Advanced Studies in Pure Mathematics 64, 2015
Nonlinear Dynamics in Partial Differential Equations
pp. 461–468

A remark on self-similar solutions for a semilinear heat equation with critical Sobolev exponent

Yūki Naito

Abstract.

The Cauchy problem for a semilinear heat equation

$$w_t = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0, \infty)$$

with singular initial data $w(x, 0) = \lambda a\,(x/|x|)\,|x|^{-2/(p-1)}$ for $x \in \mathbf{R}^N \setminus \{0\}$ is studied, where $N > 2$, $p = (N+2)/(N-2)$, $\lambda > 0$ is a parameter, and $a \geq 0$, $a \not\equiv 0$. We investigate the asymptotic properties of the profile of positive self-similar solutions to the problem as $\lambda \to 0$ when $N = 3, 4, 5$.

§1. Introduction

We consider the Cauchy problem for a semilinear heat equation with singular initial data:

$$(1) \qquad \begin{cases} w_t = \Delta w + w^p & \text{in } \mathbf{R}^N \times (0, \infty), \\ w(x, 0) = \lambda a\,(x/|x|)\,|x|^{-2/(p-1)} & \text{in } \mathbf{R}^N \setminus \{0\}, \end{cases}$$

where $N > 2$, $p = (N+2)/(N-2)$, $a : S^{N-1} \to \mathbf{R}$, and $\lambda > 0$ is a parameter. We assume that $a \in L^\infty(S^{N-1})$ and $a \geq 0$, $a \not\equiv 0$. The equation in (1) is invariant under the similarity transformation

$$w(x, t) \mapsto w_\mu(x, t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$

A solution w is said to be *self-similar*, when $w(x, t) = w_\mu(x, t)$ for all $\mu > 0$. It can be easily checked that w is a forward self-similar solution to (1) if and only if w has the form

$$w(x, t) = t^{-1/(p-1)} u(x/\sqrt{t}) \quad \text{for } x \in \mathbf{R}^N,\ t > 0,$$

Received December 14, 2011.
2010 *Mathematics Subject Classification.* 35K15, 35J60.
Key words and phrases. Self-similar solutions, critical Sobolev exponent.

where u is a solution of the problem

$$(2)\quad \begin{cases} \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + u^p = 0 & \text{in } \mathbf{R}^N, \\ \lim_{r\to\infty} r^{2/(p-1)}u(r\omega) = \lambda a(\omega) & \text{for a.e. } \omega \in S^{N-1}. \end{cases}$$

(More precisely, see [7, Lemma B.1 in Appendix B].

First we recall the results in [7] and [8] for the multiple existence of positive solutions of (2). We call a positive minimal solution $\underline{u}_\lambda$ of (2), if $\underline{u}_\lambda$ satisfies $\underline{u}_\lambda \le u_\lambda$ for any positive solution u_λ of (2).

Theorem A ([7, Theorem 1]). *There exists a constant $\lambda^* > 0$ such that,*

(i) *for $0 < \lambda < \lambda^*$, the problem (2) has a positive minimal solution $\underline{u}_\lambda \in C^2(\mathbf{R}^N)$; $\underline{u}_\lambda$ is increase with respect to λ and satisfies $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} = O(\lambda)$ as $\lambda \to 0$.*

(ii) *for $\lambda > \lambda^*$, there are no positive solutions $u \in C^2(\mathbf{R}^N)$ of (2).*

Theorem B ([8, Theorem 1.2]). *Let $N = 3, 4, 5$. Then, for $0 < \lambda < \lambda^*$, the problem (2) has a positive solution $\overline{u}_\lambda \in C^2(\mathbf{R}^N)$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$.*

Remark. (i) In the case $a \equiv 1$ in (2), the multiple existence of positive solution of (2) was studied in [9] by employing ODE shooting argument.

(ii)For the existence of self-similar solutions of (1), we refer to [1], [4], [5].

In this note we consider the asymptotic properties of the second positive solution $\overline{u}_\lambda$ as $\lambda \to 0$.

Theorem 1. *Let $N = 3, 4, 5$, and let $\overline{u}_\lambda$ be the positive solution obtained in Theorem B for $0 < \lambda < \lambda^*$. Then*

$$(3)\quad \overline{u}_\lambda \to 0 \quad \text{a.e. in } \mathbf{R}^N \quad \text{and} \quad \|\overline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} \to \infty \quad \text{as } \lambda \to 0.$$

Remark. (i) When $N \ge 6$ and $a \equiv 1$ in (2), it was shown by [8, Theorem 1.3] that (2) has no positive radially symmetric solutions u with $u \not\equiv \underline{u}_\lambda$ for $0 < \lambda < \lambda_*$ with some $\lambda_* \in (0, \lambda^*)$.

(ii) In the case $(N+2)/N < p < (N+2)/(N-2)$ it was shown by [7, Theorem 2] that the problem (2) has at least two positive solutions $\overline{u}_\lambda$ and $\underline{u}_\lambda$ with $\overline{u}_\lambda > \underline{u}_\lambda$ for $0 < \lambda < \lambda^*$, and $\overline{u}_\lambda \to u_0$ as $\lambda \to 0$, where u_0 is the unique positive solution of (2) with $\lambda = 0$. It was shown by [6] that $u_0(x)$ is radially symmetric about the origin, and has an exponential decay at $|x| = \infty$. The uniqueness of positive solution u_0 was shown by [10] and [2].

In order to investigate properties of the second positive solution to the problem (2), we introduce the following problem

$$(4)\qquad \begin{cases} \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + (u + \underline{u}_\lambda)^p - \underline{u}_\lambda^p = 0 \quad \text{in } \mathbf{R}^N, \\ \lim_{|x|\to\infty} |x|^{2/(p-1)}u(x) = 0, \end{cases}$$

for $0 < \lambda < \lambda^*$. We see that, if u_λ is a positive solution of (4), then $\overline{u}_\lambda = \underline{u}_\lambda + u_\lambda$ is the second solution of (2). In the proof of Theorem 1, we will investigate some properties of the solution u_λ obtained by the variational argument. For more precise asymptotic properties of solutions we will study in the forthcoming paper.

§2. Proof of Theorem 1

We first introduce some notations. Set $\rho(x) = e^{|x|^2/4}$. We define

$$L^q_\rho(\mathbf{R}^N) = \left\{ u \in L^q(\mathbf{R}^N) : \int_{\mathbf{R}^N} u^q \rho dx < \infty \right\} \quad \text{for } 1 \le q < \infty,$$

and

$$H^1_\rho(\mathbf{R}^N) = \left\{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} (|\nabla u|^2 + u^2)\rho dx < \infty \right\}.$$

The norms in $L^q_\rho(\mathbf{R}^N)$ and $H^1_\rho(\mathbf{R}^N)$, respectively, are defined by

$$\|u\|_{L^q_\rho} = \left(\int_{\mathbf{R}^N} u^q \rho dx \right)^{1/q} \quad \text{and} \quad \|u\|_{H^1_\rho} = \|\nabla u\|_{L^2_\rho} + \|u\|_{L^2_\rho}.$$

We consider the problem

$$(5)\qquad \begin{cases} \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + g(u, \underline{u}_\lambda) = 0 \quad \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 \quad \text{in } \mathbf{R}^N, \end{cases}$$

where $g(t,s) = (t+s)^p - s^p$. Define the corresponding variational functional of (5) by

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1}u^2 \right) \rho dx - \int_{\mathbf{R}^N} G(u, \underline{u}_\lambda) \rho dx$$

with $u \in H^1_\rho(\mathbf{R}^N)$, where

$$G(t,s) = \frac{1}{p+1}(t+s)^{p+1} - \frac{1}{p+1}s^{p+1} - s^p t.$$

We recall the existence of positive solution to the problem (5).

Proposition 1. *Let* $N = 3, 4, 5$. *For* $\lambda \in (0, \lambda^*)$ *there exists a positive solution* $u_\lambda \in H^1_\rho(\mathbf{R}^N) \cap C^2(\mathbf{R}^N)$ *of* (5) *satisfying*

$$0 < I_\lambda(u_\lambda) < \frac{1}{N} S_\rho^{N/2} \quad \text{and} \quad \liminf_{\lambda \to 0} I_\lambda(u_\lambda) > 0, \tag{6}$$

where

$$S_\rho = \inf_{u \in H^1_\rho(\mathbf{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbf{R}^N} |\nabla u|^2 \rho dx}{\left(\displaystyle\int_{\mathbf{R}^N} |u|^{2N/(N-2)} \rho dx\right)^{(N-2)/N}}.$$

Since the existence of positive solution $u_\lambda \in H^1_\rho(\mathbf{R}^N) \cap C^2(\mathbf{R}^N)$ of (5) was shown by [8] (see the proof of Proposition 3.2 in [8]), it suffices to show that u_λ satisfies (6) for the proof of Proposition 1. We show the following lemma.

Lemma 1. *Let* $\lambda \in (0, \lambda^*)$. *Then there exist some constants* $\delta = \delta(\lambda) > 0$ *and* $\eta = \eta(\lambda) > 0$ *such that*

$$I_\lambda(u) \geq \eta(\lambda) > 0 \tag{7}$$

for all $u \in H^1_\rho(\mathbf{R}^N)$ *with* $\|\nabla u\|_{L^2_\rho} = \delta(\lambda)$. *Furthermore,* $\eta(\lambda)$ *satisfies* $\liminf_{\lambda \to 0} \eta(\lambda) > 0$.

Proof. We note that the conclusion of Lemma 5.5 in [7] still holds when $p = (N+2)/(N-2)$. Then, for each $\lambda \in (0, \lambda^*)$, there exist constants $\eta(\lambda)$ and $\delta(\lambda)$ such that (7) holds for all $u \in H^1_\rho(\mathbf{R}^N)$ with $\|\nabla u\|_{L^2_\rho} = \delta(\lambda)$.

Let $\lambda_0 \in (0, \lambda^*)$ be fixed, and let $u \in H^1_\rho(\mathbf{R}^N)$. Now we will show that

$$I_\lambda(u) \geq I_{\lambda_0}(u) \quad \text{for } \lambda \in (0, \lambda_0]. \tag{8}$$

We see that $G(t, s)$ is increasing in $s > 0$ for each fixed $t > 0$. Since $\underline{u}_\lambda$ is increasing in $\lambda > 0$, $G(u, \underline{u}_\lambda)$ is increasing in $\lambda > 0$ for each $u \in H^1_\rho(\mathbf{R}^N)$. Thus (8) holds. Now, put $\eta(\lambda) = \eta(\lambda_0)$ and $\delta(\lambda) = \delta(\lambda_0)$ for $\lambda \in (0, \lambda_0]$. Then (7) holds for $\lambda \in (0, \lambda_0]$, and we obtain $\liminf_{\lambda \to 0} \eta(\lambda) = \eta(\lambda_0) > 0$. Q.E.D.

Proof of Proposition 1. Following the proof of Proposition 3.2 in [8], there exists a weak solution $u_\lambda \in H^1_\rho(\mathbf{R}^N)$ of (5), and u_λ satisfies

$$0 < \eta(\lambda) \leq I_\lambda(u_\lambda) < \frac{1}{N} S_\rho^{N/2} \quad \text{for each } \lambda \in (0, \lambda^*).$$

By Lemma 1 we have $\liminf_{\lambda\to 0} I_\lambda(u_\lambda) \geq \liminf_{\lambda\to 0} \eta(\lambda) > 0$. Thus (6) holds. Q.E.D.

For $\lambda \in (0, \lambda^*)$, let u_λ be a solution of (5) obtained in Proposition 1, and let $\{\lambda_k\}$ be a sequence such that $\lambda_k > \lambda_{k+1}$ and $\lambda_k \to 0$ as $k \to \infty$. For simplicity, one sets $u_k = u_{\lambda_k}$ and $\underline{u}_k = \underline{u}_{\lambda_k}$. We will show the following

Proposition 2. *There exists a subsequence, still denoted by $\{u_k\}$, such that, as $k \to \infty$,*

(i) $u_k \rightharpoonup 0$ *weakly in* $H^1_\rho(\mathbf{R}^N)$, $u_k \to 0$ *strongly in* $L^2_\rho(\mathbf{R}^N)$, *and* $u_k \to 0$ *a.e. in* $\mathbf{R}^N$;

(ii) $\|u_k\|_{L^\infty(\mathbf{R}^N)} \to \infty$.

To prove Proposition 2, we show the following lemma.

Lemma 2. *Assume that $u_k \rightharpoonup u_0$ weakly in $H^1_\rho(\mathbf{R}^N)$ as $k \to \infty$ for some $u_0 \in H^1_\rho(\mathbf{R}^N)$. Then, for any $\phi \in H^1_\rho(\mathbf{R}^N)$,*

$$\int_{\mathbf{R}^N} g(u_k, \underline{u}_k)\phi\rho dx \to \int_{\mathbf{R}^N} u_0^p \phi\rho dx \quad \text{as } k \to \infty. \tag{9}$$

Proof. By the argument in the proof of Lemma 2.4 in [8], for any fixed integer k_0, we have

$$\int_{\mathbf{R}^N} \left| g(u_k, \underline{u}_{k_0}) - g(u_0, \underline{u}_{k_0}) \right| \phi\rho dx \to 0 \quad \text{as } k \to \infty. \tag{10}$$

From $\lambda_k > \lambda_{k+1}$, $k = 1, 2, \ldots$, it follows that

$$\underline{u}_k \leq \underline{u}_{k_0} \quad \text{for } k \geq k_0. \tag{11}$$

Since $|g(t_1, s) - g(t_2, s)| = |(t_1 + s)^p - (t_2 + s)^p|$ is nondecreasing in $s > 0$ for each fixed $t_1,\ t_2 > 0$, we obtain

$$|g(u_k, \underline{u}_k) - g(u_0, \underline{u}_k)| \leq |g(u_k, \underline{u}_{k_0}) - g(u_0, \underline{u}_{k_0})| \quad \text{for } k \geq k_0. \tag{12}$$

Then, from (10) and (12), we deduce that

$$\int_{\mathbf{R}^N} |g(u_k, \underline{u}_k) - g(u_0, \underline{u}_k)| \phi\rho dx \to 0 \quad \text{as } k \to \infty. \tag{13}$$

By Lemma 2.1 in [8], for $s_0 > 0$, there exists a constant $C = C(s_0) > 0$ such that

$$g(t, s) \leq C(t + t^p) \quad \text{for } t \geq 0,\ 0 \leq s \leq s_0.$$

Put $s_0 = \|\underline{u}_{k_0}\|_{L^\infty(\mathbf{R}^N)}$. From (11) we obtain

$$g(u_0, \underline{u}_k) \leq C(u_0 + u_0^p) \quad \text{for } k \geq k_0.$$

Note that C is independent of k, and that $g(u_0, \underline{u}_k) \to u_0^p$ a.e. in $\mathbf{R}^N$ as $k \to \infty$. Then, by Lebesgue convergence theorem, we have

$$\int_{\mathbf{R}^N} g(u_0, \underline{u}_k)\phi\rho dx \to \int_{\mathbf{R}^N} u_0^p \phi\rho dx \quad \text{as } k \to \infty. \tag{14}$$

Combining (13) and (14) we obtain (9). Q.E.D.

Proof of Proposition 2. (i) Proposition 1 implies that $I_{\lambda_k}(u_k)$ is bounded for $k = 1, 2, \ldots$. By the same argument as in the first step of Proof of Proposition 5.2 in [7], we deduce that $\{u_k\}$ is bounded in $H_\rho^1(\mathbf{R}^N)$. Thus there exist a subsequence, still denoted by $\{u_k\}$, and some $u_0 \in H_\rho^1(\mathbf{R}^N)$ such that, as $k \to \infty$,

$$u_k \rightharpoonup u_0 \quad \text{weakly in } H_\rho^1(\mathbf{R}^N),$$

$$u_k \to u_0 \quad \text{strongly in } L_\rho^2(\mathbf{R}^N),$$

$$u_k \to u_0 \quad \text{a.e. in } \mathbf{R}^N.$$

We note that u_k satisfies

$$\int_{\mathbf{R}^N} \left(\nabla u_k \cdot \nabla\phi - \frac{1}{p-1} u_k \phi\right) \rho dx - \int_{\mathbf{R}^N} g(u_k, \underline{u}_k)\phi\rho dx = 0 \tag{15}$$

for any $\phi \in H_\rho^1(\mathbf{R}^N)$. Letting $k \to \infty$, by Lemma 2 we obtain

$$\int_{\mathbf{R}^N} \left(\nabla u_0 \cdot \nabla\phi - \frac{1}{p-1} u_0 \phi\right) \rho dx - \int_{\mathbf{R}^N} u_0^p \phi\rho dx = 0,$$

that is, $u_0 \in H_\rho^1(\mathbf{R}^N)$ is a nonnegative solution of (5) with $\underline{u}_\lambda \equiv 0$. By Proposition 4.3 in [3] we have $u_0 \equiv 0$. Thus (i) holds.

(ii) Assume to the contrary that $\liminf_{k\to\infty} \|u_k\|_{L^\infty(\mathbf{R}^N)} < \infty$. Then there exist a subsequence, still denoted by $\{u_k\}$, and a constant $M > 0$ such that $\|u_k\|_{L^\infty(\mathbf{R}^N)} \leq M$ for $k = 1, 2, \ldots$. Then it follows that

$$\int_{\mathbf{R}^N} u_k^{p+1} \rho dx \leq M^{p-1} \int_{\mathbf{R}^N} u_k^2 \rho dx.$$

Since $\|u_k\|_{L_\rho^2} \to 0$ as $k \to \infty$ by (i) of this proposition, we obtain

$$\int_{\mathbf{R}^N} u_k^{p+1} \rho dx \to 0 \quad \text{as } k \to \infty. \tag{16}$$

Put $c_k = I_{\lambda_k}(u_k)$. Then, by Proposition 1, we have

$$(17) \qquad 0 < c_k < \frac{1}{N} S^{N/2} \quad \text{and} \quad \liminf_{k\to\infty} c_k > 0.$$

Define $h(t,s)$ and $H(t,s)$, respectively, by

$$h(t,s) = g(t,s) - t^p \quad \text{and} \quad H(t,s) = G(t,s) - \frac{1}{p+1} t^{p+1}.$$

Putting $\phi = u_k$ in (15) we obtain

(18)
$$\int_{\mathbf{R}^N} \left(|\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx - \int_{\mathbf{R}^N} u_k^{p+1} \rho dx - \int_{\mathbf{R}^N} h(u_k, \underline{u}_k) u_k \rho dx = 0.$$

We remark here that $c_k = I_{\lambda_k}(u_k)$ can be written by

$$(19) \qquad \begin{aligned} c_k \;=\; & \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx \\ & - \frac{1}{p+1} \int_{\mathbf{R}^N} u_k^{p+1} \rho dx - \int_{\mathbf{R}^N} H(u_k, \underline{u}_\lambda) \rho dx. \end{aligned}$$

Since $u_k \to 0$ strongly in $L^2_\rho(\mathbf{R}^N)$ as $k \to \infty$, we may assume that $0 \le u_k \le U$ a.e. in $\mathbf{R}^N$ for some $U \in L^2_\rho(\mathbf{R}^N)$. Now, let $k \to \infty$ in (18) and (19), respectively. By applying Lemma 2.4 in [8] with $u_0 \equiv 0$, we have

$$\lim_{k\to\infty} \int_{\mathbf{R}^N} h(u_k, \underline{u}_k) u_k \rho dx = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_{\mathbf{R}^N} H(u_k, \underline{u}_k) \rho dx = 0.$$

Then we deduce, respectively, that

$$\int_{\mathbf{R}^N} |\nabla u_k|^2 \rho dx - \int_{\mathbf{R}^N} u_k^{p+1} \rho dx = o(1) \quad \text{as } k \to \infty$$

and

$$\frac{1}{2} \int_{\mathbf{R}^N} |\nabla u_k|^2 \rho dx - \frac{1}{p+1} \int_{\mathbf{R}^N} u_k^{p+1} \rho dx = c_k + o(1) \quad \text{as } k \to \infty.$$

From (16) we obtain

$$\int_{\mathbf{R}^N} |\nabla u_k|^2 \rho dx \to 0 \quad \text{and} \quad c_k \to 0 \quad \text{as } k \to \infty.$$

This contradicts (17). Thus (ii) holds. Q.E.D.

Proof of Theorem 1. We observe that, for any sequence $\{u_{\lambda_k}\}$ with $\lambda_k \to 0$, there exists a subsequence satisfying the properties (i) and (ii) in Proposition 2. This implies that u_λ satisfies

$$u_\lambda \to 0 \quad \text{a.e. in } \mathbf{R}^N \quad \text{and} \quad \|u_\lambda\|_{L^\infty(\mathbf{R}^N)} \to \infty$$

as $\lambda \to 0$. Recall that $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} \to 0$ as $\lambda \to 0$, and that the second positive solution $\overline{u}_\lambda$ given by $\overline{u}_\lambda = \underline{u}_\lambda + u_\lambda$. Thus we obtain (3) in Theorem 1. Q.E.D.

References

[1] T. Cazenave and F. B. Weissler, Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations, Math. Z., **228** (1998), 83–120.

[2] C. Dohmen and M. Hirose, Structure of positive radial solutions to the Haraux–Weissler equation, Nonlinear Anal., **33** (1998), 51–69.

[3] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions for the heat equation, Nonlinear Anal., **11** (1987), 1103–1133.

[4] V. A. Galaktionov and J. L. Vazquez, Continuation of blowup solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math., **50** (1997), 1–67.

[5] H. Kozono and M. Yamazaki Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data, Comm. Partial Differential Equations, **19** (1994), 959–1014.

[6] Y. Naito and T. Suzuki, Radial symmetry of self-similar solutions for semilinear heat equations, J. Differential Equations, **163** (2000), 407–428.

[7] Y. Naito, Non-uniqueness of solutions to the Cauchy problem for semilinear heat equations with singular initial data, Math. Ann., **329** (2004), 161–196.

[8] Y. Naito, Self-similar solutions for a semilinear heat equation with critical Sobolev exponent, Indiana Univ. Math. J., **57** (2008), 1283–1315.

[9] P. Souplet and F. B. Weissler, Regular self-similar solutions of the nonlinear heat equation with initial data above the singular steady state, Ann. Inst. H. Poincaré Anal. Non Linéaire, **20** (2003), 213–235.

[10] E. Yanagida, Uniqueness of rapidly decaying solutions to the Haraux–Weissler equation, J. Differential Equations, **127** (1996), 561–570.

Department of Mathematics
Ehime University
Matsuyama 790-8577
Japan
E-mail address: ynaito@ehime-u.ac.jp