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Large time behavior of solutions to symmetric hyperbolic systems with non-symmetric relaxation

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Abstract.

This article is concerned with the decay structure for linear symmetric hyperbolic systems with relaxation. When the relaxation matrix is symmetric, the dissipative structure of the systems is completely characterized by the Kawashima–Shizuta stability condition formulated in [8], [4]. However, some physical models which satisfy the stability condition have non-symmetric relaxation term (cf. the Timoshenko system and the Euler–Maxwell system). Therefore our purpose of this paper is to formulate a new structural condition and to analyze the weak dissipative structure for general systems with non-symmetric relaxation. This article is a survey of the paper [5].

§1. Introduction and main results

In this article, we consider the Cauchy problem for the first-order linear symmetric hyperbolic system of equations with relaxation:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0 \tag{1}$$

with

$$u|_{t=0} = u_0. \tag{2}$$

Here $u = u(t,x) \in \mathbb{R}^m$ over $t > 0$, $x \in \mathbb{R}^n$ is an unknown function, $u_0 = u_0(x) \in \mathbb{R}^m$ over $x \in \mathbb{R}^n$ is a given function, and A^j

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($j = 0, 1, \cdots, n$) and L are $m \times m$ real constant matrices, where integers $m \geq 1$, $n \geq 1$ denote dimensions. Throughout this article, it is assumed that all A^j ($j = 0, 1, \cdots, n$) are symmetric, A^0 is positive definite and L is nonnegative definite with a nontrivial kernel. Notice that L is not necessarily symmetric.

When the degenerate relaxation matrix L is symmetric, Umeda–Kawashima–Shizuta [8] proved the large-time asymptotic stability of solutions for a class of equations of hyperbolic-parabolic type with applications to both electro-magneto-fluid dynamics and magnetohydrodynamics. The key idea in [8] and the later generalized work [4] that first introduced the so-called Kawashima–Shizuta condition is to design the compensating matrix to capture the dissipation of systems over the degenerate kernel space of L. For clearness and for later use let us precisely recall the results in [8], [4] mentioned above. Taking the Fourier transform of (1) with respect to x yields

$$A^0\hat{u}_t + i|\xi|A(\omega)\hat{u} + L\hat{u} = 0. \tag{3}$$

Here and hereafter, $\xi \in \mathbb{R}^n$ denotes the Fourier variable, $\omega = \xi/|\xi| \in S^{n-1}$ is the unit vector whenever $\xi \neq 0$, and we define $A(\omega) := \sum_{j=1}^n A^j\omega_j$ with $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. The following two conditions for the coefficient matrices are needed:

Condition (A)$_0$: For the coefficient matrices, it is assumed that

$$(A^j)^T = A^j \quad \text{for} \quad j = 0, 1, \cdots, n, \qquad L^T = L,$$
$$A^0 > 0, \quad L \geq 0 \quad \text{on} \quad \mathbb{C}^m, \qquad \mathrm{Ker}(L) \neq \{0\}.$$

Here and in the sequel, the superscript T stands for the transpose of matrices, and given a matrix X, $X \geq 0$ means that $\mathrm{Re}\,\langle Xz, z\rangle \geq 0$ for any $z \in \mathbb{C}^m$, while $X > 0$ means that $\mathrm{Re}\,\langle Xz, z\rangle > 0$ for any $z \in \mathbb{C}^m$ with $z \neq 0$, where $\langle \cdot, \cdot \rangle$ denotes the standard complex inner product in $\mathbb{C}^m$. Also, for simplicity of notations, given a real matrix X, we use X_1 and X_2 to denote the symmetric and skew-symmetric parts of X, respectively, namely, $X_1 = (X + X^T)/2$ and $X_2 = (X - X^T)/2$.

Condition (K): There is a real compensating matrix $K(\omega) \in C^\infty(S^{n-1})$ with the following properties: $K(-\omega) = -K(\omega)$, $(K(\omega)A^0)^T = -K(\omega)A^0$ and

$$(K(\omega)A(\omega))_1 > 0 \quad \text{on} \quad \mathrm{Ker}(L) \tag{4}$$

for each $\omega \in S^{n-1}$.

Under the conditions (A)$_0$ and (K) one has:

Theorem 1 (Decay property of the standard type ([8], [4])). *Assume that both the conditions* $(\mathrm{A})_0$ *and* (K) *hold. Then the Fourier image* $\hat{u}$ *of the solution* u *to the Cauchy problem* (1)–(2) *satisfies the pointwise estimate:*

$$|\hat{u}(t,\xi)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \tag{5}$$

where $\rho(\xi) := |\xi|^2/(1+|\xi|^2)$. *Furthermore, let* $s \ge 0$ *be an integer and suppose that the initial data* u_0 *belong to* $H^s \cap L^1$. *Then the solution* u *satisfies the decay estimate:*

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \tag{6}$$

for $k \le s$. *Here* C *and* c *are positive constants.*

Unfortunately, when the degenerate relaxation matrix L is not symmetric, Theorem 1 can not be applied any longer. In fact, this is the case for some concrete systems, for example, the Timoshenko system [2], [3] and the Euler–Maxwell system [1], [7], [6], where the linearized relaxation matrix L indeed has a nonzero skew-symmetric part while it was still proved that solutions decay in time in some different way. Therefore, our purpose of this article is to formulate some new structural conditions in order to extend Theorem 1 to the general system (1) when L is not symmetric, which can include both the Timoshenko system and the Euler–Maxwell system.

Moer presicely, for the symmetric hyperbolic system with relaxation (1), we suppose as follows.

Condition (A): For the coefficient matrices, it is assumed that

$$(A^j)^T = A^j \quad \text{for} \quad j = 0, 1, \cdots, n,$$
$$A^0 > 0, \quad L \ge 0 \quad \text{on} \quad \mathbb{C}^m, \qquad \mathrm{Ker}(L) \ne \{0\}.$$

Then we introduce a constant matrix S which satisfies the following properties in Condition (S) with $(\mathrm{S})_1$ or $(\mathrm{S})_2$.

Condition (S): There is a real constant matrix S with the following properties: $(SA^0)^T = SA^0$ and

$$(SL)_1 + L_1 \ge 0 \quad \text{on} \quad \mathbb{C}^m, \qquad \mathrm{Ker}((SL)_1 + L_1) = \mathrm{Ker}(L). \tag{7}$$

Condition $(\mathbf{S})_1$: For each $\omega \in S^{n-1}$, the matrix S in the condition (S) satisfies

$$i(SA(\omega))_2 \ge 0 \quad \text{on} \quad \mathrm{Ker}(L_1).$$

Condition (S)$_2$: For each $\omega \in S^{n-1}$, the matrix S in the condition (S) satisfies

$$i(SA(\omega))_2 \geq 0 \quad \text{on} \quad \mathbb{C}^m.$$

Under the above structural conditions, we can state our main results on the decay property for the system (1). The first one uses the condition (S)$_1$.

Theorem 2 (Decay property of the regularity-loss type). *Assume that the conditions* (A), (S), (S)$_1$ *and* (K) *hold. Then the Fourier image* $\hat{u}$ *of the solution* u *to the Cauchy problem* (1)–(2) *satisfies the pointwise estimate:*

$$|\hat{u}(t,\xi)| \leq Ce^{-c\eta(\xi)t}|\hat{u}_0(\xi)|, \tag{8}$$

where $\eta(\xi) := |\xi|^2/(1+|\xi|^2)^2$. *Moreover, let* $s \geq 0$ *be an integer and suppose that the initial data* u_0 *belong to* $H^s \cap L^1$. *Then the solution* u *satisfies the decay estimate:*

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}u_0\|_{L^2} \tag{9}$$

for $k+\ell \leq s$. *Here* C *and* c *are positive constants.*

Remark 1. Noting that these estimates (8) and (9) are weaker than (5) and (6), respectively. The decay estimate (9) is of the regularity-loss type because we have the decay rate $(1+t)^{-\ell/2}$ only by assuming the additional ℓ-th order regularity on the initial data.

Our second main result uses the stronger condition (S)$_2$ instead of (S)$_1$ and gives the decay estimate of the standard type.

Theorem 3 (Decay property of the standard type). *If the condition* (S)$_1$ *in Theorem* 2 *is replaced by the stronger condition* (S)$_2$, *then the pointwise estimate* (8) *and the decay estimate* (9) *in Theorem* 2 *can be refined as* (5) *and* (6) *in Theorem* 1, *respectively.*

In Theorems 2 and 3, the decay estimates (9) and (6) can be derived by using the pointwise estimates (8) and (5), respectively. For the detail, we refer the readers to [5].

It should be pointed out that Theorem 3 is a direct extension of Theorem 1 and is applicable to the system (1) with a non-symmetric relaxation matrix L. More specifically, we have:

Claim 1. *Theorem* 1 *holds as a corollary of Theorem* 3. *In other words, when* L *is real symmetric, Theorem* 3 *is reduced to Theorem* 1.

Notations. For a nonnegative integer k, we denote by ∂_x^k the totality of all the k-th order derivatives with respect to $x = (x_1, \cdots, x_n)$.

Let $1 \le p \le \infty$. Then $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space over $\mathbb{R}^n$ with the norm $\|\cdot\|_{L^p}$. For a nonnegative integer s, $H^s = H^s(\mathbb{R}^n)$ denotes the s-th order Sobolev space over $\mathbb{R}^n$ in the L^2 sense, equipped with the norm $\|\cdot\|_{H^s}$. We note that $L^2 = H^0$.

Finally, in this article, we use C or c to denote various positive constants without confusion.

§2. Energy method in the Fourier space

The aim of this section is to prove the pointwise estimates stated in Theorems 2 by employing the energy method in the Fourier space (for the other proofs, see [5]). To this end, we employ the following remark.

Remark 2. Under the conditions (A) and (S), the positivity (4) in the condition (K) holds if and only if

$$\alpha(K(\omega)A(\omega))_1 + (SL)_1 + L_1 > 0 \quad \text{on} \quad \mathbb{C}^m \tag{10}$$

for each $\omega \in S^{n-1}$, where α is a suitably small positive constant.

Proof of the pointwise estimate (8) *in Theorem* 2. We derive the energy estimate for the system (3) in the Fourier space. Taking the inner product of (3) with $\hat{u}$, and taking the real part for the resultant equation, we get the basic energy equality

$$\frac{1}{2}\frac{d}{dt}E_0 + \langle L_1\hat{u}, \hat{u}\rangle = 0, \tag{11}$$

where $E_0 := \langle A^0\hat{u}, \hat{u}\rangle$. Next we create the dissipation terms. For this purpose, we multiply (3) by the matrix S in the condition (S) and take the inner product with $\hat{u}$. Then the real part of the resultant equality is described as

$$\frac{1}{2}\frac{d}{dt}E_1 + |\xi|\langle i(SA(\omega))_2\hat{u}, \hat{u}\rangle + \langle (SL)_1\hat{u}, \hat{u}\rangle = 0, \tag{12}$$

where $E_1 := \langle SA^0\hat{u}, \hat{u}\rangle$. Moreover, letting $K(\omega)$ be the compensating matrix in the condition (K), we multiply (3) by $-i|\xi|K(\omega)$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the equality, we obtain

$$-\frac{1}{2}|\xi|\frac{d}{dt}E_2 + |\xi|^2\langle (K(\omega)A(\omega))_1\hat{u}, \hat{u}\rangle - |\xi|\langle i(K(\omega)L)_2\hat{u}, \hat{u}\rangle = 0, \tag{13}$$

where $E_2 := \langle iK(\omega)A^0\hat{u}, \hat{u}\rangle$.

Now we combine the energy equalities (11), (12) and (13). First, letting α be the positive number in Remark 2, we multiply (12) and (13) by $1+|\xi|^2$ and $\alpha_2\alpha$, respectively, and add these two equalities, where α_2 is a positive constant to be determined. This yields

$$\frac{1}{2}(1+|\xi|^2)\frac{d}{dt}\mathcal{E} + (1+|\xi|^2)\langle (SL)_1\hat{u}, \hat{u}\rangle + \alpha_2|\xi|^2\langle \alpha(K(\omega)A(\omega))_1\hat{u}, \hat{u}\rangle$$
$$= -|\xi|(1+|\xi|^2)\langle i(SA(\omega))_2\hat{u}, \hat{u}\rangle + \alpha_2|\xi|\langle i\alpha(K(\omega)L)_2\hat{u}, \hat{u}\rangle,$$

where $\mathcal{E} := E_1 - \frac{\alpha_2|\xi|}{1+|\xi|^2}\alpha E_2$. Furthermore, we multiply (11) and the above equality by $(1+|\xi|^2)^2$ and α_1, respectively, and add the resulting two equalities, where α_1 is a positive constant to be determined. Then this yields

$$\frac{1}{2}\frac{d}{dt}E + D_1 + D_2 = G, \tag{14}$$

where we have defined E, D_1, D_2 and G as

$$E := E_0 + \frac{\alpha_1}{1+|\xi|^2}\,\mathcal{E} = E_0 + \frac{\alpha_1}{1+|\xi|^2}\Big(E_1 + \frac{\alpha_2|\xi|}{1+|\xi|^2}\,\alpha E_2\Big),$$
$$(1+|\xi|^2)^2 D_1 := (1+|\xi|^2)^2\langle L_1\hat{u}, \hat{u}\rangle$$
$$\qquad + \alpha_1\big\{(1+|\xi|^2)\langle (SL)_1\hat{u}, \hat{u}\rangle + \alpha_2|\xi|^2\langle \alpha(K(\omega)A(\omega))_1\hat{u}, \hat{u}\rangle\big\},$$
$$(1+|\xi|^2)^2 D_2 := \alpha_1|\xi|(1+|\xi|^2)\langle i(SA(\omega))_2 P_1\hat{u}, P_1\hat{u}\rangle,$$
$$(1+|\xi|^2)^2 G := \alpha_1\alpha_2|\xi|\langle i\alpha(K(\omega)L)_2\hat{u}, \hat{u}\rangle$$
$$\qquad - \alpha_1|\xi|(1+|\xi|^2)\big\{\langle i(SA(\omega))_2\hat{u}, \hat{u}\rangle - \langle i(SA(\omega))_2 P_1\hat{u}, P_1\hat{u}\rangle\big\}.$$

We estimate each term in (14). Because of the positivity of A^0, for suitably small $\alpha_1 > 0$ and $\alpha_2 > 0$, we see that

$$c_0|\hat{u}|^2 \le E \le C_0|\hat{u}|^2, \tag{15}$$

where c_0 and C_0 are positive constants not depending on (α_1, α_2). On the other hand, we can rewrite D_1 as

$$(1+|\xi|^2)^2 D_1 = \alpha_1\alpha_2|\xi|^2\langle (\alpha(K(\omega)A(\omega))_1 + (SL)_1 + L_1)\hat{u}, \hat{u}\rangle$$
$$\qquad + \alpha_1((1+|\xi|^2) - \alpha_2|\xi|^2)\langle ((SL)_1 + L_1)\hat{u}, \hat{u}\rangle$$
$$\qquad + (1+|\xi|^2)((1+|\xi|^2) - \alpha_1)\langle L_1\hat{u}, \hat{u}\rangle.$$

Here, using the positivity (10) which is based on the condition (K), (7) in the condition (S), and the fact that $L_1 \geq 0$ on $\mathbb{C}^m$ which is due to the condition (A), we have

$$
\begin{aligned}
(1+|\xi|^2)^2 D_1 \geq \alpha_1\alpha_2 c_1 |\xi|^2|\hat{u}|^2 + \alpha_1 c_2 (1+|\xi|^2)|(I-P)\hat{u}|^2 \\
+ c_3(1+|\xi|^2)^2|(I-P_1)\hat{u}|^2,
\end{aligned} \tag{16}
$$

where c_1, c_2 and c_3 are positive constants not depending on (α_1, α_2), and P and P_1 denote the orthogonal projections onto $\mathrm{Ker}(L)$ and $\mathrm{Ker}(L_1)$, respectively. Also we see that $D_2 \geq 0$ by the condition $(\mathrm{S})_1$.

Finally, we estimate each term in G. Noting $LP = 0$, from the Cauchy–Schwarz inequality, we have

$$
|\xi||\langle i\alpha(K(\omega)L)_2\hat{u}, \hat{u}\rangle| \leq \epsilon|\xi|^2|\hat{u}|^2 + C_\epsilon|(I-P)\hat{u}|^2 \tag{17}
$$

for any $\epsilon > 0$, where C_ϵ is a constant depending on ϵ. For the remaining term in G, we estimate it by the Cauchy–Schwarz inequality as

$$
\begin{aligned}
&|\xi|(1+|\xi|^2)\big|\langle i(SA(\omega))_2\hat{u}, \hat{u}\rangle - \langle i(SA(\omega))_2 P_1\hat{u}, P_1\hat{u}\rangle\big| \\
&\quad \leq \delta|\xi|^2|\hat{u}|^2 + C_\delta(1+|\xi|^2)^2|(I-P_1)\hat{u}|^2
\end{aligned} \tag{18}
$$

for any $\delta > 0$, where C_δ is a constant depending on δ. Consequently, we obtain

$$
\begin{aligned}
(1+|\xi|^2)^2|G| \leq \alpha_1(\alpha_2\epsilon + \delta)|\xi|^2|\hat{u}|^2 + \alpha_1\alpha_2 C_\epsilon|(I-P)\hat{u}|^2 \\
+ \alpha_1 C_\delta (1+|\xi|^2)^2|(I-P_1)\hat{u}|^2.
\end{aligned}
$$

We now choose $\epsilon > 0$ and $\delta > 0$ such that $\epsilon = c_1/4$ and $\delta = \alpha_2 c_1/4$. For this choice of (ϵ, δ), we take $\alpha_2 > 0$ and $\alpha_1 > 0$ so small that $\alpha_2 C_\epsilon \leq c_2/2$ and $\alpha_1 C_\delta \leq c_3/2$. Then, by using (16), (17) and (18), we conclude that $|G| \leq D_1/2$ and

$$
D_1 \geq c\Big\{\frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{u}|^2 + \frac{1}{1+|\xi|^2}|(I-P)\hat{u}|^2 + |(I-P_1)\hat{u}|^2\Big\}, \tag{19}
$$

where c is a positive constant. Consequently, (14) yields

$$
\frac{d}{dt}E + D_1 + 2D_2 \leq 0. \tag{20}
$$

Moreover, it follows from (15) and (19) that $D_1 \geq c\eta(\xi)E$, where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$, and c is a positive constant. Also we have $D_2 \geq 0$. Thus (20) leads to the estimate $\frac{d}{dt}E + c\eta(\xi)E \leq 0$. Solving this differential inequality, we get $E(t,\xi) \leq e^{-c\eta(\xi)t}E(0,\xi)$, which together with (15) gives the desired pointwise estimate (8). This completes the proof of Theorem 2. Q.E.D.

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