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On the computation of algebraic local cohomology classes associated with semi-quasihomogeneous singularities

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Abstract.

In this paper, a new effective algorithm for computing algebraic local cohomology classes associated with semi-quasihomogeneous singularities, is presented. The key ingredients of the proposed algorithm are weighted-degrees and Poincaré polynomials for algebraic local cohomology. An extension of the algorithm to parametric cases is also discussed.

§1. Introduction

A new algorithm for computing algebraic local cohomology associated with semi-quasihomogeneous singularities is proposed. Local cohomology is key ingredients in algebraic geometry and commutative algebra, and hence provides fundamental tools for applications in several fields both inside and outside mathematics [2], [7], [10]. For isolated singularity cases, the concept of algebraic local cohomology is also useful to analyze properties of singularities [4], [17], [25]. Therefore, it is meaningful to provide an efficient algorithm for computing algebraic local cohomology classes. In the previous works [18], [21], the computational methods for computing algebraic local cohomology classes associated with isolated singularities, were introduced. As a byproduct, a new efficient method to compute standard bases of zero-dimensional ideals was constructed [21]. These algorithms have also been used to analyze holonomic $\mathcal{D}_X$-modules attached to hypersurface isolate singularities [13], [19], [20].

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In this paper, we consider a computation method for the cases of semi-quasihomogeneous singularities. There are close relations between properties of semi-quasihomogeneous singularities and their quasihomogeneous parts [1], [16], [23]. Some of properties of semi-quasihomogeneous singularities are determined by only their quasihomogeneous parts [5], [26]. However, a lot of analytic properties of semi-quasihomogeneous singularities are not decided by only their quasihomogeneous parts [3], [6], [8], [9]. In order to analyze such subtle phenomena, it is desirable to construct an algorithm of computing algebraic local cohomology classes with respect to a monomial order compatible with a weight filtration. For semi-quasihomogeneous singularities, the list of weighted-degrees of a basis of algebraic local cohomology classes, is completely determined by a Poincaré polynomial associated with a weight vector. By exploiting this result, we are able to construct an efficient and simple algorithm for computing algebraic local cohomology classes. In 1996, Traverso [22] gave, among other results, an algorithm of computing standard bases that utilize Poincaré polynomials to detect unnecessary computation of S-polynomials. In contrast, the proposed algorithm that is based on the Grothendieck local duality theorem, utilize Poincaré polynomial more directly to construct algebraic local cohomology classes.

The proposed algorithm has been implemented in a computer algebra system. Computational experiments in this paper suggest that the proposed algorithm is superior in practice in comparison to other existing algorithm.

The paper organized as follows. Section 2 briefly reviews algebraic local cohomology, and gives notations and definitions that used in this paper. Section 3 is the discussion of the new algorithm for semi-quasihomogeneous singularities. Section 4 illustrates the possibility to generalize the new algorithm to the parametric cases.

§2. Preliminaries

In this section, first we briefly review a Čech cohomology representation of algebraic local cohomology classes. We introduce "polynomial representation" to treat, on computer, Čech cohomology classes suitably. Second, we introduce notions of weighted degrees and Poincaré polynomials, which will be exploited several times in this paper.

2.1. Algebraic local cohomology

We use the notation x as the abbreviation of n variables $x_1, \dots, x_n$. The set of natural numbers $\mathbb{N}$ includes zero. K is the field of rational numbers $\mathbb{Q}$ or the field of complex numbers $\mathbb{C}$.

Let $H^n_{[O]}(K[x])$ denote the set of algebraic local cohomology classes, defined by

$$H^n_{[O]}(K[x]) := \lim_{k\to\infty} Ext^n_{K[x]}(K[x]/\langle x_1, x_2, .., x_n\rangle^k, K[x])$$

where $\langle x_1, x_2, \ldots, x_n\rangle$ is the maximal ideal generated by $x_1, x_2, \ldots, x_n$.

Let X be a neighbourhood of the origin O of $\mathbb{C}^n$. Consider the pair $(X, X - O)$ and its relative Čech covering. Then, any section of $H^n_{[O]}(K[x])$ can be represented as an element of relative Čech cohomology. We use the notation $\sum c_\lambda \left[\frac{1}{x^{\lambda+1}}\right]$ for representing algebraic local cohomology classes in $H^n_{[O]}(K[x])$ where $c_\lambda \in K$, $x^{\lambda+1} = x_1^{\lambda_1+1}x_2^{\lambda_2+1}\cdots x_n^{\lambda_n+1}$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n$. Note that the multiplication is defined as

$$x^\alpha \left[\frac{1}{x^{\lambda+1}}\right] = \begin{cases} \left[\dfrac{1}{x^{\lambda+1-\alpha}}\right] & \lambda_i \geq \alpha_i, i = 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\lambda+1-\alpha = (\lambda_1+1-\alpha_1, \ldots, \lambda_n+1-\alpha_n)$.

We represent an algebraic local cohomology class $\sum c_\lambda \left[\frac{1}{x^{\lambda+1}}\right]$ as a n variables polynomial $\sum c_\lambda \xi^\lambda$ to manipulate algebraic local cohomology classes efficiently (on computer), where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. We call this representation "**polynomial representation**". For example, let $\psi = \left[\frac{4}{x^3y^4}\right] + \left[\frac{5}{x^2y^3}\right]$ be an algebraic local cohomology class where x, y are variables. Then, the polynomial representation of ψ, is $4\xi^2\eta^3 + 5\xi\eta^2$ where variables (ξ, η) are corresponding to variables (x, y). That is, we have the following table for n variables:

Čech representation		**polynomial representation**
$\sum c_\lambda \left[\dfrac{1}{x_1^{\lambda_1+1}x_2^{\lambda_2+1}\cdots x_n^{\lambda_n+1}}\right]$	$\longleftrightarrow$	$\sum c_\lambda \xi_1^{\lambda_1}\xi_2^{\lambda_2}\cdots\xi_n^{\lambda_n}$

where $c_\lambda \in K$. The multiplication for polynomial representation is defined as follows:

$$x^\alpha * \xi^\lambda = \begin{cases} \xi^{\lambda-\alpha} & \lambda_i \geq \alpha_i, i = 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, and $\lambda - \alpha = (\lambda_1 - \alpha_1, \ldots, \lambda_n - \alpha_n)$. (We use " $*$ " for polynomial representation.)

After here, we adapt polynomial representation to represent an algebraic local cohomology class.

2.2. Weighted-degrees and Poincaré polynomials

Let us fix a **weight** vector $\mathbf{w} = (w_1, w_2, \dots, w_n)$ in $\mathbb{N}^n$ for a fixed coordinate system $x = (x_1, \dots, x_n)$.

Definition 2.1 ([1]). (1) We define a **weighted degree** of the monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, with respect to $\mathbf{w}$ by $|x^\alpha|_{\mathbf{w}} := \sum_{i=1}^n w_i\alpha_i$.

(2) A nonzero polynomial $f \in K[x]$ is **quasihomogeneous of type** $(d;\mathbf{w})$ if all monomials of f have the same weighted degree d with respect to $\mathbf{w}$, i.e., $f = \sum_{|x^\alpha|_{\mathbf{w}}=d} c_\alpha x^\alpha$ where $c_\alpha \in K$. Also, we define a weighted degree of f by

$$\deg_{\mathbf{w}}(f) := \max\{|x^\alpha|_{\mathbf{w}} : x^\alpha \text{ is a monomial of } f\}.$$

(3) Let $f \in K[x]$ be a polynomial. We define $\mathrm{ord}_{\mathbf{w}}(f) = \min\{|x^\alpha|_{\mathbf{w}} : x^\alpha \text{ a monomial of } f\}$ ($\mathrm{ord}_{\mathbf{w}}(0) := -1$). The polynomial f is called **semi-quasihomogeneous of type** $(d;\mathbf{w})$ if f is of the form $f = f_0 + g$ where f_0 is a quasihomogeneous polynomials of type $(d;\mathbf{w})$ with an isolated singularity at the origin, $f = f_0$ or $\mathrm{ord}_{\mathbf{w}}(f - f_0) > d$.

In the next definition, we recall a Poincaré polynomial for the Jacobi ideal of qusihomogeneous function f_0, which plays a key role in our algorithm.

Definition 2.2 ([1],[15]). The **Poincaré polynomial** of type $(d;\mathbf{w})$ is defined by

$$P_{(d;\mathbf{w})}(t) = \frac{t^{d-w_1}-1}{t^{w_1}-1} \cdot \frac{t^{d-w_2}-1}{t^{w_2}-1} \cdots \frac{t^{d-w_n}-1}{t^{w_n}-1}.$$

The Poincaré polynomial of type $(d;\mathbf{w})$ is a polynomial over $\mathbb{N}$. We give an example of a Poincaré polynomial. A polynomial $f = x^3y + xy^4$ is a quasihomogeneous polynomial of type $(11;(3,2))$. Then, the Poincaré polynomial of type $(11;(3,2))$ is

$$\begin{aligned} P_{(11;(3,2))}(t) &= \frac{t^{11-3}-1}{t^3-1} \cdot \frac{t^{11-2}-1}{t^2-1} \\ &= 1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + t^8 + t^9 + t^{10} + t^{12}. \end{aligned}$$

Throughout this paper, we use the following monomial order to compute algebraic local cohomology classes.

Definition 2.3 (a weighted monomial order)**.** For two multi-indices $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ in $\mathbb{N}^n$, we denote

$$\xi^{\lambda'} \prec \xi^{\lambda} \text{ or } \lambda' \prec \lambda$$

if $|\xi^{\lambda'}|_{\mathbf{w}} < |\xi^{\lambda}|_{\mathbf{w}}$, or if $|\xi^{\lambda'}|_{\mathbf{w}} = |\xi^{\lambda}|_{\mathbf{w}}$ and there exists $j \in \mathbb{N}$ so that $\lambda'_i = \lambda_i$ for $i < j$ and $\lambda'_j < \lambda_j$.

For a given algebraic local cohomology class h of the form (in polynomial representation),

$$h = c_\lambda \xi^\lambda + \sum_{\lambda' \prec \lambda} c_{\lambda'} \xi^{\lambda'}, \ c_\lambda \neq 0$$

we call ξ^λ the **head monomial** and $\xi^{\lambda'}$, $\lambda' \prec \lambda$ the **lower monomials**. We denote the head monomial of a cohomology class h by $\mathrm{hm}(h)$. For a given algebraic local cohomology class of the form

$$\sum_{|\xi^\lambda|_{\mathbf{w}} = a} c_\lambda \xi^\lambda + \sum_{|\xi^{\lambda'}|_{\mathbf{w}} < a} c_{\lambda'} \xi^{\lambda'}$$

we call $\sum_{|\xi^\lambda|_{\mathbf{w}} = a} c_\lambda \xi^\lambda$ the **head part** and $\sum_{|\xi^{\lambda'}|_{\mathbf{w}} < a} c_{\lambda'} \xi^{\lambda'}$ the **lower part**.

§3. Computation of algebraic local cohomology

Let $f := f_0 + g$ be a semi-quasihomogeneous polynomial of type $(d; \mathbf{w})$ where f_0 is a quasihomogeneous polynomial of type $(d; \mathbf{w})$ and defines an isolated singularity at the origin.

We define a vector space H_f to be the set of algebraic local cohomology classes, in polynomial representation, in $K[\xi]$ that are annihilated by Jacobi ideal $\langle \frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \rangle$,

$$H_f := \left\{ h \in K[\xi] \;\middle|\; \frac{\partial f}{\partial x_1}(x) * h = \frac{\partial f}{\partial x_2}(x) * h = \cdots = \frac{\partial f}{\partial x_n}(x) * h = 0 \right\}.$$

In this section we describe an algorithm for computing a basis of the vector space H_f.

The new algorithm consists of the following two parts.

(1) Compute a basis Q of H_{f_0} by using a Poincaré polynomial.

(2) Compute a basis of H_f by using the result Q.

First, we consider how to obtain a basis Q of a vector space H_{f_0}. In order to construct an algorithm for computing the basis Q, we require two lemmas.

Let us recall the following lemma which follows from the fact that if $h \in H_{f_0}$, so is $x_i * h \in H_{f_0}$ for each $i = 1, 2, \ldots, n$.

Lemma 1 ([21]). *Let Λ_{H_0} denote the set of exponents of head monomials in H_{f_0} and $\Lambda_{H_0}^{(\lambda)}$ denote a subset of Λ_{H_0}:*
$\Lambda_{H_0} = \{\lambda \in \mathbb{N}^n | \exists h \in H_{f_0}$ such that $\mathrm{hm}(h) = \xi^\lambda\}$ and $\Lambda_{H_0}^{(\lambda)} = \{\lambda' \in \Lambda_{H_0} | \lambda' \prec \lambda\}$.
Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$. **(C):** *"If $\lambda \in \Lambda_{H_0}$, then, for each $j = 1, 2, \ldots, n$, $(\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \ldots, \lambda_n)$ is in $\Lambda_H^{(\lambda)}$, provided $\lambda_j \geq 1$."*

The condition above, denoted by (C), is used in Algorithm 1 to select candidates of head monomials.

Let $P_{(d;\mathbf{w})}(t) = \sum_{i=1}^{\ell} m_i t^{d_i}$ be the Poincaré polynomial of type $(d; \mathbf{w})$ where $m_i \in \mathbb{N}$ and Q be a set in $K[\xi]$. We introduce the set D_P of weighted-degrees as

$$D_P := \bigcup_{i=1}^{\ell} \{\underbrace{d_i, d_i, \ldots, d_i}_{m_i \text{ elements}}\}.$$

For a subset Q in $K[\xi]$, we define D_Q to be the set of weighted-degrees of elements of Q:

$$D_Q := \{\deg_{\mathbf{w}}(q) \in \mathbb{N} | q \in Q\}.$$

The vector space H_{f_0} is the dual space of $K[[x]]/\mathcal{J}_0$, where $K[[x]]$ stands for the space of formal power series with coefficients in K and $\mathcal{J}_0 = \langle \frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n} \rangle$. The duality is induced by the Grothendieck local residue paring $K[[x]]/\mathcal{J}_0 \times H_{f_0} \to K$. The following lemma, given in [12], follows immediately from the non-degenerateness of the Grothendieck local residue paring.

Lemma 2 ([12]). *There exists a basis Q of H_{f_0} which satisfies the following conditions*

(1) *Q consists of quasihomogeneous polynomials.*
(2) *$D_Q = D_P$.*

Lemma 2 together with Lemma 1 allows us to design an efficient algorithm to compute a basis Q of H_{f_0}.

As a Poincaré polynomial gives us the set D_Q by Lemma 2, we only select monomials whose weighted-degrees belong to D_Q, to construct a basis of H_{f_0}. Namely, a number of selecting monomials becomes less than our previous algorithm's one [21] (non-special cases). In Algorithm

1, LL means the set of these selected monomials. Since all monomial elements of the basis are found first in Algorithm 1, every element in LL is not a monomial element of the basis. Therefore, we must find an element whose form is $\sum c_\lambda \xi^\lambda$. As the head monomial is not known, we must check whether every monomial in LL is the head monomial or not.

Now, we are ready to introduce a new algorithm for computing a basis of H_{f_0}.

Theorem 3. *The following algorithm outputs a basis of H_{f_0} and terminates.*

Algorithm 1. (a basis of H_{f_0}).

Input: f_0: a quasihomogeneous polynomial, $\mathbf{w}$, $\prec$,
Output: Q: a basis of H_{f_0} (in polynomial representation).
BEGIN
$G \leftarrow \{x^\alpha |$ a monomial of $\frac{\partial f_0}{\partial x_i}$ for each $1 \le i \le n\}$; $G' \leftarrow \{\xi^\alpha | x^\alpha \in G\}$
$Q \leftarrow$ Compute the monomial basis of $K[\xi]/\langle G' \rangle$. (*1)
$D_Q \leftarrow \{\deg_{\mathbf{w}}(q) | q \in Q\}$
$D_P \leftarrow$ Compute a set D_P (of Lemma 2 (2)). ; $D \leftarrow D_P \backslash D_Q$
While $D \neq \emptyset$ **Do**
  $N \leftarrow$ All minimal elements of D ; $D \leftarrow D \backslash N$; $k \leftarrow$ Select an element from N.
  LL $\leftarrow \{\xi^\lambda | \deg_{\mathbf{w}}(\xi^\lambda) = k \ \wedge \ \xi^\lambda \notin Q\}$
  $L \leftarrow$ Select the lowest and second lowest elements in LL w.r.t. $\prec$.
  LL $\leftarrow$ LL $\backslash L$; $j \leftarrow \sharp N$ (cardinality of N)
    while $j \neq 0$ **do**
    $\xi^\lambda \leftarrow$ Take the highest element in L w.r.t $\prec$
    If λ satisfies the condition (C) **then**
    $u \leftarrow$ Set $\xi^\lambda + \sum_{\lambda' \in L \backslash \{\xi^\lambda\}, \lambda' \prec \lambda} c_{\lambda'} \xi^{\lambda'}$ where $c_{\lambda'}$ is an undetermined coefficient. (*2)

Make a system of $c^{\lambda'}$s linear equations from the condition $\frac{\partial f_0}{\partial x_1} * u = \frac{\partial f_0}{\partial x_2} * u = \cdots = \frac{\partial f_0}{\partial x_n} * u = 0$, and solve the system.	(*3)

      if the solution exists **then**
    $q \leftarrow$ Substitute the solution into $c_{\lambda'}$s of u. ; $Q \leftarrow Q \cup \{q\}$; $L \leftarrow L \backslash \{\xi^\lambda\}$; $j \leftarrow j - 1$
      end-if
    end-If
    $\xi^{\lambda'} \leftarrow$ Take the lowest element in LL w.r.t. $\prec$. ; $L \leftarrow L \cup \{\xi^{\lambda'}\}$; LL $\leftarrow$ LL $\backslash \{\xi^{\lambda'}\}$
    end-while
end-While
return Q
END

(*Termination*) As f_0 defines isolated singularity at the origin, the Jacobi ideal $\langle \frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n} \rangle$ is zero-dimensional. Hence, D_Q and D_P have finte elements. By this fact and Lemma 2 (2), D is a finite set. Theorefore, obviously, two **while-loops** must be terminated. This implies that this algorithm terminates.

(*Corectness*) At $(*1)$, every element of Q is monomial whose degree is lower than an arbitrary element of G'. This means that for all $q \in Q$, $\frac{\partial f_0}{\partial x_i} * q = 0$ for each $i = 1, \ldots, n$, which yields that Q is included in a basis of H_{f_0}. At $(*3)$, an each proper linear form of an algebraic local cohomology class, is determined. Hence, by Lemma 2, this algorithm outputs a basis of H_{f_0}.

In [21] we have already intoroduced conditions of head monomials and lower monomials. In order to keep the presentation simple, we have deliberately avoided tricks and optimizations such as applying the conditions at $(*2)$. All the tricks suggested in [21] can be used here as well. In fact, our implementations fully incorporate these optimizations.

We give an example to illustrate Algorithm 1.

Example 4. *A polynomial* $f = x^3y + x^2y^3 + y^7 + 2y^8 \in K[x, y]$ *is semi-quasihomogeneous polynomial of type* $(7; (2, 1))$. *Set* $f_0 = x^3y + x^2y^3 + y^7$ *which is quasihomogeneous, and* $g = 2y^8$. *Consider a basis of* H_{f_0}. *The partial derivatives of* f_0 *are* $\frac{\partial f_0}{\partial x} = 3x^2y + 2xy^3$ *and* $\frac{\partial f_0}{\partial y} = x^3+3x^2y^2+7y^6$. *Then, the monomial basis of* $K[\xi, \eta]/\langle \xi^2\eta, \xi\eta^3, \xi^3, \xi^2\eta^2, \eta^6\rangle$ *is given by the following ten monomials*

$$Q = \{1, \eta, \eta^2, \xi^2, \eta^3, \xi^2\eta, \eta^4, \xi\eta^2, \xi^2, \eta^5\}$$

where variables (ξ, η) *are corresponding to variables* (x, y). *These monomials can be easily computed from the reduced Gröbner basis of* $\langle \xi^2\eta, \xi\eta^3, \xi^3, \xi^2\eta^2, \eta^6\rangle$. **Fig. 1** *means the set of weighted-degrees of* Q, *i.e.,* $D_Q = \{0, 1, 2, 2, 3, 3, 4, 4, 4, 5\}$. *Since the Poincaré polynomial of type* $(7; (2, 1))$ *is*

$$P_{(7;(2,1))}(t) = \frac{t^{7-2} - 1}{t^2 - 1} \cdot \frac{t^{7-1} - 1}{t - 1} = 1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8,$$

we obtain $D_P = \{0, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 7, 8\}$. *As* $D = D_P \backslash D_Q = \{5, 6, 6, 7, 8\}$, *we only need to consider monomials whose weighted degrees belong to* D, *for obtaining five more elements of the basis.*

The minimal element in D *is* 5. $N = \{5\}$ *and* $D = D\backslash\{5\} = \{6, 6, 7, 8\}$. *In* **Fig. 2,** $\bullet$ *means an element in* Q *and a number means its weighted-degree which is lower than* 8. *We must select monomials whose weighted degree is* 5. (*We do not need to consider monomials in* Q.) *Then,* LL $= \{\xi\eta^3, \xi^2\eta\}$ *and* $L = \{\xi\eta^3, \xi^2\eta\}$. (*By Lemma 2, we need only quasihomogeneous polynomials as elements of the basis.*) *As* $\xi\eta^3 \prec \xi^3\eta$ *and* $\xi^2\eta$ *satisfies the condition* (C), *set* $u = \xi^3\eta + c_{(1,3)}\xi\eta^3$ *and check* $\frac{\partial f_0}{\partial x} * u = 3+2c_{(1,3)} = 0$, $\frac{\partial f_0}{\partial y} * u = 0$. *Then, we get the solution* $c_{(1,3)} = -\frac{3}{2}$. *Therefore,* $\underline{\xi^2\eta - \frac{3}{2}\xi\eta^3}$ *is a member of the basis.*

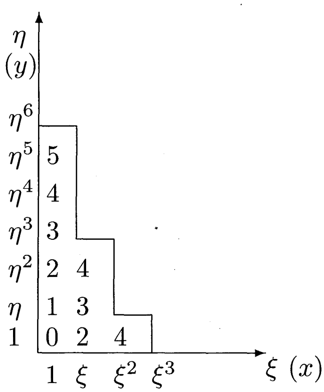

Fig. 1

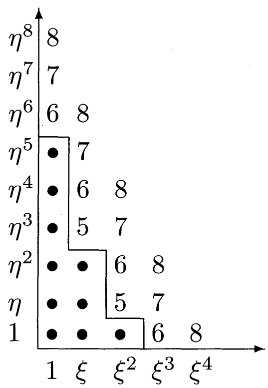

Fig. 2

The minimal elements in D are 6*. $N = \{6, 6\}$ and $D = D\backslash\{6, 6\} = \{7, 8\}$. We must select monomials whose weighted degree is* 6*. Then,* $\mathrm{LL} = \{\eta^6, \xi\eta^4, \xi^2\eta^2, \xi^3\}$ *and $L = \{\eta^6, \xi\eta^4\}$. In this case, $\xi\eta^4$ does not satisfy the condition* (C)*. Thus, we renew the set L as $\{\eta^6, \xi\eta^4\} \cup \{\xi^2\eta^2\}$. Set $u = \xi^2\eta^2 + c_{(1,4)}\xi\eta^4 + c_{(0,6)}\eta^6$ and check $\frac{\partial f_0}{\partial x} * u = 3\eta + 2c_{(1,4)}\eta = 0$, $\frac{\partial f_0}{\partial y} * u = 3 + 7c_{(0,6)} = 0$. The system of linear equations*

$$3 + 2c_{(1,4)} = 0, \quad 3 + 2c_{(0,6)} = 0$$

has the solution $\{c_{(1,4)} = -\frac{3}{2}, c_{(1,7)} = -\frac{3}{7}\}$. Hence, $\underline{\xi^2\eta^2 - \frac{3}{2}\xi\eta^4 - \frac{3}{7}\eta^6}$ is a member of the basis. We need one more element whose weighted degree of the head monomial is 6*. We also renew L as $(L\backslash\{\xi^2\eta^2\}) \cup \{\xi^3\}$. As ξ^3 satisfies the condition* (C)*, set $u = \xi^3 + c_{(1,1)}\xi\eta^4 + c_{(0,6)}\eta^6$ and check $\frac{\partial f_0}{\partial x} * u = 2c_{(1,4)}\eta = 0$, $\frac{\partial f_0}{\partial y} * u = 1 + 7c_{(0,6)} = 0$. Then, we obtain the solution $\{c_{(1,4)} = 0, c_{(0,6)} = -\frac{1}{7}\}$, and $\underline{\xi^3 - \frac{1}{7}\eta^6}$ is a member of the basis.*

The minimal element in D is 7*. $N = \{7\}$ and $D = D\backslash\{7\} = \{8\}$. We must select monomials whose weighted degree is* 7*. Then,* $\mathrm{LL} = \{\eta^7, \xi\eta^5, \xi^2\eta^3, \xi^3\eta\}$ *and $L = \{\eta^7, \xi\eta^5\}$. As $\xi\eta^5$ does not satisfy the condition* (C)*, we renew the set L as $\{\eta^7, \xi\eta^5, \xi^2\eta^3\}$. In this case, $\xi^2\eta^3$ does not satisfy the condition* (C)*, again. Renew the set L as $\{\eta^7, \xi\eta^5, \xi^2\eta^3, \xi^3\eta\}$. Since, $\xi^3\eta$ satisfies the condition* (C)*, set $u = \xi^3\eta + c_{(2,3)}\xi^2\eta^3 + c_{(1,5)}\xi\eta^5 + c_{(0,7)}\eta^7$ and check $\frac{\partial f_0}{\partial x} * u = 3\xi + 3c_{(2,3)}\eta^3 + 2c_{(2,3)}\xi + 2c_{(1,5)}\eta^2 = 0$, $\frac{\partial f_0}{\partial y} * u = \eta + 3c_{(2,3)}\eta + 7c_{(0,7)}\eta = 0$. The solution of the system of linear equations*

$$3 + 2c_{(2,3)} = 0, \ 3c_{(2,3)} + 2c_{(1,5)} = 0, \ 1 + 3c_{(2,3)} + 7c_{(0,7)} = 0$$

is $\{c_{(2,3)} = -\frac{3}{2}, c_{(1,5)} = -\frac{9}{4}, c_{(0,7)} = \frac{1}{2}\}$. *Thus, a member of the basis is* $\underline{\xi^3\eta - \frac{3}{2}\xi^2\eta^3 + \frac{9}{4}\xi\eta^5 + \frac{1}{2}\eta^7}$.

The minimal element in D *is* 8. $N = \{8\}$ *and* $D = D\backslash\{8\} = \emptyset$. *We must select monomials whose weighted degree is* 8. *Then,* $\mathrm{LL} = \{\eta^8, \xi\eta^6, \xi^2\eta^4, \xi^3\eta^2, \xi^4\}$. *As the monomials* $\eta^8, \xi\eta^6, \xi^2\eta^4$ *do not satisfy the condition* (C), *set* $L = \{\eta^8, \xi\eta^6, \xi^2\eta^4\}$, $u = \xi^3\eta^2 + c_{(2,4)}\xi^2\eta^4 + c_{(1,6)}\xi\eta^6 + c_{(0,8)}\eta^8$ *and check* $\frac{\partial f_0}{\partial x} * u = 0$, $\frac{\partial f_0}{\partial y} * u = 0$. *In this case,* u *is not a member of the basis. We renew* L *as* $\{\eta^8, \xi\eta^6, \xi^2\eta^4, \xi^3\eta^2\}$. *Then, we obtain* $\underline{\xi^4 - \frac{4}{75}\xi^2\eta^3 + \frac{2}{25}\xi^2\eta^4 - \frac{3}{25}\xi^2\eta^6 - \frac{2}{75}\eta^8}$ *which is a member of the basis. As* $D = \emptyset$, *the computation terminates.*

Conclusively, we obtain a basis of H_{f_0} *as follows:*
$\{1, \eta, \eta^2, \xi^2, \eta^3, \xi^2\eta, \eta^4, \xi\eta^2, \xi^2, \eta^5, \xi^2\eta - \frac{3}{2}\xi\eta^3, \xi^2\eta^2 - \frac{3}{2}\xi\eta^4 - \frac{3}{7}\eta^6, \xi^3 - \frac{1}{7}\eta^6, \xi^3\eta - \frac{3}{2}\xi^2\eta^3 + \frac{9}{4}\xi\eta^5 + \frac{1}{2}\eta^7, \xi^4 - \frac{4}{75}\xi^2\eta^3 + \frac{2}{25}\xi^2\eta^4 - \frac{3}{25}\xi^2\eta^6 - \frac{2}{75}\eta^8\}$.

Next, we consider semi-quasihomogeneous cases. The following theorem shows the relation between a basis of H_{f_0} and a basis of H_f.

Theorem 5 ([12] Proposition 3.2.). *Let* $Q = \{q_1, q_2, \dots, q_\mu\}$ *be a basis of* H_{f_0} *which is given as an output of Algorithm* 1. *Then, for each* $i = 1, \dots, \mu$, *there uniquely exists* r_i *such that* $\deg_{\mathbf{w}}(q_i) > \deg_{\mathbf{w}}(r_i)$ *and* $h_i = q_i + r_i$ *is an element of* H_f. *Namely, the set* $\{h_1, \dots, h_\mu\}$ *is a basis of* H_f.

Since Algorithm 1 outputs head parts of H_f, we only need to compute their lower parts. By this theorem, we can construct an algorithm for computing a basis of H_f, which is the following.

Theorem 6. *The following algorithm outputs a basis of* H_f *and terminates.*

Algorithm 2. (a basis of H_f).

Input: $f := f_0 + g$: a semi-quasihomogeneous polynomial of $(d; \mathbf{w})$, $\mathbf{w}$, $\prec$
Output: H: a basis of a vector space H_f.
BEGIN
$G \leftarrow \{x^\alpha |$ a monomial of $\frac{\partial f_0}{\partial x_i}$ for each $1 \le i \le n\}$; $G' \leftarrow \{\xi^\alpha | x^\alpha \in G\}$
$H \leftarrow$ Compute the monomial basis of $K[\xi]/\langle G' \rangle$. $(*1)$
$Q \leftarrow$ Compute a basis of H_{f_0} by Algorithm 1. ; $Q \leftarrow Q\backslash H$
while $Q \neq \emptyset$ **do**
  $q \leftarrow$ Select the minimal polynomial in Q w.r.t. $\prec$. ; $Q \leftarrow Q\backslash\{q\}$
  $L \leftarrow \{\xi^\lambda | \deg_{\mathbf{w}}(\xi^\lambda) < \deg_{\mathbf{w}}(q) \ \wedge \ \xi^\lambda \notin \{\mathrm{hm}(h) | h \in H\}\}$ $(*2)$
  $h \leftarrow$ Set $u = q + \sum_{\lambda' \in L} c_{\lambda'}\xi^{\lambda'}$, check $\frac{\partial f}{\partial x_i} * u = 0$ and determine $c_{\lambda'}$. $(*3)$
  $H \leftarrow H \cup \{h\}$
end-while
return H
END

(*Termination*) At $(*1)$, by the same reason of Algorithm 1, H is a finite set. As Algorithm 1 outputs a finite set, $Q \backslash H$ is a finite or empty set. Theorefore, the **while-loop** must be terminated. This algorithm terminates.
(*Corestness*) As all head parts are known by Algorithm 1, the algorithm determines their lower parts at $(*3)$. Hence, by Theorem 5, this algorithm outputs a basis of H_f.

Note that we compute the set L whose elements are lower monomials of q at $(*2)$. In order to keep the presentation simple, we have deliberately avoided tricks and optimizations such as applying the condition of lower monomials at $(*2)$. All the tricks suggested in [21] can be used here as well. In fact, our implementations fully incorporate these optimizations.

We give an example to illustrate Algorithm 2.

Example 7. *Let consider Example* 4, *again. A polynomial* $f := f_0 + g \in K[x, y]$ *is semi-quasihomogeneous of type* $(7; (2, 1))$ *where* $f_0 = x^3y + x^2y^3 + y^7$ *and* $g = 2y^8$. *In Example* 4, *we obtained a basis of a vector space* H_{f_0}. *Here, we follow Algorithm* 2 *to obtain a basis of* H_f.

The partial derivatives of f *are* $\frac{\partial f}{\partial x} = 3x^2y + 2xy^3$ *and* $\frac{\partial f}{\partial y} = x^3 + 3x^2y^2 + 7y^6 + 16y^7$. *Then, the monomial bais of* $K[\xi, \eta]/\langle \xi^2\eta, \xi\eta^3, \xi^3, \xi^2\eta^2, \eta^6, \eta^7\rangle$ *is the same as the case of* H_{f_0}, *i.e.,* $H = \{1, \eta, \eta^2, \xi^2, \eta^3, \xi^2\eta, \eta^4, \xi\eta^2, \xi^2, \eta^5\}$. *By Algorithm* 1, *we obtain a set* Q *which is a basis of* H_{f_0}. *The set* Q *is already known in Example* 4, *so* $Q = Q \backslash H = \{\xi^3\eta - \frac{3}{2}\xi\eta^3, \xi^2\eta^2 - \frac{3}{2}\xi\eta^4 - \frac{3}{7}\eta^6, \xi^3 - \frac{1}{7}\eta^6, \xi^3\eta - \frac{3}{2}\xi^2\eta^3 + \frac{9}{4}\xi\eta^5 + \frac{1}{2}\eta^7, \xi^3\eta^2 - \frac{3}{2}\xi^2\eta^4 - \frac{3}{7}\xi\eta^6 + \frac{1}{2}\eta^8\}$. *As an each element of* Q *is a head part, we have to decide an each lower part.*

The minimal element in Q *is* $q = \xi^2\eta - \frac{3}{2}\xi\eta^3$ *whose weighted degree is* 5. $Q = Q \backslash \{\xi^2\eta - \frac{3}{2}\xi\eta^3\}$. *Next, we must select monomials whose weighted degree are lower than* $\deg_{\mathbf{w}}(q)$ (*except for elements in* $\{\mathrm{hm}(h) | h \in H\}$). *In this case, there is no monomial. Thus,* $\xi^2\eta - \frac{3}{2}\xi\eta^3$ *is a member of the basis and* $H = H \cup \{\xi^2\eta - \frac{3}{2}\xi\eta^3\}$.

The minimal element in Q *is* $q = \xi^2\eta^2 - \frac{3}{2}\xi\eta^4 - \frac{3}{7}\eta^6$ *whose weighted degree is* 6, *and* $Q = Q \backslash \{\xi^2\eta^2 - \frac{3}{2}\xi\eta^4 - \frac{3}{7}\eta^6\}$. *Then,* $L = \{\xi^\lambda | \deg_{\mathbf{w}}(\xi^\lambda) < 6 \ \wedge \ \xi^\lambda \notin \{\mathrm{hm}(h) | h \in H\}\} = \{\xi\eta^3\}$ *and set* $h = q + c_{(1,3)}\xi\eta^3$. *Solve a system of linear equations which is from the condition* $\frac{\partial f}{\partial x} * h = \frac{\partial f}{\partial y} * h = 0$. *The solution is* $c_{(1,3)} = 0$. *Thus,* $\xi^2\eta^2 - \frac{3}{2}\xi\eta^4 - \frac{3}{7}\eta^6$ *is a member of the basis and* $H = H \cup \{\xi^2\eta^2 - \frac{3}{2}\xi\eta^4 - \frac{3}{7}\eta^6\}$.

Next, we consider $\xi^3 - \frac{1}{7}\eta^6$. $Q = Q \backslash \{\xi^3 - \frac{1}{7}\eta^6\}$. *Repeat the same procedure. Then, we obtain* $\xi^3 - \frac{1}{7}\eta^6$ *as a member of the basis and* $H = H \cup \{\xi^3 - \frac{1}{7}\eta^6\}$.

The minimal element in Q is $q = \xi^3\eta - \frac{3}{2}\xi^2\eta^3 + \frac{9}{4}\xi\eta^5 + \frac{1}{2}\eta^7$ whose weighted degree is 7, and renew the set Q as $Q\backslash\{q\}$. Then, $L = \{\xi^\lambda | \deg_{\mathbf{w}}(\xi^\lambda) < 7 \ \wedge \ \xi^\lambda \notin \{\mathrm{hm}(h)|h \in H\}\} = \{\xi\eta^3, \xi\eta^4, \eta^6\}$ and set $h = q + c_{(1,3)}\xi\eta^3 + c_{(1,4)}\xi\eta^4 + c_{(0,6)}\eta^6$. By checking the condition $\frac{\partial f}{\partial x} * h = \frac{\partial f}{\partial y} * h = 0$, we obtain the solution $\{c_{(1,3)} = c_{(1,4)} = 0, c_{(0,6)} = -\frac{8}{7}\}$. Thus, $q - \frac{8}{7}\eta^6$ is a member of the basis and $H = H \cup \{q - \frac{8}{7}\eta^6\}$.

Finally, we consider the last element $q = \xi^4 - \frac{4}{75}\xi^2\eta^3 + \frac{2}{25}\xi^2\eta^4 - \frac{3}{25}\xi^2\eta^6 - \frac{2}{75}\eta^8$ whose weighted degree is 8. Then, $L = \{\xi^\lambda | \deg_{\mathbf{w}}(\xi^\lambda) < 8 \ \wedge \ \xi^\lambda \notin \{\mathrm{hm}(h)|h \in H\}\} = \{\xi\eta^3, \xi\eta^4, \eta^6, \xi^2\eta^3, \xi\eta^5, \eta^7\}$ and set $h = q + c_{(1,3)}\xi\eta^3 + c_{(1,4)}\xi\eta^4 + c_{(0,6)}\eta^6 + c_{(2,3)}\xi^2\eta^3 + c_{(1,5)}\xi\eta^5 + c_{(0,7)}\eta^7$. By checking the condition $\frac{\partial f}{\partial x} * h = \frac{\partial f}{\partial y} * h = 0$, we obtain the solution $\{c_{(1,3)} = c_{(1,4)} = c_{(2,3)} = c_{(1,5)} = 0, c_{(0,6)} = -\frac{512}{3675}, c_{(0,7)} = \frac{32}{525}\}$. Thus, $q + \frac{32}{525}\eta^7 - \frac{512}{3675}\eta^6$ is a member of the basis.

Conclusively, we obtain a basis of H_f as follows:

$$\{1, \eta, \eta^2, \xi^2, \eta^3, \xi^2\eta, \eta^4, \xi\eta^2, \xi^2, \eta^5, \xi^2\eta - \tfrac{3}{2}\xi\eta^3, \xi^2\eta^2 - \tfrac{3}{2}\xi\eta^4 - \tfrac{3}{7}\eta^6, \ \xi^3 - \tfrac{1}{7}\eta^6, \xi^3\eta - \tfrac{3}{2}\xi^2\eta^3 + \tfrac{9}{4}\xi\eta^5 + \tfrac{1}{2}\eta^7 - \tfrac{8}{7}\eta^6, \xi^4 - \tfrac{4}{75}\xi^2\eta^3 + \tfrac{2}{25}\xi^2\eta^4 - \tfrac{3}{25}\xi^2\eta^6 - \tfrac{2}{75}\eta^8 + \tfrac{32}{525}\eta^7 - \tfrac{512}{3675}\eta^6\}.$$

In semi-quasihomogeneous cases, as a Poincaré polynomial tells us candidates of head monomials and number of elements of a basis, the computation cost of selecting candidates of head monomials and lower monomials, becomes smaller than the our previous one. In this point, our new algorithm is more efficient than the our previous one.

We have implemented Algorithm 1 and Algorithm 2 in the computer algebra system Risa/Asir[1] version 20091015 (Kobe Distribution) and have executed some computation.

Here, we give the results of benchmark tests. The Table 1 shows a comparison of the implemantation of Algorithm 1 with our previous implementation (non-special cases) [21]. That is, in Table 1, quasihomoheneous polynomials are computed. (x, y, z are variables.) The Table 2 shows a comparison of the implemantation of Algorithm 2 with our previous implementation (non-special cases) [21]. In Table 2, proper semi-quasihomoheneous polynomials are computed. ($f_1, \dots, f_{10}$ are from Table 1.)

[1]Risa/Asir is an open source general computer algebra system [14]. Kobe distribution is being developed by OpenXM committers. The original Risa/Asir is developed at Fujitsu Labs LTD.
`http://www.math.kobe-u.ac.jp/Asir/asir.html`

Quasihomogeneous	**Milnor no.**	**Algorithm**	**cpu time**
$f_1 := (x^4+y^6+x^2y^3)^2$ $+x^2y^9$	**77**	**Algorithm 1**	0.031
		[21]	0.936
$f_2 := (x^5+y^7)^2+3y^{14}$	**117**	**Algorithm 1**	0.015
		[21]	1.56
$f_3 := (y^{13}+x^3)^2+x^6$	**125**	**Algorithm 1**	0.016
		[21]	1.669
$f_4 := (x^4+y^6+x^2y^3)^3$ $+x^8y^6$	**187**	**Algorithm 1**	0.171
		[21]	22.09
$f_5 := (x^2z+yz^2+y^5+y^3z)^2$ $+z^5+x^6y$	**204**	**Algorithm 1**	3.135
		[21]	941.4
$f_6 := (x^2y+z^4+y^5)^2+x^5$ $+y^5z^4$	**252**	**Algorithm 1**	0.983
		[21]	1014.4
$f_7 := (y^4+xz^3+x^3)^2$ $+y^8+z^9$	**280**	**Algorithm 1**	0.718
		[21]	1151.5
$f_8 := (x^3y+y^7+x^2y^3)^4$ $+x^{14}$	**351**	**Algorithm 1**	1.217
		[21]	285.1
$f_9 := (x^4+y^9)^4+3x^{16}$	**525**	**Algorithm 1**	0.577
		[21]	378.6
$f_{10} := (x^3+xz^2+xy^3+zy^3)^3$ $+xz^8+xy^{12}$	**800**	**Algorithm 1**	1.619×10^4
		[21]	1.987×10^5

Table 1: Timings

Semi-quasihomo.	**Milnor no.**	**Algorithm**	**cpu time**
$f_1+3x^3y^8$	**77**	**Algorithm 2**	0.359
		[21]	1.263
$f_2+x^{10}y^5+xy^{14}$	**117**	**Algorithm 2**	0.905
		[21]	1.607
$f_3-2x^3y^{20}$	**125**	**Algorithm 2**	0.827
		[21]	1.544
$f_4+2x^{11}y^2$	**187**	**Algorithm 2**	7.94
		[21]	38.63
$f_5+x^3y^2z^2$	**204**	**Algorithm 2**	143.2
		[21]	2023.6
$f_6+x^2y^3z^3$	**252**	**Algorithm 2**	356.1
		[21]	1393.8
f_7+xy^7	**280**	**Algorithm 2**	415.9
		[21]	1876.3
$f_8+x^{13}y^3$	**351**	**Algorithm 2**	93.85
		[21]	893.3
$f_9+x^{15}y^3$	**525**	**Algorithm 2**	287.8
		[21]	1244
$f_{10}+4x^2y^{10}z$	**800**	**Algorithm 2**	1.003×10^5
		[21]	3.167×10^5

Table 2: Timings

We used a PC [OS: Windows 7 (64bit), CPU: Intel(R) Xeon(R) CPU X5650@ 2.67 GHz 2.66 GHz, RAM: 64 GB]. The time is given in second.

As is evident from Table 1, Algorithm 1 generates head parts efficiently, which results the better performance of Algorithm 2 in contrast to our previous algorithm.

§4. Algebraic local cohomology with parameters

Here, we consider how to generalize Algorithm 2 to manipulate the parametric cases. Let f_0 be a quasihomogeneous polynomial type $(d; \mathbf{w})$ with parameters and **generically** define an isolated singularity at the origin. If f_0 has the isolated singularity, then let us fix the form $f := f_0 + g$ as a semi-quasihomogeneous polynomials of type $(d; \mathbf{w})$. (A parameter means that a parameter can take an arbitrary value from $\mathbb{C}$ and parameters are in cofficients.) In parametric cases, there is a possibility that f_0 have a non-isolated singularity for some values of parameters. In order to generalize Algorithm 2, we have to take away these values of parametes because Algorithm 2 assumes that f_0 has the isolated singularity.

How do we compute these values of parameters?
This classification is **possible** by computing a comprehensive Gröbner system of the Jacobi ideal of f_0 [24]. There is a computer algebra technique "classification of dimensions for parametric ideals." By using this technique, one can detect and discard unnecessary values of parametes.

After the classification, we follow Algorithm 2 to obtain bases of H_f. In Algorithm 2, systems of parametric linear equations apper. There are also computer algebra techniques and implementations for solving systems of parametric linear equations. Therefore, **it is possible to generalize Algorithm 2 to the parametric cases.** The detail of the algorithm is described in our preprint [11].

Concluding this section, we give an example for a parametric case.

Example 8. *A polynomial* $f = x^4 + y^6 + tx^2y^3 + ay^7$ *is semi-quasihomogeneous polynomial of type* $(12; (3, 2))$ *where* x, y *are variables and* t, a *are parameters. Set* $f_0 = x^4 + y^6 + tx^2y^3$ *which is quasihomogeneous, and* $g = ay^7$*. First, take away unnecessary values by using a comprehensive Gröbner system of* $\langle \frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y} \rangle$*. Then, we obtain* $t = 2, -2$*. We take away the cases* $t = 2$ *and* $t = -2$*. Next, we follow Algorithm 2 which needs techniques for sloving systems of parametric linear equations. We can obtain the following bases for* H_{f_0}*. (Variables* (ξ, η) *are corresponding to variables* (x, y)*.)*

- *If* $t = 0$*, then a basis of* H_{f_0} *is*
  $\{1, \eta, \eta^2, \eta^3, \eta^4, \xi, \xi\eta, \xi\eta^2, \xi^2, \xi^2\eta, \xi^2\eta^4, \xi^2\eta^3, \xi\eta^4, \xi^2\eta^2, \xi\eta^3\}$.
- *If* $t^3 - 4t \neq 0$*, then a basis of* H_{f_0} *is*
  $\{1, \eta, \eta^2, \eta^3, \eta^4, \xi, \xi\eta, \xi\eta^2, \xi^2, \xi^2\eta, \xi^3 - \frac{2}{t}\xi\eta^3, \xi^4\eta - \frac{2}{t}\xi^2\eta^4 + \eta^7,$
  $\xi^4 - \frac{2}{t}\eta^3\xi^2 + \eta^6, \xi^3\eta - \frac{2}{t}\xi\eta^4, \xi^2\eta^2 - \frac{1}{2}t\eta^5\}$.

By using these parametric bases, the generalized Algorithm 2 outputs the following.

- *If* $t = 0$*, then a basis of* H_f *is*
  $\{1, \eta, \eta^2, \eta^3, \eta^4, \xi, \xi\eta, \xi\eta^2, \xi^2, \xi^2\eta, \xi^2\eta^4, \xi^2\eta^3, \xi\eta^4, \xi^2\eta^2, \xi\eta^3\}$.
- *If* $t^3 - 4t \neq 0$*, then a basis of* H_f *is*
  $\{1, \eta, \eta^2, \eta^3, \eta^4, \xi, \xi\eta, \xi\eta^2, \xi^2, \xi^2\eta, \xi^3 - \frac{2}{t}\xi\eta^3, \xi^4\eta - \frac{2}{t}\xi^2\eta^4 + \eta^7 -$
  $\frac{7}{6}a\eta^6 + \frac{49}{36}a^2\eta^5, \xi^4 - \frac{2}{t}\xi^2\eta^3 + \eta^6 - \frac{7}{6}a\eta^5, \xi^3\eta - \frac{2}{t}\xi\eta^4, \xi^2\eta^2 - \frac{1}{2}t\eta^5\}$.

The method in [21] that utilize a basis of H_f *for computing standard bases is also applicable to parametric cases. One obtains, by this method, a parametric standard bais of* $\mathcal{J} := \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ *w.r.t. the local weighted degree reverse lexicographic order as follows:*

- *If* $t = 0$*, then a standard basis of* $\mathcal{J}$ *is*
  $\{x^3, y^5\}$.
- *If* $t^3 - 4t \neq 0$*, then a standard basis of* $\mathcal{J}$ *is*
  $\{x^5, x^3y^2, y^5 - \frac{49}{36}a^2x^4y + \frac{7}{6}ax^4 + \frac{1}{2}tx^2y^2, xy^3 + \frac{2}{t}x^3\}$.

§5. Conclusion

A new effective method that utilize Poincaré polynomials for constructing algebraic local cohomology classes associated with semiquasihomogeneous isolated singularities, is proposed. The resulting algorithms efficiently compute algebraic local cohomology classes with respect to a weighted monomial order. Computer experiments show that the algorithms are superior in practice in comparison to our previous algorithms.

A generalization of the proposed method to parametric cases is discussed. It is shown that the algorithms can be also extendable to handle parametric cases.

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