}_0 + t\epsilon\psi)\cdot\psi \Big] dt \Big) dx.$$

We use (1.4) and (1.8) to obtain that

$$|DW_1(\nabla \mathbf{u}_0 + t\epsilon\nabla\psi)| + |\nabla_u W_2(x, \mathbf{u}_0 + t\epsilon\psi)|$$
$$\leq 2C_1\Big(|\nabla \mathbf{u}_0 + t\epsilon\nabla\psi|^{p-1} + |\mathbf{u}_0 + t\epsilon\psi|^{p-1} + 2\Big).$$

Then use the fact that ψ and $\nabla\psi$ are uniformly bounded to obtain a constant $C > 0$ such that

$$|\langle DW_1(\nabla \mathbf{u}_0 + t\epsilon\nabla\psi), \nabla\psi\rangle| + |\nabla_u W_2(x, \mathbf{u}_0 + t\epsilon\psi)\cdot\psi|$$
$$\leq C\Big(|\nabla \mathbf{u}_0|^{p-1} + |\mathbf{u}_0|^{p-1} + 1\Big) \in L^1(\Omega).$$

Thus, the expression

$$\int_0^1 \Big(\langle DW_1(\nabla \mathbf{u}_0 + t\epsilon\nabla\psi), \nabla\psi\rangle + \nabla_u W_2(x, \mathbf{u}_0 + t\epsilon\psi)\cdot\psi \Big) dt,$$

which appears above in $f(\epsilon) - f(0)$, is bounded by an integrable function independent of ϵ. Using the fact that 0 minimizes f, we conclude that

$$0 = \lim_{\epsilon\to 0} \frac{f(\epsilon) - f(0)}{\epsilon} = \int_\Omega \Big(\langle DW_1(\nabla \mathbf{u}_0), \nabla\psi\rangle + \nabla_u W_2(x, \mathbf{u}_0)\cdot\psi \Big) dx.$$

Thus, (ii) holds.

Conversely, suppose that (ii) holds and let $\mathbf{u} \in \mathcal{A}$. The growth condition in (1.4) ensures that

$$DW_1(\nabla \mathbf{u}_0) \in L^q(\Omega, \mathbb{R}^{d\times d})$$

and so, by Young's inequality

$$\langle DW_1(\nabla \mathbf{u}_0), \nabla \mathbf{u} - \nabla \mathbf{u}_0 \rangle \in L^1(\Omega).$$

Similarly, (1.8) yields that

$$\nabla_u W_2(x, \mathbf{u}_0) \cdot (\mathbf{u} - \mathbf{u}_0) \in L^1(\Omega).$$

Thanks to the convexity property of W_1, we exploit (2.4) and use an analogous inequality for $W_2(x, \cdot)$ to conclude that

(2.13)
$$I(\mathbf{u}) \geq I(\mathbf{u}_0) + \int_\Omega \Big(\langle DW_1(\nabla \mathbf{u}_0), \nabla \mathbf{u} - \nabla \mathbf{u}_0 \rangle + \nabla_u W_2(x, \mathbf{u}_0) \cdot (\mathbf{u} - \mathbf{u}_0) \Big) dx.$$

We use the fact that $\mathbf{u} - \mathbf{u}_0 \in W_0^{1,p}(\Omega, \mathbb{R}^{d\times d})$ and that (2.12) holds to conclude that

$$0 = \int_\Omega \Big(\langle DW_1(\nabla \mathbf{u}_0), \nabla \mathbf{u} - \nabla \mathbf{u}_0 \rangle + \nabla_u W_2(x, \mathbf{u}_0) \cdot (\mathbf{u} - \mathbf{u}_0) \Big) dx.$$

This, together with (2.13) yields $I(\mathbf{u}) \geq I(\mathbf{u}_0)$ and so, (i) holds. Q.E.D.

2.2. The quasiconvex integrand case

In this subsection, we assume that W_1 is a real valued quasiconvex function defined on $\mathbb{R}^{d\times d}$ (cf. (1.12)). It is well-known that every quasiconvex function is rank-one-convex: given $A \in \mathbb{R}^{d\times d}$, $t \to W_1(A + t\xi)$ is convex for every matrix $\xi \in \mathbb{R}^{d\times d}$ of rank one (cf. e.g. [11]). In particular, W_1 is a convex function of each variable ξ_{ij} and so, W is locally Lipschitz.

Since W_1 is quasiconvex, the sole assumption (1.3) does not guaranty the lower semicontinuity property of I, unlike the case when W_1 is convex. A central result in the calculus of variations, due to Morrey [15], reduces to the following lemma when W assumes the special form in (2.1).

Lemma 2.3. *Suppose the growth conditions in* (1.2), (1.4), (1.5), (1.6) *and* (1.9) *hold. Then,*

(i) *the functional I is weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^d)$.*

(ii) *Further assume that* (1.2) *holds and*

$$||a||_{L^\infty(\Omega)} C_P^p < C_0.$$

If I is not indentically ∞ on $\mathcal{A}$, then it admits a minimizer on $\mathcal{A}$.

Proof. We refer the reader to [11] and [15] for the proof of (i). The proof of (ii) is identical to that of Lemma 2.1 (ii). Q.E.D.

Since W_1 is not assumed to be convex, the full analogue of Lemma 2.2 cannot be expected to hold. In other words, solutions of (2.14) are not known to be minimizers of I over $\mathcal{A}$, and only the following fact is known.

Lemma 2.4. *Suppose $W_1 \in C^1(\mathbb{R}^{d\times d})$,* (1.2), (1.4) *hold and W_2 is a Borel function that satisfies* (1.7) *and* (1.8). *If $\mathbf{u}_0 \in \mathcal{A}$ minimizes I over $\mathcal{A}$, then*

$$-\text{div}\Big(DW_1(\nabla \mathbf{u}_0)\Big) = \nabla_u W_2(\cdot, \mathbf{u}_0) \qquad \mathcal{D}'(\Omega, \mathbb{R}^d). \tag{2.14}$$

Proof. Note that in Lemma 2.2, we have proven that (i) implies (ii) without using the any convexity property of W_1 or $W_2(x, \cdot)$. Q.E.D.

§3. Minimization involving a polyconvex integrand

Let ω be a strictly convex differentiable function defined on the set of $d \times d$ matrices and let $h \in C^1(0, \infty)$ be a convex functional such that

$$\lim_{t\to 0^+} h(t) = \lim_{t\to\infty} \frac{h(t)}{t} = \infty.$$

Define

$$W_1(A) = \begin{cases} \omega(A) + h(\det A) & \text{if} \quad \det A > 0 \\ \infty & \text{if} \quad \det A \leq 0. \end{cases} \tag{3.1}$$

Let $\mathcal{A}_0$ be the set of Borel maps $\mathbf{u} : \Omega \to \Lambda$ such that $\mathbf{u} \in W^{1,p}(\Omega, \Lambda)$ and $\Lambda \setminus \mathbf{u}(\Omega)$ is of null measure. Set

$$I_\tau[\mathbf{u}] := \int_\Omega \Big(W_1(\nabla \mathbf{u}) + \frac{1}{2\tau}|\mathbf{u}_0 - \mathbf{u}|^2\Big) dx.$$

In other words, we have set

$$W_2(x, u) = \frac{1}{2\tau}|\mathbf{u}_0(x) - u|^2$$

The current state of the art in the calculus of variations dramatically fails to predict whether or not I_T has minimizers over $\mathcal{A}_0$. Indeed, $\mathcal{A}_0$ is not closed in the weak topology and $W_1(A)$ fails to be bounded above by an expression of the form $C_1(|A|^p + 1)$, and so, the known methods (cf. e.g. [1], [4], [15]) cannot be applied. Even if we assume we could find a minimizer, the Euler–Lagrange equations it satisfies remains an open outstanding problem in the calculus of variations (cf. e.g., [5], [6], [7]), due to the fact that $\lim_{t\to 0^+} h(t) = \infty$.

3.1. A relaxation into Young type measures

We next show that in some sense, Problem (1.22) is a relaxation of Problem (1.17).

Since by (1.1) the Lebesgue measure of Ω has been normalized to 1, (1.18) ensures that every measure $\gamma \in \Gamma$ is a probability measure. We apply the disintegration theorem to γ (cf. [10] III–70) to obtain a Borel map $x \to \gamma^x$ of Ω into the set of Borel probability measures on

$$D = (0, \infty) \times \bar{\Lambda} \times \mathbb{R}^{d\times d}$$

such that

$$\int_C L(x,t,u,\xi)\gamma(dx,dt,du,d\xi) = \int_\Omega dx \int_D L(x,t,u,\xi)\gamma^x(dt,du,d\xi)$$

for all $L \in C_c(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d})$. Thanks to (1.19), we may apply Jensen's inequality to deduce that the Borel map

$$x \to U_\gamma(x) = \int_D \xi L(x,t,u,\xi)\gamma^x(dt,du,d\xi)$$

is well-defined almost everywhere. Since ω is convex, using Jensen's inequality we have

$$\begin{aligned}\int_\Omega \omega(U_\gamma(x))dx &\le \int_\Omega \Big(\int_D \omega(\xi)\gamma^x(dt,du,d\xi)\Big)dx \\ &= \int_C \omega(\xi)\gamma(dx,dt,du,d\xi) < \infty.\end{aligned}$$

This, together with the growth condition (1.14) on ω implies that $U_\gamma \in L^p(\Omega, \mathbb{R}^{d\times d})$. Similarly, the fact that the support of γ in the u variables is contained in $\bar{\Omega}^*$, which is a convex set, is used to show that the Borel function

$$x \to \mathbf{u}_\gamma(x) = \int_D uL(x,t,u,\xi)\gamma^x(dt,du,d\xi)$$

maps Ω into $\bar{\Lambda}$ up to a set of zero measure.

Note that (1.21) implies $\mathbf{u}_\gamma \in W^{1,p}(\Omega, \mathbb{R}^d)$ and $\nabla \mathbf{u}_\gamma \equiv U_\gamma$. In conclusion, there is a natural embedding from the set

$$\Gamma_0 = \{(\beta, \mathbf{u}) \mid \mathbf{u} \in W^{1,p}(\Omega, \Omega^*), \mathbf{u}_{\#}\beta = \chi_{\Omega^*}\}$$

to the set Γ. Indeed, define the measure $\gamma_{(\beta,\mathbf{u})}$ as follows: if $L \in C_c(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d})$, then

$$\int_C L(x,t,u,\xi)\gamma_{(\beta,\mathbf{u})}(dx,dt,du,d\xi) = \int_\Omega L(x,\beta(x),\mathbf{u}(x),\nabla\mathbf{u}(x))dx.$$

Observe that if $(\beta, \mathbf{u}) \in \Gamma_0$, then $\gamma_{(\beta,\mathbf{u})} \in \Gamma$ and

$$E(\beta,\mathbf{u}) = \bar{E}(\gamma_{(\beta,\mathbf{u})}).$$

Consequently,

$$\inf_{(\beta,\mathbf{u})\in\mathcal{V}} E(\beta,\mathbf{u}) = \inf_{(\beta,\mathbf{u})\in\Gamma_0} \bar{E}(\gamma_{(\beta,\mathbf{u})}).$$

This shows that Problem (1.22) is a relaxation of Problem (1.17).

3.2. Duality result; Characterization of optima; Uniqueness

Let μ_0 be the restriction of the Lebesgue measure to Ω, let ω^* be the Legendre transform of ω and recall that $q = p/(p-1)$. Suppose $\mathbf{u}_0 \in L^1(\Omega, \mathbb{R}^d)$ is a Borel map. Let $\mathcal{C}$ be the set of (k, l, ψ) such that $\psi \in L^q(\Omega, \mathbb{R}^{d\times d})$, $k \in C(\mathbb{R}^d)$ is convex, $l \in C(\Lambda)$ and

$$k(v) + t\tau l(u) + \tau h(t) + \frac{1}{2}|u|^2 \geq u \cdot v$$

for all $(t,u,v) \in (0,\infty) \times \Lambda \times \mathbb{R}^d$. In particular, given $\psi \in L^q(\Omega, \mathbb{R}^{d\times d})$ and $l \in C(\Lambda)$, setting $k = l^\sharp$ where

$$l^\sharp(v) = \sup_{u\in\Lambda, t>0} \{u\cdot v - t\tau l(u) - \tau h(t) - \frac{1}{2}|u|^2\},$$

we have that $(k,l,\psi) \in \mathcal{C}$.

Set

$$J(k,l,\psi) = \int_\Lambda l(y)dy + \int_\Omega \Big(\tau\omega^*\Big(\frac{\psi}{\tau}\Big) + k(\mathbf{u}_0 + \operatorname{div}\psi)\Big)dx.$$

Theorem 3.1. *We have the duality result*

$$\tau \min_{\gamma\in\Gamma} \bar{E}(\gamma) = \frac{1}{2}||\mathbf{u}_0||^2_{L^2(\Omega)} + \max_{(k,l,\psi)\in\mathcal{C}} -J(k,l,\psi).$$

Theorem 3.2. *Suppose that* $\mathbf{u}_0$ *is non degenerate in the sense that* $\mathbf{u}_0\,_{\#}\mu_0$ *is absolutely continuous.*

(i) *If* γ_0 *and* γ_0^* *minimize* $\bar{E}$ *over* Γ, *then* $\mathbf{u}_{\gamma_0} = \mathbf{u}_{\gamma_0^*}$.

(ii) *If there is no gap between problems* (1.17) *and* (1.22) *in the sense that*

$$\inf_{(\beta,\mathbf{u})\in\mathcal{U}} E(\beta,\mathbf{u}) = \inf_{\gamma\in\Gamma} \bar{E}(\gamma),$$

then there is a unique $(\mathbf{u}^*,\beta^*)$ *which minimizes* $E(\beta,\mathbf{u})$ *over* $\mathcal{U}$.

(iii) *When* (ii) *holds, the Euler–Lagrange equations satisfied by* $(\mathbf{u}^*, \beta^*)$ *is the following complete characterization of minimizers: there exists* $(k,l,\psi)\in\mathcal{C}$ *such that*

$$\mathbf{u}^* = \nabla k(\mathbf{u}_0 + \operatorname{div}\psi), \quad \tau h'(\beta^*) = -l(\mathbf{u}^*), \quad \psi = \tau D\omega(\nabla\mathbf{u}^*), \quad k = l^{\sharp}.$$

Theorems 3.1 and 3.2 can be proven by exploiting the theory developed in the work in progress [3].

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