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Behaviors of a front-back pulse arising in a bistable medium with jump-type heterogeneity

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Abstract.

We consider the dynamics of an oscillatory pulse (standing breather, SB) of front-back type, in which the motion of two interfaces that interact through a continuous field is described by a mixed ODE-PDE system. We carry out a center manifold reduction around the Hopf singularity of a stationary pulse solution, which provides us insight into the underlying mechanism for a sliding motion of SB in a jump-type spatial heterogeneous medium.

§1. Introduction

Spatially localized moving patterns such as pulses or spots typically arise in reaction-diffusion (RD) systems. In heterogeneous media, such moving patterns exhibit a variety of behavior, not observed in spatially homogeneous media. In excitable systems, for instance, traveling pulses exhibit various outputs such as repulsion, pinned-oscillation and splitting when they encounter heterogeneities [9], [10], [11], [12]. On the other hand, bistable systems usually have a front solution that connects two spatially uniform states. We are especially interested in a concatenated type of pulse solutions called a front-back pulse, namely two interfaces (front and back) are concatenated to form a pulse shape. Such a pulse solution exhibits not just traveling but also oscillatory motion of interfaces depending on the parameters.

In the paper [8], we investigated the behavior of a front-back pulse in a certain bistable system with a jump-type spatial heterogeneity. Our main concern is to study the influence of heterogeneity on the dynamics of localized patterns from a dynamical system point of view. One of

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the heterogeneity-induced behavior is shown in Fig. 1, in which an oscillatory pulse (standing breather, SB) initially placed away from the jump point never stays at one point but slowly moves to the negative direction of x-axis. The duration time before crossing the jump point goes to infinity with increasing initial distance between SB and the jump point. In this short article, we focus on this sliding motion of an SB and clarify analytically its underlying mechanism by deriving the equations of motion by means of center manifold reduction.

The model system employed here is a mixed ODE-PDE system in one space dimension

$$\text{(1)} \qquad \begin{cases} \dot{\phi_2} = -\dfrac{v(\phi_2)}{\sqrt{2}\tau}, \quad \dot{\phi_1} = \dfrac{v(\phi_1)}{\sqrt{2}\tau}, \\ v_t = Dv_{xx} + u(x;\phi_2,\phi_1) - v + \theta(x), \end{cases}$$

where the component $v = v(t,x)$ depends on time t and space x. This mixed ODE-PDE system, called a hybrid system (HS), is obtained as a reduced system of a certain two-component bistable RD system [4] by representing the locations of the two interfaces of a front-back pulse by $\phi_2(t)$ and $\phi_1(t)$ ($\phi_2 > \phi_1$). The function $u(x;\phi_2,\phi_1)$, which corresponds to the profile of the activator component in terms of the original RD system, is given by

$$u(x;\phi_2,\phi_1) := H(x-\phi_1) - H(x-\phi_2) - 1/2,$$

where $H(x) = 0 \ (x<0), 1 \ (x \geq 0)$ is the Heaviside function. The two interfaces form a localized domain of high activator concentration of $u = +1/2$ and hence a high concentration domain in the slowly varying v field of the inhibitor. The front and back interfaces interact each other through the spatially varying field of v thus formed. The parameters $\tau > 0$, $0 < \epsilon \ll 1$ and $D > 0$ are fixed at some appropriate values. The well-posedness of such a mixed ODE-PDE system is proved in [6]. The function $\theta(x)$ denotes the spatial heterogeneity of jump-type:

$$\text{(2)} \qquad \theta(x) = \theta^L + \zeta_H - \eta_2 + (\eta_2 - \eta_1)H(-x + \bar{x}).$$

The parameter $\theta(x)$ jumps from $\theta^L + \zeta_H - \eta_1$ to $\theta^L + \zeta_H - \eta_2$ at $x = \bar{x}$, $\eta_1 - \eta_2$ being the jump height of the heterogeneity. Let us briefly explain more about the notations in Eq. (2). For the homogeneous case $\theta(x) \equiv \theta^L + \zeta$ with some constant ζ, we can analytically obtain the stationary pulse (SP) solution of Eqs. (1) which are stable for ζ sufficiently large [7]. When ζ is decreased, the SP loses its stability via Hopf bifurcation at $\zeta = \zeta_H$. Thus, Eq. (2) represents the situation where $\theta(x)$ is slightly deviated from the bifurcation point $\theta^L + \zeta_H$ by η_1

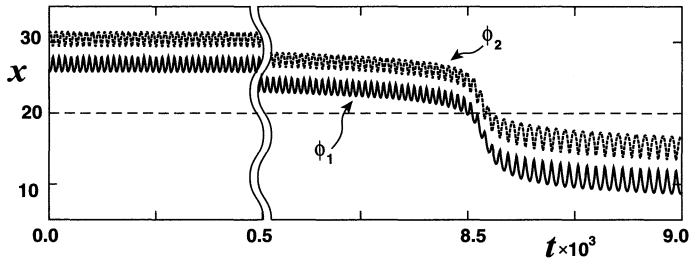

Fig. 1. Sliding motion of a standing breather in Eqs.(1) with (2) for $\eta_1 - \eta_2 > 0$. The black and gray curves indicate the interface positions of ϕ_1 and ϕ_2, respectively. The horizontal broken line indicates the jump point of $\overline{x} = 20$

for $x < \bar{x}$ and by η_2 for $x > \bar{x}$. We illustrate our problem setting in Fig. 2(a) and suppose $0 < \eta_1, \eta_2 \ll 1$ and $\eta_1 - \eta_2 > 0$, which corresponds to a jump-up situation. For this heterogeneous case, we numerically observe an SB solution that slowly slides to the negative direction of x-axis (See Fig. 1). However, the sliding velocity is so slow that it is difficult to judge only by numerical simulation whether such a sliding motion really occurs when we initially set an SB farther away from the jump point.

In what follows, we derive a system of finite dimensional ordinary differential equations (ODEs) by performing center manifold reduction [1], [5] for the pulse motion, which inherits the essential features of the pulse dynamics near the onset of the bifurcation. In Section 2, we analytically confirm the super-criticality of the Hopf bifurcation and hence the existence of a stable SB near the onset. In Section 3, we shall see that following a similar reduction procedure leads us to ODEs in the presence of a jump-type heterogeneity.

§2. Reduction from the hybrid system to ODEs near Hopf singularity

In this section, we formally perform a center manifold reduction for the homogeneous system of (1) with $\theta(x) \equiv \theta^L + \zeta$ at the Hopf bifurcation point $\zeta = \zeta_H$ of an SP solution, assuming the existence and (some finite) smoothness of a center manifold at the singularity.

Let ζ be slightly deviated from the Hopf bifurcation point ζ_H as $\zeta = \zeta_H + \eta$ where $|\eta| \ll 1$. Let us rewrite Eqs.(1) with $\theta(x) = \theta^L + \zeta$

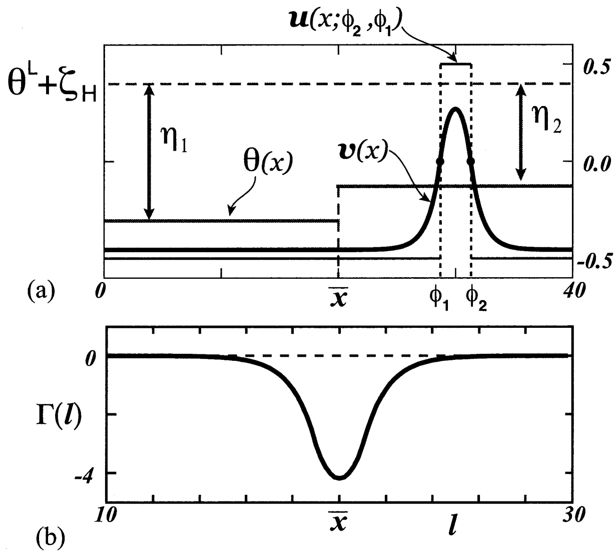

Fig. 2. (a) Schematic illustration of our problem setting. The thick curve and thin lines indicate the pulse profile given by the v-field and the function $u(x;\phi_2,\phi_1)$, respectively. We initially set the front-back pulse at the upper side domain of $x \geq \overline{x}$. The gray lines indicate a jump type heterogeneity function $\theta(x)$. (b) Typical profile of $\Gamma(l)$ of Eq.(6) with $(\tau, D) = (0.170, 1.0)$

formally into

$$\dot{U} = F(U;\zeta) = F(U;\zeta_H) + \eta\, G(U;\zeta), \tag{3}$$

where $U = (\phi_2(t), \phi_1(t), v(x,t))$ and $\eta G(U;\zeta) := F(U;\zeta_H+\eta) - F(U;\zeta_H)$. Let P_0 denote the SP at $\zeta = \zeta_H$ which satisfies $F(P_0;\zeta_H) = 0$. Assuming that the solution for Eq.(3) can be approximated by $U = P_0 + P$, $F(U;\zeta_H)$ is expanded around P_0 as $F(U;\zeta_H) = \mathbf{L}P + \mathbf{M}P^2 + \mathbf{N}P^3 + \cdots$.

On the spectrum of $\mathbf{L}$, the following two properties are assumed:
(H1) Let Σ_c denote the spectrum of $\mathbf{L}$. $\Sigma_c = \Sigma_1 \cup \Sigma_2$ where $\Sigma_1 := \{0, \pm i\Omega\}$ $(\Omega > 0)$ and $\Sigma_2 \subset \{z \in \mathbf{C}; Re(z) < -\gamma_0\}$ for some positive constant γ_0.
(H2) Let Φ and Ψ denote the eigenfunctions associated with 0 and $i\Omega$, respectively, such that $\mathbf{L}\Phi = 0$ and $\mathbf{L}\Psi = i\Omega\Psi$. The adjoint operator $\mathbf{L}^*$ satisfies similar relation: There exist Φ^* and Ψ^* such that $\mathbf{L}^*\Phi^* = 0$ and $\mathbf{L}^*\Psi^* = -i\Omega\,\Psi^*$. The eigenfunctions Φ and Φ^* are odd, whereas Ψ

and Ψ^* are even.

The eigenfunctions are arbitrary by factors of constant, so that we may set $\langle \Phi^*, \Phi \rangle = \langle \Psi^*, \Psi \rangle = 1$ to eliminate the arbitrariness. Note that $\langle \Phi^*, \Psi \rangle = \langle \Phi^*, \bar{\Phi} \rangle = \langle \Psi^*, \Phi \rangle = \langle \Psi^*, \bar{\Psi} \rangle = 0$ automatically hold. Let the eigenspace associated with Σ_1 be $E = E_1 \oplus E_2$ where $E_1 = \text{span}\{\Phi\}$ and $E_2 = \text{span}\{\Psi, \bar{\Psi}\}$. Under these assumptions, we can derive ODEs for pulse motion based on the center manifold theory [1], [5].

Proposition 2.1 Assume that **(H1)** and **(H2)** hold for (1). Furthermore, assume that the solution to Eq.(3) is close to the SP solution $P_0(x)$ and given by $U(x,t) = T(l(t)) \cdot \{P_0(x) + z(t)\Psi(x) + \bar{z}(t)\bar{\Psi}(x) + W(x,t)\}$ where $l(t) \in \mathbf{R}$, $z(t) \in \mathbf{C}$ and $W(x,t) \in E^{\perp}$ is the residual term of the order of $\mathcal{O}(|z|^2 + |\eta|)$. The operator $T(l)$ translates the pulse location (i.e., $T(l) \cdot P_0(x) \equiv P_0(x - l)$). Then the equations for l and z near the Hopf bifurcation point ζ_H are locally topologically equivalent to the system

$$\begin{cases} \dot{l} = 0, \\ \dot{w} = (\alpha\eta + i)\, w + \beta w^2 \bar{w}, \end{cases} \tag{4}$$

where α and β are real constants and $w = z + \mathcal{O}(\bar{z}\eta)$.

Proof. Since the proof can be done in a parallel way to [2], [3], we sketch its derivation briefly here. See for details [8]. Expand $W(x,t)$ further as $W(x,t) = z^2 H_{200}(x) + z\bar{z}H_{110}(x) + \bar{z}^2 \bar{H}_{200}(x) + \eta H_{001}(x) + W^*(x,t)$ where $W^*(x,t) = \mathcal{O}(|z|^3 + |\eta|^2)$. H_{ijk} are determined by solving

$$\begin{aligned} (\mathbf{L} - 2i\Omega)H_{200} + \Pi_{200} &= \alpha_{200}\Psi + \bar{\alpha}_{200}\bar{\Psi}, \\ \mathbf{L}H_{110} + \Pi_{110} &= \alpha_{110}\Psi + \bar{\alpha}_{110}\bar{\Psi}, \\ \mathbf{L}H_{001} + \Pi_{001} &= \alpha_{001}\Psi + \bar{\alpha}_{001}\bar{\Psi}, \end{aligned}$$

where $\Pi_{200} = \mathbf{M}\Psi^2$, $\Pi_{110} = 2\mathbf{M}\Psi\bar{\Psi}$, $\Pi_{001} = G(P_0; \zeta_H)$. The constants α_{ijk} satisfy the following equations:

$$\begin{aligned} \langle \Psi^*, \Pi_{200} - \alpha_{200}\Psi \rangle &= 0, \quad \langle \Psi^*, \Pi_{110} - \alpha_{110}\Psi \rangle = 0, \\ \langle \Psi^*, \Pi_{001} - \alpha_{001}\Psi \rangle &= 0. \end{aligned}$$

Substituting the above ansatz into Eq. (3) and taking inner products with Φ^* and Ψ^*, we obtain

$$\begin{cases} \dot{l} = & 0 + \mathcal{O}(|z|^3 + |\eta|^2), \\ \dot{z} = & i\Omega z + g_{101} z\eta + \bar{g}_{101}\bar{z}\eta + g_{300}z^3 + g_{210}z^2\bar{z} + \bar{g}_{210}z\bar{z}^2 + \bar{g}_{300}\bar{z}^3 \\ & +\mathcal{O}(|z|^4 + |\eta|^2), \end{cases}$$

where g_{ijk} are complex constants independent of η. Especially, g_{210} and g_{101} are given by

$$\begin{aligned} g_{210} &= \left\langle \Psi^*, 3\mathbf{N}\Psi^2\bar{\Psi} + 2\mathbf{M}\Psi H_{110} + 2\mathbf{M}\bar{\Psi}H_{200} \right\rangle, \\ g_{101} &= \left\langle \Psi^*, 2\mathbf{M}\Psi H_{001} \right\rangle. \end{aligned}$$

By appropriate variable transformation and time rescaling, the equations can be rewritten as

$$\begin{cases} \dot{l} = & 0 + \mathcal{O}(|w|^4 + |\eta|^2), \\ \dot{w} = & (\alpha\eta + i)\, w + \beta w^2 \bar{w} + \mathcal{O}(|w|^4 + |\eta|^2), \end{cases}$$

where $\alpha = Re\{g_{101}\}/\Omega$ and $\beta = Re\{g_{210}\}/\Omega$. Q.E.D.

The resulting ODEs tell us about the asymptotic behavior of the pulse solution near Hopf singularity as well as its bifurcation properties for the original system. The information about the original system is encoded in the coefficients α and β, which are analytically confirmed as $\alpha < 0$ and $\beta < 0$ for the hybrid system. This result indicates that a stable SB appears for $\eta < 0$ via the super-critical Hopf bifurcation. The details of the calculations are shown in [8], which completes the above proof. In the next section, we extend the center manifold reduction procedure to the case in the presence of the jump-type heterogeneity and investigate the resulting ODEs, which reveals the underlying mechanism behind the sliding motion of an SB.

§3. Mechanism for sliding motion of standing breather

In the previous section, it was shown under natural assumptions that, for the homogeneous case, a stable SB exists near the Hopf bifurcation point. In this section, we consider how the motion of an SB is influenced by the jump-type heterogeneity. Recall that the deviations from the Hopf bifurcation point, η_1 and η_2, are small in both regions for $x > \bar{x}$ and $x < \bar{x}$, and that the difference $\eta_1 - \eta_2$ is sufficiently small (See Fig. 2(a)). This condition allows us to regard the term $(\eta_2 - \eta_1)H(-x + \bar{x})$ in Eq.(2) as a perturbation to the homogeneous system around the Hopf singularity at $\zeta = \zeta_H$, so that the calculations demonstrated in the previous section are readily extended to the heterogeneous case to derive the ODEs which describe the motion of an SB in the heterogeneous medium.

Proposition 3.1 Suppose that $\theta(x)$ in Eqs.(1) is given by $\theta(x) = \theta^L + \zeta_H - \eta_2 + (\eta_2 - \eta_1)H(-x + \bar{x})$ where $H(x)$ is the Heaviside function.

If $|\eta_1|, |\eta_2| \ll 1$, then the asymptotic motion of the pulse location $l(t)$ is given by

$$\dot{l} = (\eta_1 - \eta_2)\Gamma(l), \tag{5}$$

where $\Gamma(l)$ is a negative-valued function symmetric about $l = \bar{x}$. It has one minimum at $l = \bar{x}$ and decays exponentially fast as $|l| \to \infty$. We can easily prove this proposition by replacing η in the proof of Proposition 2.1 by $-\eta_2 + (\eta_2 - \eta_1)H(-x + \bar{x})$. See [8] for details.

The function $\Gamma(l)$ is explicitly calculated for the hybrid system as

$$\Gamma(l) = k_2 \times \begin{cases} \dfrac{-e^{\{l-(\bar{x}-\phi_1^{(0)})\}/\sqrt{D}} + e^{\{l-(\bar{x}-\phi_2^{(0)})\}/\sqrt{D}}}{2\sqrt{2\tau}}, & \left(l \le \bar{x} - \phi_2^{(0)}\right), \\ \dfrac{-e^{\{l-(\bar{x}-\phi_1^{(0)})\}/\sqrt{D}} - e^{-\{l-(\bar{x}-\phi_2^{(0)})\}/\sqrt{D}}}{2\sqrt{2\tau}} + \dfrac{2}{2\sqrt{2\tau}}, & \\ & \left(\bar{x} - \phi_2^{(0)} \le l \le \bar{x} - \phi_1^{(0)}\right), \\ \dfrac{e^{-\{l-(\bar{x}-\phi_1^{(0)})\}/\sqrt{D}} - e^{-\{l-(\bar{x}-\phi_2^{(0)})\}/\sqrt{D}}}{2\sqrt{2\tau}}, & \left(l \ge \bar{x} - \phi_1^{(0)}\right), \end{cases}$$

where k_2 is some negative constant and $\phi_1^{(0)}$ and $\phi_2^{(0)}$ are the locations of the SP solution at the bifurcation point. We see that $\Gamma(l)$ is axisymmetric about $l = l^* := \bar{x} - \left(\phi_1^{(0)} + \phi_2^{(0)}\right) / 2$ and negative for $^\forall l \in \mathbf{R}$ that decays exponentially fast as $l \to \pm\infty$. We can set $\phi_2^{(0)} = -\phi_1^{(0)}$ without loss of generality, thus putting $l^* = \bar{x}$ simply. The typical profile of $\Gamma(l)$ is shown in Fig. 2(b).

The equation (5) for the pulse location, together with the explicit form of $\Gamma(l)$, shows us that the SB always slides to the left ($\dot{l} < 0$ for all l), and that the velocity of the sliding motion becomes fast in the region around the jump point of $l = \bar{x}$ and exponentially slower as the pulse moves away from it, in qualitatively good agreement with numerical observation of Fig. 1. Therefore, we conclude that the sliding motion of an SB occurs as long as there exists a jump step of heterogeneity, i.e., $\eta_1 - \eta_2 \neq 0$, however small it is.

The underlying mechanism for a sliding motion of SB is summarized as follows: The heterogeneity breaks the translational invariance that an SB holds in a homogeneous medium, preventing it from staying at one point. Thus the SB feels a force that drives it to the negative direction of x-axis for the jump-up case.

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