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Analytic and algebraic conditions for bifurcations of homoclinic orbits in reversible systems

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Abstract.

We study bifurcations of homoclinic orbits to hyperbolic saddles and saddle-centers in reversible systems analytically by Melnikov-type methods and algebraically by differential Galois theory.

§1. Introduction

Differential Galois theory is an extended version of the classical Galois theory, which treats the solvability of algebraic equations, for differential equations and deals with the problem of integrability by quadrature for them. It was also used to obtain necessary conditions for integrability (i.e., sufficient conditions for nonintegrability) of Hamiltonian systems in [1] and to discuss bifurcations of homoclinic orbits and an eigenvalue problem of Sturm–Liouville type on the infinite interval recently in [2], [3]. Especially, it was shown in [2] that variational equations around homoclinic orbits to hyperbolic equilibria are integrable in the meaning of the differential Galois theory if their saddle-node or pitchfork bifurcations occur in four-dimensional systems under some additional conditions. These bifurcations are of codimension two in general but of codimension one in Hamiltonian systems, and can also be detected by an extended version of Melnikov's method [4].

In this paper we extend the result of [2] to symmetric homoclinic orbits to hyperbolic saddles and saddle-centers in reversible systems, using the differential Galois theory and generalizing a Melnikov-type method of [5]. Their saddle-node and pitchfork bifurcations, which are of codimension one and two for the former and latter cases, respectively, are

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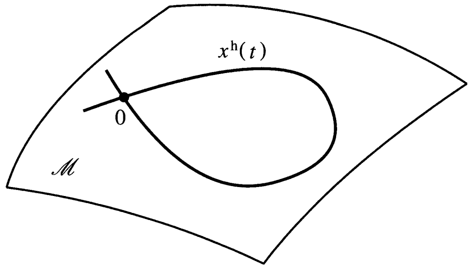

Fig. 1. Assumption (A3)

described. Homoclinic orbits in ordinary differential equations are frequently related to pulses in partial differential equations, the dynamics of which have attracted much attention in the field of dynamical systems. Reversible systems often appear in applications including water waves, nonlinear optics and celestial mechanics. The details of the results will be reported elsewhere [6].

§2. Set-up

We consider four-dimensional systems of the form

$$\dot{x} = f(x;\mu), \quad x \in \mathbb{R}^4, \quad \mu \in \mathbb{R}^m, \tag{1}$$

where $f : \mathbb{R}^4 \times \mathbb{R}^m \to \mathbb{R}^4$ is analytic, μ represents a parameter vector and $m = 1$ or 2. Now we state our assumptions on (1).

(A1) The system (1) is *reversible*, i.e., there exists a (linear) involution $R : \mathbb{R}^4 \to \mathbb{R}^4$ such that $f(Rx;\mu) + Rf(x;\mu) = 0$ for all $(x,\mu) \in \mathbb{R}^4 \times \mathbb{R}^2$. Moreover, $\dim \mathrm{Fix}(R) = 2$, where $\mathrm{Fix}(R) = \{x \in \mathbb{R}^4 \mid Rx = x\}$.

A fundamental characteristic of reversible systems is that if $x(t)$ is a solution, then so is $Rx(-t)$. We call a solution (and the corresponding orbit) *symmetric* if $x(t) = Rx(-t)$. It is a well-known fact that an orbit is symmetric if and only if it intersects the space $\mathrm{Fix}(R)$.

(A2) The origin O is an equilibrium in (1), i.e., $f(0;\mu) = 0$, for all $\mu \in \mathbb{R}^2$;

(A3) When $\mu = 0$, there exists a two-dimensional analytic invariant manifold $\mathscr{M}$ containing a symmetric homoclinic orbit $x^{\mathrm{h}}(t)$ to $x = 0$. See Fig. 1.

Finally, we assume one of the followings.

(A4) $D_x f(0;0)$ has real eigenvalues $\pm\lambda_1, \pm\lambda_2$ such that $0 < \lambda_1 \leq \lambda_2$ (hyperbolic saddle);

or

(A4') it has real eigenvalues $\pm\lambda$ and imaginary eigenvalues $\pm i\omega$ (saddle-center).

Assumption (A4) (resp. (A4')) means that the equilibrium O is a *hyperbolic saddle* (resp. *saddle-center*) and has two-dimensional (resp. one-dimensional) stable and unstable manifolds, which are denoted by $W^s_\mu(O)$ and $W^u_\mu(O)$, respectively, near $\mu = 0$. The reversibility of the system implies that $W^u_\mu(O) = RW^s_\mu(O)$ and $W^s_\mu(O) = RW^u_\mu(O)$.

The variational equation (VE) of (1) around $x = x^{\mathrm{h}}(t)$ at $\mu = 0$ is given by

$$\dot{\xi} = D_x f(x^{\mathrm{h}}(t);0)\xi, \quad \xi \in \mathbb{R}^4. \tag{2}$$

We easily see that if $\xi(t)$ is a solution, then so are $\pm R\xi(-t)$ and that $\xi = \dot{x}^{\mathrm{h}}(t)$ is a symmetric bounded solution with $\dot{x}^{\mathrm{h}}(t) = -R\dot{x}^{\mathrm{h}}(-t)$, i.e., $\dot{x}^{\mathrm{h}}(0) \in \mathrm{Fix}(-R)$. The adjoint variational equation (AVE) of (1) around $x = x^{\mathrm{h}}(t)$ at $\mu = 0$ is given by

$$\dot{\eta} = -D_x f(x^{\mathrm{h}}(t);0)^*\eta, \quad \eta \in \mathbb{R}^4, \tag{3}$$

where '$*$' represents the transpose operator.

§3. Main results

We begin with the case in which the origin O is a hyperbolic saddle, i.e., assumptions (A1)–(A4) hold, and take $m = 1$. Consider the following condition.

(C) The VE (2) has another symmetric bounded solution $\varphi(t)$ independent of $\xi = \dot{x}^{\mathrm{h}}(t)$ with $\varphi(0) \in \mathrm{Fix}(-R)$.

Using some techniques similar to those of [2], we can prove the following result.

Theorem 1. *Suppose that condition* (C) *holds as well as assumptions* (A1)–(A4). *Then a saddle-node or pitchfork bifurcation of symmetric homoclinic orbits occurs under some nondegenerate conditions.*

This theorem shows that condition (C) provides a criterion for bifurcations of symmetric homoclinic orbits in (1). The nondegenerate conditions are precisely stated by using some integrals [6]. A similar result was obtained by Knobloch [7] earlier although no computable condition was given. We can prove the same statement for general $2n$-dimensional systems with $n > 2$.

Let $\Gamma_0 = \{x = x^{\mathrm{h}}(t) \,|\, t \in \mathbb{R}\} \cup \{0\}$. The curve Γ_0 in the complex space $\mathbb{C}^4$ consists of the homoclinic orbit $x^{\mathrm{h}}(t)$ and saddle $x = 0$, and it is "singular" at $x = 0$. We introduce two points 0^+ and 0^- corresponding to the origin for desingularizing the curve Γ_0. The points 0_+ and 0_- are represented in the temporal parameterization by $t = +\infty$ and $t = -\infty$, respectively. Denote by Γ_1 the nonsingular curve and let Γ be the Riemann surface defined by the curve Γ_1. We transform the VE (2) onto Γ. Choose a sufficiently narrow domain Γ_{loc} on the Riemann surface Γ such that the curve Γ_1 is contained but singular points of the transformed VE are only $0_\pm$. Applying arguments in [2], we can also obtain the following result.

Theorem 2. *Suppose that condition* (C) *holds as well as assumptions* (A1)–(A4). *Then the VE* (2) *has a triangularizable differential Galois group when regarded as a complex differential equation with meromorphic coefficients in* Γ_{loc}.

This theorem gives an algebraic condition for bifurcations of homoclinic orbits to hyperbolic saddles in (1) and means that the VE (2) is integrable in the meaning of differential Galois theory if such a bifurcation occurs. Thus, there is a relationship between analytic and algebraic conditions for the hyperbolic saddle case.

We turn to the case in which the origin O is a saddle-center, i.e., assumptions (A1)–(A3) and (A4') hold, and take $m = 2$. We easily see that the AVE (3) has a bounded solution $\psi(t)$ with $\psi(0) \in \mathrm{Fix}(-R^*)$. Let

$$a_i = \int_{-\infty}^{\infty} \langle \psi(t), \mathrm{D}_{\mu_i} f(x^{\mathrm{h}}(t); 0) \mathrm{d}t, \quad i = 1, 2,$$

where $\langle \cdot, \cdot \rangle$ represents the inner product. Using the result of [5], we obtain the following result.

Theorem 3. *Suppose that assumptions* (A1)–(A3) *and* (A4) *hold. If* $a_i = 0$ *for* $i = 1, 2$, *then a saddle-node or pitchfork bifurcation of homoclinic orbits occurs under some nondegenerate conditions.*

This theorem gives analytic condition for bifurcations of homoclinic orbits to saddle-centers in (1). The nondegenerate conditions are also stated by using some integrals. We have no corresponding algebraic condition in general, but the VE (2) is often integrable in the meaning of differential Galois theory if such a bifurcation occurs. See [6] for more details.

§4. Example

We consider a system

$$(4)\qquad \begin{aligned} \dot{x}_1 = x_2, \quad \dot{x}_2 =& x_1 - (x_1^2 + 8x_3^2)x_1 - 2\mu_1 x_1 x_3, \\ \dot{x}_3 = x_4, \quad \dot{x}_4 =& s x_3 - \mu_2(x_1^2 + 2x_3^2)x_3 - \mu_1(\epsilon_1 x_1^2 + \epsilon_2 x_2^2), \end{aligned}$$

which comes from a model of a nonlinear optical medium with both quadratic and cubic nonlinearities [8], where s and ϵ_j, $j = 1, 2$, are constants. Equation (4) is reversible with the involution

$$R : (x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, x_3, -x_4)$$

and an equilibrium at the origin O, which is a saddle for $s > 0$ and a saddle-center for $s < 0$. There are a pair of homoclinic orbits,

$$x_\pm^h(t) = (\pm\sqrt{2}\operatorname{sech} t, \mp\sqrt{2}\operatorname{sech} t \tanh t, 0, 0),$$

to O at $\mu_1 = 0$. The VE along $x_\pm^h(t)$ at $\mu_1 = 0$ is given by

$$(5)\qquad \begin{aligned} \dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 =& (1 - 6\operatorname{sech}^2 t)\xi_1, \\ \dot{\xi}_3 = \xi_4, \quad \dot{\xi}_4 =& (s - 2\mu_2 \operatorname{sech}^2 t)\xi_3. \end{aligned}$$

The case of $s < 0$ was studied earlier in [5].

Theorems 1 and 2 can be applied to (4) for $s > 0$, while Theorem 3 for $s < 0$. We also show that equation (5) has a triangularizable differential Galois group if and only if

$$(6)\qquad \mu = \frac{(2\sqrt{s} + 2\ell + 1)^2 - 1}{8}$$

for $s > 0$ and

$$(7)\qquad \mu = \frac{\ell(\ell - 1)}{2}$$

for $s < 0$, where $\ell \in \mathbb{Z}$. Hence, when $s > 0$, a saddle-node or pitchfork bifurcations of homoclinic orbits can occur only if condition (6) holds. However, when $s < 0$, a saddle-node or pitchfork bifurcations of homoclinic orbits can occur even if condition (7) does not hold, although for $\epsilon_2 = 0$ they can only if it does. See [6] for the details.

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