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On the solutions for nonlinear wave equations with localized dissipations in exterior domains

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Abstract.

The Cauchy problem for nonlinear wave equations with localized dissipation is considered in exterior domains outside of compact obstacles in three spatial dimensions. Under the null conditions for the quadratic nonlinear terms, the global solutions are shown for sufficiently small data. The solutions which have different propagation speeds are considered.

§1. Introduction

Let $\mathcal{K}$ be any fixed compact domain in three dimensional Euclidean space $\mathbb{R}^3$ with smooth boundary. Without loss of generality, we assume that $0 \in \mathcal{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}$ by the shift and scaling argument. Let $a = a(x)$ be a nonnegative smooth function on $\mathbb{R}^3$ with compact support in $\{x \in \mathbb{R}^3 : |x| < 1\}$. We assume that there exists a positive constant C for which $a \geq C$ on a domain which contains the closure of $\{x \in \partial\mathcal{K} : x \cdot \nu(x) < 0\}$, where $\nu(x)$ is the outward unit normal to $\mathcal{K}$ at a point $x \in \partial\mathcal{K}$. This assumption means that we can consider any shape for the boundary of the obstacle, but we expect the local effect expressed by a in the neighborhood of the boundary where it is not star-shaped.

Let Δ be the Laplacian $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$. We consider the Cauchy problem for a system of nonlinear wave equations with localized dissipation $a(x)\partial_t$ and $D \geq 1$ propagation speeds $\{c_I\}_{1 \leq I \leq D}$, $c_I > 0$,

(1)
$$\begin{cases} (\partial_t^2 - c_I^2\Delta + a(x)\partial_t)u_I(t,x) = F_I(u, u', u'')(t,x) & \text{for } t \in \mathbb{R}_+, x \in \mathbb{R}^3 \backslash \mathcal{K} \\ u_I(t,x)|_{x \in \partial\mathcal{K}} = 0 & \text{for } t \in [0, \infty) \\ u_I(0,x) = f_I(x),\ \partial_t u_I(0,x) = g_I(x) & \text{for } x \in \mathbb{R}^3 \backslash \mathcal{K}, \end{cases}$$

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where $1 \leq I \leq D$, we denote $(u_1, \cdots, u_D)$ by u, the first derivatives $\{\partial_j u\}_{0\leq j\leq 3}$ by u', and the second derivatives $\{\partial_j\partial_k u\}_{0\leq j,k\leq 3}$ by u'' with $\partial_0 = \partial_t$. We denote the nonlinear terms $(F_1, \cdots, F_D)$ by F, and we assume that F vanishes to second order and has the form

$$F_I(u, u', u'') = B_I(u') + Q_I(u', u'') + P_I(u, u') + R_I(u, u', u''), \tag{2}$$

where

$$B_I(u') := \sum_{\substack{1\leq J,K\leq D\\ 0\leq j,k\leq 3}} B_I^{JKjk}\partial_j u_J \partial_k u_K \tag{3}$$

$$Q_I(u', u'') := \sum_{\substack{1\leq J,K\leq D\\ 0\leq j,k,l\leq 3}} Q_I^{JKjkl}\partial_j u_J \partial_k\partial_l u_K \tag{4}$$

$$R_I(u, u', u'') := \sum_{\substack{1\leq K\leq D\\ 0\leq k,l\leq 3}} R_I^{Kkl}(u, u')\partial_k\partial_l u_K \tag{5}$$

and $R_I^{Kkl}(u, u') = O(|u|^2 + |u'|^2)$ and $P_I(u, u') = O(|u|^3 + |u'|^3)$ near $(u, u') = 0$. Here, $\{B_I^{JKjk}\}_{\substack{1\leq I,J,K\leq D\\ 0\leq j,k\leq 3}}$ and $\{Q_I^{JKjkl}\}_{\substack{1\leq I,J,K\leq D\\ 0\leq j,k,l\leq 3}}$ are real constants and $\{R_I^{Kkl}(u, u')\}_{\substack{1\leq I,K\leq D\\ 0\leq k,l\leq 3}}$ are real-valued polynomials which satisfy the symmetry conditions

$$B_I^{JKjk} = B_I^{JKkj} = B_I^{KJjk} \tag{6}$$

$$Q_I^{JKjkl} = Q_I^{JKjlk} = Q_I^{JKkjl} = Q_I^{KJjkl} \tag{7}$$

$$R_I^{Kkl}(u, u') = R_I^{Klk}(u, u'). \tag{8}$$

These symmetry conditions are used to derive the energy estimates.

We assume that the propagation speeds $\{c_I\}_{1\leq I\leq D}$ are positive and distinct with $0 < c_1 < \cdots < c_D$ for convenience. Straightforward modifications can be made to allow variant components to have the same speed. To show the global solutions of our problem, we also assume that the quadratic terms satisfy the null conditions

$$\sum_{0\leq j,k\leq 3} B_I^{IIjk}\xi_j\xi_k = \sum_{0\leq j,k,l\leq 3} Q_I^{IIjkl}\xi_j\xi_k\xi_l = 0 \tag{9}$$

for any $1 \leq I \leq D$ and any $(\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$ which satisfy $\xi_0^2 = c_I^2(\xi_1^2 + \xi_2^2 + \xi_3^2)$.

Since (1) is the initial and boundary value problem, the initial data $f = (f_1, \cdots, f_D)$ and $g = (g_1, \cdots, g_D)$ must satisfy the following compatibility conditions. For any natural number k, let $\overline{\partial_x}^k u := \{\partial_x^\alpha u \mid |\alpha| \leq k\}$. For the solution u of (1), we can write $\partial_t^k u(0, \cdot) = \psi_k(\overline{\partial_x}^k f, \overline{\partial_x}^{k-1} g)$, where ψ_k is called the compatibility function, which depends on f and g and F. Our Dirichlet condition requires some conditions on the compatibility functions. Since we are considering smooth solutions, we require the compatibility conditions of infinite order, namely, $\psi_k(\overline{\partial_x}^k f, \overline{\partial_x}^{k-1} g)|_{\partial\mathcal{K}} = 0$ holds for any $k \geq 1$.

Our result is the following theorem.

Theorem 1. *Let $\mathcal{K}$, a and F be as above. Let f and g be smooth functions which satisfy the compatibility conditions of infinite order. Then there exist a positive real number $\varepsilon_0 > 0$ and a positive large natural number N such that if*

$$\sum_{|\alpha| \leq N} \|\langle x \rangle^{|\alpha|} \partial_x^\alpha f\|_{L^2(\mathbb{R}^3 \backslash \mathcal{K})} + \sum_{|\alpha| \leq N-1} \|\langle x \rangle^{|\alpha|+1} \partial_x^\alpha g\|_{L^2(\mathbb{R}^3 \backslash \mathcal{K})} \leq \varepsilon_0, \tag{10}$$

then (1) *has a unique global solution $u \in C^\infty([0, \infty) \times \mathbb{R}^3 \backslash \mathcal{K})$.*

We use the following notations. The Lebesgue space of p order in the exterior domain is denoted by $L^p(\mathbb{R}^3 \backslash \mathcal{K})$, or L_x^p or L^p for simplicity. And for any $R > 0$, $L^p(|x| < R)$ denotes $L^p(\{x \in \mathbb{R}^3 \backslash \mathcal{K} : |x| < R\})$. We put $\| \cdot \|_2 := \| \cdot \|_{L^2(\mathbb{R}^3 \backslash \mathcal{K})}$. For $c > 0$, $\Box_c := \partial_t^2 - c^2 \Delta$ denotes the D'Alembertian with c propagation speed. Since our estimates for $\Box_c$ are easily reduced to the case of $\Box = \Box_1$, we abbreviate the index c when it is not essential in our estimates. The notation $a \lesssim b$ denotes the inequality $a \leq Cb$ for a positive constant C which is not essential for our arguments. We put $r = |x|$ for the spatial variable $x \in \mathbb{R}^3$.

We use the method of commuting vector fields introduced by John and Klainerman [8], [9], [15]. See also Keel, Smith and Sogge [12] for exterior domains. We denote $\partial_0 = \partial_t$, ∂_1, ∂_2, ∂_3 and the angular derivatives $x_j \partial_k - x_k \partial_j$, $1 \leq j \neq k \leq 3$, by Z, and the scaling operator $t\partial_t + r\partial_r$ by L. These operators have the commuting properties with $\Box_c$ such as

$$\Box_c Z = Z \Box_c, \qquad \Box_c L = (L + 2)\Box_c. \tag{11}$$

Note that the operator $t\partial_j + x_j \partial_t$ is not commutable with $\Box_c$ except for $c = 1$, and we do not use this. Although these operators are not commutable with $\Box_c + a\partial_t$, we can construct the energy estimates and the pointwise estimates which contain Z and L by the properties that the function $a(x)$ is not negative and has the compact support. This support condition enables us to use the standard cut-off arguments in exterior domains and to show the required estimates. The most important component is the local energy decay estimates by Nakao [23]. He has considered Kirchhoff-type wave equations [25] and wave equations of power type nonlinearities [24]. See [26] for more references.

When the dissipation is given by $a(x) = 1$ in $\mathbb{R}^3 \backslash \mathcal{K}$, Shibata [27] has proved the Cauchy problems has small global solutions regardless of the geometry of $\mathcal{K}$. When $\mathcal{K} = \phi$ and $a(x) = 0$, the problem (1) is nonlinear wave equations in $[0, \infty) \times \mathbb{R}^3$ and it is known that any nontrivial solutions blow up in finite time in general for quadratic nonlinearities (see John [7]), while the null conditions guarantee the global solutions (see Christodoulou [1] and Klainerman [16]). In this sense, our problem is critical for global solutions. And we have already shown that the solutions exist almost globally for sufficiently small data in [21] without the null conditions. Here, almost global means that the lifespan of solutions is guaranteed from below by $C \exp(c/\varepsilon)$ with positive constants C, c and an adequate norm of the initial data ε (see Klainerman [14], John and Klainerman [9] for the boundaryless case $[0, \infty) \times \mathbb{R}^3$, and see also Klainerman and Sideris [17] for the multiple speed case). We show the solutions are global if we put the null conditions on the quadratic nonlinearities.

We note that our conditions for the quadratic terms exclude the term u^2. Du, Ma and Yao have shown that the existence time of solutions for u^2 is longer than C/ε^2 in [2], which is sharp by the result in [6, Theorem III] and the finite propagation speeds of (1).

When $a(x) = 0$, the problems have been considered for several types of the obstacles $\mathcal{K}$. There is a series of papers on almost global and global solutions in this case. See Keel, Smith and Sogge [10] for convex obstacles, [12] for nontrapping obstacles, [11] and [13] for star-shaped obstacles. See also [19], [20] and [22] for Ikawa's type trapping obstacles (see Ikawa [4] and [5] for the details on the obstacles).

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