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p-adic multiple zeta values, p-adic multiple L-values, and motivic Galois groups

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Abstract.

We will give a survey on the theory of p-adic multiple zeta values and p-adic multiple L-values.

CONTENTS

§0. Introduction

This article is an English translation of the author's Japanese article (2005) with some extensions. In this article, we will give an introductory

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survey on the theory of p-adic multiple zeta values and related topics. For positive integers $k_1, \ldots, k_{d-1} \geq 1$, $k_d \geq 2$, the infinite sum

$$\zeta(k_1, \ldots, k_d) := \sum_{n_1 < \ldots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} (= \lim_{\mathbb{C} \ni z \to 1} \mathrm{Li}_{k_1, \ldots, k_d}(z)) \in \mathbb{R}$$

absolutely converges, and we call it **multiple zeta value**. (Some people use the convention of $\zeta(k_d, \ldots, k_1)$.) Here,

$$\mathrm{Li}_{k_1, \ldots, k_d}(z) := \sum_{n_1 < \ldots < n_d} \frac{z^{n_d}}{n_1^{k_1} \cdots n_d^{k_d}}$$

is a multiple polylogarithm function. We call $k_1 + \cdots + k_d$ its weight and d its depth (This weight times -2 corresponds to the weight of a motive). It was Euler who first studied the multiple zeta values ($d = 2$ case), and Zagier refound them in the modern era (Some people call the multiple zeta values Euler–Zagier sums). Now, they attract mathematicians and physicists, because we found that they were related with many areas of mathematics and physics.

For positive integers $k_1, \ldots, k_d \geq 1$ and N-th roots of unity $\zeta_1, \ldots, \zeta_d$ satisfying $(k_d, \zeta_d) \neq (1, 1)$, we can define a variant of multiple zeta values, which are called **multiple L-values** by the following converging infinite sum:

$$L(k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d) := \sum_{n_1 < \ldots < n_d} \frac{\zeta_1^{-n_1} \zeta_2^{n_1 - n_2} \cdots \zeta_d^{n_{d-1} - n_d}}{n_1^{k_1} \cdots n_d^{k_d}}$$
$$\left(= \lim_{\mathbb{C} \ni z \to 1} \mathrm{Li}_{k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d}(z) \right) \in \mathbb{C}.$$

Here, $\mathrm{Li}_{k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d}(z) := \sum_{n_1 < \ldots < n_d} \frac{\zeta_1^{-n_1} \zeta_2^{n_1 - n_2} \ldots \zeta_d^{n_{d-1} - n_d} z^{n_d}}{n_1^{k_1} \cdots n_d^{k_d}}$ is the twisted multiple polylogarithm function.

Now, we would like to consider a p-adic analogue of multiple zeta values and multiple L-values. The infinite sum

$$\sum_{n_1 < \ldots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

does *not* p-adically converge. On the other hand, the multiple polylogarithm $\mathrm{Li}_{k_1, \ldots, k_d}(z)$ has the iterated integral expression by using inductively the following formula:

$$\frac{d\mathrm{Li}_{k_1, \ldots, k_d}(z)}{dz} = \begin{cases} \frac{1}{z} \mathrm{Li}_{k_1, \ldots, k_d - 1}(z) & \text{if } k_d > 1, \\ \frac{1}{1-z} \mathrm{Li}_{k_1, \ldots, k_{d-1}}(z) & \text{if } k_d = 1, \text{and } d > 1, \\ \frac{1}{1-z} & \text{if } k_d = 1, \text{and } d = 1. \end{cases}$$

H. Furusho constructed a p-adic multiple polylogarithm function

$$\mathrm{Li}^a_{k_1,\ldots,k_d}(z)$$

as a p-adic analogue of the above iterated integral expression by using Coleman's p-adic integration theory (Here, a is a branching parameter. We will see the details later.), defined a p-adic multiple zeta value as the limit value at 1,

$$\zeta_p(k_1,\ldots,k_d) := \lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}_{k_1,\ldots,k_d}(z) \in \mathbb{Q}_p$$

and studied their properties (*cf.* [Fu1], [Fu2], [Fu3], [BF], [FJ]). Here, $\mathbb{C}_p$ denote the p-adic completion of an algebraic closure of $\mathbb{Q}_p$. We will see the meaning of $\lim'$ later. Similarly, we can define a p-adic multiple L-value

$$L_p(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d) := \lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z) \in \mathbb{Q}_p(\mu_N)$$

by using a twisted p-adic polylogarithm function $\mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z)$ for $p \nmid N$.

We will give a survey on these in this short article. In Chapter 1, we will give a short review of Coleman's theory of p-adic integration. In Chapter 2, we will define the (twisted) p-adic multiple polylogarithms and the p-adic multiple zeta values (L-values). In Chapter 3, we will explain their relation with p-adic KZ equation and p-adic Drinfel'd associator. In Chapter 4, we will give explicit relations among the p-adic multiple zeta values. In Chapter 5, we will explain a Tannakian interpretation of multiple polylogarithms, p-adic multiple polylogarithms, and their variants. In Chapter 6, we will give a upper bound of the dimensions of p-adic multiple zeta value (L-values) spaces. In this short article, we do not give proofs.

§1. Coleman's p-adic integration theory

R. F. Coleman established his p-adic integration theory for curves and defined p-adic polylogarithms in his studies of a p-adic analogue of Bloch's results on the dilogarithm and the regulator (*cf.* [C], [Bl]).

In this chapter, we will give a brief review of Colman's p-adic integration theory. We need and expect the following two properties for p-adic integration theory:

(A) The "theorem of identity" holds. Thus, we can consider the analytic continuation, and

(B) We can locally integrate any differential forms, and it is unique up to a constant.

Coleman constructed p-adic integration theory for curves by using Tate's rigid analysis. We do not give a review of Tate's rigid analysis here. For an affinoid U, put $A(U) := \Gamma(U, \mathcal{O}_U)$. In Tate's rigid analysis, the above (A) holds, however, the above (B) does not hold in general (for example, when the point $t = 0$ is removed for a local coordinate t, we cannot integrate dt/t in the affinoid algebra). So, we will extend $A(U)$ so that we can integrate any differential forms in a bigger ring.

Let X be a proper smooth curve over $O_{\mathbb{C}_p}$, $D \subset X$ a closed subscheme which is étale over $O_{\mathbb{C}_p}$, $Y := X \setminus D$, and $j : Y_{\overline{\mathbb{F}_p}} \hookrightarrow X_{\overline{\mathbb{F}_p}}$. (Here, $O_{\mathbb{C}_p}$ is the valuation ring of $\mathbb{C}_p$.) Put $X_{\overline{\mathbb{F}_p}} \setminus Y_{\overline{\mathbb{F}_p}} = \{e_1, \dots, e_s\}$. For $0 \leq r < 1$, put $U_r := X(\mathbb{C}_p)^{\mathrm{an}} \setminus \cup_{i=1}^s D^+(\widetilde{e}_i, r)$. Here, $\widetilde{e}_i \in X(\mathbb{C}_p)$ is a lift of e_i, $D^+(\widetilde{e}_i, r)$ is the closed disk centered at $\widetilde{e}_i$ with radius r. (We do not mind the choice of a lift, since we will take a limit $r \to 1$ later.) For $S \subset X(\overline{\mathbb{F}_p})$, let $]S[:= \mathrm{sp}^{-1}(S) \subset X(\mathbb{C}_p)$ denote its tubular neighbourhood, where sp is the specialization map $X(\mathbb{C}_p) = X(O_{\mathbb{C}_p}) \stackrel{\text{reduction}}{\longrightarrow} X(\overline{\mathbb{F}_p})$. Let $a \in \mathbb{C}_p$ be fixed, and $\log^a : \mathbb{C}_p^\times \to \mathbb{C}_p$ the p-adic logarithm function with $\log^a p = a$ (the value $\log p$ decides p-adic logarithm function by the requirement $\log x + \log y = \log xy$). We call a **branch** of the p-adic logarithm. Coleman's p-adic integration theory depends on the choice of the branch of the p-adic logarithm.

Remark . The cardinality of the branch of the logarithm is countable ($\#\mathbb{Z}$) in the case over $\mathbb{C}$, and it is uncountable ($\#\mathbb{C}_p$) in the case over $\mathbb{C}_p$. So, one might see the situation is so different. However, the "branch" in the case over $\mathbb{C}_p$ does *not* correspond to a monodromy, but corresponds to a vector in the tangent space, which is $\mathbb{C}_p$. We have *no* monodromy in Coleman's theory. In the case over $\mathbb{C}$, we have $\mathbb{C}^\times$ as non-zero vectors in the tangent space. On the other hand, in the case over $\mathbb{C}_p$, we have $\mathbb{C}_p^\times$ as non-zero vectors in the tangent space, and we can extend it to $\mathbb{C}_p$ since we have *no* monodromy in his theory.

Besser's Frobenius invariant path (we will explain later) from 0 to 1 in $\mathbb{P}^1$ does not depend on the choice of branch of the p-adic logarithm, however, the one from 0 to 1 in $\mathbb{P}^1 \setminus \{0, 1\}$ *does* depend on the choice of branch of the p-adic logarithm. This fact also corresponds that the branch of the p-adic logarithm corresponds to a tangent vector.

Furthermore, we can understand that we have no monodromy in Coleman's theory by seeing the sum of "the interior angles" of "the triangle" is 0 in the 3-cycle relation, and "$2\pi\sqrt{-1} = 0$" in the p-adic analytic world.

If we use Fontaine's p-adic period ring B_{crys}, not $\mathbb{C}_p$, then we can see "monodromy" and that "$2\pi\sqrt{-1}$" corresponds $t \in B_{\mathrm{crys}}$. (Then the branch is parameterized by $\mathbb{Z}$.) See also [Y2]. Note that any variety with good reduction is simply connected in the p-adic analytic geometry (In particular $\mathbb{P}^1$ minus finitely many points is simply connected). (Coleman's p-adic integration theory treats only the good reduction case.)

Now, we define

$$A^a_{\log}(U_x) := \begin{cases} A(]x[) & \text{if } x \in Y(\overline{\mathbb{F}_p}), \\ \lim_{r \to 1} A(]x[\cap U_r)[\log^a z_x] & \text{if } x \in \{e_1, \dots, e_s\}, \end{cases}$$

$$\Omega^a_{\log}(U_x) := A^a_{\log}(U_x) dz_x,$$

where z_x is a local coordinate around x z_x $:]x[\cap Y(\mathbb{C}_p) \xrightarrow{\cong} D^-(0,1)$, and we define

$$A^a_{\mathrm{loc}} := \prod_{x \in X(\overline{\mathbb{F}_p})} A^a_{\log}(U_x), \ \Omega^a_{\mathrm{loc}} := \prod_{x \in X(\overline{\mathbb{F}_p})} \Omega^a_{\log}(U_x).$$

These do not depend on the choice of z_x. We can define a derivative $d : A^a_{\mathrm{loc}} \to \Omega^a_{\mathrm{loc}}$ in the natural way. We define **the ring of overconvergent functions** as $A^\dagger := \Gamma(]Y_{\overline{\mathbb{F}_p}}[, j^\dagger \mathcal{O}_{]Y_{\overline{\mathbb{F}_p}}[})$. Then, A^a_{loc} and Ω^a_{loc} are $A^\dagger$-algebra, and $A^\dagger$-module respectively. Now, $d : A^a_{\mathrm{loc}} \to \Omega^a_{\mathrm{loc}}$ is surjective. So, we can integrate any elements in Ω^a_{loc} (*i.e.*, (B) holds). However, the kernel of d is $\prod_{x \in X(\overline{\mathbb{F}_p})} \mathbb{C}_p$, and we do not have the notion of the analytic continuation yet (*i.e.*, (A) does not hold). Thus, we will define a subalgebra $A^a_{\mathrm{Col}} \subset A^a_{\mathrm{loc}}$ and a submodule $\Omega^a_{\mathrm{Col}} \subset \Omega^a_{\mathrm{loc}}$ to get the notion of the analytic continuation, keeping the surjectivity of d as follows:

First, we state the expected properties of the integration $\int$:

(1) $d \int \omega = \omega$,
(2) (Frobenius invariance) $\int \phi^* \omega = \phi^* \int \omega$ (where ϕ^* is Frobenius homomorphism), and
(3) $\int dg = g + \text{const.}$ for $g \in A^\dagger$.

Thus, if we have a polynomial $P(t)$ with coefficients in $\mathbb{C}_p$ such that we already know that $P(\phi^*)\omega$ can be integrated, then we have $P(\phi^*) \int \omega$ up to a constant. If $P(t)$ has no roots of unity, then we can get $\int \omega$ up to a constant by the Frobenius invariance. The principal idea is to extend the classes of the integrable differential forms from $d(A^\dagger)$ by using Frobenius invariance like this. Note that we extend the classes of integrable differential forms so that the integration is unique up to a constant in $\mathbb{C}_p$, not in $\prod_{x \in X(\overline{\mathbb{F}_p})} \mathbb{C}_p$.

We omit the details of extending. We define **the ring of Coleman functions** A^a_{Col} and **the module of Coleman** 1**-forms** Ω^a_{Col} as follows:

$$A^a_{\mathrm{Col}} := \cup_{n\geq 1} A^a_{\mathrm{Col}}(n),\ \ \Omega^a_{\mathrm{Col}} := \cup_{n\geq 0}\Omega^a_{\mathrm{Col}}(n),$$

$$A^a_{\mathrm{Col}}(n) := A^\dagger \int(\Omega^a_{\mathrm{Col}}(n-1)),$$

$$\Omega^a_{\mathrm{Col}}(n) := \begin{cases} A^a_{\mathrm{Col}}(n)\Omega^\dagger & \text{if } n\geq 1,\\ d(A^\dagger) & \text{if } n=0,\end{cases}$$

where $\Omega^\dagger := \Gamma(]Y_{\overline{\mathbb{F}_p}}[, j^\dagger\Omega_{]Y_{\overline{\mathbb{F}_p}}[})$. Then, we have the following short exact sequence:

$$0\to\mathbb{C}_p\to A^a_{\mathrm{Col}}\to\Omega^a_{\mathrm{Col}}\to 0.$$

In other words, we can integrate any Coleman 1-form $\omega\in\Omega^a_{\mathrm{Col}}$, and it is unique up to a constant. We got a p-adic integration theory satisfying (A) and (B) like this.

§2. (twisted)p-adic multiple polylogarithms, and p-adic multiple zeta values (L-values)

Fix $a\in\mathbb{C}_p$. In this chapter, we define (twisted) p-adic multiple polylogarithms and p-adic multiple zeta values (L-values) by using Coleman's p-adic integration theory for $\mathbb{U}(\mathbb{C}_p) := \mathbb{P}^1(\mathbb{C}_p)\setminus\{0,1,\infty\}$ (in the case of p-adic multiple zeta values) or $\mathbb{U}_N(\mathbb{C}_p) := \mathbb{P}^1(\mathbb{C}_p)\setminus\{0,\infty\}\cup\mu_N$ $(p\nmid N)$ (in the case of p-adic multiple L-values). *When we consider p-adic multiple L-values in this article, we always fix an integer N which is not divisible by p, and an embedding $\mathbb{Q}_p(\mu_N)\hookrightarrow\mathbb{C}_p$.*

Definition 2.1. (Furusho [Fu1]) For positive integers $k_1,\dots,k_d\geq 1$, we define p**-adic multiple polylogarithm function** $\mathrm{Li}^a_{k_1,\dots,k_d}(z)\in A^a_{\mathrm{Col}} = A^a_{\mathrm{Col}}(\mathbb{U}(\mathbb{C}_p))$ as follows:

$$\mathrm{Li}^a_{k_1,\dots,k_d}(z) := \begin{cases} \int_0^z \frac{1}{z}\mathrm{Li}^a_{k_1,\dots,k_d-1}(z)dz & \text{if } k_d>1,\\ \int_0^z \frac{1}{1-z}\mathrm{Li}^a_{k_1,\dots,k_{d-1}}(z)dz & \text{if } k_d=1, \text{and } d>1,\\ \int_0^z \frac{1}{1-z}dz := -\log^a(1-z) & \text{if } k_d=1, \text{and } d=1.\end{cases}$$

Definition 2.2. Similarly, for positive integers $k_1,\dots,k_d\geq 1$ and for N-th roots of unity $\zeta_1,\dots,\zeta_d$, we define **twisted** p**-adic multiple polylogarithm function** $\mathrm{Li}^a_{k_1,\dots,k_d;\zeta_1,\dots,\zeta_d}(z)\in A^a_{\mathrm{Col}} = A^a_{\mathrm{Col}}(\mathbb{U}_N(\mathbb{C}_p))$

as follows ([Y1]):

$$\mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z) :=$$
$$\begin{cases} \int_0^z \frac{1}{z}\mathrm{Li}^a_{k_1,\ldots,k_d-1;\zeta_1,\ldots,\zeta_d}(z)dz & \text{if } k_d > 1, \\ \int_0^z \frac{1}{\zeta_d - z}\mathrm{Li}^a_{k_1,\ldots,k_{d-1};\zeta_1,\ldots,\zeta_{d-1}}(z)dz & \text{if } k_d = 1, \text{and } d > 1, \\ \int_0^z \frac{1}{\zeta_1 - z}dz := -\log^a(\zeta_1 - z) & \text{if } k_d = 1, \text{and } d = 1. \end{cases}$$

We call $k_1 + \cdots + k_d$ its **weight** and d its **depth.**

By definition, we can see that $\mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z)$ satisfies

$$\mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z)|_{]0[} \in A(]0[),$$
$$\mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z)|_{]\zeta[} \in A(]\zeta[)[\log^a(z-\zeta)] \ (\zeta \in \mu_N), \text{ and}$$
$$\mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z)|_{]\infty[} \in A(]\infty[)[\log^a(1/t)].$$

Theorem 2.1. (Furusho [Fu1]) *The convergence of*

$$\lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}^a_{k_1,\ldots,k_d}(z)$$

does not depend on the choice of a. Furthermore, the limit value does not depend on the choice of a either when it converges. Here, we write $\lim'_{z\to c} f(z)$ *when for any sequence* $\{z_i\}_{i=1}^\infty$ *converging to c such that the ramification index of* $\mathbb{Q}_p(z_1, z_2, \ldots)/\mathbb{Q}_p$ *is finite, the limit* $\lim_{i\to\infty} f(z_i)$ *exists and does not depend on the choice of the sequence* $\{z_i\}_{i=1}^\infty$.

The same thing holds for $\lim_{\mathbb{C}_p \ni z\to 1}{}'\mathrm{Li}^a_{k_1,\ldots,k_d}(z)$ ([Y1]).

Definition 2.3. (Furusho [Fu1]) When $\lim'_{\mathbb{C}_p \ni z\to 1} \mathrm{Li}^a_{k_1,\ldots,k_d}(z)$ converges, we call it $\zeta_p(k_1, \ldots, k_d)$ **p-adic multiple zeta value**.

By the same way, when $\lim'_{\mathbb{C}_p \ni z\to 1} \mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z)$ converges, we call it $L_p(k_1, \ldots, k_d)$ **p-adic multiple L-value** ([Y1]).

Theorem 2.2. (Furusho [Fu1]) *For* $k_d > 1$, $\lim'_{\mathbb{C}_p \ni z\to 1} \mathrm{Li}^a_{k_1,\ldots,k_d}(z)$ *converges.*

Remark . It may converge even when $k_d = 1$. Then, the limit value is a normalized p-adic multiple zeta value (we will explain it later). In particular, it is a linear combination of p-adic multiple zeta values of the same weight ([Fu1]).

In the case of p-adic multiple L-values, for $(k_d, \zeta_d) \neq (1, 1)$,

$$\lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z)$$

converges ([Y1]).

Example 2.3. (Coleman) For $n > 1$,

$$\zeta_p(n) = \frac{1}{1-p^{-n}} L_p(n, \omega^{1-n}).$$

Here, L_p is Kubota–Leopoldt's p-adic L function, and ω is the Teichmuüller character. In particular, $\zeta_p(2n) = 0$ for $n \geq 1$. Note that Kubota–Leopoldt's p-adic L function is characterized by the p-adic interpolation of L-values at **negative** integers, and that the above comparison between the p-adic polylogarithm and the p-adic L function is at **positive** integers. On the other hand, Furusho showed $\zeta_p(2n) = 0$ by using 2- and 3-cycle relations. This comes from the fact that the summation of "the interior angles of the triangle is 0" in the p-adic analytic world, and we call it "$2\pi\sqrt{-1}$ is 0" in the p-adic analytic world.

The values at odd integers are more difficult. In fact, $\zeta_p(2n+1) \neq 0 \Leftrightarrow L_p(2n+1, \omega^{-2n}) \neq 0 \Leftrightarrow H^2(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(-n)) = 0$ (*higher Leopoldt conjecture*) for $n \geq 1$. This holds if p is a regular prime or if n is divisible by $p-1$. The general case is not settled yet.

Theorem 2.4. (Furusho [Fu1]) *We have* $\zeta_p(k_1, \dots, k_d) \in \mathbb{Q}_p$.

By the same way, $L_p(k_1, \dots, k_d; \zeta_1, \dots, \zeta_d) \in \mathbb{Q}_p(\mu_N)$ ([Y1]).

Question 2.5. (Furusho) *When does* $\zeta_p(k_1, \dots, k_d)$ *land in* $\mathbb{Z}_p$*?*

We have no results on this question except the case of the value is 0 like $\zeta_p(2n)$, $\zeta_p(2n, \dots, 2n)$, and $\zeta_p(3, 1, \dots, 3, 1)$. By the same way, we can also consider the question when $L_p(k_1, \dots, k_d; \zeta_1, \dots, \zeta_d) \in \mathbb{Z}_p[\mu_N]$?

For $w > 0$,we define $Z^p_w[N] \subset \mathbb{Q}_p$ as follows:

$$Z^p_w[N] :=$$
$$\left\langle L_p(k_1, \dots, k_d; \zeta_1, \dots, \zeta_d) \,\middle|\, \begin{array}{l} d \geq 1, k_i \geq 1, \zeta_i \in \mu_N \text{ for } i = 1, \dots, d, \\ k_1 + \cdots + k_d = w, (k_d, \zeta_d) \neq (1,1) \end{array} \right\rangle_{\mathbb{Q}},$$

(the $\mathbb{Q}$-vector space generated by $L_p(k_1, \dots, k_d; \zeta_1, \dots, \zeta_d)$'s), and put $Z^p_0[N] := \mathbb{Q}$, and $Z^p_\bullet[N] := \oplus_w Z^p_w[N]$. Put also $Z^p_w := Z^p_w[1]$, $Z^p_\bullet := Z^p_\bullet[1]$. We call Z^p_w (resp. $Z^p_w[N]$) **p-adic multiple zeta value space** (resp. **p-adic multiple L-value space**) of weight w. We will discuss the dimensions of these spaces in Chapter 6.

§3. p-adic KZ equation, and p-adic Drinfel'd associator

The Drinfel'd associator was originally introduced in [Dr] related with the associativity constraint in a quasi-tensor category (in other words, braded tensor category), and Drinfel'd gave another (clearer) proof of Kohno's theorem which says that all representations of quantum groups can be constructed as the monodromy representations of KZ equation. Furthermore, he studied the pentagon axiom and the hexagon axiom $(+x)$ which the unit element, the associativity constraint and the commutativity constraint should satisfy, he got the notion of Grothendieck–Teichmüller group, and showed that it was closely related with the absolute Galois group of $\mathbb{Q}$. We call the pentagon axiom, the hexagon axiom $(+x)$ and some modifications of them the **associator relations**, or **2-, 3-, 5-cycle relations**.

In this chapter, we explain p-adic KZ equation, and p-adic Drinfel'd associator. We do not discuss the braded tensor category.

Let $\mathbb{C}_p\langle\langle A,B\rangle\rangle$ denote the non-commutative formal power series ring with variables A and B. For a word W in $\mathbb{C}_p\langle\langle A,B\rangle\rangle$, we call the number of letters in W its **weight**, and the number of B in W its **depth**.

Definition 3.1. (Furusho [Fu1]) We call the following p-adic differential equation for a p-adically analytic function $G(z)$ valued in $\mathbb{C}_p\langle\langle A,B\rangle\rangle$ **p-adic KZ equation**[1]:

$$\frac{dG}{dz}(z)=\left(\frac{A}{z}+\frac{B}{z-1}\right)G(z),$$

$z\in\mathbb{U}(\mathbb{C}_p)$. Here, a function valued in $\mathbb{C}_p\langle\langle A,B\rangle\rangle$ is called p-adically analytic, if the coefficients of any words of A and B are p-adically analytic functions.

Similarly, for the p-adic multiple L-value case, we consider the following p-adic differential equation for a p-adically analytic function $G(z)$ $(z\in\mathbb{U}_N(\mathbb{C}_p))$ valued in $\mathbb{C}_p\langle\langle A,\{B_\zeta\}_\zeta\rangle\rangle$ ([Y1]):

$$\frac{dG}{dz}(z)=\left(\frac{A}{z}+\sum_{\zeta\in\mu_N}\frac{B_\zeta}{z-\zeta}\right)G(z),$$

Theorem 3.1. (Furusho [Fu1]) *There exists a unique solution $G_0^a(z)$ (resp. $G_1^a(z)$) $\in A^a_{\mathrm{Col}}\langle\langle A,B\rangle\rangle$ of the p-adic KZ equation with the following boundary condition:*

$$G_0^a(z)\approx z^A:=\sum_{n=0}^{\infty}\frac{1}{n!}(\log^a zA)^n \quad (z\to 0),$$

[1]The author thanks the referee for kindly informing him that this equation and its generalization are studied also in [W1].

$$(resp.\ G_1^a(z) \approx (1-z)^B \quad (z \to 1)\ ),$$

where $G_0^a(z) \approx z^A \ (z \to 0)$ *means that* $G_0^a(z)z^{-A}|_{]0[}$ *is in*

$$A(]0[)\langle\langle A,B\rangle\rangle A + A(]0[)\langle\langle A,B\rangle\rangle B$$

and that its value at $z=0$ *is* 1 (*similarly, for* $G_1^a(z) \approx (1-z)^B \quad (z \to 1)$) (*Note that* z^A *depends on the choice of the branch of the* p*-adic logarithm.*).

We have the same result for the p-adic multiple L-value case ([Y1]).

Theorem 3.2. (Furusho [Fu1]) *The element*

$$\Phi_{\mathrm{KZ}}^p(A,B) := G_1^a(z)^{-1}G_0^a(z)$$

does not depend on z *or* a, *and lands in* $\mathbb{C}_p\langle\langle A,B\rangle\rangle^\times$.

We have the same result for the p-adic multiple L-value case ([Y1]).

Definition 3.2. (Furusho [Fu1]) We call $\Phi_{\mathrm{KZ}}^p(A,B) \in \mathbb{C}_p\langle\langle A,B\rangle\rangle^\times$ **p-adic Drinfel'd associator**.

In the p-adic multiple L-value case, we define p-adic Drinfel'd associator $\Phi_{\mathrm{KZ}}^p(A,\{B_\zeta\}_\zeta)$ in the same way ([Y1]).

Theorem 3.3. (explicit formulae, Furusho [Fu1]) *Put* $\Phi_{\mathrm{KZ}}^p(A,B) = 1 + \sum_W I_p(W)W$, *where* W *runs all words of* A *and* B. *Take* $W = B^rVA^s$, $V \in A\mathbb{C}_p\langle\langle A,B\rangle\rangle B$ *or* $V=1$, *then we have*

$$I_p(W) = (-1)^{\mathrm{depth}(W)} \sum_{0\le a\le r, 0\le b\le s} (-1)^{a+b} Z_p(f(B^a ⧢ B^{r-a}VA^{s-b} ⧢ A^b)),$$

where f *is the composition*

$$\begin{aligned}\mathbb{C}_p\langle\langle A,B\rangle\rangle &\twoheadrightarrow \mathbb{C}_p\langle\langle A,B\rangle\rangle/(B\mathbb{C}_p\langle\langle A,B\rangle\rangle + \mathbb{C}_p\langle\langle A,B\rangle\rangle A)\\ &\xrightarrow{\cong} \mathbb{C}_p\cdot 1 + A\mathbb{C}_p\langle\langle A,B\rangle\rangle B \hookrightarrow \mathbb{C}_p\langle\langle A,B\rangle\rangle,\end{aligned}$$

⧢ *is the shuffle product of words* (*i.e., inductively defined as*

$$XW ⧢ YW' := X(W ⧢ YW') + Y(XW ⧢ W')$$

for $X,Y = A$ *or* B, *and* $W ⧢ 1 = 1 ⧢ W := W$), *and* $Z_p(W)$ *is defined by* $Z_p(A^{k_d-1}B\cdots A^{k_1-1}B) = \zeta_p(k_1,\ldots,k_d)$ *and its extension to* $\mathbb{C}_p\cdot 1 + A\mathbb{C}_p\langle\langle A,B\rangle\rangle B$ *by the linearity. In particular, we have* $I_p(W) = (-1)^{\mathrm{depth}(W)}Z_p(W)$ *for* $W \in A\mathbb{C}_p\langle\langle A,B\rangle\rangle B$.

We have a similar result for p-adic multiple L-values ([Y1]).

In general, we have $(-1)^{\mathrm{depth}(W)} I_p(A^{k_d-1}B\cdots A^{k_1-1}B)$ even in the case of $k_d = 1$ in the above definition, and we call them **normalized p-adic multiple zeta values**. In particular, normalized p-adic multiple zeta values are linear combinations of p-adic multiple zeta values of the same weight ([Fu1]). Furthermore, if $\lim'_{\mathbb{C}_p \ni z \to 1} \mathrm{Li}^a_{k_1,\ldots,k_d}(z)$ exists in the case of $k_d = 1$ (it can happen as mentioned in the previous chapter), then the value coincides with the normalized p-adic multiple zeta values ([Fu1]). In particular, when we define Z^p_w as

$$Z^p_w := \left\langle \zeta_p(k_1,\ldots,k_d) \,\middle|\, d \geq 1, k_1,\ldots,k_d \geq 1, \lim_{\mathbb{C}_p \ni z \to 1} {}'\mathrm{Li}^a_{k_1,\ldots,k_d}(z) \text{ exists} \right\rangle_{\mathbb{Q}},$$

then this Z^p_w coincides with the old one. The same result holds for the p-adic multiple L-values ([Y1]).

We define a $\mathbb{C}_p$-algebra homomorphism

$$\Delta : \mathbb{C}_p\langle\langle A, B\rangle\rangle \to \mathbb{C}_p\langle\langle A, B\rangle\rangle \hat{\otimes} \mathbb{C}_p\langle\langle A, B\rangle\rangle$$

by $\Delta(A) = A \otimes 1 + 1 \otimes A$ and $\Delta(B) = B \otimes 1 + 1 \otimes B$.

Theorem 3.4. ([Fu1]) *We have* $\Delta(\Phi^p_{\mathrm{KZ}}) = \Phi^p_{\mathrm{KZ}} \hat{\otimes} \Phi^p_{\mathrm{KZ}}$.

Corollary 3.5. (integral shuffle product, [Fu1]) *We have*

$$Z_p(W) \cdot Z_p(W') = Z_p(W ш W')$$

for $W, W' \in A\mathbb{C}_p\langle\langle A, B\rangle\rangle B$. In particular, $Z^p_\bullet$ is a graded ring, i.e., we have $Z^p_w \cdot Z^p_{w'} \subset Z^p_{w+w'}$.

In the p-adic multiple L-value case, we also have

$$\Delta(\Phi^p_{\mathrm{KZ}}) = \Phi^p_{\mathrm{KZ}} \hat{\otimes} \Phi^p_{\mathrm{KZ}},$$

$L_p(W) \cdot L_p(W') = L_p(W ш W')$, and $Z^p_w[N] \cdot Z^p_{w'}[N] \subset Z^p_{w+w'}[N]$ ([Y1]).

We omit the explicit formulae of $G^a_0(z)$ and the functional equation with respect to $z \leftrightarrow 1 - z$ here (see [Fu1]).

§4. Relations among p-adic multiple zeta values (L-values)

We have enormous relations like $\zeta(3) = \zeta(1,2)$, $\zeta(4) = \zeta(1,1,2) = 4\zeta(1,3) = \frac{4}{3}\zeta(2,2)$ among the multiple zeta values, and it is one of the most interesting research areas of the multiple zeta values, because these relations are related with many areas, for example, the hypergeometric differential equations, the quasi-tensor category (in other words, braded tensor category), the representations of quasi-triangular quasi-Hopf quantized universal enveloping algebras, the quantum invariants of knots, the integrable lattice models, the moduli of curves, the category of mixed Tate motives, and the algebraic K-theory.

In this chapter, we introduce the known relations among p-adic multiple zeta values.

Theorem 4.1. (double shuffle relations, Besser–Furusho [BF]) *For $W, W' \in A\mathbb{C}_p\langle\langle A, B\rangle\rangle B$, we have $Z_p(W * W') = Z_p(W)Z_p(W') = Z_p(W ш W')$, where $*$ means the series (or harmonic) shuffle product.*

Remark . We mentioned the integral shuffle product in the previous chapter. The series (or harmonic) shuffle product is non-trivial, because we have no description of the p-adic multiple zeta values in terms of converging infinite sums. They are defined only by the limit values. In the proof, they used **p-adic multiple polylogarithms of two variables**, Besser's higher dimensional generalization of Coleman's p-adic integration for curves (see, [B]), and higher dimensional version of tangential base points.

Remark . We also have the regularized version of double shuffle relations in the case where the last index is 1 by suitably formulating the series regularized p-adic multiple zeta values and the integral regularized p-adic multiple zeta values in $\mathbb{Q}_p[T]$, where T is a formal variable (see [FJ]). The **integral regularized p-adic multiple zeta values** $\zeta_p^I(k_1, \ldots, k_d) \in \mathbb{Q}_p[T]$ is defined by $\zeta_p^I(k_1, \ldots, k_d) := \sum_{i=0}^m b_i(0)T^i$ for $\mathrm{Li}_{k_1,\ldots,k_d}(1-\epsilon) = \sum_{i=0}^m b_i(\epsilon)(\log^a \epsilon)^i$, (for example, $\zeta_p^I(1) = -T$) and the **series regularized p-adic multiple zeta values** $\zeta_p^S(k_1, \ldots, k_d) \in \mathbb{Q}_p[T]$ is defined by $\zeta_p^S(k_1, \ldots, k_d) := \mathrm{Li}_{k_1,\ldots,k_d}^{a,(D_0)}(\bar{x}_1, \ldots, \bar{x}_d)\Big|_L$, where

$$\mathrm{Li}_{k_1,\ldots,k_d}^{a,(D_0)}(\bar{x}_1, \ldots, \bar{x}_d)$$

is the analytic continuation of the p-adic multiple polylogarithm of several variables to a part of the normal bundle of certain divisor in the stable compactification $\overline{\mathcal{M}}_{0,d+3}$ of the moduli of curves of type $(0, d+3)$, L is a tangent line in the divisor at certain point, and $T = \log^a t$. For the details, see [FJ]. The **regularized double shuffle relations** ([FJ]) is as follows:

$$\zeta_p^S(k_1, \ldots, k_d) = \mathbb{L}_p(\zeta_p^I(k_1, \ldots, k_d)),$$

where $\mathbb{L}_p$ a $\mathbb{Q}_p$-linear map from $\mathbb{Q}_p[T]$ to itself characterized by

$$\sum_{n=0}^{\infty} \mathbb{L}_p(T^n)\frac{u^n}{n!} = \exp\left(-\sum_{n=1}^{\infty} \frac{\zeta_p^I(n)}{n} u^n\right).$$

This is a p-adic analogue of the regularized double shuffle relations in [IKZ]. A similar result for the p-adic multiple L-values is not known yet.

Furusho announced the following theorem (he is preparing the third article in the series papers):

Theorem 4.2. (Furusho) *The p-adic Drinfel'd associator $\Phi^p_{\mathrm{KZ}}(A,B)$ satisfies the 2-, 3-, and 5-cycle relations, i.e.,*

$$\begin{aligned}
&\Phi^p_{\mathrm{KZ}}(A,B)\Phi^p_{\mathrm{KZ}}(B,A) = 1,\\
&\Phi^p_{\mathrm{KZ}}(C,A)\Phi^p_{\mathrm{KZ}}(B,C)\Phi^p_{\mathrm{KZ}}(A,B) = 1, \ \textit{where } ABC = 1, \ \textit{and}\\
&\Phi^p_{\mathrm{KZ}}(x_{12},x_{23})\Phi^p_{\mathrm{KZ}}(x_{34},x_{45})\Phi^p_{\mathrm{KZ}}(x_{51},x_{12})\Phi^p_{\mathrm{KZ}}(x_{23},x_{34})\Phi^p_{\mathrm{KZ}}(x_{45},x_{51})\\
&= 1, \ \textit{in } \widehat{P}_5.
\end{aligned}$$

Here, $\widehat{P}_5$ denotes the Malcev completion of the pure sphere braid group with 5 strings, x_{ij}'s are its standard generators, and for any unipotent group U over $\mathbb{Q}_p$ and for any elements $a, b \in U(\mathbb{Q}_p)$, let $\Phi^p_{\mathrm{KZ}}(a,b)$ denote the image of $\Phi^p_{\mathrm{KZ}}(A,B) \in \widehat{F}_2(\mathbb{Q}_p)$ by the unique homomorphism $\widehat{F}_2 \to U$ sending A and B to a and b respectively, where $\widehat{F}_2$ denotes the Malcev completion of the free group of rank 2.

Remark . Furusho also proved that the associator relations implied the regularized double shuffle relations (see, [Fu5], [T2]). Combining this and the above theorem, we will have another proof of the regularized double shuffle relations.

Remark . The 2-cycle relation comes from the symmetry of $\mathbb{U}$ with respect to $z \leftrightarrow 1-z$, and the 3-cycle relation comes from the symmetry of $\mathbb{U}$ with respect to $z \mapsto \frac{1}{1-z} \mapsto 1-\frac{1}{z} \mapsto z$ (by which $0,1,\infty$ are sent to $1,\infty,0$, then to $\infty,0,1$). The variety $\mathbb{U}$ is the moduli space $\mathcal{M}_{0,4}$ of curves of genus 0 with 4 (ordered) marked points, and we have 10 morphisms from it to the boundaries of the stable compactification $\overline{\mathcal{M}_{0,5}}$ of the moduli space of curves of genus 0 with 5 (ordered) marked points. In this way, we get 12 pentagons on it. The 5-cycle relation comes from the symmetry of these pentagons (or one of these pentagons).

Remark . We do not have the symmetry on $\mathbb{U}_N$ ($N > 1$) with respect to $z \leftrightarrow 1-z$ or $z \mapsto \frac{1}{1-z} \mapsto 1-\frac{1}{z} \mapsto z$. Thus, we cannot expect a naive analogue of 2- or 3-cycle relations for p-adic multiple L-values. We have a relation corresponding to the symmetry of $\mathbb{U}_N$ with respect to $z \leftrightarrow \frac{1}{z}$ ([Y1]). We can also expect the relation corresponding to the symmetry with respect to $1 \mapsto \zeta_N \mapsto \zeta_N^2 \mapsto \cdots \mapsto \zeta_N^{N-1} \mapsto 1$. In the case of $N = 4$, we have the special symmetry, *i.e.*, $\{0, 1, \sqrt{-1}, -1, -\sqrt{-1}, \infty\}$ forms a regular octahedron. So, we can expect a special relation corresponding to it in the case of $N = 4$. The author does not know that $\mathbb{U}_N$ $(N > 1)$ has a suitable moduli interpretation for an analogue of the 5-cycle relation.

By the above theorem, we have the crystalline side of the relations with Grothendieck–Teichmüller groups, which fits into the following picture (we omit the details here):

- $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \overset{\hat{\Phi}}{\hookrightarrow} \widehat{\mathrm{GT}}$ for the profinite side (here, the injectivity is Belyi's theorem),
- $\underline{\mathrm{Gal}_{\mathbb{Q}}^{(\ell)}}(\mathbb{Q}) \overset{\Phi_{\mathbb{Q}}^{(\ell)}}{\hookrightarrow} \underline{\mathrm{GT}_1}(\mathbb{Q})$ for the ℓ-adic Galois side,
- $\mathrm{Hom}_{\mathbb{Q}\text{-alg.}}(Z_\bullet/\pi^2, \mathbb{Q}) \overset{\Phi_{\mathrm{KZ}}}{\hookrightarrow} \underline{\mathrm{GRT}_1}(\mathbb{Q})$ for the Hodge side,
- $\mathrm{Hom}_{\mathbb{Q}\text{-alg.}}(Z_\bullet^p, \mathbb{Q}) \overset{\Phi_{\mathrm{KZ}}^p}{\hookrightarrow} \underline{\mathrm{GRT}_1}(\mathbb{Q})$ for the crystalline side, and
- $\mathrm{Gal}(\mathrm{MT}(\mathbb{Z}), \omega) \overset{\Phi^{\mathcal{M}}}{\hookrightarrow} \underline{\mathrm{GRT}}$ for the motivic side (here, the injectivity is highly non-trivial and due to F. Brown [Br]. we will give some explanations in the next chapter).

As a digress, we would like to remark that 2-, 3-, 5-cycle relations were originally studied by Drinfel'd as (a modification) of the following commutativity diagrams which the associativity constraint and the commutativity constraint $(+x)$ in a quasi-tensor category (or braided tensor category) should satisfy:

$$\begin{tikzcd}
((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[r, "\Phi_{1,2,3}\otimes \mathrm{id}_4", "\cong"'] \arrow[dd, "\Phi_{1\otimes 2,3,4}"', "\cong"] & (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d, "\cong"', "\Phi_{1,2\otimes 3,4}"] \\
 & V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[d, "\cong"', "\mathrm{id}_1 \otimes \Phi_{2,3,4}"] \\
(V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[r, "\cong", "\Phi_{1,2,3\otimes 4}"'] & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)),
\end{tikzcd}$$

$$\begin{tikzcd}
(V_1 \otimes V_2) \otimes V_3 \arrow[r, "R_{1,2}\otimes \mathrm{id}_3", "\cong"'] \arrow[d, "\Phi_{1,2,3}"', "\cong"] & (V_2 \otimes V_1) \otimes V_3 \arrow[r, "\Phi_{2,1,3}", "\cong"'] & V_2 \otimes (V_1 \otimes V_3) \arrow[d, "\mathrm{id}_2\otimes R_{1,3}"', "\cong"] \\
V_1 \otimes (V_2 \otimes V_3) \arrow[r, "\cong", "R_{1,2\otimes 3}"'] & (V_2 \otimes V_3) \otimes V_1 \arrow[r, "\cong", "\Phi_{2,3,1}"'] & V_2 \otimes (V_3 \otimes V_1),
\end{tikzcd}$$

$$\begin{tikzcd}
(V_1 \otimes V_2) \otimes V_3 \arrow[r, "R_{1\otimes 2,3}", "\cong"'] \arrow[d, "\Phi^{-1}_{1,2,3}"', "\cong"] & V_3 \otimes (V_1 \otimes V_2) \arrow[r, "\Phi^{-1}_{1,2,3}", "\cong"'] & (V_3 \otimes V_1) \otimes V_2 \arrow[d, "\cong"', "R_{3,1}\otimes \mathrm{id}_2"] \\
V_1 \otimes (V_2 \otimes V_3) \arrow[r, "\cong", "\mathrm{id}_1\otimes R_{2,3}"'] & V_1 \otimes (V_3 \otimes V_2) \arrow[r, "\cong", "\Phi^{-1}_{1,3,2}"'] & (V_1 \otimes V_3) \otimes V_2.
\end{tikzcd}$$

Here, R was originally found as an R-matrix in the studies of integrable lattice models, and these relations yield the Yang–Baxter equation in the studies of braids and integrable lattice models. Such views of quasi-tensor category (or braided tensor category) are also interesting. However, we do not investigate this direction in this short article.

Returning to the p-adic multiple zeta values, we have the following conjecture on the relations among them:

Conjecture 4.3. (Furusho) *All $\mathbb{Q}$-linear relations among the p-adic multiple zeta values are generated by 2-, 3-, and 5-cycle relations.*

Conjecture 4.4. (p-adic isobar conjecture, Furusho) *All $\mathbb{Q}$-linear relations among p-adic multiple zeta values are generated by the relations of the same weights, i.e., we have*

$$Z^p_\bullet := \bigoplus_w Z^p_w = \sum_w Z^p_w \quad (\textit{summation in } \mathbb{Q}_p).$$

For the p-adic multiple L-values, it is conjectured that all $\mathbb{Q}$-linear relations are generated by the relations of the same weights ([Y1]).

§5. Tannakian interpretation and a variant of (p-adic) multiple polylogarithms

In this chapter, we explain the Tannakian interpretations of the multiple polylogarithms and the p-adic multiple polylogarithms. Many contents of this chapter came from Furusho's Japanese article [Fu3]. See also [Y2].

First, we will give a brief review on Bloch–Zagier's variant of polylogarithms. For an odd (resp. even) integer $k \geq 1$, we define

$$P_k(z) := \mathrm{Re}(\text{resp. Im}) \left(\sum_{a=0}^{k-1} \frac{B_a}{a!} (\log |z|^2)^a \mathrm{Li}_{k-a}(z) \right),$$

where B_a is the a-th Bernoulli number, *i.e.*, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

For example, we have

$$P_1(z) = -\log|1-z|,$$
$$P_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z|\arg(1-z).$$

These functions have no monodromy, that is, they are **single valued** functions on $\mathbb{U}(\mathbb{C})$ (for example, when z goes around 1 anticlockwise, then $P_2(z)$ becomes $\mathrm{Im}(\mathrm{Li}_2(z) + 2\pi i \log z) + \log|z|(\arg(1-z) - 2\pi) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z|\arg(1-z) = P_2(z)$. So $P_2(z)$ is a single-valued function). Bloch used this variant for $k = 2$ (the dilogarithm) to calculate regulators from the algebraic K-theory ([Bl]). After that, Zagier used this variant to formulate a conjecture (so-called Zagier's conjecture, [Z]), and Beilinson–Deligne gave a Hodge-theoretic interpretation of this variant ([BD]).

Furusho constructed a single valued variant of complex multiple polylogarithms and an overconvergent variant of p-adic multiple polylogarithms by the Tannakian interpretations.

Put $\mathbb{U} := \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$. We do not give a review on the tangential base point $\overrightarrow{01}$ in this short article.

First, we begin with the complex case. Fix $z \in \mathbb{U}(\mathbb{C})$. We consider the Tannakian category of unipotent local systems on $\mathbb{U}(\mathbb{C})$ with the fiber functors corresponding to $\overrightarrow{01}$ and z. We call the corresponding prounipotent groupoid

$$\pi_1^{\mathrm{B}}(\mathbb{U}(\mathbb{C}); \overrightarrow{01}, z)$$

over $\mathbb{Q}$ the **Betti prounipotent fundamental groupoid** of $\mathbb{U}(\mathbb{C})$. Similarly, we consider the Tannakian category of coherent $\mathcal{O}_{\mathbb{U}}$ modules with unipotent integrable connections on $\mathbb{U}$ with the fiber functors corresponding to $\overrightarrow{01}$ and z. We call the corresponding prounipotent groupoid

$$\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, z)$$

over $\mathbb{Q}$ the **de Rham prounipotent fundamental groupoid** of $\mathbb{U}$. Put

$$\pi_1^{\mathrm{B}}(\mathbb{U}(\mathbb{C}), \overrightarrow{01}) := \pi_1^{\mathrm{B}}(\mathbb{U}(\mathbb{C}); \overrightarrow{01}, \overrightarrow{01}),$$

and

$$\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}, \overrightarrow{01}) := \pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, \overrightarrow{01}).$$

Choose a topological path b_z from $\overrightarrow{01}$ to z on $\mathbb{U}(\mathbb{C})$. We can consider b_z to be an element in $\pi_1^{\mathrm{B}}(\mathbb{U}(\mathbb{C}); \overrightarrow{01}, z)$. On the other hand, all $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}, \overrightarrow{01})$-torsors are trivial, because of $H^1(\mathbb{U}, \mathcal{O}_{\mathbb{U}}) = 0$. Therefore, we have a canonical element d_z in $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, z)$. We also have a canonical element $d_{\overline{z}}$ in $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, \overline{z})$ in the same way, where $\overline{z}$ is the complex conjugate of z.

We define ϕ_∞ to be the inverse of the isomorphism

$$\pi_1^{\mathrm{B}}(\mathbb{U}(\mathbb{C}); \overrightarrow{01}, z) \xrightarrow{\cong} \pi_1^{\mathrm{B}}(\mathbb{U}(\mathbb{C}); \overrightarrow{01}, \overline{z}),$$

which is induced by the complex conjugate. Now, by using the comparison isomorphism

$$\pi_1^{\mathrm{B}}(\mathbb{U}(\mathbb{C}); \overrightarrow{01}, z)(\mathbb{C}) \cong \pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, z)(\mathbb{C}),$$

we can consider b_z to be an element in $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, z)(\mathbb{C})$. In the same way, by the comparison isomorphism, we can consider ϕ_∞ to be the following isomorphism:

$$\phi_\infty : \pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, \overline{z})(\mathbb{C}) \xrightarrow{\cong} \pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, z)(\mathbb{C}).$$

Therefore, now we have three special elements in $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, z)(\mathbb{C})$:

$$b_z, d_z, \phi_\infty(d_{\overline{z}}) \in \pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}; \overrightarrow{01}, z)(\mathbb{C}).$$

Theorem 5.1. *We embed $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}, \overrightarrow{01})(\mathbb{C})$ into $\mathbb{C}\langle\langle A, B\rangle\rangle$ by sending the dual of $\frac{dz}{z}$ and $\frac{dz}{z-1}$ to $e^A := \sum_{n\geq 0} A^n/n!$ and $e^B := \sum_{n\geq 0} B^n/n!$ respectively. Then we have the following:*

(1) (Chen) *By the embedding*

$$\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q}, \overrightarrow{01})(\mathbb{C}) \hookrightarrow \mathbb{C}\langle\langle A, B\rangle\rangle,$$

the element $(d_z)^{-1} b_z$ is sent to

$$G_0(z) = \sum_W (-1)^{\mathrm{depth}(W)} \mathrm{Li}_W(z) W,$$

where $\mathrm{Li}_W(z)$ is the multiple polylogarithm defined for the word W in the same way. The choice of b_z corresponds to the branch of $\mathrm{Li}_W(z)$.

(2) (Furusho) *We define $\exp(G_0^-(z))$ to be the image of $(d_z)^{-1}\phi_\infty(d_{\overline{z}})$ by the above embedding, and put*

$$G_0^-(z) = \sum_W (-1)^{\mathrm{depth}(W)} \ell_W(z) W.$$

Then, $\ell_{A^{k-1}B}(z)$ coincides with $P_k(z)$, and $\ell_W(z)$ is a single-valued function for any word W. We call it a **single-valued variant of multiple polylogarithm**[2].

(3) (Differential equation, Furusho) *Here, $G_0^-(z)$ satisfies the following differential equation:*

$$\begin{aligned}\frac{dG}{dz} &= \left(\frac{A}{z}+\frac{B}{z-1}\right)G(z)\\ &\quad - G(z)\left(\frac{d\overline{z}}{z}(-A)+\frac{d\overline{z}}{z-1}\Phi_{\mathrm{KZ}}^-(A,B)^{-1}(-B)\Phi_{\mathrm{KZ}}^-(A,B)\right),\end{aligned}$$

where we do not explain the details of $\Phi_{\mathrm{KZ}}^-(A,B)$ (cf. [Fu2]).

(4) (Relation, Furusho) *We have*

$$G_0^-(z) = G_0(A,B)(z)\left[G_0(-A,\Phi_{\mathrm{KZ}}^-(A,B)^{-1}(-B)\Phi_{\mathrm{KZ}}^-(A,B))(\overline{z})\right]^{-1}.$$

Remark . We fixed $z \in \mathbb{U}(\mathbb{C})$, and did not consider $\ell_W(z)$'s as the functions of z. However, by using the above (4), we can describe $\ell_W(z)$ by the multiple polylogarithms. Thus, we can see that $\ell_W(z)$ is a real analytic function on $\mathbb{U}(\mathbb{C})$ (they are not holomorphic, since $\overline{z}$ appears there.).

Remark . The differential equation (3) comes from the *compatibility of the complex conjugate and the connections.*

Remark . The relation (4) comes from

$$(d_z)^{-1}\phi_\infty(d_{\overline{z}}) = (d_z)^{-1}b_z[\phi_\infty((d_{\overline{z}})^{-1}\overline{b_z})]^{-1}.$$

Next, we consider the p-adic case. Fix $z \in \mathbb{U}(\mathbb{Q}_p) \subset \mathbb{P}^1(\mathbb{Q}_p) = \mathbb{P}^1(\mathbb{Z}_p)$ satisfying $z_0 := z \bmod p \in \mathbb{U}(\mathbb{F}_p)$. We consider the Tannakian category of unipotent overconvergent isocrystals on $\mathbb{U}_{\mathbb{F}_p}$ with the fiber functors corresponding to $\overrightarrow{01}$ and z_0 (we can omit the overconvergence here, because Chiarellotto–Le Stum proved that any unipotent isocrystals are overconvergent). Then, we call the corresponding prounipotent groupoid

$$\pi_1^{\mathrm{rig}}(\mathbb{U}_{\mathbb{F}_p}/\mathbb{Q}_p;\overrightarrow{01},z_0)$$

over $\mathbb{Q}_p$ the **rigid prounipotent fundamental groupoid** of $\mathbb{U}_{\mathbb{F}_p}$. Put $\pi_1^{\mathrm{rig}}(\mathbb{U}_{\mathbb{F}_p}/\mathbb{Q}_p,\overrightarrow{01}) := \pi_1^{\mathrm{rig}}(\mathbb{U}_{\mathbb{F}_p}/\mathbb{Q}_p;\overrightarrow{01},\overrightarrow{01})$. By Besser's theorem ([B]), we

[2]The author thanks the referee for kindly informing him that the same or related single valued function are constructed in [W2] (see Proposition 10.1 of this), and also in [W3] (see Lemma 2.8.2 and 2.8.4 of this).

have the unique Frobenius invariant path c_{z_0} from $\overrightarrow{01}$ to z_0 in

$$\pi_1^{\mathrm{rig}}(\mathbb{U}_{\mathbb{F}_p}/\mathbb{Q}_p;\overrightarrow{01},z_0).$$

On the other hand, we have the canonical element d_{z^p} in $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q};\overrightarrow{01},z^p)$.

We define ϕ_p to be the inverse of the isomorphism

$$\pi_1^{\mathrm{rig}}(\mathbb{U}_{\mathbb{F}_p}/\mathbb{Q}_p;\overrightarrow{01},z_0)\xrightarrow{\cong}\pi_1^{\mathrm{rig}}(\mathbb{U}_{\mathbb{F}_p}/\mathbb{Q}_p;\overrightarrow{01},z_0),$$

which is induced by the Frobenius. Now, by using the comparison isomorphism

$$\pi_1^{\mathrm{rig}}(\mathbb{U}_{\mathbb{F}_p}/\mathbb{Q}_p;\overrightarrow{01},z_0)(\mathbb{Q}_p)\cong\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q};\overrightarrow{01},z)(\mathbb{Q}_p),$$

we can consider c_{z_0} to be an element in $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q};\overrightarrow{01},z)(\mathbb{Q}_p)$. In the same way, by the comparison isomorphism, we can consider ϕ_p to be the following isomorphism:

$$\phi_p:\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q};\overrightarrow{01},z^p)(\mathbb{Q}_p)\xrightarrow{\cong}\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q};\overrightarrow{01},z)(\mathbb{Q}_p).$$

Therefore, now we have three special elements in $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q};\overrightarrow{01},z)(\mathbb{Q}_p)$:

$$c_{z_0},d_z,\phi_p(d_{z^p})\in\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q};\overrightarrow{01},z)(\mathbb{Q}_p).$$

Theorem 5.2. *We embed $\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q},\overrightarrow{01})(\mathbb{Q}_p)$ into $\mathbb{Q}_p\langle\langle A,B\rangle\rangle$ by sending the dual of $\frac{dz}{z}$ and $\frac{dz}{z-1}$ to $e^A:=\sum_{n\geq 0}A^n/n!$ and $e^B:=\sum_{n\geq 0}B^n/n!$ respectively. Then we have the following:*

(1) (Furusho) *By the embedding*

$$\pi_1^{\mathrm{dR}}(\mathbb{U}/\mathbb{Q},\overrightarrow{01})(\mathbb{Q}_p)\hookrightarrow\mathbb{Q}_p\langle\langle A,B\rangle\rangle,$$

the element $(d_z)^{-1}c_{z_0}$ is sent to

$$G_0(z)=\sum_W(-1)^{\mathrm{depth}(W)}\mathrm{Li}_W^a(z)W,$$

where $\mathrm{Li}_W^a(z)$ is the p-adic multiple polylogarithm defined for the word W in the same way.

(2) (Furusho) *We define $\exp(G_0^\dagger(z))$ to be the image of $(d_z)^{-1}\phi_p(d_{z^p})$ by the above embedding, and put*

$$G_0^\dagger(z)=\sum_W(-1)^{\mathrm{depth}(W)}\ell_W^a(z)W.$$

Then, $\ell_W^a(z)$ is an element in $A^\dagger(\mathbb{U}_{\mathbb{Q}_p})$. We call it an **overconvergent variant of p-adic multiple polylogarithm**.

(3) (p-adic differential equation, Furusho–Y.) *Here, $G_0^\dagger(z)$ satisfies the following p-adic differential equation:*

$$\frac{dG}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right) G(z)$$
$$- G(z)\left(\frac{dz^p}{z}(p^{-1}A) + \frac{dz^p}{z^p-1}\Phi_{\mathrm{D}}^p(A,B)^{-1}(p^{-1}B)\Phi_{\mathrm{D}}^p(A,B)\right),$$

where we do not explain the details of $\Phi_{\mathrm{D}}^p(A,B)$ (cf. [Y1], [Fu2]).

(4) (Relation, Furusho) *We have*

$$G_0^\dagger(z) = G_0(A,B)(z)\left[G_0(p^{-1}A, \Phi_{\mathrm{D}}^p(A,B)^{-1}(p^{-1}B)\Phi_{\mathrm{D}}^p(A,B))(z^p)\right]^{-1}.$$

Remark . We fixed $z \in \mathbb{U}(\mathbb{Q}_p)$, and did not consider $\ell_W^a(z)$'s as the functions of z. However, by using the above (4), we can describe $\ell_W^a(z)$ by the p-adic multiple polylogarithms. Thus, we can see that $\ell_W^a(z)$ can be extended to a p-adically analytic function on $\mathbb{P}^1(\mathbb{C}_p) -]1,\infty[$. We can also see that none of c_{z_0}, $\mathrm{Li}_W^a(z)$ and $\ell_W^a(z)$ depends on the choice of the branch a of the p-adic logarithm on the region $z \in \mathbb{P}^1(\mathbb{C}_p) -]1,\infty[$.

Remark . The function $\ell_W^a(z)$ is an element of $A^\dagger(\mathbb{U}_{\mathbb{Q}_p})$, that is, an overconvergent function. So, it can be p-adic analytically extended to a region which is $\mathbb{P}^1(\mathbb{C}_p)$ minus open disks centered at 1 and ∞ with smaller radii than 1, and $\ell_W^a(z)$ does not depend on the choice of the branch a of the p-adic logarithm on this region. However, we can p-adic analytically extend it to a larger region, and it does depend on the choice of the branch a of the p-adic logarithm on this larger region.

Remark . The p-adic differential equation (3) comes from the *compatibility of Frobenius and the connections.* This method of yielding the p-adic differential equation from the compatibility of Frobenius and the connections came from Deligne's paper [D1, (19.6.2)] (in which he studied the meta-abelian quotients).

Remark . The relation (4) comes from

$$(d_z)^{-1}\phi_p(d_{z^p}) = (d_z)^{-1}c_{z_0}[\phi_p((d_{z^p})^{-1}c_{z_0})]^{-1}.$$

The twisted p-adic multiple polylogarithms also have the Tannakian interpretation and the p-adic differential equation ([Y1]).

§6. The upper bounds of the dimensions and the motivic Galois groups

In this chapter, we will discuss the dimensions of the p-adic multiple zeta value (resp. L-value) spaces, and the relation with the motivic

Galois groups of the mixed Tate motives over $\mathbb{Z}$ (resp. over the ring of S-integers of the cyclotomic fields).

We define the spaces of the multiple zeta values and L-values Z_w, $Z_w[N]$ by the same way as Z_w^p and $Z_w^p[N]$ respectively. First, we have the following conjecture on the dimensions of the complex multiple zeta value spaces.

Conjecture 6.1. (Dimension conjecture, Zagier) *We define a sequence $\{D_n\}_n$ by $D_0 = 1$, $D_1 = 0$, $D_2 = 1$, $D_{n+3} = D_{n+1} + D_n$ for $n \geq 0$ (In terms of the generating function, it is defined by $\sum_{n=0}^{\infty} D_n t^n = 1/(1-t^2-t^3)$). Then, we have $\dim_{\mathbb{Q}} Z_w = D_w$ for $w \geq 0$.*

The following celebrated theorem of Goncharov, Terasoma, and Deligne–Goncharov says the upper bound:

Theorem 6.2. (Goncharov, Terasoma, Deligne–Goncharov [G1], [T1], [DG]) *We have $\dim_{\mathbb{Q}} Z_w \leq D_w$ for $w \geq 0$.*

This says that there exist *enormous $\mathbb{Q}$-linear relations* among the multiple zeta values (the difference between the upper bound on the weight $w \geq 0$ and the number of the indices $(k_1, \ldots, k_d)$ of the weight $w = k_1 + \cdots + k_d$ grows exponentially when w goes to the infinity). The other inequality is a kind of transcendental number theoretic problem, and it seems that algebraic geometric approaches are not sufficient. On the multiple L-values, we have the following theorem:

Theorem 6.3. (Deligne–Goncharov [DG]) *For $N = 2$ (resp. $N > 2$), we define a sequence $\{D_n[N]\}_n$ by a generating function $1/(1-t-t^2) = \sum_{n\geq 0}^{\infty} D_n[2] t^n$ (resp. $1/(1-(\frac{\varphi(N)}{2}+\nu)t+(\nu-1)t^2) = \sum_{n\geq 0}^{\infty} D_n[N]t^n$, where $\varphi(N) := \#(\mathbb{Z}/N\mathbb{Z})^{\times}$, and ν is the number of the prime numbers dividing N). Then, we have $\dim_{\mathbb{Q}} Z_w[N] \leq D_w[N]$ for $w \geq 0$.*

Remark . In the case where $N > 4$ and N is a prime number, then the equality does not hold (Goncharov [G2]). The gap comes from the relation with the motivic Galois group (we will explain it later), and a part of the gap is related with the space of cusp forms of weight 2 with the level $\Gamma_1(N)$ (*loc. cit.*). The gap is not fully explained by the terms of cusp forms. It might be related with a kind of "iterated integrals of cusp forms".

Next, we consider the p-adic case. We have a p-adic analogue of Zagier's dimension conjecture:

Conjecture 6.4. (Dimension conjecture, Furusho–Y.) *We define a sequence $\{d_n\}_n$ by $d_0 = 1$, $d_1 = 0$, $d_2 = 0$, $d_{n+3} = d_{n+1} + d_n$ for*

$n \geq 0$ (*In terms of the generating function, it is defined by* $\sum_{n=0}^{\infty} d_n t^n = (1-t^2)/(1-t^2-t^3)$). *Then, we have* $\dim_{\mathbb{Q}} Z_w^p = d_w$ *for* $w \geq 0$.

Theorem 6.5. (Y. [Y1]) *For* $N = 2$ (*resp.* $N > 2$), *we define a sequence* $\{d_n[N]\}_n$ *by a generating function* $(1-t^2)/(1-t-t^2)$ (*resp.* $(1-t)/(1-(\frac{\varphi(N)}{2}+\nu)t+(\nu-1)t^2)$ *where* $\varphi(N) := \#(\mathbb{Z}/N\mathbb{Z})^\times$, *and* ν *is the number of the prime numbers dividing* N). *Then, we have* $\dim_{\mathbb{Q}} Z_w^p[N] \leq d_w[N]$ *for* $w \geq 0$.

This theorem also says that there exist *enormous* $\mathbb{Q}$*-linear relations* among the p-adic multiple zeta values (the difference between the upper bound on the weight $w \geq 0$ and the number of the indices $(k_1, \ldots, k_d)$ of the weight $w = k_1 + \cdots + k_d$ grows exponentially when w goes to the infinity). The other inequality is a kind of p-adic transcendental number theoretic problem, and it seems that algebraic geometric approaches are not sufficient in this case either.

Remark . It is *not* known that $\dim_{\mathbb{Q}} Z_w^p[N]$ is independent of p, and it seems difficult (*cf.* higher Leopoldt conjecture in Example 2.3).

Remark . In the case where $N > 4$ and N is a prime number, the equality does not hold by the same reason as in the complex case. The gap comes from the relation with the motivic Galois group (we will explain it later), and a part of the gap is also related with the space of the cusp forms of weight 2 with the level $\Gamma_1(N)$ in this case too.

First, we will explain the K-theoretic meaning of these sequences (In fact, all of the above upper bounds are proved by the relation with the algebraic K-theory). For example, we have

$$\frac{1}{1-t^2-t^3} = \frac{1}{1-t^2}\frac{1}{1-\frac{t^3}{1-t^2}} = \frac{1}{1-t^2}\frac{1}{1-(t^3+t^5+t^7+\cdots)}.$$

Here, $1/(1-t^2)$ corresponds to π^2 in the weight 2, and $t^3+t^5+t^7+\cdots$ corresponds to

$$\operatorname{rank} K_{2n-1}(\mathbb{Z}) = \begin{cases} 0 & \text{for } n : \text{even or } n = 1, \\ 1 & \text{for } n : \text{odd and } n \neq 1. \end{cases}$$

In this way, we use

$$\begin{aligned} &\operatorname{rank} K_{2n-1}\left(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]\right) \\ &= \begin{cases} r_1 + r_2 - 1 + \nu & \text{for } n = 1, \\ r_2 & \text{for } n : \text{even}, \\ r_1 + r_2 & \text{for } n : \text{odd and } n \neq 1. \end{cases} \end{aligned}$$

in the case of $N > 1$. In the p-adic case, the generating function is $(1-t^2)/(1-t^2-t^3)$, and it loses the factor $1/(1-t^2)$. This corresponds to the fact "$\pi^2 = 0$ in the p-adic analytic world" (See also [Y2]). In the case of $N > 2$, the difference between the complex case and the p-adic case is not $1/(1-t^2)$, but $1/(1-t)$. This corresponds to the fact that $-\log(1-\zeta) + \log(1-\zeta^{-1}) = -\log(-\zeta) \in \mathbb{Q}\cdot\pi$ in the weight 1, and that it disappears, since "$\pi = 0$ in the p-adic analytic world".

We will give a brief sketch of the proof of the upper bounds of [DG] and [Y1]. For the simplicity, we assume that $N = 1$ (In the general case, we use $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$ and $\mathbb{U}_N := \mathbb{P}^1 \setminus \{0, \infty\} \cup \mu_N$ *etc.* instead). The key ingredients are as follows:

- Deligne–Goncharov's Tannakian category $\mathrm{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$,
- Deligne–Goncharov's motivic fundamental groupoids

$$\{\pi_1^{\mathcal{M}}(\mathbb{U}; a, b)\}_{a,b=\vec{01},\dots},$$

- Tannakian interpretation of complex (resp. p-adic) multiple zeta values due to Chen and Drinfel'd (resp. Furusho),
- Borel's calculation of the rank of the algebraic K-groups

$$K_{2n-1}(\mathbb{Z}),$$

- in the p-adic case, Besser's Frobenius invariant path [B], and
- in the p-adic case, p-adic Hodge theory for open varieties [Y3], (and Bloch–Kato's H^1_f. see [Y1]).

First, we consider the Tannakian category $\mathrm{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$. This category is unconditional (First, we consider the subtriangulated category $\mathrm{DMT}(k)$ generated by the Tate objects in Voevodsky's triangulated category of (geometric) mixed motives $\mathrm{DM}_{\mathrm{gm}}(k) \otimes \mathbb{Q}$ over a field k (see [V]) tensored with $\mathbb{Q}$. Next, M. Levine defined subcategories $\mathrm{DMT}(k)^{\geq 0}$ and $\mathrm{DMT}(k)^{\leq 0}$ of $\mathrm{DMT}(k)$, and he showed that these give a t-structure on $\mathrm{DMT}(k)$ if Beilinson–Soulé's vanishing conjecture holds for k (see [L]). In particular, if k is a number field, then the conjecture holds, and we can get an abelian category $\mathrm{MT}(k)$ by taking the heart. It is easy to see that it is a Tannakian category. Finally, Deligne–Goncharov defined a sub-Tannakian category $\mathrm{MT}(\mathbb{Z})$ in $\mathrm{MT}(\mathbb{Q})$ by putting the condition "unramified" at all prime p (see [DG])). We have a canonical fiber functor

$$\omega : \mathrm{MT}(\mathbb{Z}) \to \mathrm{Vect}_{\mathbb{Q}},$$

which sends M to $\oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{MT}(\mathbb{Z})}(\mathrm{gr}^W_{-2n}M, \mathbb{Q}(n))$. Then, we have the Tannakian fundamental group $G_\omega := \underline{\mathrm{Aut}}(\omega)$ of $\mathrm{MT}(\mathbb{Z})$ with respect to

ω. We call it the **motivic Galois group of** MT($\mathbb{Z}$) (with respect to ω). This is a pro-algebraic group over $\mathbb{Q}$, and it is known that G_ω is a semi-direct product $G_\omega = \mathbb{G}_m \ltimes U_\omega$ of $\mathbb{G}_m$ with a prounipotent group U_ω, and that ω is equal to the de Rham realization functor (see [DG]), where the homomorphism $G_\omega \to \mathbb{G}_m$ corresponds to the action of G_ω on $\omega(\mathbb{Q}(1))$. We call U_ω the pro-unipotent part of the motivic Galois group of MT($\mathbb{Z}$) (with respect to ω). Let τ denote the splitting $\mathbb{G}_m \to G_\omega$. The functorial comparison isomorphisms $\mathbb{C}\otimes_{\mathbb{Q}} M_{\mathrm{B}} \cong \mathbb{C}\otimes_{\mathbb{Q}} M_{\mathrm{dR}}$ for $M \in$ MT($\mathbb{Z}$) between the Betti realization M_{B} and the de Rham realization M_{dR} give us an element $a_\sigma \in G_\omega(\mathbb{C})$ which is unique up to $G_\omega(\mathbb{Q})$ (Here, σ is an embedding $\mathbb{Q}(\mu_N) \hookrightarrow \mathbb{C}$. Now we are assuming $N = 1$. So we have only one such element). In other words, the element $a_\sigma \in G_\omega(\mathbb{C})$ expresses the difference of the $\mathbb{Q}$-structure of the Betti realizations and the one of the de Rham realizations. The period of $\mathbb{Q}(1)$ is $2\pi i$. So, $a_\sigma^0 := a_\sigma \tau(2\pi i)^{-1}$ is in $U_\omega(\mathbb{C})$.

In the p-adic case, we have functorial comparison isomorphisms $M_{\mathrm{crys}} \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} M_{\mathrm{dR}}$ (see [Y1]), and the Frobenius action F_p^{-1} on M_{crys}. By functorially transporting F_p^{-1} to $\mathbb{Q}_p \otimes_{\mathbb{Q}} M_{\mathrm{dR}}$, we get an element $F_p^{-1} \in G_\omega(\mathbb{Q}_p)$. In other words, the element $F_p^{-1} \in G_\omega(\mathbb{Q}_p)$ expresses the difference of the $\mathbb{Q}$-structure of the de Rham realizations and the one of the de Rham realizations twisted by the crystalline Frobenius (via the comparison isomorphisms). The action of F_p^{-1} of $\mathbb{Q}(1)$ is given by p. So, $\varphi_p := F_p^{-1}\tau(p)^{-1}$ is in $U_\omega(\mathbb{Q}_p)$.

Remark . In the complex case, the comparison isomorphisms $\mathbb{C}\otimes_{\mathbb{Q}} M_{\mathrm{B}} \cong \mathbb{C}\otimes_{\mathbb{Q}} M_{\mathrm{dR}}$ are given by the iterated integrals, two realizations give two $\mathbb{Q}$-structures M_{B} and M_{dR}, and the difference between them gives the multiple zeta values via the iterated integrals for $M = \pi_1^{\mathcal{M}}(\mathbb{U}, \overrightarrow{01})$.

However, in the p-adic case, the comparison isomorphisms $M_{\mathrm{crys}} \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} M_{\mathrm{dR}}$ are *not* given by the iterated integrals, or we have *no* natural $\mathbb{Q}$-structure on M_{crys} other than M_{dR} via the comparison isomorphism. In the p-adic case, the Frobenius action on M_{crys} gives the p-adic iterated integrals (This corresponds to the fact that the p-adic integration theory was constructed by the p-adic analytic continuation via Frobenius.), and the difference between M_{dR} and $F_p^{-1}(M_{\mathrm{dR}})$ (via the comparison isomorphism) gives the p-adic multiple zeta values via the iterated integrals for $M = \pi_1^{\mathcal{M}}(\mathbb{U}, \overrightarrow{01})$ ([Fu2], [Y1]).

We have the following relation with the algebraic K-theory and the theory of motives (see [L], [DG]):

$$\mathrm{Lie}(U_\omega^{\mathrm{ab}}) = \prod_n \mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n))^\vee \cong \prod_n K_{2n-1}(\mathbb{Z})^\vee_{\mathbb{Q}}.$$

Next, we consider the motivic fundamental groupoids

$$\{\pi_1^{\mathcal{M}}(\mathbb{U}; a, b)\}_{a,b=\overrightarrow{01},\dots},$$

and its realization $\pi_1^{\omega}(\mathbb{U}; a, b) := \omega(\pi_1^{\mathcal{M}}(\mathbb{U}; a, b))$ with respect to ω. Here, $\pi_1^{\mathcal{M}}(\mathbb{U}; a, b)$ is a pro-object in $\mathrm{MT}(\mathbb{Z})$. We have a pro-algebraic group H_ω of automorphisms of $\{\pi_1^{\omega}(\mathbb{U}; a, b)\}_{a,b=\overrightarrow{01},\dots}$. It is also known that H_ω is a semi-direct product $H_\omega = \mathbb{G}_m \ltimes V_\omega$ of $\mathbb{G}_m$ with a prounipotent group V_ω. Roughly speaking, V_ω is big and it has no power to give upper bounds of the space of (p-adic) multiple zeta values (In fact, V_ω is the pro-vector space subscheme of $\mathbb{Q}\langle\langle A, B\rangle\rangle$ which consists of group-like elements, and $\Phi_{\mathrm{KZ}} \in \mathbb{C}\langle\langle A, B\rangle\rangle$ or $\Phi_{\mathrm{KZ}}^p \in \mathbb{Q}_p\langle\langle A, B\rangle\rangle$ does not give any information on the upper bounds). On the other hand, U_ω is enough small by the above relation with algebraic K-groups of $\mathbb{Z}$ and Borel's calculation of their ranks.

The fact that $\pi_1^{\omega}(\mathbb{U}; a, b)$ comes from $\pi_1^{\mathcal{M}}(\mathbb{U}; a, b)$ gives a homomorphism

$$\iota : G_\omega \to H_\omega, \quad \text{and} \quad \iota : U_\omega \to V_\omega.$$

Finally, we can relate the element $\iota((a_\sigma^0, \tau(2\pi i))) \in \iota(U_\omega \rtimes \mathbb{G}_m)(\mathbb{C}) \subset V_\omega(\mathbb{C}) \rtimes \mathbb{G}_m(\mathbb{C})$ (resp. $\iota(\varphi_p) \in \iota(U_\omega)(\mathbb{Q}_p) \subset V_\omega(\mathbb{Q}_p)$) with the Drinfel'd associator Φ_{KZ} (resp. the p-adic Drinfel'd associator Φ_{KZ}^p), roughly because the period (resp. the Frobenius action) of $\pi_1^{\omega}(\mathbb{U}, \overrightarrow{01})$ is given by the multiple zeta values (resp. p-adic multiple zeta values). Pro-algebraic varieties are defined by defining equations. So, $\iota((a_\sigma^0, \tau(2\pi i)))$ (resp. $\iota(\varphi_p)$) should satisfy the defining equations of $\iota(U_\omega \rtimes \mathbb{G}_m)(\mathbb{C})$ (resp. $\iota(U_\omega)(\mathbb{Q}_p)$), these yield enormous relations among the multiple zeta values (resp. the p-adic multiple zeta values) via the relation with Φ_{KZ} (resp. Φ_{KZ}^p), and this gives the required upper bounds (We have no need of knowing concrete descriptions of the defining equations. It is just a general theory). This is the rough sketch of the proof. As a summary, the upper bounds come in the following way:

$$\begin{aligned} K_{2n-1}(\mathbb{Z})_{\mathbb{Q}} &\longleftrightarrow U_\omega, \text{where the differences of realizations live} \\ &\qquad i.e., (U_\omega \rtimes \mathbb{G}_m)(\mathbb{C}) \ni (a_\sigma^0, \tau(2\pi i)) \quad (\text{resp. } U_\omega(\mathbb{Q}_p) \ni \varphi_p) \\ &\qquad \longleftrightarrow \Phi_{\mathrm{KZ}} \quad (\text{resp. } \Phi_{\mathrm{KZ}}^p) \\ &\qquad \longleftrightarrow \text{MZV's} \quad (\text{resp. } p\text{-adic MZV's}). \end{aligned}$$

We return to the general case of N. We have some remarks on $\iota : G_\omega \to H_\omega$, $a_\sigma \in G_\omega(\mathbb{C})$, and $\varphi_p \in U_\omega(\mathbb{Q}_p(\mu_N))$.

Theorem 6.6. *The homomorphism $\iota : G_\omega \to H_\omega$ is injective in the following cases:*

- (Deligne [D2]) $N = 2, 3, 4, 8$, *and*
- (Brown [Br]) $N = 1$.

Remark . In the case where $N > 4$ and N is a prime number, then the injectivity does not hold (Goncharov [G2]). In fact, he showed that the restriction of ι to $\mathrm{Lie}(U_\omega)_{\deg=1} \wedge \mathrm{Lie}(U_\omega)_{\deg=1}$ ($\subset \mathrm{Lie}(U_\omega)_{\deg=2}$) is not injective. The gaps explained in the remarks after Theorem 6.3 and Theorem 6.5 come from this non-injectivity. The injectivity holds in the above exceptional cases, and we cannot expect it for other N's.

Remark . The injectivity for $N = 1$ (resp. $N = 2, 3, 4, 8$) implies the following (in fact, each of (1) and (2) is equivalent to the injectivity):

- $\mathrm{MT}(\mathbb{Z})$ (resp. $\mathrm{MT}(\mathbb{Z}[\mu_N, 1/N])$) is generated by $\pi_1^{\mathcal{M}}(\mathbb{U}, \overrightarrow{01})$ (resp. $\pi_1^{\mathcal{M}}(\mathbb{U}_N, \overrightarrow{01})$),
- the motivic multiple zeta values (resp. the motivic multiple L-values) generate the coordinate ring of G_ω.
- All the periods of the motives in $\mathrm{MT}(\mathbb{Z})$ (resp. $\mathrm{MT}(\mathbb{Z}[\mu_N, 1/N])$) are linear combinations of the multiple zeta values (resp. the multiple L-values for N), and
- so-called **Deligne-Ihara's problem** (resp. an analogue of Deligne–Ihara's problem) [3]. *i.e.* we consider the filtration I on $G_\ell := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty}))$ induced by the lower central series on $\pi_1^\ell(\mathbb{U}_{\overline{\mathbb{Q}}}, \overrightarrow{01})$ (resp. $\pi_1^\ell(\mathbb{U}_{\overline{\mathbb{Q}}}, \overrightarrow{01})$) via the natural action, where π_1^ℓ is the pro-ℓ fundamental group. Then, $\oplus_{n\geq 0}(\mathrm{gr}_I^n G_\ell) \otimes \mathbb{Q}$ is a free graded Lie algebra with one generator in each odd degree > 1 (resp. with one generator in each odd degree for $N = 2$, resp. with $\varphi(N)/2 - 1 + \nu = \varphi(N)/2$ generators in degree 1, $\varphi(N)/2$ generators in each degree > 1 for $N = 3, 4, 8$).

Remark . The proof in the case of $N = 2, 3, 4, 8$, and the one in the case of $N = 1$ are different. The latter one is more difficult. The difficulty for $N = 1$ comes from the fact that *the lower central series of* Lie U_ω *does not coincide with the depth filtration.* This non-coincidence was first observed by Ihara in [I]. The coincidence for $N = 2, 3, 4, 8$ comes from the proof of the injectivity theorem. We will briefly explatin their proofs.

First, we consider Deligne's proof for $N = 2, 3, 4, 8$ (He also has a similar result for $N = 6$). He got an explicit description of a basis in the image of Lie U_ω^{ab} to the first graded quotient of the depth

[3]The author thanks the referee for kindly informing him that some results in this direction are also in [W4] (see Theorem 15.4.7, Theorem 15.5.3, Corollary 15.6.4, and 15.6.5 of this).

filtration by using the *distribution properties* in the depth 1 in [DG]. For $N = 2, 3, 4$ (resp. for $N = 8$), he considered a homomorphism $\pi_1^\omega(\mathbb{G}_m - \mu_N) \to \pi_1^\omega(\mathbb{G}_m - \{1\}) \ltimes \pi_1^\omega(\mathbb{G}_m - \{\zeta_N^{-1}\})$ (resp. $\pi_1^\omega(\mathbb{G}_m - \mu_N) \to \pi_1^\omega(\mathbb{G}_m - \{1, -1\}) \ltimes \pi_1^\omega(\mathbb{G}_m - \{\zeta_N^{-1}, -\zeta_N^{-1}\})$) and the first graded quotient of the depth filtration of the Lie algebras of the above homomorphisms, and took the reduction modulo p for $p = 2$ and $p = 3$ in the case of $N = 2, 4, 8$, and $N = 3$ respectively after suitably taking an integral model (which does not come from a geometry). Then, from the explicit description of the above basis, we can see that the reduction modulo p is injective, and then we have the required injectivity by considering the ranks of free Lie algebras.

On the other hand, in the proof of $N = 1$, Brown used Goncharov's motivic multiple zeta values defined by using the theory of the motivic iterated integrals [G3]. We have the inclusions

$$
\begin{aligned}
&\Gamma(\mathbb{A}^1 \times U_\omega, \mathcal{O}_{U_\omega}) \\
&\supset \{\text{subalgebra generated by the motivic MZV's}\} \\
&\supset \{\text{subalgebra generated by the motivic MZV's with indices 2 and 3}\}.
\end{aligned}
$$

The first inclusion corresponds to the full subcategory generated by $\pi_1^{\mathcal{M}}(\mathbb{U}, \overrightarrow{01})$:

$$\mathrm{MT}(\mathbb{Z}) \supset \langle \pi_1^{\mathcal{M}}(\mathbb{U}, \overrightarrow{01}) \rangle.$$

It is easy to show that the number of indices of weight n consisting only of 2 and 3 (we call it the **motivic Hoffman basis**) is the same as the dimension of $\Gamma(\mathbb{A}^1 \times U_\omega, \mathcal{O}_{U_\omega})$ of weight n. His strategy is to show the former one is linearly independent (So, the motivic Hoffman basis is a basis as the vector space).

One of the key ingredients is Goncharov's coproduct formula in [G3], or its slight generalization to the coaction by considering "π^2" (or f_2). He studied the derivation version of coaction formula and its kernels. A determination of its kernels is useful for the induction in the proof. He used the induction on an increasing filtration given by the number of 3 in the indices, which is called a **level filtration**. The level filtration is stable under the derivation version of the coaction. By taking the graded quotients, we get square matrices. The last step of the proof is to show that these matrices are invertible. In this step, he needed Zagier's analytic linear relations of the multiple zeta values. By combining Zagier's relations and the determination of the kernel of the derivation version of the coaction, he lifted Zagier's relations to the relations among the motivic multiple zeta values. Then, seeing 2-adically certain coefficients of the relations gives us the invertibility of the matrices.

We finish this article by giving remarks on the elements $a_\sigma \in G_\omega(\mathbb{C})$ and $\varphi_p \in U_\omega(\mathbb{Q}_p(\mu_N))$.

Conjecture 6.7. (Grothendieck [DG]) *The element $a_\sigma \in G_\omega(\mathbb{C})$ is $\mathbb{Q}$-Zariski dense. In other words, this element gives an injection $\Gamma(G_\omega, \mathcal{O}_{G_\omega}) \hookrightarrow \mathbb{C}$.*

In the case of $N = 1, 2, 3, 4, 8$, by using Theorem 6.6, this conjecture is equivalent to the combination of Zagier's dimension conjecture 6.1 and the isobar conjecture (*cf.* p-adic isobar conjecture 4.4).

In the p-adic case, we can formulate a p-adic analogue of the above Grothendieck's conjecture on a special element in the motivic Galois group (See also [Y2]).

Conjecture 6.8. (Y. [Y1]) *The element $\varphi_p \in U_\omega(\mathbb{Q}_p(\mu_N))$ is $\mathbb{Q}$-Zariski dense. In other words, this element gives an injection*

$$\Gamma(U_\omega, \mathcal{O}_{U_\omega}) \hookrightarrow \mathbb{Q}_p(\mu_N).$$

In the case of $N = 1, 2, 3, 4, 8$, by using Theorem 6.6, this conjecture is equivalent to the combination of Furusho–Y's dimension conjecture 6.4 and the p-adic isobar conjecture 4.4 by the same way.

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