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Existence of traveling wave solution in a diffusive predator-prey model with Holling type-III functional response

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Abstract.

In this work, we show the existence of traveling wave solution of a diffusive predator-prey model with Holling type III functional response. The analysis is based on *Wazewski*'s principle in the four-dimensional phase space of the nonlinear ordinary differential equation system given by the diffusive predator-prey system under the moving coordinates.

§1. Introduction

We consider a special type of reaction-diffusion system based on a predator-prey interaction model with Holling-type III functional response [4]:

$$(1)\qquad \begin{cases} u_t = & d_1 u_{xx} + Au(1-\frac{u}{K}) - B\frac{u^2 w}{1+Eu^2}, \\ w_t = & d_2 w_{xx} - Dw + C\frac{u^2 w}{1+Eu^2}, \end{cases}$$

where all parameters in (1) are positive. The functions $u(x,t)$ and $w(x,t)$ are the species densities of the prey and predator, respectively. The parameter E measures the *satiation effect* [4], [8].

The existence of traveling wave solutions of the diffusive predator-prey system with various type of reaction term has been studied by many researchers. Dunbar [1], [2] investigated the existence of traveling wave solutions for a diffusive Lotka–Volterra model with or without limited carrying capacity of the prey. The proof is based on *Wazewski*'s principle which is an extension of shooting methods in higher dimension.

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After that, Dunbar[3] further consider the existence of traveling wave solutions for a diffusive predator-prey model with Holling type II functional response which includes the effects of predator satiation. The diffusion coefficient of prey is set to be zero. Then the traveling wave solutions connecting two equilibria was established in the similar manner. Huang, Lu and Ruan [5] generalized the result of existence of the traveling wave connecting two rest states for a nonzero diffusion coefficient. Li and Wu [7] prove the similar results for a predator-prey system with Holling type-III functional response in zero diffusive rate of prey.

Our main goal is to generalize the Li and Wu's results to the nonzero diffusive rate of prey. More precisely, we establish the existence of traveling wave solution of a nonzero diffusive predator-prey system with Holling type-III functional response. By changing the following variables,

$$\begin{aligned} &u^* = \sqrt{E}u,\ w^* = Bw/(\sqrt{E}D),\ \tilde{x} = \sqrt{D/d_2}x,\ \tilde{t} = Dt,\\ &d = d_1/d_2,\ \alpha = A/(\sqrt{E}DK),\ \gamma = \sqrt{E}K,\ \beta = C/(DE), \end{aligned}$$

we can get a more simple systems (use the same notations for simplicity).

$$\tag{2} \begin{cases} u_t = & du_{xx} + \alpha u(\gamma - u) - \frac{u^2 w}{1+u^2},\\ w_t = & w_{xx} - w + \beta \frac{u^2 w}{1+u^2}. \end{cases}$$

There are several reasonable restrictions on parameters. First, we require that $\gamma > 1$, i.e. $E > 1/K^2$, so that the satiation effect is great enough. We also require that $\beta > (1+\gamma^2)/\gamma^2 > 1$, which ensures that equation (2) has a positive equilibrium point corresponding to constant coexistence of the two species.

The rest of this paper is organized as follows. In Section 2, we recall the *Wazewski*'s principle and state the main result on the existence of traveling wave solution. Section 3 is devoted to prove the main theorems.

§2. Main results

By simple calculations, system (2) has three spatially uniform equilibria given by $E_0 = (0,0)$, $E_1 = (\gamma, 0)$, and $E = (u_s, w_s)$, where $u_s = 1/\sqrt{\beta - 1}$ and $w_s = \alpha(\gamma - u_s)(1 + u_s^2)/u_s$. The aim of this work is to show the existence of the traveling wave solution connecting the equilibria $E_1 = (\gamma, 0)$ and $E = (u_s, w_s)$. A traveling wave solution is a solution of (2) of special form $u(x,t) = u(x+ct) = u(s)$ and $w(x,t) = w(x+ct) = w(s)$ where c is a positive constant wave speed and s is the so-called moving coordinate. The system (2) becomes

$$\tag{3} \begin{cases} cu' = & du'' + \alpha u(\gamma - u) - \frac{u^2 w}{1+u^2}, \\ cw' = & w'' - w + \beta \frac{u^2 w}{1+u^2}. \end{cases}$$

Here " $'$ " denotes the differentiation with respect to the moving coordinate s. We require that the traveling wave solutions u and w are nonnegative for natural ecological restriction and satisfy the asymptotic boundary conditions

$$\tag{4} u(-\infty) = \gamma, w(-\infty) = 0, u(\infty) = u_s, \text{ and } w(\infty) = w_s.$$

Rewrite system (3) and (4) as a system of first order ODEs in $\mathbb{R}^4$,

$$\tag{5} \begin{cases} u' = v, \quad v' = & \frac{c}{d} v + \frac{\alpha}{d} u(u - \gamma) + \frac{u^2 w}{d(1+u^2)}, \\ w' = z, \quad z' = & cz + w - \beta \frac{u^2 w}{1+u^2}, \end{cases}$$

and the boundary conditions

$$\tag{6} \begin{cases} u(-\infty) = \gamma, \ v(-\infty) = 0, \ w(-\infty) = 0, \ z(-\infty) = 0, \\ u(\infty) = u_s, \ v(\infty) = 0, \ w(\infty) = w_s, \ z(\infty) = 0. \end{cases}$$

Recall the *Wazewski*'s principle. Consider the differential equation: $y' = f(y), ' = d/ds, y \in \mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and satisfies the Lipschitz condition. Let $y(s, y_0)$ be the unique solution satisfying $y(0, y_0) = y_0$. For convenience, we denote $y(s, y_0)$ by $y_0 \cdot s$ and let $Y \cdot S$ be the set of points $y \cdot s$, where $y \in Y$ and $s \in S$. Given $W \subseteq \mathbb{R}^n$, define $W^- = \{y_0 \in W | \forall s > 0, y_0 \cdot [0, s) \not\subseteq W\}$, the immediate exit set of W. Given $\Sigma \subseteq W$, let $\Sigma^0 = \{y_0 \in \Sigma | \exists s_0 = s_0(y_0) \text{ such that } y_0 \cdot s_0 \notin W\}$. For $y_0 \in \Sigma$, define $T(y_0) \equiv \sup\{s | y_0 \cdot [0, s] \subset W\}$, the exit time of y_0.

Lemma 1. ([2] *Section* 3.A.) *Suppose that*

(i) *if* $y_0 \in \Sigma$ *and* $y_0 \cdot [0, s] \subseteq cl(W)$, *then* $y_0 \cdot [0, s] \subseteq W$;

(ii) *if* $y_0 \in \Sigma$, $y_0 \cdot s \in W$, $y_0 \cdot s \notin W^-$, *then there is an open set* V_s *about* $y_0 \cdot s$ *disjoint from* W^-;

(iii) $\Sigma = \Sigma_0$, Σ *is a compact set and intersects a trajectory of* $y' = f(y)$ *only once.*

Then the mapping $F(y_0) = y_0 \cdot T(y_0)$ *is a homeomorphism from* Σ *to its image on* W^-.

Now we state the main results as follows.

Theorem 2. (i) *If* $0 < c < \sqrt{4(\frac{\beta\gamma^2 - \gamma^2 - 1}{1+\gamma^2})}$, *then there are no nonnegative solutions of system* (5) *satisfying the boundary conditions* (6).

(ii) *If* $\frac{\gamma^2+1}{\gamma^2} < \beta < \gamma^2 + 1$ *and* $c > \sqrt{4\left(\frac{\beta\gamma^2-\gamma^2-1}{1+\gamma^2}\right)}$, *then there are nonnegative solutions of* (5) *satisfying the boundary conditions* (6), *which correspond to traveling wave solutions of system* (2).

§3. Proof of Theorem 2

First we note that $\big(u(s), v(s)\big)$ is a solution of the system

$$u' = v, \quad v' = cv/d + \alpha u(u-\gamma)/d \tag{7}$$

if and only if $\big(u(s), v(s), 0, 0\big)$ is a solution of (5). That is, the set $N = \{(u,v,w,z)|w = 0, z = 0\}$ is a 2-dimensional invariant submanifold in the 4-dimensional phase space of the system (5). Similarly, the set $H = \{(u,v,w,z)|u = 0, v = 0\}$ is a 2-dimensional invariant submanifold. It is a routine work to calculate that the eigenvalues of linearized system (5) at $(\gamma, 0, 0, 0)$ are

$$\lambda_1 = (c - \sqrt{c^2 + 4\,\alpha\, d\, \gamma})/(2d), \qquad \lambda_2 = \left(c - \sqrt{c^2 - 4\left(\tfrac{\beta\gamma^2-\gamma^2-1}{1+\gamma^2}\right)}\right)/2,$$
$$\lambda_3 = \left(c + \sqrt{c^2 - 4\left(\tfrac{\beta\gamma^2-\gamma^2-1}{1+\gamma^2}\right)}\right)/2, \quad \lambda_4 = (c + \sqrt{c^2 + 4\,\alpha\, d\, \gamma})/(2d).$$

The eigenvalues of the linearization of (7) at $(\gamma, 0)$ are the λ_1 and λ_4. An eigenvector associated with λ_4 for (7) is $(-1, -\lambda_4)$. Any nontrivial trajectory of solutions of (7) which approach $(\gamma, 0)$ tangent to $(-1, -\lambda_4)$ as $s \to -\infty$. It is clearly that this trajectory is contained in the invariant submanifold N. Then a solution corresponding to this trajectory cannot approach $(u_s, 0, w_s, 0)$ as $s \to \infty$, so it is not the desired traveling wave solution of (5). If $0 < c < 2\sqrt{\frac{\beta\gamma^2-\gamma^2-1}{1+\gamma^2}}$, we have two complex eigenvalues, λ_2 and λ_3, with positive real part. Any solution of (5) which is not contained in N and approaches $(\gamma, 0, 0, 0)$ as $s \to -\infty$ must spiral in toward $(\gamma, 0, 0, 0)$. This spiraling of solutions cannot take place with $w \geq 0$. This violates the requirement that traveling wave front solutions be nonnegative. This prove the part 1 of the main theorem.

We only need to consider the case that $c > 2\sqrt{\frac{\beta\gamma^2-\gamma^2-1}{1+\gamma^2}}$ in the following discussions. It is easy to see that $\lambda_1 < 0 < \lambda_2 < \lambda_3 < \lambda_4$. The eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ associated with λ_1, λ_2, λ_3, λ_4, respectively, are

$$\mathbf{e}_1 = (1, \lambda_1, 0, 0), \qquad \mathbf{e}_2 = (1, \lambda_2, p(\lambda_2), \lambda_2 p(\lambda_2)),$$
$$\mathbf{e}_3 = (1, \lambda_3, , p(\lambda_3), \lambda_3 p(\lambda_3)), \quad \mathbf{e}_4 = (1, \lambda_4, 0, 0),$$

where $p(\lambda) = \frac{1+\gamma^2}{\gamma^2}\left[(d-1)\lambda^2 - \alpha\gamma - \frac{d(\beta\gamma^2-\gamma^2-1)}{1+\gamma^2}\right]$.

Define $W \equiv \mathbb{R}^4 \setminus (P \cup Q \cup R \cup S)$, where

$$\begin{aligned} P &= \{(u,v,w,z) | u < u_s, w > w_s, z > 0\}, \\ Q &= \{(u,v,w,z) | u > u_s, w < w_s, z < 0\}, \\ R &= \{(u,v,w,z) | u < u_s, \alpha(u-\gamma) + \frac{uw}{1+u^2} < 0, v < 0\}, \\ S &= \{(u,v,w,z) | u > u_s, \alpha(u-\gamma) + \frac{uw}{1+u^2} > 0, v > 0\}. \end{aligned}$$

By the definition, it is easy to see that

$$\partial W = (\partial P \setminus R) \cup (\partial Q \setminus S) \cup (\partial S \setminus Q) \cup (\partial R \setminus P),$$

$P \cap R \neq \emptyset$, and $Q \cap S \neq \emptyset$. We need the following technical result. It can be directly verified, so we omit it.

Lemma 3. *If $1 < \gamma < 3\sqrt{3}$, then the function $f(u) = \alpha(\gamma - u)(1 + u^2)/u$ is strictly monotone decreasing on the interval $(0, \gamma)$.*

Lemma 4. *If $1 < \gamma < 3\sqrt{3}$, then*

$$W^- = \partial W \setminus (\{(u_s, 0, w_s, 0)\} \cup J_1 \cup J_2),$$

where

$$\begin{aligned} J_1 =& \{(u,v,w,z) : u < 0, \alpha(u-\gamma) + \frac{uw}{1+u^2} \leq 0, v = 0\} \\ & \dot{\cup} \{(u,v,w,z) : u = 0, \alpha(u-\gamma) + \frac{uw}{1+u^2} < 0, v = 0\} \\ & \dot{\cup} \{(u,v,w,z) : u < 0, \alpha(u-\gamma) + \frac{uw}{1+u^2} = 0, v < 0, \\ & \quad \frac{\alpha v}{u(1+u^2)} f(u) + \frac{uz}{1+u^2} \geq 0\}, \\ J_2 =& \{(u,v,w,z) | u = u_s, w > w_s, z = 0, v < 0\} \\ & \dot{\cup} \{(u,v,w,z) | u \leq -u_s, w \geq w_s, z = 0, v \geq 0\}. \end{aligned}$$

Here the notation $\dot{\cup}$ means disjoint union.

By the standard Stable Manifold Theorems, we can find the strongest unstable manifold Ω_1 tangent to $\mathbf{e}_4$ at $(\gamma, 0, 0, 0)$, and a parametric representation for the 1-dimensional strongest unstable manifold Ω_1 in a small neighborhood of $(\gamma, 0, 0, 0)$ is $f_1(m) = (\gamma, 0, 0, 0) + m\mathbf{e}_4 + O(|m|^2)$. There is also a 2-dimensional strongly unstable manifold Ω_2 tangent to the linear subspace of span of $\mathbf{e}_4$ and $\mathbf{e}_3$ at $(\gamma, 0, 0, 0)$, and a parametric representation for the 2-dimensional strongly unstable manifold Ω_2 in a small neighborhood of $(\gamma, 0, 0, 0)$ is $f_2(m, n) = (\gamma, 0, 0, 0) + m\mathbf{e}_4 +$

$n\mathbf{e}_3 + O(|m|^2 + |n|^2)$. Finally, there is a 3-dimensional unstable manifold Ω_3 tangent to the linear subspace of span of $\mathbf{e}_4$, $\mathbf{e}_3$ and $\mathbf{e}_2$ at $(\gamma, 0, 0, 0)$, and a parametric representation for the 3-dimensional unstable manifold Ω_3 in a small neighborhood of $(\gamma, 0, 0, 0)$ is $f_3(m, n, \ell) = (\gamma, 0, 0, 0) + m\mathbf{e}_4 + n\mathbf{e}_3 + \ell\mathbf{e}_2 + O(|m|^2 + |n|^2 + |\ell|^2)$.

The part 2 of the main theorem will be established by a series of lemmas as follows. We construct a simply connected subset Σ of Ω_3 by a series of lemmas (Lemma 4 to Lemma 11). Hence any trajectory starting from Σ will approach to $(\gamma, 0, 0, 0)$ as $s \to -\infty$. By applying *Wazewski*'s principle (Lemma 1), we prove that there must be a trajectory through Σ which does not leave W in Lemmas 12. Then from Lemma13 to Lemma 15 a bounded subset Ω of W contained this trajectory is defined. Finally, a Lyapunov function is constructed on Ω and we use LaSalle's Invariance Principle to show that the trajectory approaches $(u_s, 0, w_s, 0)$ in Lemma 16. For conveniently, throughout the remainder of this paper we use the notation $u(s, \mathbf{y}_0)$ for the first coordinate function of $\mathbf{y}(s, \mathbf{y}_0)$, and similarly for the other three coordinate functions, v, w, and z.

For the solution proceeding from Ω_1, we have the following two results.

Lemma 5. *Consider a solution $\mathbf{y}(s, \mathbf{y}_0)$ with initial condition $\mathbf{y}_0 = (u_0, v_0, w_0, z_0) \in \Omega_1$ and $u_0 < \gamma$. Then there is a finite s_0 such that $u(s_0, \mathbf{y}_0) < u_s$, $v(s_0, \mathbf{y}_0) < 0$. That is, the solution enters the region R.*

Lemma 6. (i) *A solution $\mathbf{y}(s, \mathbf{y}_0)$ on Ω_1 which approaches $(\gamma, 0, 0, 0)$ as $s \to -\infty$ in the region $u > \gamma$,$v > 0$ will remain inside for all s.*

(ii) *Any trajectory with initial point $\mathbf{y}_0 = (u_0, v_0, w_0, z_0)$ such that $0 < u_s < \gamma$, $w_0 > 0$, and $z_0 > (c/2)w_0$ will have $w(s) > 0$ and $z(s) > (c/2)w(s)$ for all $s > 0$ such that $0 < u(s) < \gamma$. In particular, this is true for trajectories on Ω_2 approaching $(\gamma, 0, 0, 0)$ tangent to the vector $\mathbf{e}_3$ in the region $u < \gamma$.*

Consider a small circle on Ω_2 parametrically given by

$$g(\theta) = \big(\gamma, 0, 0, 0\big) + \varepsilon\cos(\theta + \psi)\mathbf{e}_3 + \varepsilon\sin(\theta + \psi)\mathbf{e}_4. \tag{8}$$

The phase ψ is fixed so that $g(0)$ is on Ω_1 in the region $u < \gamma$, and the parameter $\theta \in [0, 2\pi]$. Choose g so that as θ increases from 0, $\gamma + \varepsilon\cos(\theta+\psi) + \varepsilon\sin(\theta+\psi) + O(\varepsilon)$ decreases and $p(\lambda_3)\varepsilon\sin(\theta+\psi) + O(\varepsilon)$ increases from 0. Let A be the connected component of the set $\{\theta \in [0, 2\pi) : \text{there exists } s_0 \text{ such that } u(s_0, g(\theta)) = u_s, v(s, g(\theta)) < 0, s \le s_0\}$. By Lemma 5 and 6, A is nonempty and bounded. Let $\theta_1 = \sup A$ and $\mathbf{y}_1 = g(\theta_1)$. Since $g(0)$ is on Ω_1 with $u < \gamma$, Lemma 6 shows that if $u(s_0, g(0)) = u_s$ then $v(s_0, g(0)) = (d/ds)u(s_0, g(0)) < 0$. By Implicit Function Theorem, $u(s, g(0)) = u_s$ for $s = s_0(\theta)$ for θ in a small neighborhood of $\theta = 0$. Therefore $\theta_1 \neq 0$. Observe the sign of $w + f(u)$ when u-coordinate touches u_s, then we have the following two case.

Lemma 7. *Suppose* $\mathbf{y}_0 = g(\theta)$ *for some* $\theta \in (0, \theta_1)$. *Then* $\mathbf{y}(s; \mathbf{y}_0)$ *will leave* W *through the boundary of* P *or* R.

Lemma 8. *There exists an* s_0 *such that*

$$u(s_0, \mathbf{y}_1) = u_s, w(s_0, \mathbf{y}_1) > w_s, \text{ and } v(s_0, \mathbf{y}_1) = 0.$$

Proof. We have $u(0, \mathbf{y}_1) \in (u_s, \gamma)$, $v(0, \mathbf{y}_1) \leq 0$ and $w(0, \mathbf{y}_1) > 0$. The lemma can be proved in the following four steps.

1. Suppose $u(s; \mathbf{y}_1) > u_s$ and $v(s; \mathbf{y}_1) < 0$ for all $s > 0$. Then we have $u(\infty; \mathbf{y}_1) \geq u_s$ and $v(\infty; \mathbf{y}_1) = 0$. By Lemma 6, we have $w(\infty; \mathbf{y}_1) = \infty$ and $v'(\infty; \mathbf{y}_1) = \infty$. It contradicts $v(\infty; \mathbf{y}_1) = 0$. Hence $u(s_0; \mathbf{y}_1) \leq u_s$ or $v(s_0; \mathbf{y}_1) \geq 0$ for some $s_0 > 0$.
2. Suppose $u(s_0; \mathbf{y}_1) = u_s$ and $v(s; \mathbf{y}_1) < 0$ for $s \in (0, s_0]$. By the Implicit Function Theorem and the continuous dependence of the solution on θ, there are $\theta \gtrsim \theta_1$ satisfying

$$v(s_0; g(\theta)) < 0 \text{ on } (0, s_0(\theta)] \text{ and } u(s_0(\theta); g(\theta)) = u_s.$$

This fact contradicts the definition of θ_1. Thus $v(s_0; \mathbf{y}_1) = 0, v(s; \mathbf{y}_1) < 0$ for $s \in (0, s_0)$, and $u(s_0; \mathbf{y}_1) \geq u_s$.
3. Since $v(s_0; \mathbf{y}_1) = 0$ and $v(s; \mathbf{y}_1) < 0$ on $s \in (0, s_0)$, we have $v'(s_0; \mathbf{y}_1) \geq 0$ and $w(s_0; \mathbf{y}_1) + f(u(s_0; \mathbf{y}_1)) \geq 0$. Suppose $w(s_0; \mathbf{y}_1) + f(u(s_0; \mathbf{y}_1) = 0$, then $dv''(s_0; \mathbf{y}_1) > 0$ by Lemma 6. This implies that $v(s; \mathbf{y}_1) \geq 0$ for $s \approx s_0$, which contradicts the definition of s_0. Therefore $w(s_0; \mathbf{y}_1) + f(u(s_0; \mathbf{y}_1)) > 0$ and $v'(s_0; \mathbf{y}_1) > 0$.
4. Suppose $u(s_0; \mathbf{y}_1) > u_*$. Since $v(s_0; \mathbf{y}_1) = 0$ and $v'(s_0; \mathbf{y}_1) > 0$, by the Implicit Function Theorem and the continuous dependence of the solution on θ, there exists a function $s_0(\theta)$ for $\theta \approx \theta_1$ such that

$$v(s_0(\theta); g(\theta)) = 0, v'(s_0(\theta); g(\theta)) > 0; v(s; g(\theta)) < 0 \text{ on } s \in (0, s_0(\theta));$$

and $u(s_0(\theta), g(\theta)) > u_s$ for $\theta \approx \theta_1$. Thus $\theta \notin A$ for $\theta \approx \theta_1$, a contradiction. Hence $u(s_0; \mathbf{y}_1) = u_s$. It follows from $w(s_0; \mathbf{y}_1) + f(u(s_0; \mathbf{y}_1)) > 0$ that $w(s_0; \mathbf{y}_1) > w_s$. The proof is complete. Q.E.D.

Lemma 9. *There is a value* θ_2 *such that the* v *coordinate of* $g(\theta_2)$ *is zero and, moreover,* $\theta_2 > \theta_1$. *Here, we denote* $\mathbf{y}_2 = g(\theta_2)$.

Proof. Since $g(0) \in \Omega_1$ in the region $u < \gamma$, the v coordinate of $g(0)$ is negative. There is a θ^*, $0 < \theta^* < 2\pi$, such that $g(\theta^*)$ is on Ω_1

with $u > \gamma$. The v coordinate of $g(\theta^*)$ is positive. Then there is a $\theta_2 > 0$ such that the v coordinate of $g(\theta_2)$ is zero. The proof of Lemma 7, part 2, says that for $s \geq 0$ the trajectory through $g(\theta_1)$ can never have a v coordinate equal to 0 if $u > \gamma$. Therefore $\theta_2 > \theta_1$. This completes the proof. Q.E.D.

On Ω_3, we consider a small sphere centered at $(\gamma, 0, 0, 0)$ with radius ε, which is parameterized by

$$S(\theta, \varphi) = f_3(\varepsilon \cos(\theta + \psi) \sin \varphi, \varepsilon \sin(\theta + \psi) \sin \varphi, \varepsilon \cos \varphi),$$

where $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$ and the constant phase ψ is the one in (8). By the Implicit Function Theorem, we have the following two results.

Lemma 10. *The sphere S intersects the hyperplane defined by $v = 0$ in a smooth closed curve.*

Lemma 11. *The sphere S intersects the hyperplane defined by $z = 0$ in a smooth closed curve.*

We denote the intersection of $\{v = 0\}$ and $\{z = 0\}$ on the sphere S by $\mathbf{y}_3$. Let S^+ be the hemisphere of S with the range of φ such that $\cos \varphi \geq 0$.

Notation 12. (1). *Let $\mathbf{y}_0 := g(0)$ be to the intersection of the sphere with Ω_1 in the region $0 < u < \gamma$.*

(2). *Denote by $\widehat{\mathbf{y}_0\mathbf{y}_i}$, $i = 1, 2$ the portion of the circle with $\theta \in (0, \theta_i)$.*

(3). *Denote by $\widehat{\mathbf{y}_2\mathbf{y}_3}$ the portion of the intersection of the hemisphere S^+ with $\{v = 0\}$ lying between (not including) $\mathbf{y}_2$ and $\mathbf{y}_3$.*

(4). *Denote by $\widehat{\mathbf{y}_3\mathbf{y}_0}$ the portion of the intersection of hemisphere S^+ with $\{z = 0\}$ lying between (not including) $\mathbf{y}_3$ and $\mathbf{y}_0$.*

(5). *Let B be a small ball around $\mathbf{y}_0$ in the space spanned by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Let $\mathbf{y}_4$ and $\mathbf{y}_5$ be the interaction points of B with $\widehat{\mathbf{y}_3\mathbf{y}_0}$ and $\widehat{\mathbf{y}_2\mathbf{y}_2}$ respectively. Denote by $\widehat{\mathbf{y}_4\mathbf{y}_5}$ the portion of interaction of the hemisphere with B (not including $\mathbf{y}_4$, $\mathbf{y}_5$).*

Now we construct Σ as the closed topological quadrangle in the hemisphere, whose sides consist of the closure of the arcs $\widehat{\mathbf{y}_i\mathbf{y}_{i+1}}$, $i = 1, 2, 3, 4$ and $\widehat{\mathbf{y}_5\mathbf{y}_1}$. To check Σ connected but $F(\Sigma)$ not connected is tedious, we omit here. By the *Wazewski*'s Theorem, we get an invariant orbit.

Lemma 13. *There exists $\mathbf{y}^* \in \Sigma$ such that the solution $\mathbf{y}(s, \mathbf{y}^*) = \bar{\mathbf{y}}(s) = (\bar{u}(s), \bar{v}(s), \bar{w}(s), \bar{z}(s))$ remains the region W.*

Further, we have the upper and lower bound of $\bar{u}$ and $\bar{w}$.

Lemma 14. *The coordinate functions $\bar{u}(s)$ and $\bar{w}(s)$ are positive for all time.*

Lemma 15. *The coordinate functions $\bar{u}(s)$ and $\bar{w}(s)$ are bounded above by γ, L, respectively, where L is a positive constant.*

It is easy to check that $\bar{\mathbf{y}}(s)$ is a bounded orbit.

Lemma 16. *Let*

$$\Omega = \{(u,v,w,z)|0 < u < \gamma,\ 0 < w < L, \\ -w/c < z < qw,\ -K_1 u < v < K_2 u\}$$

where $q > \max\{\frac{-c+\sqrt{c^2+4}}{2}, \lambda_2, \lambda_3\}$, $K_1 > \max\{\frac{L}{c}, \frac{\sqrt{c^2+4dL}-c}{2d}\}$, and $K_2 > (-c+\sqrt{c^2+4dL})/2$. Then the solution $\mathbf{y}(s,\mathbf{y}^*) = \bar{\mathbf{y}}(s)$ remains in Ω for all s.

Lemma 17. *This trajectory $\mathbf{y}(s,\mathbf{y}_*)$ approaches $(u_s,0,w_s,0)$ as s approaches ∞.*

Proof. We construct a *Lyapunov* function as follows,

$$V = cu - dv + c\frac{u_s^2}{u} + d\frac{u_s^2 v}{u^2} + u_e^2[c(w-w_s) - z] + u_s^2 w_s[\frac{z}{w} - c\log\frac{w}{w_s}].$$

By the LaSalle's Invariant Principle [6], the ω-limit set of $\mathbf{y}(s,\mathbf{y}_*)$ is contained in the largest invariant subset of $\{y \in \Omega : dV/ds = 0\}$, which is the singleton $(u_s,0,w_s,0)$. It follows that $\mathbf{y}(\infty,\mathbf{y}_*) = (u_s,0,w_s,0)$. Q.E.D.

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