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The gradient flow for the modified one-dimensional Willmore functional defined on planar curves with infinite length

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Abstract.

In this paper, we are interested in a motion of non-closed planar curves with infinite length. The motion is governed by a gradient flow for the modified one-dimensional Willmore functional. We shall prove a long time existence of solution of an initial boundary value problem for the flow.

§1. Introduction

In this paper, we deal with a geometric evolution equation for non-closed planar curves with infinite length. In particular, we focus on planar curves whose starting point is always fixed at the origin. The geometric evolution equation is defined as a steepest descent flow for the modified one-dimensional Willmore functional (see (1.3)).

The geometric evolution equations for curves have been studied by many researchers. One of the most famous geometric evolution equation for curves is curve shortening flow ([1], [2], [3], [4], [7], [8], [9], etc.). The curve shortening flow is given by a steepest descent flow for the length functional of curve:

$$\mathcal{L}(\gamma) = \int_\gamma ds, \tag{1.1}$$

where s denotes the arc length parameter of curve γ. Another well-known geometric evolution equation for curves is curve straightening flow ([10], [11], [14], [15], [18], [19], etc.). The curve straightening flow

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is defined by a steepest descent flow for the total squared curvature

$$\mathcal{W}(\gamma) = \int_\gamma \kappa^2 \, ds, \tag{1.2}$$

where κ is the curvature of γ. The functional is also called one-dimensional Willmore functional. In this paper, we consider a steepest descent flow for the modified one-dimensional Willmore functional which is defined by (1.1) and (1.2). To begin with, we introduce the geometric functional and our problem.

For a given constant $\lambda \neq 0$, the modified one-dimensional Willmore functional is defined as

$$E(\gamma) = \mathcal{W}(\gamma) + \lambda^2 \mathcal{L}(\gamma). \tag{1.3}$$

The steepest descent flow for E is written as follows:

$$\partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \tag{1.4}$$

where $\boldsymbol{\nu}$ is the unit normal vector of γ pointing in the direction of the curvature.

We mention the known results of the flow. First it has been proved by A. Polden ([17]) that the equation (1.4) admits smooth solutions globally defined in time, when the initial curve is smooth, closed, and has finite length. In [6], G. Dziuk, E. Kuwert, and R. Schätzle extended Polden's result ([17]) to closed curves with finite length in $\mathbb{R}^n$. However there are no results for *non-closed* curves with *infinite length.* One of our purpose of this paper is to extend Polden's result to non-compact case.

We are interested in the following problem:

Problem 1.1. *What is a dynamics of non-closed planar curves with infinite length governed by shortening-straightening flow?*

Regarding the problem, we find a smooth solution of (1.4) starting from an initial curve γ_0 with infinite length. Indeed let $\gamma_0(x) : [0, \infty) \to \mathbb{R}^2$ be a non-closed smooth planar curve satisfying the following:

(i) $\gamma_0(0) = (0,0), \quad \kappa_0(0) = 0, \quad |\gamma_0{}'(x)| \equiv 1,$

(ii) $\gamma_0(x)$ approaches a straight line in C^1 sense as $x \to \infty$,

where κ_0 denotes the curvature of γ_0. The precise conditions of the initial curve are stated in Section 2.

We state the main result of this paper in a concise form:

Theorem 1.1. *Let $\gamma_0(x)$ be a smooth non-closed planar curve satisfying* (i)–(ii)*. Then there exists a classical solution of* (1.4) *starting from γ_0 for any finite time.*

Remark 1.1. We can not prove an uniqueness of solution in Theorem 1.1. Furthermore, the solution in Theorem 1.1 is obtained as a solution of initial boundary value problem for (1.4). For, we have to impose a certain boundary condition on the flow (1.4) as the compatibility condition. We shall show the precise initial boundary value problem in Section 2.

The paper is organized as follows: In Section 2, first we formulate Problem 1.1 as a certain initial boundary value problem for the flow (1.4). And then we state the main result of this paper in a precise form. Finally we shall state an outline of proof of Theorem 1.1 in Section 3.

§2. Formulation and preliminaries

To begin with, we formulate Problem 1.1 as an initial boundary value problem for (1.4). We start with the initial condition. Let $\gamma_0(x) = (\phi(x), \psi(x)) : [0, \infty) \to \mathbb{R}^2$ be a smooth non-closed curve and satisfy the following conditions:

$$\text{(A1)} \qquad \gamma_0(0) = (0,0), \quad \kappa_0(0) = 0, \quad |\partial_x \gamma_0(x)| \equiv 1,$$

$$\text{(A2)} \qquad \kappa_0,\ \partial_x^m \kappa_0 \in L^2(0,\infty) \quad \text{for all } m \in \mathbb{N},$$

$$\text{(A3)} \qquad \lim_{x\to\infty} \phi(x) = \infty, \quad \lim_{x\to\infty} \phi'(x) = 1,$$

$$\text{(A4)} \quad \psi(x) = O(x^{-\alpha}) \text{ for some } \alpha > \frac{1}{2} \text{ as } x \to \infty, \quad \psi' \in L^2(0,\infty),$$

where κ_0 denotes the curvature. The definition of γ_0 and condition (A1) imply that the length of γ_0 is infinite. Moreover, from the conditions (A3)–(A4), we deduce that γ_0 approaches the axis in C^1 sense as $x \to \infty$. Indeed, for sufficiently small $\rho > 0$, there exists a constant $M > 0$ such that

(2.1)

$$\sup_{x\in(M,\infty)} ||\phi'(x)| - 1| < \rho, \qquad \sup_{x\in(M,\infty)} |\psi(x)| < \rho, \qquad \sup_{x\in(M,\infty)} |\psi'(x)| < \rho.$$

Concerning Problem 1.1, we consider the following initial-boundary value problem:

$$\text{(SS)} \qquad \begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \\ \gamma(0,t) = (0,0), \quad \kappa(0,t) = 0, \\ \gamma(x,0) = \gamma_0(x). \end{cases}$$

In the rest of this section, we shall mention several lemmas needed later. For a precise proof of these lemmas, see [16]. Let us set

$$F^{\lambda} = 2\partial_s^2 \kappa + \kappa^3 - \lambda^2 \kappa.$$

Then the gradient flow (1.4) is written as

$$\partial_t \gamma = -F^{\lambda} \boldsymbol{\nu}.$$

Lemma 2.1. *Under* (1.4), *the following commutation rule holds*:

$$\partial_t \partial_s = \partial_s \partial_t - \kappa F^{\lambda} \partial_s.$$

Lemma 2.1 gives us the following:

Lemma 2.2. *Let* γ *satisfy* (1.4). *Then the curvature* κ *satisfies*

$$\begin{aligned} \partial_t \kappa &= -\partial_s^2 F^{\lambda} - \kappa^2 F^{\lambda} \\ &= -2\partial_s^4 \kappa - 5\kappa^2 \partial_s^2 \kappa + \lambda^2 \partial_s^2 \kappa - 6\kappa (\partial_s \kappa)^2 - \kappa^5 + \lambda^2 \kappa^3. \end{aligned} \tag{2.2}$$

Furthermore, the line element ds *of* γ *satisfies*

$$\partial_t ds = \kappa F^{\lambda} ds = (2\kappa \partial_s^2 \kappa + \kappa^4 - \lambda^2 \kappa^2) ds. \tag{2.3}$$

Let us define the following notation:

Definition 2.1. ([5]) *We use a symbol* $\mathfrak{q}^r(\partial_s^l \kappa)$ *for a polynomial as follows*:

$$\mathfrak{q}^r(\partial_s^l \kappa) = \sum_m C_m \prod_{i=1}^{N_m} \partial_s^{c_{m_i}} \kappa$$

with all the c_{m_i} *less than or equal to* l *and*

$$\sum_{i=1}^{N_m} (c_{m_i} + 1) = r$$

for every m, *where* C_m *are constant coefficients.*

By virtue of Lemmas 2.1 and 2.2, we have

Lemma 2.3. *For any* $j \in \mathbb{N}$, *the following formula holds*:

$$\partial_t \partial_s^j \kappa = -2\partial_s^{j+4} \kappa - 5\kappa^2 \partial_s^{j+2} \kappa + \lambda^2 \partial_s^{j+2} \kappa + \lambda^2 \mathfrak{q}^{j+3}(\partial_s^j \kappa) + \mathfrak{q}^{j+5}(\partial_s^{j+1} \kappa).$$

Let us define L^p norm with respect to the arc length parameter of γ. For a function $f(s)$ defined on γ, we write

$$\|f\|_{L^p_s} = \left\{ \int_\gamma |f(s)|^p \, ds \right\}^{\frac{1}{p}},$$
$$\|f\|_{L^\infty_s} = \sup_{s \in [0, \mathcal{L}(\gamma)]} |f(s)|,$$

where $\mathcal{L}(\gamma)$ denotes the length of γ.

§3. Main result and outline of proof

The main result of this paper is stated as follows:

Theorem 3.1. *Let $\gamma_0(x)$ be a smooth planar curve satisfying* (A1)–(A4). *Then there exists a classical solution of* (SS) *for any finite time $t > 0$.*

As we have already mentioned above, the solution γ of (SS) is fixed at the origin. In [13], we will extend Theorem 3.1 to more general case.

As a first step in proving Theorem 3.1, we consider a certain compact case with fixed boundary. Let $\Gamma_0(x) : [0, L] \to \mathbb{R}^2$ be a smooth planar curve and satisfy the conditions

(C)
$$|\Gamma_0'(x)| \equiv 1, \quad \Gamma_0(0) = (0,0), \quad \Gamma_0(L) = (R, 0), \quad k_0(0) = k_0(L) = 0,$$

where L and R are certain positive constants, and $k_0(x)$ is the curvature. For the initial curve Γ_0, we solve the following initial boundary value problem:

$$\text{(SSC)} \quad \begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \\ \gamma(0,t) = (0,0), \quad \gamma(L,t) = (R,0), \quad \kappa(0,t) = \kappa(L,t) = 0, \\ \gamma(x,0) = \Gamma_0(x). \end{cases}$$

A short time existence of solution of (SSC) is followed from a standard argument. In order to show a long time existence of solution to (SSC), we have to prove that the curvature and its derivatives are bounded for any finite time.

Since (1.4) is the steepest descent flow for (1.3), we deduce an estimate of the curvature.

Lemma 3.1. *Let γ be a solution of* (SSC) *and κ denote the curvature. Then it holds that*

$$\text{(3.1)} \qquad \|\kappa\|^2_{L^2_s} \le \|k_0\|^2_{L^2(0,L)} + \lambda^2 \left(\mathcal{L}(\Gamma_0) - R \right).$$

By virtue of Lemmas 2.2 and 2.3, we observe the following:

Lemma 3.2. *Let κ be the curvature of γ satisfying* (SSC). *Then, for any $m \in \mathbb{N}$, it holds that*

$$\partial_s^{2m}\kappa(0,t) = \partial_s^{2m}\kappa(L,t) = 0. \tag{3.2}$$

In order to estimate any order derivative of κ with respect to its arc length, we need a Gagliardo–Nirenberg type interpolation inequality.

Lemma 3.3. *Let $\gamma(x,t)$ be a solution of* (SSC). *Let $u(x,t)$ be a function defined on γ and satisfy*

$$\partial_s^{2m}u(0,t) = \partial_s^{2m}u(L,t) = 0$$

for any $m \in \mathbb{N}$. Then, for integers $0 \le p < q < r$, it holds that

$$\|\partial_s^q u\|_{L_s^2} \le \|\partial_s^p u\|_{L_s^2}^{\frac{r-q}{r-p}} \|\partial_s^r u\|_{L_s^2}^{\frac{q-p}{r-p}}. \tag{3.3}$$

Moreover, for integers $0 \le p \le q < r$, it holds that

$$\|\partial_s^q u\|_{L_s^\infty} \le \sqrt{2}\,\|\partial_s^p u\|_{L_s^2}^{\frac{2(r-q)-1}{2(r-p)}} \|\partial_s^r u\|_{L_s^2}^{\frac{2(q-p)+1}{2(r-p)}}. \tag{3.4}$$

With the aid of Lemma 3.2, we can apply (3.3) and (3.4) to κ. Then we obtain the following estimate:

Lemma 3.4. *For any $j \in \mathbb{N}$, we have*

$$\frac{d}{dt}\|\partial_s^j \kappa\|_{L_s^2}^2 \le C\,\|\kappa\|_{L_s^2}^{4j+6} + C\,\|\kappa\|_{L_s^2}^{4j+10}.$$

It is followed from Lemmas 3.1 and 3.4 that the curvature κ and its derivatives are bounded for any finite time. Concerning the problem (SSC), we obtain the following result:

Theorem 3.2. ([16]) *Let Γ_0 be a smooth planar curve satisfying the condition* (C). *Then there exists a unique classical solution $\gamma(x,t)$ of* (SSC) *for any time $t > 0$. Moreover there exist a sequence $\{t_i\}_{i=0}^\infty$ with $t_i \to \infty$ and a stationary solution $\hat{\gamma}$ of* (SSC) *such that $\gamma(\cdot,t_i)$ converges to $\hat{\gamma}(\cdot)$ in the C^∞ topology as $t_i \to \infty$.*

Remark 3.1. If we are able to know a stability of a stationary solution, then we can comprehend a dynamical aspect of solution (SSC) in a neighborhood of the stationary solution. The representation formula of the stationary solutions is given by A. Linnér ([12]). However its stability is an outstanding question.

By virtue of Theorem 3.2, we are able to construct an "approximate solution" of (SS). Indeed, using a cut-off function $\eta_r(x) \in C_c^\infty(0,\infty)$ with $\eta_r(x) \equiv 1$ on $[0, r-1]$ and $\eta_r(x) \equiv 0$ on $[r, \infty)$, we define a smooth planar curve $\Gamma_{0,r}(x) : [0,r] \to \mathbb{R}^2$ as

$$\Gamma_{0,r}(x) = (\phi(x), \eta_r(x)\psi(x)) \big|_{0 \le x \le r} .$$

It is easy to check that the curve $\Gamma_{0,r}(x)$ satisfies the conditions

$$\Gamma_{0,r}(0) = (0,0), \quad \Gamma_{0,r}(r) = (\phi(r), 0), \quad k_{0,r}(0) = k_{0,r}(r) = 0, \tag{3.5}$$

where $k_{0,r}(x)$ denotes the curvature of $\Gamma_{0,r}(x)$. For the initial curve $\Gamma_{0,r}$, we solve the following problem:

$$(\mathrm{SS}_r) \quad \begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \\ \gamma(0,t) = (0,0), \quad \gamma(r,t) = (\phi(r), 0), \quad \kappa(0,t) = \kappa(r,t) = 0, \\ \gamma(x,0) = \Gamma_{0,r}(x). \end{cases}$$

By virtue of (3.5) and Theorem 3.2, we observe the following:

Lemma 3.5. *Let $r > M$. Then there exists a unique classical solution $\gamma_r(x,t)$ of (SS_r) for any time $t > 0$. Moreover there exist a sequence $\{t_i\}_{i=0}^\infty$ with $t_i \to \infty$ and a stationary solution $\hat{\gamma}_r$ of (SS_r) such that $\gamma_r(\cdot, t_i)$ converges to $\hat{\gamma}_r(\cdot)$ in the C^∞ topology as $t_i \to \infty$.*

We close this paper by showing a key to proving Theorem 3.1. By applying Arzelà–Ascoli's theorem to the sequence $\{\gamma_r\}_{r>M}$, we construct a solution of (SS). To do so, we have to observe that the sequence $\{\gamma_r\}_{r>M}$ is uniformly bounded and equicontinuous. The point is to prove an uniform boundedness of the sequence. By virtue of Lemma 3.4, we see that it is sufficient to prove a uniform boundedness of κ_r.

Lemma 3.6. ([13]) *There exists a positive constant C being independent of r such that*

$$\sup_{r \in (M,\infty)} \|\kappa_r(t)\|_{L_s^2} < C \tag{3.6}$$

for any $t > 0$.

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