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Low Mach number limit for the compressible non-isentropic magnetohydrodynamic equations

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Abstract.

We summarize our recent results on the low Mach number limit for the compressible non-isentropic magnetohydrodynamic equations. We consider two cases: (i) small variations on density and temperature with well-prepared initial data; (ii) large variations on density and temperature with ill-prepared initial data. In both cases, we establish the limits rigorously.

§1. Introduction

The magnetohydrodynamic (MHD) equations govern the motion of compressible quasi-neutrally ionized fluids under the influence of electromagnetic fields. The full three-dimensional compressible MHD equations read as (see [10], [11])

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = (\nabla\times\mathbf{H})\times\mathbf{H} + \operatorname{div}\Psi, \tag{1.2}$$

$$\partial_t\mathbf{H} - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \quad \operatorname{div}\mathbf{H} = 0, \tag{1.3}$$

$$\begin{aligned}\partial_t\mathcal{E} + \operatorname{div}(\mathbf{u}(\mathcal{E}' + p)) &= \operatorname{div}((\mathbf{u}\times\mathbf{H})\times\mathbf{H}) \\ &\quad + \operatorname{div}\big(\nu\mathbf{H}\times(\nabla\times\mathbf{H}) + \mathbf{u}\Psi + \kappa\nabla\theta\big).\end{aligned} \tag{1.4}$$

Here the unknowns ρ denotes the density, $\mathbf{u}\in\mathbb{R}^3$ the velocity, $\mathbf{H}\in\mathbb{R}^3$ the magnetic field, and θ the temperature, respectively; Ψ is the viscous stress tensor given by

$$\Psi = 2\mu\mathbb{D}(\mathbf{u}) + \lambda\operatorname{div}\mathbf{u}\,\mathbf{I}_3$$

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with $\mathbb{D}(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^\top)/2$, and $\mathbf{I}_3$ being the 3×3 identity matrix, and $\nabla\mathbf{u}^\top$ the transpose of the matrix $\nabla\mathbf{u}$; $\mathcal{E}$ is the total energy given by $\mathcal{E} = \mathcal{E}' + |\mathbf{H}|^2/2$ and $\mathcal{E}' = \rho\left(e + |\mathbf{u}|^2/2\right)$ with e being the internal energy, $\rho|\mathbf{u}|^2/2$ the kinetic energy, and $|\mathbf{H}|^2/2$ the magnetic energy. The viscosity coefficients λ and μ of the flow satisfy $2\mu + 3\lambda > 0$ and $\mu > 0$; $\nu > 0$ is the magnetic diffusion coefficient of the magnetic field, and $\kappa > 0$ is the heat conductivity. For simplicity, we assume that μ, λ, ν and κ are constants. The equations of state $p = p(\rho, \theta)$ and $e = e(\rho, \theta)$ relate the pressure p and the internal energy e to the density ρ and the temperature θ of the flow.

The MHD equations have attracted a lot of attention of physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges, see, for example, [2], [4], [5], [10], [8], [11], [13] and the references cited therein. One of the important topics on the equations (1.1)–(1.4) is to study its low Mach number limit. We consider two cases: (1) small variations on density and temperature with well-prepared initial data; (2) large variations on density and temperature with ill-prepared initial data. In both cases, we establish the limits rigorously.

First we rewrite the energy equation (1.4) in the form of the internal energy. Multiplying (1.2) by $\mathbf{u}$ and (1.3) by $\mathbf{H}$, and summing them together, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{H}|^2\Big) + \frac{1}{2}\mathrm{div}\big(\rho|\mathbf{u}|^2\mathbf{u}\big) + \nabla p\cdot\mathbf{u} \\
&\quad = \mathrm{div}\Psi\cdot\mathbf{u} + (\nabla\times\mathbf{H})\times\mathbf{H}\cdot\mathbf{u} + \nabla\times(\mathbf{u}\times\mathbf{H})\cdot\mathbf{H} \\
&\qquad - \nu\nabla\times(\nabla\times\mathbf{H})\cdot\mathbf{H}.
\end{aligned}
\tag{1.5}
$$

Using the identities

$$
\begin{aligned}
&\mathrm{div}(\mathbf{H}\times(\nabla\times\mathbf{H})) = |\nabla\times\mathbf{H}|^2 - \nabla\times(\nabla\times\mathbf{H})\cdot\mathbf{H}, \\
&\mathrm{div}((\mathbf{u}\times\mathbf{H})\times\mathbf{H}) = (\nabla\times\mathbf{H})\times\mathbf{H}\cdot\mathbf{u} + \nabla\times(\mathbf{u}\times\mathbf{H})\cdot\mathbf{H}
\end{aligned}
\tag{1.6}
$$

and subtracting (1.5) from (1.4), we obtain the internal energy equation

$$
\partial_t(\rho e) + \mathrm{div}(\rho\mathbf{u}e) + (\mathrm{div}\mathbf{u})p = \nu|\nabla\times\mathbf{H}|^2 + \Psi:\nabla\mathbf{u} + \kappa\Delta\theta,
\tag{1.7}
$$

where $\Psi : \nabla\mathbf{u}$ denotes the scalar product of two matrices:

$$
\Psi:\nabla\mathbf{u} = \sum_{i,j=1}^{3}\frac{\mu}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right)^2 + \lambda|\mathrm{div}\mathbf{u}|^2 = 2\mu|\mathbb{D}(\mathbf{u}^\epsilon)|^2 + \lambda(\mathrm{tr}\mathbb{D}(\mathbf{u}^\epsilon))^2.
$$

In this paper, we shall focus our study on the ionized fluid obeying the perfect gas relations

$$p = \mathfrak{R}\rho\theta, \quad e = c_V\theta, \tag{1.8}$$

where the constants $\mathfrak{R}$ and $c_V > 0$ are the gas constant and the heat capacity at constant volume, respectively.

§2. Small variations on density and temperature with well-prepared initial data

In this section we study the low Mach number limit of the system (1.1)–(1.3) and (1.7) in the framework of classical solutions with small density and temperature variations. We use its appropriate dimensionless form as follows (see the Appendix of [6] for the details)

$$\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0, \tag{2.1}$$

$$\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) + \frac{\nabla(\rho\theta)}{\epsilon^2} = (\nabla\times\mathbf{H})\times\mathbf{H} + \mathrm{div}\Psi, \tag{2.2}$$

$$\partial_t\mathbf{H} - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \quad \mathrm{div}\mathbf{H} = 0, \tag{2.3}$$

$$\begin{aligned}&\rho(\partial_t\theta + \mathbf{u}\cdot\nabla\theta) + (\gamma-1)\rho\theta\mathrm{div}\mathbf{u}\\ &\qquad = \epsilon^2\nu|\nabla\times\mathbf{H}|^2 + \epsilon^2\Psi : \nabla\mathbf{u} + \kappa\Delta\theta,\end{aligned} \tag{2.4}$$

where $\epsilon = M$ is the Mach number and the coefficients μ, λ, ν and κ are the scaled parameters. $\gamma = 1 + \mathfrak{R}/c_V$ is the ratio of specific heats. Note that we have used the same notations and assumed that the coefficients μ, λ, ν and κ are independent of ϵ for simplicity.

We shall study the limit as $\epsilon \to 0$ of the solutions to (2.1)–(2.4). We further restrict ourselves to the small density and temperature variations, i.e.

$$\rho = 1 + \epsilon q, \quad \theta = 1 + \epsilon\phi. \tag{2.5}$$

We first give a formal analysis. Putting (2.5) and (1.8) into the system (2.1)–(2.4), and using the identities

$$\begin{aligned}&\mathrm{curl}\,\mathrm{curl}\,\mathbf{H} = \nabla\,\mathrm{div}\mathbf{H} - \Delta\mathbf{H},\\ &\nabla(|\mathbf{H}|^2) = 2\mathbf{H}\cdot\nabla\mathbf{H} + 2\mathbf{H}\times\mathrm{curl}\,\mathbf{H},\end{aligned} \tag{2.6}$$

$$\mathrm{curl}\,(\mathbf{u}\times\mathbf{H}) = \mathbf{u}(\mathrm{div}\mathbf{H}) - \mathbf{H}(\mathrm{div}\mathbf{u}) + \mathbf{H}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{H}, \tag{2.7}$$

then we can rewrite (2.1)–(2.4) as

$$\partial_t q^\epsilon + \mathbf{u}^\epsilon\cdot\nabla q^\epsilon + \frac{1}{\epsilon}(1+\epsilon q^\epsilon)\mathrm{div}\mathbf{u}^\epsilon = 0, \tag{2.8}$$

$$(1+\epsilon q^\epsilon)(\partial_t \mathbf{u}^\epsilon + \mathbf{u}^\epsilon\cdot\nabla\mathbf{u}^\epsilon) + \frac{1}{\epsilon}\left[(1+\epsilon q^\epsilon)\nabla\phi^\epsilon + (1+\epsilon\phi^\epsilon)\nabla q^\epsilon\right]$$
$$-\mathbf{H}^\epsilon\cdot\nabla\mathbf{H}^\epsilon + \frac{1}{2}\nabla(|\mathbf{H}^\epsilon|^2) = 2\mu\mathrm{div}(\mathbb{D}(\mathbf{u}^\epsilon)) + \lambda\nabla(\mathrm{tr}\mathbb{D}(\mathbf{u}^\epsilon)), \tag{2.9}$$
$$(1+\epsilon q^\epsilon)(\partial_t\phi^\epsilon + \mathbf{u}^\epsilon\cdot\nabla\phi^\epsilon) + \frac{\gamma-1}{\epsilon}(1+\epsilon q^\epsilon)(1+\epsilon\phi^\epsilon)\mathrm{div}\mathbf{u}^\epsilon$$
$$= \kappa\Delta\phi^\epsilon + \epsilon(2\mu|\mathbb{D}(\mathbf{u}^\epsilon)|^2 + \lambda(\mathrm{tr}\mathbb{D}(\mathbf{u}^\epsilon))^2) + \nu\epsilon|\nabla\times\mathbf{H}^\epsilon|^2, \tag{2.10}$$
$$\partial_t\mathbf{H}^\epsilon + \mathbf{u}^\epsilon\cdot\nabla\mathbf{H}^\epsilon + \mathrm{div}\mathbf{u}^\epsilon\mathbf{H}^\epsilon - \mathbf{H}^\epsilon\cdot\nabla\mathbf{u}^\epsilon = \nu\Delta\mathbf{H}^\epsilon, \quad \mathrm{div}\mathbf{H}^\epsilon = 0. \tag{2.11}$$

Here we have added the superscript ϵ on the unknowns to stress the dependence of the parameter ϵ. Therefore, the formal limit as $\epsilon\to 0$ of (2.8)–(2.11) is the following incompressible MHD equations (suppose that the limits $\mathbf{u}^\epsilon\to\mathbf{w}$ and $\mathbf{H}^\epsilon\to\mathbf{B}$ exist.)

$$\partial_t\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{w} + \nabla\pi + \frac{1}{2}\nabla(|\mathbf{B}|^2) - \mathbf{B}\cdot\nabla\mathbf{B} = \mu\Delta\mathbf{w}, \tag{2.12}$$
$$\partial_t\mathbf{B} + \mathbf{w}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{w} = \nu\Delta\mathbf{B}, \tag{2.13}$$
$$\mathrm{div}\mathbf{w} = 0, \quad \mathrm{div}\mathbf{B} = 0. \tag{2.14}$$

The system (2.8)–(2.11) is equipped with the initial data

$$(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \phi^\epsilon)|_{t=0} = (q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x), \mathbf{H}_0^\epsilon(x), \phi_0^\epsilon(x)). \tag{2.15}$$

We shall show that for sufficiently small Mach number, the compressible flows admit a smooth solution on the time interval where the smooth solution of the incompressible MHD equations exists.

We first recall the local existence of strong solutions to the incompressible MHD equations (2.12)–(2.14) in the whole space $\mathbb{R}^3$. The proof can be found in [12].

Proposition 2.1 ([12]). *Let $s > 3/2+2$. Assume that the initial data $(\mathbf{w},\mathbf{B})|_{t=0} = (\mathbf{w}_0,\mathbf{B}_0)$ satisfy $\mathbf{w}_0\in H^s, \mathbf{B}_0\in H^s$, and* $\mathrm{div}\,\mathbf{w}_0 = 0$, $\mathrm{div}\mathbf{B}_0 = 0$*. Then, there exist a $\hat{T}^*\in(0,\infty]$ and a unique solution $(\mathbf{w},\mathbf{B})\in L^\infty(0,\hat{T}^*;H^s)$ to the incompressible MHD equations* (2.12)–(2.14) *satisfying* $\mathrm{div}\,\mathbf{w}=0$ *and* $\mathrm{div}\mathbf{B}=0$*, and for any $0<T<\hat{T}^*$,*

$$\sup_{0\le t\le T}\left\{||(\mathbf{w},\mathbf{B})(t)||_{H^s} + ||(\partial_t\mathbf{w},\partial_t\mathbf{B})(t)||_{H^{s-2}} + ||\nabla\pi(t)||_{H^{s-2}}\right\} \le C.$$

The main result in this section reads as follows.

Theorem 2.2 ([6]). *Let $s>3/2+2$. Suppose that the initial data* (2.15) *satisfy*

$$||q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x) - \mathbf{w}_0(x), \mathbf{H}_0^\epsilon(x) - \mathbf{B}_0(x), \phi_0^\epsilon(x)||_s = O(\epsilon).$$

Let $(\mathbf{w}, \mathbf{B}, \pi)$ *be a smooth solution to* (2.12)–(2.14) *obtained in Proposition* 2.1. *If* $(\mathbf{w}, \pi) \in C([0,T^*], H^{s+2}) \cap C^1([0,T^*], H^s)$ *with* $T^* > 0$ *finite, then there exists a constant* $\epsilon_0 > 0$ *such that, for all* $\epsilon \le \epsilon_0$, *the system* (2.8)–(2.11) *with initial data* (2.15) *has a unique smooth solution* $(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \phi^\epsilon) \in C([0,T^*], H^s)$. *Moreover, there exists a positive constant* $K > 0$, *independent of* ϵ, *such that, for all* $\epsilon \le \epsilon_0$,

$$\sup_{t\in[0,T^*]} \left\| (q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \phi^\epsilon) - \left(\frac{\epsilon}{2}\pi, \mathbf{w}, \mathbf{B}, \frac{\epsilon}{2}\pi\right) \right\|_s \le K\epsilon.$$

The proof of Theorem 2.2 is based on the energy estimates for symmetrizable quasilinear hyperbolic-parabolic systems and the convergence—stability lemma for singular limit problems [3], see [6] for details.

§3. Large variations on density and temperature with ill-prepared initial data

In this section we study the low Mach number limit of the system (1.1)–(1.3) and (1.7) in the framework of classical solutions with large variations on density and temperature. Let ϵ be the Mach number, which is a dimensionless number. Consider the system (1.1)–(1.3), (1.7) in the physical regime:

$$p \sim p_0 + O(\epsilon), \quad \mathbf{u} \sim O(\epsilon), \quad \mathbf{H} \sim O(\epsilon), \quad \nabla\theta \sim O(1),$$

where $p_0 > 0$ is a certain given constant which will be normalized to be one. Thus we consider the case when the pressure p is a small perturbation of the given state 1 while the temperature θ has a finite variation. As in [1], we introduce the following transformation to ensure the positivity of p and θ

$$p(x,t) = e^{\epsilon p^\epsilon(x,\epsilon t)}, \quad \theta(x,t) = e^{\theta^\epsilon(x,\epsilon t)}, \tag{3.1}$$

where a longer time scale $t = \tau/\epsilon$ (still denote τ by t later for simplicity) is introduced in order to seize the evolution of the fluctuations. Note that (1.8) and (3.1) imply that $\rho(x,t) = e^{\epsilon p^\epsilon(x,\epsilon t) - \theta^\epsilon(x,\epsilon t)}$ by taking $\mathfrak{R} \equiv c_V \equiv 1$. Set

$$\mathbf{H}(x,t) = \epsilon \mathbf{H}^\epsilon(x,\epsilon t), \quad \mathbf{u}(x,t) = \epsilon \mathbf{u}^\epsilon(x,\epsilon t), \tag{3.2}$$

and

$$\mu = \epsilon\mu^\epsilon, \quad \lambda = \epsilon\lambda^\epsilon, \quad \nu = \epsilon\nu^\epsilon, \quad \kappa = \epsilon\kappa^\epsilon. \tag{3.3}$$

Under these changes of variables and coefficients, the system, (1.1)–(1.3), (1.7) with (1.8), takes the following equivalent form:

$$\partial_t p^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) p^\epsilon + \frac{1}{\epsilon}\mathrm{div}(2\mathbf{u}^\epsilon - \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon}\nabla\theta^\epsilon)$$

$$= \epsilon e^{-\epsilon p^\epsilon}[\nu^\epsilon|\mathrm{curl}\,\mathbf{H}^\epsilon|^2 + \Psi(\mathbf{u}^\epsilon) : \nabla\mathbf{u}^\epsilon] + \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon}\nabla p^\epsilon \cdot \nabla\theta^\epsilon, \tag{3.4}$$

$$e^{-\theta^\epsilon}[\partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\mathbf{u}^\epsilon] + \frac{\nabla p^\epsilon}{\epsilon}$$

$$= e^{-\epsilon p^\epsilon}[(\mathrm{curl}\,\mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon + \mathrm{div}\Psi^\epsilon(\mathbf{u}^\epsilon)], \tag{3.5}$$

$$\partial_t \mathbf{H}^\epsilon - \mathrm{curl}\,(\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) - \nu^\epsilon \Delta \mathbf{H}^\epsilon = 0, \quad \mathrm{div}\mathbf{H}^\epsilon = 0, \tag{3.6}$$

$$\partial_t \theta^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\theta^\epsilon + \mathrm{div}\mathbf{u}^\epsilon - \kappa^\epsilon e^{-\epsilon p^\epsilon}\mathrm{div}(e^{\theta^\epsilon}\nabla\theta^\epsilon)$$

$$= \epsilon^2 e^{-\epsilon p^\epsilon}[\nu^\epsilon|\mathrm{curl}\,\mathbf{H}^\epsilon|^2 + \Psi^\epsilon(\mathbf{u}^\epsilon) : \nabla\mathbf{u}^\epsilon], \tag{3.7}$$

where $\Psi^\epsilon(\mathbf{u}^\epsilon) = 2\mu^\epsilon\mathbb{D}(\mathbf{u}^\epsilon) + \lambda^\epsilon \mathrm{div}\mathbf{u}^\epsilon\,\mathbf{I}_3$, and the identity $\mathrm{curl}\,(\mathrm{curl}\,\mathbf{H}^\epsilon) = \nabla\mathrm{div}\mathbf{H}^\epsilon - \Delta\mathbf{H}^\epsilon$ and the constraint $\mathrm{div}\mathbf{H}^\epsilon = 0$ are used.

Formally, as ϵ goes to zero, if the sequence $(\mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ converges strongly to a limit $(\mathbf{w}, \mathbf{B}, \vartheta)$ in some sense, and $(\mu^\epsilon, \lambda^\epsilon, \nu^\epsilon, \kappa^\epsilon)$ converges to a constant vector $(\bar\mu, \bar\lambda, \bar\nu, \bar\kappa)$, then taking the limit to (3.4)–(3.7), we have

$$\mathrm{div}(2\mathbf{w} - \bar\kappa\, e^{\vartheta}\nabla\vartheta) = 0, \tag{3.8}$$

$$e^{-\vartheta}[\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w}] + \nabla\pi = (\mathrm{curl}\,\mathbf{B}) \times \mathbf{B} + \mathrm{div}\Phi(\mathbf{w}), \tag{3.9}$$

$$\partial_t \mathbf{B} - \mathrm{curl}\,(\mathbf{w} \times \mathbf{B}) - \bar\nu \Delta \mathbf{B} = 0, \quad \mathrm{div}\mathbf{B} = 0, \tag{3.10}$$

$$\partial_t \vartheta + (\mathbf{w} \cdot \nabla)\vartheta + \mathrm{div}\mathbf{w} = \bar\kappa\,\mathrm{div}(e^{\vartheta}\nabla\vartheta), \tag{3.11}$$

with some function π, where $\Phi(\mathbf{w})$ is defined by

$$\Phi(\mathbf{w}) = 2\bar\mu\mathbb{D}(\mathbf{w}) + \bar\lambda \mathrm{div}\mathbf{w}\,\mathbf{I}_3. \tag{3.12}$$

We supplement the system (3.4)–(3.7) with the following initial data

$$(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)|_{t=0} = (p^\epsilon_{\mathrm{in}}(x), \mathbf{u}^\epsilon_{\mathrm{in}}(x), \mathbf{H}^\epsilon_{\mathrm{in}}(x), \theta^\epsilon_{\mathrm{in}}(x)), \quad x \in \mathbb{R}^3. \tag{3.13}$$

For simplicity of presentation, assume that $\mu^\epsilon \equiv \bar\mu > 0$, $\nu^\epsilon \equiv \bar\nu > 0$, $\kappa^\epsilon \equiv \bar\kappa > 0$, and $\lambda^\epsilon \equiv \bar\lambda$.

As in [1], we will use the notation $\|v\|_{H^\sigma_\eta} := \|v\|_{H^{\sigma-1}} + \eta\|v\|_{H^\sigma}$ for any $\sigma \in \mathbb{R}$ and $\eta \geq 0$. For each $\epsilon > 0$, $t \geq 0$ and $s \geq 0$, we will also use the following norm:

$$\|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon - \bar\theta)(t)\|_{s,\epsilon} :=$$
$$\sup_{\tau\in[0,t]} \left\{\|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(\tau)\|_{H^s} + \|(\epsilon p^\epsilon, \epsilon\mathbf{u}^\epsilon, \epsilon\mathbf{H}^\epsilon, \theta^\epsilon - \bar\theta)(\tau)\|_{H^{s+2}_\epsilon}\right\}$$

$$+\left\{\int_0^t [\|\nabla(p^\epsilon,\mathbf{u}^\epsilon,\mathbf{H}^\epsilon)\|_{H^s}^2+\|\nabla(\epsilon\mathbf{u}^\epsilon,\epsilon H^\epsilon,\theta^\epsilon)\|_{H_\epsilon^{s+2}}^2](\tau)d\tau\right\}^{1/2}.$$

Then, the main result of this section reads as follows.

Theorem 3.1 ([7]). *Let $s\geq 4$. Assume that the initial data $(p_{\rm in}^\epsilon, \mathbf{u}_{\rm in}^\epsilon,\mathbf{H}_{\rm in}^\epsilon,\theta_{\rm in}^\epsilon)$ satisfy*

$$\|(p_{\rm in}^\epsilon,\mathbf{u}_{\rm in}^\epsilon,\mathbf{H}_{\rm in}^\epsilon,\theta_{\rm in}^\epsilon-\bar\theta)(t)\|_{s,\epsilon}\leq L_0$$

for all $\epsilon\in(0,1]$ and two given positive constants $\bar\theta$ and L_0. Then there exist positive constants T_0 and $\epsilon_0<1$, depending only on L_0 and $\bar\theta$, such that the Cauchy problem (3.4)–(3.7), (3.13) *has a unique solution $(p^\epsilon,\mathbf{u}^\epsilon,\mathbf{H}^\epsilon,\theta^\epsilon)$ satisfying*

$$\|(p^\epsilon,\mathbf{u}^\epsilon,\mathbf{H}^\epsilon,\theta^\epsilon-\bar\theta)(t)\|_{s,\epsilon}\leq L,\quad \forall\, t\in[0,T_0],\ \forall\epsilon\in(0,\epsilon_0],$$

where L depends only on L_0, $\bar\theta$ and T_0. Moreover, assume further that the initial data satisfy the following conditions

$$|\theta_0^\epsilon(x)-\bar\theta|\leq N_0|x|^{-1-\zeta},\quad |\nabla\theta_0^\epsilon(x)|\leq N_0|x|^{-2-\zeta},\quad \forall\epsilon\in(0,1],$$
$$\left(p_{\rm in}^\epsilon,{\rm curl}\,(e^{-\theta_{\rm in}^\epsilon}\mathbf{u}_{\rm in}^\epsilon),\mathbf{H}_{\rm in}^\epsilon,\theta_{\rm in}^\epsilon\right)\to(0,\mathbf{w}_0,\mathbf{B}_0,\vartheta_0)\ \ in\ H^s(\mathbb{R}^3)$$

as $\epsilon\to 0$, where N_0 and ζ are fixed positive constants. Then the solution sequence $(p^\epsilon,\mathbf{u}^\epsilon,\mathbf{H}^\epsilon,\ \theta^\epsilon)$ converges weakly in $L^\infty(0,T_0;H^s(\mathbb{R}^3))$ and strongly in $L^2(0,T_0;\ H^{s_2}_{\rm loc}(\mathbb{R}^3))$ for all $0\leq s_2<s$ to the limit $(0,\mathbf{w},\mathbf{B},\vartheta)$, where $(\mathbf{w},\mathbf{B},\vartheta)$ satisfies the system (3.8)–(3.11) *with initial data $(\mathbf{w},\mathbf{B},\vartheta)|_{t=0}=(\mathbf{w}_0,\mathbf{B}_0,\vartheta_0)$, where $\mathbf{w}_0$ is determined by*

$${\rm div}(2\mathbf{w}_0-\bar\kappa e^{\vartheta_0}\nabla\vartheta_0=0,\ {\rm curl}\,(e^{-\vartheta_0}\mathbf{w}_0)={\rm curl}\,(e^{-\vartheta_0}\mathbf{u}_0).$$

The key point in the proof of Theorem 3.1 is to establish the uniform estimates in Sobolev norms for the acoustic components of solutions, which are propagated by wave equations whose coefficients are functions of the temperature. The strategy is to bound the norm of $(\nabla p^\epsilon,{\rm div}\mathbf{u}^\epsilon)$ in terms of the norm of $(\epsilon\partial_t)(p^\epsilon,\mathbf{u}^\epsilon,\mathbf{H}^\epsilon)$ and that of $(\epsilon p^\epsilon,\epsilon\mathbf{u}^\epsilon,\epsilon\mathbf{H}^\epsilon,\theta^\epsilon)$ through the density and the momentum equations. Once the uniform bounds of the solutions are obtained, the convergence result in Theorem 3.1 can be proved by applying the compactness arguments and the dispersive estimates on the acoustic wave equations in the whole space developed in [9], see [7] for details.

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