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The links specific to hypersurface simple $K3$ singularities

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Dedicated to Professor Shihoko Ishii on the occasion of her 60th birthday

Abstract.

We show the existence of infinitely many links of non-degenerate simple $K3$ singularities defined by non-quasi-homogeneous polynomials such that the second betti numbers of the links are 17, which do not appear in the case of the singularities defined by quasi-homogeneous polynomials.

§1. Introduction

Let $f(z_1, \ldots, z_n)$ be a non-constant polynomial over $\mathbb{C}^n$ for $n \geq 3$, and X be the hypersurface defined by the polynomial f having an isolated singularity at the origin $x = 0$ in $\mathbb{C}^n$. Then the intersection

$$L := X \cap S_\epsilon^{2n-1}$$

of the hypersurface X and a small $(2n-1)$-sphere S_ϵ^{2n-1} with the center at the origin in $\mathbb{R}^{2n}$ is called the *link* of the singularity, and the link L is a $(n-3)$-connected closed spin smooth real $(2n-3)$-manifold. The homeomorphism type of the link L and the embedding L into the sphere S_ϵ^{2n-1} determine the topological type of the isolated hypersurface singularity (see Milnor [8]). The problem is what the topology of the singularity (X, x) is for a given singularity (X, x). We focus on the topology of the

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links of *non-degenerate hypersurface simple* $K3$ *singularities.* Then the links are simply-connected closed spin smooth real 5-manifolds.

On the other hand, Boyer, Galicki and Matzeu treated the links from the viewpoint of the Sasakian structures on compact 5-manifolds (see [1], [2]). Let f be a quasi-homogeneous polynomial of degree d with weight-vector $w = (w_1, \ldots, w_4)$ such that $\sum_{i=1}^{4} w_i = d$. Let L be the link of the isolated singularity defined by f. Due to Kollár's results on 5-dimensional Seifert bundles [6], the second homology group $H_2(L, \mathbb{Z})$ is free. Further, from Smale's classification theorem on simply-connected closed spin smooth real 5-manifolds [11], we have the following theorem:

Theorem 1.1 (Boyer and Galicki [1] Propostion 9.2.4 and Theorem 10.3.8, [2])**.** *Let L be the link of the isolated singularity defined by a quasi-homogeneous polynomial of degree d with weight-vector $w = (w_1, \ldots, w_4)$ such that $\sum_{i=1}^{4} w_i = d$. Let b be the second betti number of the link L. Then the link L is diffeomorphic to the connected sum of b copies of $S^2 \times S^3$, where $2 \leq b \leq 21$.*

As a corollary, together with the results of non-degenerate hypersurface simple $K3$ singularities which will be stated in §2, we have the following:

Corollary 1.1. *Let L be the link of a non-degenerate hypersurface simple $K3$ singularity defined by a quasi-homogeneous polynomial. Let b be the second betti number of the link L. Then the link L is diffeomorphic to the connected sum of b copies of $S^2 \times S^3$, where $3 \leq b \leq 21$ and $b \neq 17$.*

Hence, an interesting question is whether $b = 2$ or 17 can occur, which is an open problem in Boyer and Galicki's book [1]. In this paper, we will give a partial answer for this problem. The main theorem is as follows:

Theorem 1.2. *There exist infinitely many links of non-degenerate simple $K3$ singularities defined by non-quasi-homogeneous polynomials such that the second betti numbers of the links are* 17.

The plan of this paper is as follows: in §2, we recall basic facts about hypersurface simple $K3$ singularities. In §3, we review the topology of the link L of an isolated hypersurface singularity [8] and Smale's classification of simply-connected closed spin smooth real 5-manifolds [11]. In §4, we prove Theorem 1.2.

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§2. Hypersurface simple $K3$ singularities

In general, a simple $K3$ singularity is defined as a Gorenstein purely elliptic singularity of type $(0,2)$ in Ishii–Watanabe [4], and its geometric characterization is that the reduced exceptional divisor is an irreducible normal $K3$ surface for a $\mathbb{Q}$-factorial terminal modification of the singularity. A $K3$ surface is a simply-connected compact complex surface with a trivial canonical line bundle. Therefore, this singularity is considered as a three-dimensional analogue of the simple elliptic singularity in dimension 2, whose exceptional divisor is an elliptic curve having a trivial canonical line bundle. In this paper, we deal with *non-degenerate* hypersurface simple $K3$ singularities, which are characterized in terms of the Newton diagram in Watanabe [13].

Let $f(z) = \sum_k a_k z^k$ be a polynomial in $\mathbb{C}[z_1, \dots, z_n]$, where $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$. Then the Newton diagram $\Gamma_+(f)$ of f is the convex hull of $\bigcup_{a_k \neq 0}(k + \mathbb{R}^n_{\geq 0})$ in $\mathbb{R}^n_{\geq 0}$ and the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$. For a face Δ of $\Gamma(f)$, we put $f_\Delta(z) := \sum_{k \in \Delta} a_k z^k$. We say that the polynomial f is *non-degenerate* if $\partial f_\Delta / \partial z_1 = \cdots = \partial f_\Delta / \partial z_n = 0$ has no solutions in $(\mathbb{C} \setminus \{0\})^n$ for any face Δ of $\Gamma(f)$. A hypersurface singularity defined by f at the origin is called *non-degenerate* if f is a non-degenerate polynomial.

The criteria for non-degenerate hypersurface simple $K3$ singularities are as follows:

Theorem 2.1 (Watanabe [13]). *Let $f = \sum a_k z^k \in \mathbb{C}[z_1, \dots, z_4]$ be a non-degenerate polynomial defining an isolated singularity at the origin of $\mathbb{C}^4$. Then the singularity is a simple $K3$ singularity if and only if the Newton boundary $\Gamma(f)$ contains the point $(1,1,1,1)$, and the face $\Delta_0(f)$ of $\Gamma(f)$, containing the point $(1,1,1,1)$ in its relative interior, is of dimension 3.*

Definition 2.1. The *weight-vector* $\alpha(f)$ of f is the vector $\alpha(f) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Q}^4_{>0}$ with $\Sigma_{i=1}^4 \alpha_i = 1$ such that the 3-dimensional polygon $\Delta_0(f)$ is perpendicular to $\alpha(f)$ in $\mathbb{R}^4$.

Theorem 2.2 (Reid [10], Fletcher [3], and Yonemura [14]). *The number of weight-vectors of the defining polynomials of non-degenerate hypersurface simple $K3$ singularities at the origin equals 95.*

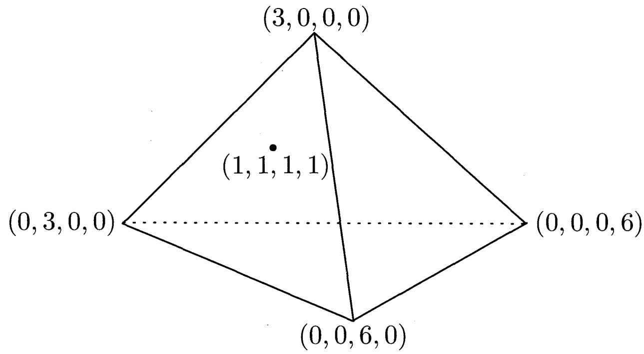

Fig. 1. YN3: $f = x^3 + y^3 + z^6 + w^6$

Throughout this paper, we use the numbering of the weight-vectors in Yonemura's list in [14]. For example, YN3: $f = x^3 + y^3 + z^6 + w^6$ means polynomial no.3 in his list (see Fig. 1).

§3. Topology of the link L

In the case of the hypersurface simple $K3$ singularity, its link is a simply-connected closed spin smooth real 5-manifold. By Smale's classification of these 5-manifolds in [11], the diffeomorphism type of the link L is only determined by the second homology group $H_2(L, \mathbb{Z})$ of the link L.

Theorem 3.1 (Smale [11]). *There exists a one-to-one correspondence φ from the set of isomorphism classes of simply-connected closed spin smooth real 5-manifolds to the set of isomorphism classes of finitely generated abelian groups.*

Let M be a simply-connected closed spin smooth real 5-manifold, and let $H_2(M)$ be $F \oplus T$, where F is the free part and T is the torsion part. Then the correspondence φ is given by $\varphi(M) := F \oplus (1/2)T$, where $T = (1/2)T \oplus (1/2)T$.

Corollary 3.1. *If the second homology group $H_2(L, \mathbb{Z})$ of the link L is free, then the link L is diffeomorphic to the connected sum of some copies of $S^2 \times S^3$.*

For the isolated singularity defined by a quasi-homogeneous polynomial, we can calculate the second betti number of the link L as follows: let $\phi : S^7 \backslash L \rightarrow S^1$ be the Milnor fibration. Let $F_\theta = \phi^{-1}(e^{i\theta}) \subset S^7 \backslash L$

be the Milnor fiber. Note that the boundary of the closure $\bar{F_\theta}$ of F_θ is L. Consider the Wang sequence corresponding to the fibration

$$H_3(F_\theta,\mathbb{Z}) \xrightarrow{h_*-I_*} H_3(F_\theta,\mathbb{Z}) \longrightarrow H_3(S^7\setminus L,\mathbb{Z}) \longrightarrow 0.$$

Here I is the identity map of the fiber F_θ and $h: F_\theta \to F_\theta$ is the characteristic map of the fibration. The homomorphism $h_*: H_3(F_\theta,\mathbb{Z}) \to H_3(F_\theta,\mathbb{Z})$ is called the monodromy. Together with the Alexander duality isomorphism and the Poincaré duality isomorphism, we have $H_3(S^7\setminus L,\mathbb{Z}) \cong H^3(L,\mathbb{Z}) \cong H_2(L,\mathbb{Z})$. It follows from the exactness that

$$H_2(L,\mathbb{Z}) \cong Coker(h_*-I_*).$$

We denote the characteristic polynomial of the monodromy by $\Delta_f(t) = \det(tI_* - h_*)$, which is a topological invariant of $S^7\setminus L$ (see [8]). Then the rank of $H_2(L,\mathbb{Z})$ is equal to the multiplicity of the root 1 of the characteristic polynomial $\Delta_f(t)$. Therefore we obtain the following lemma by using the method of Milnor and Orlik [9]: for example, consider the YN3 polynomial $f = x^3+y^3+z^6+w^6$ in [14]. By using the notations in [9], we have $\mathrm{div}\Delta_f(t) = (\Lambda_3-1)^2(\Lambda_6-1)^2 = 16\Lambda_6+\Lambda_3+1$. This means that $\Delta_f(t) = (t^6-1)^{16}(t^3-1)(t-1) = \Phi_1^{18}\Phi_2^{16}\Phi_3^{17}\Phi_6^{16}$, where Φ_n denotes the nth cyclotomic polynomial. Hence the rank of $H_2(L,\mathbb{Z})$ is equal to 18.

Lemma 3.1. *Let L be the link of the non-degenerate hypersurface simple $K3$ singularity defined by a quasi-homogeneous polynomial. Let b be the second betti number of the link L. Then $3 \le b \le 21$ and $b \ne 17$.*

b	YN
3	YN52, YN56, YN73
4	YN17, YN30, YN46, YN61, YN65, YN80, YN84, YN86, YN91
5	YN57, YN64, YN68, YN74, YN83, YN90, YN92
6	YN16, YN29, YN35, YN43, YN48, YN54, YN62, YN88, YN93, YN94
7	YN31, YN47, YN53, YN55, YN79
8	YN15, YN26, YN27, YN34, YN49, YN67, YN70, YN76, YN95
9	YN36, YN41, YN69, YN75, YN81, YN85
10	YN2, YN11, YN20, YN33, YN59
11	YN23, YN38, YN58, YN60, YN77
12	YN4, YN9, YN14, YN22, YN28, YN32, YN45, YN51, YN71, YN78, YN87
13	YN37, YN39, YN50, YN82
14	YN13, YN18, YN24, YN63, YN72, YN89
15	YN8, YN19, YN40, YN44
16	YN6, YN12
17	
18	YN3, YN25, YN66
19	YN7, YN42
20	YN10, YN21
21	YN1, YN5

Table

We have Corollary 1.1 since the defining polynomials are quasi-homogeneous, due to Kollár's results on the freeness for $H_2(L, \mathbb{Z})$ and Smale's results on the diffeomorphism types of the links.

Remark 3.1. There are many hypersurface singularities which are not simple $K3$ singularities such that their links are the connected sum of some copies of $S^2 \times S^3$: for example, let $f = x^2 + y^2 + z^c + w^d$ for $2 \leq c \leq d$ and $c, d \in \mathbb{Z}$. Then the singularity defined by f is not a simple $K3$ singularity by Watanabe's criteria [13]. However, the link is diffeomorphic to the connected sum of $(c, d) - 1$ copies of $S^2 \times S^3$, where (c, d) is their greatest common divisor (see Katanaga and Nakamoto [5]).

§4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we consider a polynomial in Yonemura's list such that the second betti number of the associated link is 18. There are three polynomials: YN3, YN25 and YN66 (see Table in Lemma 3.1). We choose the simplest polynomial YN3 : $f = x^3 + y^3 + z^6 + w^6$. By calculating the monodromy $h_* - I_*$, we have $H_2(L, \mathbb{Z}) \cong Coker(h_* - I_*) \cong \mathbb{Z}^{18}$. From Smale's results in Theorem 3.1, the link L associated with f is diffeomorphic to the connected sum of

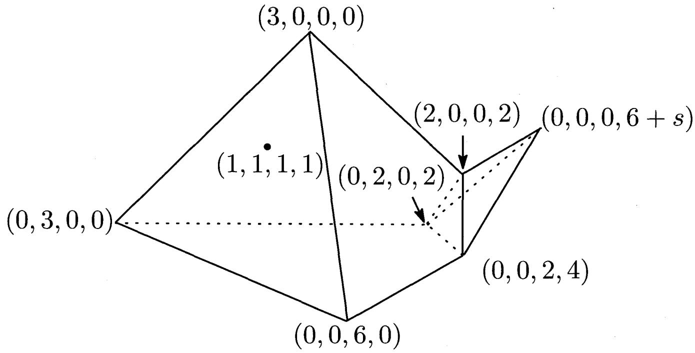

Fig. 2. $f_s = x^3 + y^3 + z^6 + (x^2 + y^2 + z^2w^2)w^2 + w^{6+s}$

18 copies of $S^2 \times S^3$ (see also [5] for more details). We change this polynomial f slightly into a new polynomial $f_s = x^3 + y^3 + z^6 + (x^2 + y^2 + z^2w^2)w^2 + w^{6+s}$ for any non-negative integer s as in Fig. 2. The new polynomial f_s is non-degenerate and satisfies Watanabe's criteria in Theorem 2.1 for having a simple $K3$ singularity at the origin. Note that f_s is non-quasi-homogeneous. Then the following proposition holds.

Proposition 4.1. *Let s be a non-negative integer, and let $f_s = x^3 + y^3 + z^6 + (x^2 + y^2 + z^2w^2)w^2 + w^{6+s}$. Then the Milnor number $\mu(f_s)$ of f_s is equal to $100 + s$, and the second betti number of the link associated with f_s is equal to 17 for s odd, and 18 for s even.*

Proof. The Milnor number

$$\mu(f_s) := \dim_{\mathbb{C}} \mathbb{C}[[z_1, \ldots, z_4]]/J_{f_s}$$

where $J_{f_s} = (\partial f_s/\partial z_1, \ldots, \partial f_s/\partial z_4)$ is the Jacobian ideal of f_s, which is a topological invariant by the result of Lê Dung Tráng [7].

In order to determine the second betti number of the link associated with f_s, we calculate the characteristic polynomial $\Delta_{f_s}(t)$ by following the method of Varchenko [12]: the characteristic polynomial $\Delta_{f_s}(t)$ is expressed by means of the zeta-function $\zeta_{f_s}(t)$ according to the formula

$$\Delta_{f_s}(t) = t^{\mu(f_s)}[(t/(t-1))\zeta_{f_s}(1/t)]^{-1}$$

where

$$\zeta_{f_s}(t) = \prod_{q \geq 0} \{\det[I^* - th^*; H^q(F_\theta, \mathbb{C})]\}^{(-1)^q}.$$

Note that $H^q(F_\theta, \mathbb{C}) = 0$ for $q \neq 0, 3$. Since f_s is non-degenerate, the zeta-function $\zeta_{f_s}(t)$ is equal to the zeta-function

$$\zeta_{\Gamma(f_s)}(t) = \prod_{l=1}^{4} (\zeta^l(t))^{(-1)^{l-1}}$$

of the Newton boundary $\Gamma(f_s)$. By calculating $\zeta^l(t)$ for $l = 1, \ldots, 4$, we have

$$\zeta^1(t) = (1-t^3)^2(1-t^6)(1-t^{6+s}),$$

$$\zeta^2(t) = \begin{cases} (1-t^3)^3(1-t^6)^{12}(1-t^{2s+12})^3 & \text{for } s \text{ odd,} \\ (1-t^3)^3(1-t^6)^{12}(1-t^{s+6})^6 & \text{for } s \text{ even,} \end{cases}$$

$$\zeta^3(t) = \begin{cases} (1-t^6)^{42}(1-t^{2s+12})^6 & \text{for } s \text{ odd,} \\ (1-t^6)^{42}(1-t^{s+6})^{12} & \text{for } s \text{ even,} \end{cases}$$

$$\zeta^4(t) = \begin{cases} (1-t^6)^{46}(1-t^{2s+12})^4 & \text{for } s \text{ odd,} \\ (1-t^6)^{46}(1-t^{s+6})^8 & \text{for } s \text{ even.} \end{cases}$$

Hence,

$$\zeta_{f_s}(t) = \zeta_{\Gamma(f_s)}(t) = \begin{cases} \dfrac{(1-t^{6+s})}{(1-t^3)(1-t^6)^{15}(1-t^{2s+12})} & \text{for } s \text{ odd,} \\[2ex] \dfrac{1}{(1-t^3)(1-t^6)^{15}(1-t^{s+6})} & \text{for } s \text{ even.} \end{cases}$$

Therefore the characteristic polynomial

$$\Delta_{f_s}(t) = \begin{cases} \Phi_1^{17}\Phi_2^{15}\Phi_3^{16}\Phi_6^{15} \prod_{d|(2s+12),\ d \nmid (s+6)} \Phi_d & \text{for } s \text{ odd,} \\[2ex] \Phi_1^{18}\Phi_2^{15}\Phi_3^{16}\Phi_6^{15} \prod_{d|(s+6),\ d \neq 1} \Phi_d & \text{for } s \text{ even.} \end{cases}$$

Since the second betti number is the exponent of Φ_1, we have the required results. Q.E.D.

From this proposition, Theorem 1.2 is proved.

Remark 4.1. In general, there is no simple relationship between the second betti numbers of the links and the Milnor numbers.

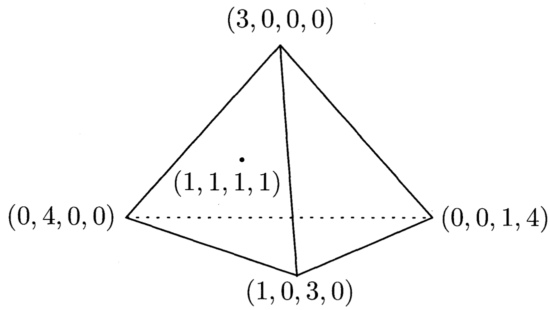

Fig. 3. YN52: $f = x^3 + y^4 + xz^3 + zw^4$

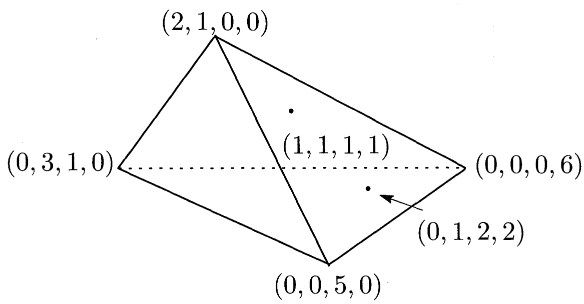

Fig. 4. YN56: $f = x^2y + y^3z + z^5 + w^6$

Remark 4.2. The method for producing the second betti number 17 does not produce the second betti number 2. In fact, there are three polynomials whose links have the second betti number 3 in Table of Lemma 3.1: YN52, YN56, YN73. However, there are no monomials which produce a new Newton boundary satisfying Watanabe's criteria for each polynomial since the number of monomials is very few (see also [14]): let $T(\alpha) = \{n \in \mathbb{Z}^4_{\geq 0} | \alpha \cdot n = 1\}$ be a set of monomials in the Newton boundary of the defining polynomial with the weight-vector α. The number of the elements of $T(\alpha)$ is denoted by $\#T(\alpha)$. For YN52, the weight-vector $\alpha = (1/3, 1/4, 2/9, 7/36)$, $\#T(\alpha) = 5$, and the Newton boundary is in Fig. 3. For YN56, the weight-vector $\alpha = (11/30, 4/15, 1/5, 1/6)$, $\#T(\alpha) = 6$, and the Newton boundary is in Fig. 4. For YN73, the weight-vector $\alpha = (1/2, 1/5, 4/25, 7/50)$, $\#T(\alpha) = 6$, and the Newton boundary is in Fig. 5.

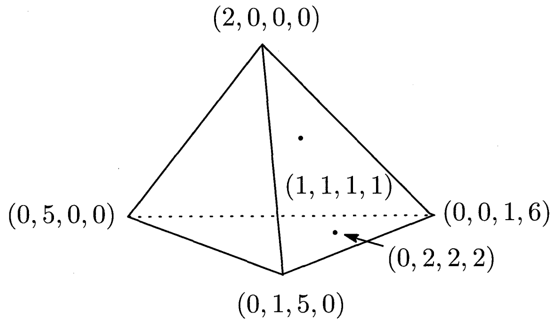

Fig. 5. YN73: $f = x^2 + y^5 + yz^5 + zw^6$

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