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# Fake liftings of Galois covers between smooth curves

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**Abstract.**

In this paper we formulate a refined version of the Oort conjecture on liftings of cyclic Galois covers between curves. We introduce the notion of fake liftings of cyclic Galois covers between curves; their existence would contradict the Oort conjecture, and we study the geometry of their semi-stable models. Finally, we introduce and investigate some examples of the smoothening process, which ultimately aims to show that fake liftings do not exist. This in turn would imply the Oort conjecture.

CONTENTS



## §0. Introduction

In what follows $R$ is a complete discrete valuation ring of unequal characteristic, $K \stackrel{\text{def}}{=} \text{Fr}(R)$ the quotient field of $R$, $\text{char}(\text{K}) = 0$, and $k$ the residue field of $R$ which we assume to be algebraically closed of characteristic $p > 0$. This paper is motivated by the following problem.

**Problem I.** Let $X$ be a proper, smooth, geometrically connected $R$-curve, and $f_k : Y_k \to X_k \stackrel{\text{def}}{=} X \times_R k$ a finite Galois cover between smooth $k$-curves with group $G$. Is it possible to lift the Galois cover

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$f_k$ to a Galois cover $f : Y \to X' \overset{\text{def}}{=} X \times_R R'$ where $R'/R$ is a finite extension and $Y$ is a smooth $R'$-curve?

We shall refer to a lifting $f$ as above, if it exists, as a smooth lifting of the Galois cover $f_k$. This problem has been considered successfully by Grothendieck in the case where $f_k$ is a tamely ramified cover. In this case a smooth lifting $f$ as above exists over $R$ (cf. [Gr]). The answer to this problem is however No in general. Indeed, in the case where $G$ is the full automorphism group of $Y_k$ there are examples where the size of $G$ exceeds the Hurwitz bound for the size of automorphism groups of curves in characteristic zero (cf. [Ro]), and the cover $f_k$ can not be lifted in this case. Also it is in general necessary to perform a finite extension of $R$ in order to solve this problem (cf. [Oo], 1). In the case where $f_k$ is wildly ramified there are non liftable examples with Galois groups as simple as $G \overset{\sim}{\to} \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (cf. [Gr-Ma], 5). See also [Oo], 1, for an example of a genus 2 curve in characteristic 5 and an automorphism group of cardinality 20 which cannot lift to characteristic 0. The following was conjectured by F. Oort.

**Oort conjecture [Conj-O]** Problem I has a positive answer if $G \overset{\sim}{\to} \mathbb{Z}/m\mathbb{Z}$ is a cyclic group. Moreover, in this case one can choose $R'$ in a solution to Problem I to be the minimal extension of $R$ which contains the $m$-th roots of 1.

In order to solve this conjecture one may reduce to the case where $G \overset{\sim}{\to} \mathbb{Z}/p^n\mathbb{Z}$ is a cyclic $p$-group (cf. Lemma 2.1.1). In this case the Oort conjecture has been verified when $n \leq 2$ (cf. [Se-Oo-Su] for the case $n = 1$, and [Gr-Ma] for the case $n = 2$). In the approach of Oort, Sekiguchi, Suwa, Green, and Matignon one uses the Oort–Sekiguchi–Suwa theory, which provides explicit equations describing the degeneration of the Kummer equations in characteristic 0 to the Artin–Schreier–Witt equations in characteristic $p > 0$. The conjecture [Conj-O] is still open. Recently, Obus and Wewers claim to have proved [Conj-O] for $n = 3$ and in several cases when $n > 3$ (cf. [Ob]). We revisit in §2, 2.1, the Oort conjecture. We formulate the following refined version of this conjecture (cf. 2.1, for more details).

**Oort Conjecture Revisited [Conj-O-Rev]** We use the same notations as in Problem I. Assume that $G \overset{\sim}{\to} \mathbb{Z}/m\mathbb{Z}$ is a cyclic group. Let $H$ be a quotient of $G$ and $g_k : Z_k \to X_k$ the Galois subcover of $f_k$ with group $H$. Then there exists a smooth Galois lifting $g : Z' \to X' \overset{\text{def}}{=} X \times_R R'$ of $g_k$ over some finite extension $R'/R$. Furthermore, for every smooth lifting $g$ of the Galois subcover $g_k$ of $f_k$ as above there exists a smooth lifting $f : Y'' \to X'' \stackrel{\text{def}}{=} X \times_R R''$ of $f_k$ over some finite extension $R''/R'$ such that $f$ dominates $g$, i.e. we have a factorisation $f : Y'' \to Z'' \stackrel{\text{def}}{=} Z' \times_{R'} R'' \stackrel{g \times_{R'} R''}{\longrightarrow} X''$. Moreover, $R''$ can be chosen to be the minimal extension of $R'$ which contains a primitive $m$-th root of 1.

As for the original Oort conjecture, to prove this revisited version one may reduce to the case where $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$. In the case $n = 1$ both [Conj-O] and [Conj-O-Rev] are clearly equivalent. In Section 2.2, and in the case where $n = 2$, we verify [Conj-O-Rev] in some cases (cf. Lemma 2.2.1, and Lemma 2.2.2). This paper is motivated by the idea of the search for a path, or a bridge, between Garuti's theory developed in [Ga] to approach Problem I and the (revisited) Oort conjecture, which may lead to the solution of this conjecture. We introduce in §2 the notion of fake liftings of cyclic Galois covers between curves with the purpose of establishing such a bridge.

Next, we explain the definition of fake liftings. Assume that $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$, $n \geq 1$. Let $H$ be the unique quotient of $G$ with cardinality $p^{n-1}$. We use the notations in Problem I and assume that $X = \mathbb{P}^1_R$. In fact one can reduce the solution of Problem I to this case (cf. 2.1, and Lemma 2.1.1). Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover with group $G$ and $g_k : X_k \to \mathbb{P}^1_K$ the Galois subcover of $f_k$ with group $H$. In order to solve [Conj-O-Rev] for the Galois cover $f_k$ and the subcover $g_k$ one may proceed by induction on the cardinality of the group $G$. The case where $G$ has cardinality $p$ is solved in [Se-Oo-Su]. So we may assume, by an induction hypothesis, that $g_k$ admits a smooth lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ defined over $R$, i.e. we assume that [Conj-O] holds for the Galois subcover $g_k$ of $f_k$. We would like to show that [Conj-O-Rev] is true for $f_k$ and the smooth lifting $g$ of the sub-cover $g_k$, i.e. show that $g$ can be dominated by a smooth lifting of $f_k$, possibly after a finite extension of $R$. Consider all possible Garuti liftings $f : \mathcal{Y} \to \mathcal{X} \xrightarrow{g} \mathbb{P}^1_R$ of $f_k$ which dominate the smooth lifting $g$ of $g_k$. A Garuti lifting $f : \mathcal{Y} \to \mathbb{P}^1_k$ of $f_k : Y_k \to \mathbb{P}^1_k$ is a finite Galois cover with group $G$ such that the special fibre $\mathcal{Y}_k$ of $\mathcal{Y}$ is irreducible (not necessarily smooth), the finite morphism $\mathcal{Y}_k \to \mathbb{P}^1_k$ is generically Galois with group $G$, and we have a factorisation $f_k : Y_k \to \mathcal{Y}_k \to \mathbb{P}^1_k$ where the morphism $Y_k \to \mathcal{Y}_k$ is a morphism of normalisation (cf. [Sa1], Definition 2.5.2 for more details. Note that a smooth lifting is a Garuti lifting). Garuti liftings $f : \mathcal{Y} \to \mathbb{P}^1_R$ as above which dominate the smooth lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ exist by the refined version of Garuti's theory established in [Sa1], Theorem 2.5.3, and are a priori defined over a finite extension of $R$. For a Garuti lifting $f$ as above, which we can assume is defined over $R$, the degree of the different in the morphism $f_K : \mathcal{Y}_K \overset{\text{def}}{=} \mathcal{Y} \times_R K \to \mathbb{P}^1_K$ between generic fibres is greater than or equal to the degree of the different in the morphism $f_k : Y_k \to \mathbb{P}^1_k$. Moreover, $\mathcal{Y}$ is smooth over $R$, which implies that [Conj-O-Rev] holds in this case, if and only if these degrees of different are equal. Next, we argue by contradiction. Assume that [Conj-O-Rev] doesn't hold for the Galois cover $f_k$ and the smooth lifting $g$ of the subcover $g_k$. In particular, for all possible Garuti liftings $f$ as above $\mathcal{Y}$ is not smooth over $R$. A Garuti lifting $f : \mathcal{Y} \to \mathbb{P}^1_R$ as above such that the degree of the different in the morphism $f_K : \mathcal{Y}_K \overset{\text{def}}{=} \mathcal{Y} \times_R K \to \mathbb{P}^1_K$ between generic fibres is minimal, among all possible $f$'s, is called a fake lifting of the Galois cover $f_k$ relative to the smooth lifting $g$ of $g_k$ (cf. Definition 2.3.2). Fake liftings won't exist if [Conj-O-Rev] is true. In fact in order to prove [Conj-O-Rev] for the Galois cover $f_k$ and the smooth lifting $g$ of $g_k$ it suffices to show that fake liftings $f$ as above do not exist (cf. Remark 2.3.3).

One expects fake liftings to have very special properties, which possibly may lead to their non existence. Special properties of fake liftings should be encoded in their semi-stable models. Let $f : \mathcal{Y} \to \mathcal{X} \overset{g}{\to} \mathbb{P}^1_R$ be a fake lifting as above, assuming it exists. In §2, we study the geometry of a minimal semi-stable model $\mathcal{Y}' \to \mathcal{Y}$ of $\mathcal{Y}$, which we suppose defined over $R$, and in which the ramified points in the morphism $f_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ specialise in smooth distinct points of $\mathcal{Y}'_k \overset{\text{def}}{=} \mathcal{Y}' \times_R k$. It turns out that these semi-stable models have indeed very specific properties, which are in some sense reminiscent to the properties of the minimal semi-stable models of smooth liftings of cyclic Galois covers between curves. We prove, among other facts, that the configuration of the special fibre $\mathcal{Y}'_k \overset{\text{def}}{=} \mathcal{Y}' \times_R k$ of the semi-stable model $\mathcal{Y}'$ of the fake lifting $f$ is tree-like (cf. Theorem 2.5.4 (i)). Moreover, all the irreducible components of positive genus in $\mathcal{Y}'_k$ which contribute to the difference between the generic and special different in the morphism $f : \mathcal{Y} \to \mathbb{P}^1_R$ are end vertices of the tree associated to $\mathcal{Y}'_k$ with special properties (cf. loc. cit). In the course of proving this result we establish some of the properties of the minimal semi-stable model of an order $p^n$ automorphism of a $p$-adic open disc, with no inertia at the level of special fibres, that were established in the case $n = 1$ in [Gr-Ma1] (cf. 2.5.3).

Finally, in §3, we introduce the smoothening process for a fake lifting $f : \mathcal{Y} \to \mathcal{X} \overset{g}{\to} \mathbb{P}^1_R$ as above. The ultimate aim of this process is to show that fake liftings do not exist. This in turn would prove [Conj-O-Rev].

The basic idea of smoothening of the fake lifting $f$ is to construct, starting from $f$, a new Garuti lifting $f_1 : \mathcal{Y}_1 \to \mathcal{X} \xrightarrow{g} \mathbb{P}^1_R$ which dominates the smooth lifting $g$ of $g_k$ and such that the degree of the different in the morphism $f_{1,K} : \mathcal{Y}_{1,K} \stackrel{\text{def}}{=} \mathcal{Y}_1 \times_R K \to \mathbb{P}^1_K$ between generic fibres is smaller than the degree of the different in the morphism $f_K : \mathcal{Y}_K \to \mathbb{P}^1_K$. We call such $f_1$ a smoothening of $f$. If this construction is possible, it would imply that the fake lifting $f$ doesn't exist. Indeed, this would contradict the minimality of the generic different in $f$, hence will prove [Conj-O-Rev] for the Galois cover $f_k$ and the smooth lifting $g$ of the subcover $g_k$. We describe a formal way, using formal patching techniques, to construct a smoothening $f_1$ of the fake lifting $f : \mathcal{Y} \to \mathbb{P}^1_R$ starting from the minimal semi-stable model $\mathcal{Y}' \to \mathcal{Y}$ of $\mathcal{Y}$ (cf. 3.1). This construction is related to the existence of (internal) irreducible components in the special fibre $\mathcal{P}_k$ of the quotient $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$ of the semi-stable model $\mathcal{Y}'$ by $G$, which satisfy certain technical conditions arising from the geometry of the semi-stable model $\mathcal{Y}'$ and the Galois cover $\mathcal{Y}' \to \mathcal{P}$. We call such a component a removable vertex of the tree associated to $\mathcal{P}_k \stackrel{\text{def}}{=} \mathcal{P} \times_R k$ (cf. Definition 3.1.2). The existence of a removable vertex in $\mathcal{P}_k$ leads immediately to the existence of a smoothening $f_1$ of the fake lifting $f$ as above (cf. Definition 3.1.3).

We show that the smoothening process is possible in the case where $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ (cf Proposition 3.2.2). This gives an alternative proof of the Oort conjecture in this case. This proof, though simple, is striking in the view of the author in many respects. First, this proof is not explicit, in the sense that it doesn't produce an explicit lifting of the Galois cover $f_k$. Second, the proof doesn't rely (in any form) on the degeneration of the Kummer equation to the Artin–Schreier equation as in [Se-Oo-Su] (cf. also [Gr-Ma]), but rather on the degeneration of the Kummer equation to a radicial equation (cf. proof of Proposition 3.2.2). This suggests the possibility of proving [Conj-O] without using the Oort–Sekiguchi–Suwa theory. In the case where $n = 2$, i.e. $G \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$, we give, in 3.3, some sufficient conditions for the existence of removable vertices which lead to the execution of the smoothening process (cf. cf. Proposition 3.3.1).

Next, we briefly review the content of each section of this paper. In §1 we collect some background material which is used in this paper. In §2 we revisit the Oort conjecture and introduce the notion of fake liftings of cyclic Galois covers between curves. We then establish the main properties of their minimal semi-stable models in 2.5.4. In §3 we introduce the notion of the smoothening process for fake liftings and we investigate on some examples, in degree $p$ and $p^2$, this process.

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## §1. Background

In this section we collect some background material which is used in this paper.

### 1.1. Formal Patching

In this subsection we explain the procedure which allows to construct (Galois) covers of curves in the setting of formal geometry, by patching covers of formal affine curves with covers of formal fibres at closed points of the special fibre (cf. [Sa], 1, for more details). Let $R$ be a complete discrete valuation ring with fraction field $K$, residue field $k$, and uniformiser $\pi$. Let $X$ be an admissible formal $R$-scheme which is an $R$-curve; meaning that the special fibre $X_k \overset{\text{def}}{=} X \times_R k$ is a reduced one-dimensional scheme of finite type over $k$. Let $Z$ be a finite set of closed points of $X$. For a point $x \in Z$ let $X_x \overset{\text{def}}{=} \operatorname{Spf} \hat{\mathcal{O}}_{\mathrm{X,x}}$ be the formal completion of $X$ at $x$, which is the formal fibre at the point $x$. Let $X'$ be a formal open subscheme of $X$ whose special fibre is $X_k \setminus Z$. For each closed point $x \in Z$ let $\{\mathcal{P}_i\}_{i=1}^n$ be the set of minimal prime ideals of $\hat{\mathcal{O}}_{X,x}$ which contain $\pi$; they correspond to the branches $\{\eta_i\}_{i=1}^n$ of the completion of $X_k$ at $x$, and let $X_{x,i} \overset{\text{def}}{=} \operatorname{Spf} \hat{\mathcal{O}}_{\mathrm{x},\mathcal{P}_{\mathrm{i}}}$ be the formal completion of the localisation of $X_x$ at $\mathcal{P}_i$. The local ring $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$ is a complete discrete valuation ring. The set $\{X_{x,i}\}_{i=1}^n$ is the set of boundaries of the formal fibre $X_x$. For each $i \in \{1, ..., n\}$ we have a canonical morphism $X_{x,i} \to X_x$.

**Definition 1.1.1.** With the same notations as above a ($G$-)cover patching data for the pair $(X, Z)$ consists of the following.
(i) A finite (Galois) cover $Y' \to X'$ (with group $G$).
(ii) For each point $x \in Z$ is given a finite (Galois) cover $Y_x \to X_x$ (with group $G$).

The above data (i) and (ii) must satisfy the following compatibility condition.
(iii) If $\{X_{x,i}\}_{i=1}^n$ are the boundaries of the formal fibre at the point $x$, then for each $i \in \{1, ..., n\}$ is given a ($G$-equivariant) $X_x$-isomorphism

$$\sigma_i : Y_x \times_{X_x} X_{x,i} \xrightarrow{\sim} Y' \times_{X'} X_{x,i}.$$

Property (iii) should hold for each $x \in Z$.

The following is the main patching result that we will use in this paper (cf [Sa] 1) for more details).

**Proposition 1.1.2.** *We use the same notations as above. Given a (G-)cover patching data as in Definition 1.1.1 there exists a unique, up to isomorphism, (Galois) cover $Y \to X$ (with group $G$) which induces the above (G-)cover in Definition 1.1.1 (i) when restricted to $X'$, and induces the above (G-)cover in Definition 1.1.1 (ii) when pulled-back to $X_x$, for each point $x \in Z$.*

1.1.3. We use the same notations as above. Let $x \in Z$ and $\tilde{X}_k$ the normalisation of $X_k$. There is a one-to-one correspondence between the set of points of $\tilde{X}_k$ above $x$ and the set of boundaries of the formal fibre at the point $x$. Let $x_i$ be the point of $\tilde{X}_k$ above $x$ which corresponds to the boundary $X_{x,i}$, for $i \in \{1, ..., n\}$. Assume that the point $x \in X_k(k)$ is rational. Then the completion of $\tilde{X}_k$ at $x_i$ is isomorphic to the spectrum of a ring of formal power series $k[[t_i]]$ in one variable over $k$, where $t_i$ is a local parameter at $x_i$. The complete local ring $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$ is a discrete valuation ring with residue field isomorphic to $k((t_i))$. Let $T_i \in \hat{\mathcal{O}}_{x,\mathcal{P}_i}$ be an element which lifts $t_i$. Such an element is called a parameter of $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$. Then there exists an isomorphism $\hat{\mathcal{O}}_{x,\mathcal{P}_i} \xrightarrow{\sim} R[[T_i]]\{T_i^{-1}\}$ where

$$R[[T]]\{T^{-1}\} \overset{\text{def}}{=} \{ \sum_{i=-\infty}^{\infty} a_i T^i, \lim_{i \to -\infty} |a_i| = 0\},$$

and $| \ |$ is a normalised absolute value of $R$.

As a direct consequence of the above patching result, and the theorems of liftings of étale covers (cf. [Gr]), one obtains the following (well-known) local-global principle for liftings of (Galois) covers of curves.

**Proposition 1.1.4.** *Let $X$ be a proper, flat, algebraic (or formal) $R$-curve and let $Z \overset{def}{=} \{x_i\}_{i=1}^n$ be a finite set of closed points of $X$. Let $f_k : Y_k \to X_k$ be a finite generically separable (Galois) cover (with group $G$) whose branch locus is contained in $Z$. Assume that for each $i \in \{1, ..., n\}$ there exists a (Galois) cover $f_i : Y_i \to \mathrm{Spf}\hat{\mathcal{O}}_{\mathrm{X,x_i}}$ (with group $G$) which lifts the (Galois) cover $\hat{Y}_{k,x_i} \to \mathrm{Spec}\hat{\mathcal{O}}_{\mathrm{X_k,x_i}}$ induced by $f_k$, where $\hat{\mathcal{O}}_{X_k,x_i}$ (resp. $\hat{Y}_{k,x_i}$) denotes the completion of $X_k$ at $x_i$ (resp. the completion of $Y_k$ above $x_i$). Then there exists a unique, up to isomorphism, (Galois) cover $f : Y \to X$ (with group $G$) which lifts the (Galois) cover $f_k$ and which is isomorphic to the cover $f_i$ when pulled back to $\mathrm{Spf}\hat{\mathcal{O}}_{\mathrm{X,x_i}}$, for each $i \in \{1, ..., n\}$.*

### 1.2. Degeneration of $\mu_p$-torsors

In this subsection we recall the (well-known) degeneration of $\mu_p$-torsors from zero to positive characteristic above the boundaries of formal fibres of formal $R$-curves at closed points. Here $R$ denotes a complete discrete valuation ring of unequal characteristic with fraction field $K$, residue field $k$ of characteristic $p > 0$, uniformiser $\pi$, and which contains $\zeta$: a primitive $p$-th root of 1. We write $\lambda \stackrel{\text{def}}{=} \zeta - 1$. We denote by $v_K$ the valuation of $K$ which is normalised by $v_K(\pi) = 1$. First, we recall the definition of a certain class of $R$-group schemes (cf. [Se-Oo-Su], for more details).

1.2.1. **Torsors under finite and flat $R$-group schemes of rank $p$: the group schemes $\mathcal{G}_n$ and $\mathcal{H}_n$.** Let $n \geq 1$ be an integer. Define the affine $R$-group scheme $\mathcal{G}_{n,R} \stackrel{\text{def}}{=} \operatorname{Spec}(\mathrm{A_n})$ as follows.
(i) $A_n \stackrel{\text{def}}{=} R[X, \frac{1}{1+\pi^n X}]$.
(ii) The comultiplication $c_n : A_n \to A_n \otimes_R A_n$ is defined by $c_n(X) \stackrel{\text{def}}{=} X \otimes 1 + 1 \otimes X + \pi^n X \otimes X$.
(iii) The coinverse $i_n : A_n \to A_n$ is defined by $i_n(X) \stackrel{\text{def}}{=} -\frac{X}{1+\pi^n X}$.
(iv) The counit $\epsilon_n : A_n \to R$ is defined by $\epsilon_n(X) \stackrel{\text{def}}{=} 0$.
One verifies that $\mathcal{G}_n \stackrel{\text{def}}{=} \mathcal{G}_{n,R}$ is an affine, commutative, and smooth $R$-group scheme with generic fibre $(\mathcal{G}_n)_K \stackrel{\sim}{\to} \mathbb{G}_{m,K}$ and special fibre $(\mathcal{G}_n)_k \stackrel{\sim}{\to} \mathbb{G}_{a,k}$. Assume that $n$ satisfies the following condition

$$(*) \qquad 0 < n(p-1) \leq v_K(p).$$

Consider the map $\phi_n : \mathcal{G}_n \to \mathcal{G}_{pn}$ given by:

$$X \mapsto \frac{(1+\pi^n X)^p - 1}{\pi^{pn}}.$$

Then $\phi_n$ is a surjective homomorphism of $R$-group schemes. Denote by $\mathcal{H}_n \stackrel{\text{def}}{=} \mathcal{H}_{n,R} \stackrel{\text{def}}{=} \operatorname{Ker}(\phi_\mathrm{n})$. The group scheme $\mathcal{H}_n$ is finite, flat, and commutative of rank $p$. Under the assumption $(*)$ one verifies that the generic fibre $\mathcal{H}_{n,K} \stackrel{\text{def}}{=} \mathcal{H}_n \otimes_R K \stackrel{\sim}{\to} \mu_{p,K}$ is étale, the special fibre $\mathcal{H}_{n,k} \stackrel{\text{def}}{=} \mathcal{H}_n \otimes_R k \stackrel{\sim}{\to} \alpha_{p,k}$ is radicial of type $\alpha_p$ if $n < v_K(\lambda)$, and $\mathcal{H}_{n,k} \stackrel{\text{def}}{=} \mathcal{H}_n \otimes_R k \stackrel{\sim}{\to} (\mathbb{Z}/p\mathbb{Z})_k$ is étale if $n = v_K(\lambda)$.

Let $\mathcal{U} \stackrel{\text{def}}{=} \operatorname{Spf}\mathrm{A}$ be a formal affine $R$-scheme and $f : \mathcal{V} \to \mathcal{U}$ a torsor under the group scheme $\mathcal{H}_n$, for some $n$ as above satisfying $(*)$. Then there exists a regular function $u \in A$ such that the image $\bar{u}$ of $u$ in $\overline{A} \stackrel{\text{def}}{=} A/\pi A$ is not a $p$ power if $n < v_K(\lambda)$, $1 + \pi^{pn} u$ is defined up to multiplication by a $p$-th power of the form $(1 + \pi^n v)^p$, and the torsor $f$ is given by an equation $(X')^p = (1 + \pi^n X)^p = 1 + \pi^{np} u$ where $X'$ and $X$ are indeterminates. Moreover, the natural morphism $f_k : \mathcal{V}_k \to \mathcal{U}_k$ between the special fibres is either the $\alpha_p$-torsor given by the equation

$x^p = \bar{u}$ where $x = X \mod \pi$, and $\bar{u} = u \mod \pi$, if $n < v_K(\lambda)$. Or is the $\mathbb{Z}/p\mathbb{Z}$-torsor given by the equation $x^p - x = \bar{u}$ where $x = X \mod \pi$, and $\bar{u} = u \mod \pi$, if $n = v_K(\lambda)$.

Next, we recall the degeneration of $\mu_p$-torsors on the boundary $\mathcal{X} \overset{\text{def}}{=} \text{SpfR}[[T]]\{T^{-1}\}$ of formal fibres of germs of formal $R$-curves. Here $R[[T]]\{T^{-1}\}$ is as in 1.1.3. Note that $R[[T]]\{T^{-1}\}$ is a complete discrete valuation ring with uniformising parameter $\pi$ and residue field $k((t))$, where $t = T \mod \pi$.

**Proposition 1.2.2.** *Let $A \overset{def}{=} R[[T]]\{T^{-1}\}$ (cf. 1.1.3) and $f$ : $\text{SpfB} \to \text{SpfA}$ a non trivial Galois cover of degree $p$. Assume that the ramification index of the corresponding extension of discrete valuation rings equals* 1. *Then $f$ is a torsor under a finite and flat $R$-group scheme $G$ of rank $p$. Let $\delta$ be the degree of the different in the above extension. The following cases occur.*

*(a) $\delta = v_K(p)$. Then $f$ is a torsor under the group scheme $G = \mu_{p,R}$ and two cases occur.*

*(a1) For a suitable choice of the parameter $T$ of $A$ the torsor $f$ is given, after possibly a finite extension of $R$, by an equation $Z^p = T^h$. In this case we say that the torsor $f$ has a degeneration of type $(\mu_p, 0, h)$.*

*(a2) For a suitable choice of the parameter $T$ of $A$ the torsor $f$ is given, after possibly a finite extension of $R$, by an equation $Z^p = 1 + T^m$ where $m$ is a positive integer prime to $p$. In this case we say that the torsor $f$ has a degeneration of type $(\mu_p, -m, 0)$.*

*(b) $0 < \delta < v_K(p)$. Then $f$ is a torsor under the group scheme $\mathcal{H}_{n,R}$, where $n$ is such that $\delta = v_K(p) - n(p-1)$. Moreover, for a suitable choice of the parameter $T$ the torsor $f$ is given, after possibly a finite extension of $R$, by an equation $Z^p = 1 + \pi^{pn}T^m$ with $m \in \mathbb{Z}$ prime to $p$. In this case we say that the torsor $f$ has a degeneration of type $(\alpha_p, -m, 0)$.*

*(c) $\delta = 0$. Then $f$ is an étale torsor under the $R$-group scheme $G = \mathcal{H}_{v_K(\lambda),R}$ and is given, after possibly a finite extension of $R$, by an equation $Z^p = 1 + \lambda^p T^m$ where $m$ is a negative integer prime to $p$, for a suitable choice of the parameter $T$ of $A$. In this case we say that the torsor $f$ has a degeneration of type $(\mathbb{Z}/p\mathbb{Z}, -m, 0)$.*

*Proof.* See [Sa], Proposition 2.3. Q.E.D.

## §2. Fake liftings of cyclic covers between smooth curves

In this section we formulate a refined version of Oort conjecture on liftings of cyclic covers between curves. We introduce the notion of fake liftings of cyclic covers between curves and study their semi-stable models.

### 2.1. The Oort conjecture

First, we recall the following main conjecture which was formulated by F. Oort and several of its variants. In what follows $R$ is as in 1.1.

**The Original Oort conjecture [Conj-O]** (cf. [Oo] and [Oo1]) Let $f_k : Y_k \to X_k$ be a finite (possibly ramified) Galois cover between smooth $k$-curves with group $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$ a cyclic group. Then there exists a finite extension $R'/R$ and a Galois cover $f' : Y' \to X'$ between smooth $R'$-curves with group $G$, such that the special fibre $X'_k \stackrel{\text{def}}{=} X' \times_R k$ (resp. $Y'_k \stackrel{\text{def}}{=} Y' \times_R k$) is isomorphic to $X_k$ (resp. is isomorphic to $Y_k$) and the natural morphism $f'_k \stackrel{\text{def}}{=} f' \times_R k : Y'_k \to X'_k$ which is induced by $f'$ on the level of special fibres is isomorphic to $f_k$.

In the original version of the conjecture one doesn't fix $R$ but fixes $k$, $f_k$, and asks for the existence of a local domain $R$ dominating the ring of Witt vectors $W(k)$ over which a lifting of $f_k$ exists as part of the conjecture (cf. [Oo]). One can formulate several variants of the above conjecture that we will list below.

**[Conj-O1]** Let $X$ be a proper, smooth, geometrically connected $R$-curve and $f_k : Y_k \to X_k \stackrel{\text{def}}{=} X \times_R k$ a finite (possibly ramified) Galois cover between smooth $k$-curves with group $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$. Then there exists a finite extension $R'/R$ and a Galois cover $f' : Y' \to X' \stackrel{\text{def}}{=} X \times_R R'$ between smooth $R'$-curves with group $G$, such that the special fibre $X'_k \stackrel{\text{def}}{=} X' \times_R k$ (resp. $Y'_k \stackrel{\text{def}}{=} Y' \times_R k$) equals $X_k$ (resp. is isomorphic to $Y_k$) and the natural morphism $f'_k \stackrel{\text{def}}{=} f' \times_R k : Y'_k \to X'_k = X_k$ which is induced by $f'$ on the level of special fibres is isomorphic to $f_k$. We call $f'$ as above a smooth lifting of $f_k$ over $R'$.

**[Conj-O2]** Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover, with $Y_k$ a smooth $k$-curve, and with group $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$. Then there exists a finite extension $R'/R$ and a finite Galois cover $f' : Y' \to \mathbb{P}^1_{R'}$ with $Y'$ a smooth $R'$-curve, with group $G$, such that the natural morphism $f'_k \stackrel{\text{def}}{=} f' \times_R k : Y'_k \to \mathbb{P}^1_k$ which is induced by $f'$ on the level of special fibres is isomorphic to $f_k$.

**[Conj-O3]** Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite Galois cover with $Y_k$ a smooth $k$-curve and with group $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$, which is (totally) ramified above a unique point $\infty \in \mathbb{P}^1_k$. Then there exists a finite extension $R'/R$, a finite Galois cover $f' : Y' \to \mathbb{P}^1_{R'}$ with $Y'$ a smooth $R'$-curve, with group $G$, and such that the natural morphism $f'_k \overset{\text{def}}{=} f' \times_R k : Y'_k \to \mathbb{P}^1_k$ which is induced by $f'$ on the level of special fibres is isomorphic to $f_k$.

**[Conj-O4]** Let $\tilde{X} \overset{\text{def}}{=} \text{SpecR[[T]]}$ and $\tilde{X}_k \overset{\text{def}}{=} \text{Speck[[t]]}$. Let $f_k : \tilde{Y}_k \to \tilde{X}_k$ be a finite morphism which is generically Galois with group $G \tilde{\to} \mathbb{Z}/m\mathbb{Z}$, with $\tilde{Y}_k$ normal and connected. Then there exists a finite extension $R'/R$ and a smooth lifting $f' : \tilde{Y}' \to \tilde{X}' \overset{\text{def}}{=} \tilde{X} \times_R R'$ of $f_k$, i.e. $\tilde{Y}' \tilde{\to}$ SpecR$'$[[T$'$]] is $R'$-smooth and the natural morphism $f'_k : \tilde{Y}'_k \to \tilde{X}'_k = \tilde{X}_k$ which is induced by $f'$ at the level of special fibres is isomorphic to $f_k$.

Moreover, in the above conjectures **[Conj-O1]**, **[Conj-O2]**, **[Conj-O3]**, and **[Conj-O4]** one predicts that $R'$ can be chosen to be the minimal extension of $R$ which contains a primitive $m$-th root of 1. In fact all the above variants of the Oort conjecture turn out to be equivalent. More precisely, we have the following.

**Lemma 2.1.1.** *The above conjectures* **[Conj-O]**, **[Conj-O1]**, **[Conj-O2]**, **[Conj-O3]**, *and* **[Conj-O4]** *are all equivalent. Moreover, in order to solve the above conjecture(s) it suffices to treat the case where* $G \tilde{\to} \mathbb{Z}/p^n\mathbb{Z}$ *is a cyclic p-group.*

*Proof.* Follows easily from the local-global principle for the lifting of Galois covers between curves (cf. Proposition 1.1.4), the result of approximation of local extensions by global extensions due to Katz, Gabber, and Harbater, (cf. [Ha], and [Ka]), and the formal patching result in Proposition 1.1.2. The last assertion can also be easily verified (see for example the arguments in [Gr-Ma], 6). Q.E.D.

Oort conjecture holds true in the case where the Galois cover $f_k$ is étale, as follows from the theorems of liftings of étale covers (cf. [Gr]). In this case the statement of the conjecture is true for any finite group $G$ (not necessarily cyclic), and a smooth lifting exists over $R$. In the case where $G \tilde{\to} \mathbb{Z}/p^n\mathbb{Z}$ is a cyclic $p$-group the conjecture has been verified in the cases where $n = 1$ and $n = 2$ (cf. [Se-Oo-Su] for the case $n = 1$ and [Gr-Ma] for the case $n = 2$). In this paper we propose the following refined version of the Oort conjecture. More precisely, we will formulate a refined version of **[Conj-O1]**.

**Oort Conjecture revisited [Conj-O1-Rev]** Let $X$ be a proper, smooth, geometrically connected $R$-curve and $f_k : Y_k \to X_k \overset{\text{def}}{=} X \times_R k$ a finite (possibly ramified) Galois cover between smooth $k$-curves with group

$G \tilde{\to} \mathbb{Z}/m\mathbb{Z}$. Let $H$ be a quotient of $G$ and $g_k : Z_k \to X_k$ the corresponding Galois subcover of $f_k$ with group $H$. Then there exists a smooth Galois lifting $g : Z' \to X' \stackrel{\text{def}}{=} X \times_R R'$ of $g_k$ over some finite extension $R'/R$, i.e. $g$ is a Galois cover with group $H$ between smooth $R'$-curves which is a lifting of $g_k$ (cf. **[Conj-O1]**). Furthermore, for every smooth lifting $g$ of the Galois subcover $g_k$ of $f_k$ as above there exists a finite extension $R''/R'$ and a finite Galois cover $f : Y'' \to X'' \stackrel{\text{def}}{=} X \times_R R''$ between smooth $R''$-curves with group $G$, which is a smooth lifting of $f_k$, and such that $f$ dominates $g$, i.e. we have a factorisation $f : Y'' \to Z'' \stackrel{\text{def}}{=} Z' \times_{R'} R'' \stackrel{g \times_{R'} R''}{\longrightarrow} X''$. Moreover, $R''$ can be chosen to be the minimal extension of $R'$ which contains a primitive $m$-th root of 1.

**Remark 2.1.2.** In a similar way as in the above discussion one can revisit the above (equivalent) variants of the original Oort conjecture, and formulate the revisited versions **[Conj-O2-Rev]**, **[Conj-O3-Rev]**, and **[Conj-O4-Rev]**, which turn out to be all equivalent to **[Conj-O1-Rev]** (use similar arguments as in the proof of Lemma 2.1.1). Moreover, in order to solve these revisited versions one can reduce to the case where $G \tilde{\to} \mathbb{Z}/p^n\mathbb{Z}$ is a cyclic $p$-group. In the case where $n = 1$ (i.e. $G$ is a cyclic group of cardinality $p$) the revisited Oort conjecture is clearly true, since the (original) Oort conjecture is true in this case (see [Se-Oo-Su]).

**2.2.**

Next, we give examples where the revisited Oort conjecture can be verified in the case where $G \tilde{\to} \mathbb{Z}/p^2\mathbb{Z}$.

Assume that $G \tilde{\to} \mathbb{Z}/p^2\mathbb{Z}$. We will work within the framework of **[Conj-O4-Rev]** (cf. Remark 2.1.2). More precisely, let $\tilde{X} \stackrel{\text{def}}{=} \text{SpecR[[T]]}$ and $\tilde{X}_k \stackrel{\text{def}}{=} \text{Speck[[t]]}$ its special fibre ($t = T \mod \pi$). Let $f_k : Z_k \to \tilde{X}_k$ be a cyclic Galois cover of degree $p^2$ with $Z_k$ normal, and $h_k : Z'_k \to \tilde{X}_k$ its unique Galois subcover of degree $p$. A smooth local lifting of $f_k$ (cf. **[Conj-O4]**) exists by [Gr-Ma], Theorem 5.5, over $R$ if $R$ contains the $p^2$-th roots of 1. Assume that $R$ contains a primitive $p^2$-th root of 1. Let $h : Z' \to \tilde{X}$ be a smooth Galois lifting of $h_k$, i.e. $h$ is a Galois cover of degree $p$, $Z' \stackrel{\text{def}}{=} \text{SpecA}'$, $A' \tilde{\to} R[[S']]$ is an open disc, and $h$ induces the Galois cover $h_k$ on the level of special fibres. Then in order to verify the **[Conj-O4-Rev]** for the Galois cover $f_k$ and the smooth lifting $h$ of $h_k$ one has to show that there exists a smooth Galois lifting $f : Z \to \tilde{X}$ of $f_k$, i.e. $f$ is a cyclic Galois cover of degree $p^2$, $Z \stackrel{\text{def}}{=} \text{SpecA}$, $A \tilde{\to} R[[S]]$ is an open disc, and $f$ induces the Galois cover

$f_k$ on the level of special fibres, which dominates $h$; i.e. such that we have a factorisation $f : Z \to Z' \overset{h}{\to} \tilde{X}$.

We use arguments similar to the ones used in [Gr-Ma]. The Galois cover $f_k$ is generically given, for an appropriate choice of the parameter $t$, by the equations:

$$(*) \qquad x_1^p - x_1 = t^{-m_1},$$

and

$$(**) \qquad x_2^p - x_2 = c(x_1^p, -x_1) + \sum_{0 \le s < m_1(p-1)} a_s t^{-s} + \sum_{0 \le j < m_1} t^{-jp} \sum_{0<i<p} (x_1^p - x_1)^i p_{j,p-i}(x_1^p - x_1)^p,$$

where $a_i \in k$, $p_{j,p-i} \in k[x]$ are polynomials of respective degrees $d_{j,p-i}$, $\gcd(m_1,p) = 1$, and $c(x,y) \overset{\text{def}}{=} \frac{(x+y)^p - x^p + (-y^p)}{p}$ (see [Gr-Ma], Lemma 5.1). Moreover, the degree of the different in the Galois cover $f_k$ is

$$d_s \overset{\text{def}}{=} (m_1+1)(p-1)p + (m_2+1)(p-1),$$

where $m_2 \overset{\text{def}}{=} \max_{\substack{0 \le j < m_1 \\ 0<i<p}} (p^2 m_1, p(jp + (i + pd_{j,p-i})m_1)) - (p-1)m_1$ (cf. loc. cit.). Let $\zeta_2 \in R$ be a primitive $p^2$-th root of 1. Let $\zeta_1 \overset{\text{def}}{=} \zeta_2^p$ and $\lambda \overset{\text{def}}{=} \zeta_1 - 1$. The smooth lifting $h : Z' \to \tilde{X}$ of $h_k$ is generically given (by the Oort-Sekiguchi-Suwa theory (cf. [Se-Oo-Su])) by an equation

$$\frac{(\lambda X_1 + 1)^p - 1}{\lambda^{\,p}} = f(T),$$

where $f(T) = \frac{h(T)}{g(T)}$, $h(T) \in R[[T]]$, $g(T) \in R[T]$ is a distinguished polynomial (i.e. its highest coefficient is a unit in $R$), the degree of $g(T)$ is $m$, the Weierstrass degree of $h(T)$ is $m'$, $m \ge m'$, and $m - m' = m_1$. Furthermore, $\frac{h(T)}{g(T)} = T^{-m_1} \mod \pi$. The smoothness of $Z'$ is equivalent, by the local criterion for smoothness (cf. [Gr-Ma], 3.4), to the fact that the Galois cover $h_K : Z'_K \to \tilde{X}_K$ which is induced by $h$ between generic fibres, and which is given by the equation $(\lambda X_1 + 1)^p = \frac{\lambda^p h(T) + g(T)}{g(T)}$, is ramified above $m_1+1$ distinct geometric points of $\tilde{X}_K$. Moreover, $X_1^{-\frac{1}{m_1}}$ is a parameter for the open disc $Z'$, as follows easily from arguments similar to the ones given in the proof of Theorem 4.1 in [Gr-Ma] (cf.

also [Gr-Ma], proof of 3.4). We will consider two cases, depending on the lift $h$ of $h_k$, where we can prove the revisited Oort conjecture **[Conj-O4-Rev]** for the smooth lifting $h : Z' \to \tilde{X}$ (i.e. we can dominate $h$ by a smooth lifting $f$ of $f_k$). These two cases are considered separately in the following Lemmas 2.2.1 and 2.2.2.

**Lemma 2.2.1.** *With the same notations as above assume that in the above second equation (**) defining the Galois cover $f_k$ we have*

$$\sum_{0\leq s<m_1(p-1)} a_s t^{-s} + \sum_{0\leq j<m_1} t^{-jp} \sum_{0<i<p} (x_1^p - x_1)^i p_{j,p-i} (x_1^p - x_1)^p = 0,$$

*and also assume that the degree of $g(T)$ above equals $m_1$. (In particular, $h(T) \in R[[T]]$ above is a unit in this case). Then there exists a smooth lifting $f$ of $f_k$ which dominates the smooth lifting $h$ of $h_k$. In particular,* **[Conj-O4-rev]** *is true under these conditions for the Galois cover $f_k$ and the smooth lifting $h$ of the subcover $h_k$.*

*Proof.* Consider the cover $f : Z \to \tilde{X}$ which is generically given by the equations

$$\text{(i)} \qquad \frac{(\lambda X_1 + 1)^p - 1}{\lambda^{\ p}} = f(T),$$

where $f(T) = h(T)/g(T)$ is as above, and

$$\text{(ii)} \qquad (\lambda X_2 + \mathrm{Exp_p}(\mu \mathrm{X_1}))^{\mathrm{p}} = (\lambda \mathrm{X_1} + 1)\mathrm{Exp_p}(\mu^{\mathrm{p}}\mathrm{Y}),$$

where $\mathrm{Exp_p X} \stackrel{\text{def}}{=} 1 + \mathrm{X} + ... + \frac{\mathrm{X^{p-1}}}{(\mathrm{p}-1)!}$ is the truncated exponential, $\mu \stackrel{\text{def}}{=} \log_p(\zeta_2) = 1 - \zeta_2 + ...... + (-1)^{p-1}\frac{\zeta_2^{p-1}}{p-1}$ ($\mathrm{Exp_p}$ and $\log_p$ denote the truncation of the exponential and the logarithm, respectively, by terms of degree $> p - 1$), and $Y \stackrel{\text{def}}{=} \frac{(\lambda X_1+1)^p - 1}{\lambda^p} = \frac{h(T)}{g(T)}$. Then $f$ is a cyclic Galois cover of degree $p^2$ which generically lifts the Galois cover $f_k$ (cf. [Gr-Ma], the discussion in the beginning of 3, and Lemma 5.2). We claim that $Z$ is smooth over $R$. Indeed, the degree of the different in the morphism $f_k : Z_k \to \tilde{X}_k$ in this case is

$$d_s = (m_1 + 1)(p - 1)p + (p^2 m_1 - (p - 1)m_1 + 1)(p - 1).$$

Moreover, the above second equation (ii) defining the lifting $f$ is

$$(X_2')^p = (\lambda X_1 + 1)\mathrm{Exp_p}(\mu^{\mathrm{p}}\mathrm{Y})$$

$$= (1 + \lambda X_1)(1 + \mu^p \frac{h(T)}{g(T)} + .... + \frac{\mu^{p(p-1)}}{(p-1)!} \frac{h(T)^{(p-1)}}{g(T)^{(p-1)}})$$

and $1+\mu^p\frac{h(T)}{g(T)}+....+\frac{\mu^{p(p-1)}}{(p-1)!}\frac{h(T)^{(p-1)}}{g(T)^{(p-1)}}$ equals

$$\frac{(p-1)!g(T)^{p-1}+\mu^p(p-1)!h(T)g(T)^{p-2}+....+\mu^{p(p-1)}h(T)^{(p-1)}}{(p-1)!g(T)^{p-1}}.$$

Furthermore, $(p-1)!\ g(T)^{p-1}+\mu^p(p-1)!\ h(T)g(T)^{p-2}+\cdots$ $+\mu^{p(p-1)}\ h(T)^{(p-1)}$ can be written as a series in $X_1^{-\frac{1}{m_1}}$ whose Weierstrass degree is $pm_1(p-1)$ (since we assumed the degree of $g(T)$ to be $m_1$). From this we deduce that the degree of the generic different $d_\eta$ in the cover $f_K: Z_K\to\tilde{X}_K$ satisfies $d_\eta\le(m_1+1)(p^2-1)+pm_1(p-1)^2$, which implies $d_\eta\le d_s$. One then concludes that $d_\eta=d_s$, hence that $Z$ is smooth over $R$, since in general we must have $d_s\le d_\eta$. Moreover, we have (by construction) a natural factorisation $f:Z\to Z'\overset{h}{\to}\tilde{X}$. Q.E.D.

**Lemma 2.2.2.** *With the same notations as above. Assume that* $g(T)=T^{m_1}$. *Thus,* $h(T)\in R[[T]]$ *is a unit. (This case is rather special, since the corresponding smooth lifting* $h$ *of the Galois subcover* $h_k$ *has the property that all branched points are equidistant in the p-adic topology of* $K$*). Then there exists a smooth lifting* $f$ *of* $f_k$ *which dominates the smooth lifting* $h$ *of* $h_k$. *In particular,* **[Conj-O4-rev]** *is true under these conditions for the Galois cover* $f_k$ *and the smooth lifting* $h$ *of the subcover* $h_k$.

*Proof.* Consider the lifting $f:Z\to\tilde{X}$ of the Galois cover $f_k: Z_k\to\tilde{X}_k$ which is generically given by the equations

$$\text{(i}'\text{)}\qquad \frac{(\lambda X_1+1)^p-1}{\lambda^{\ p}}=f(T),$$

where $f(T)=\frac{h(T)}{T^{m_1}}$ satisfies the above condition in the Lemma, and

$$\text{(ii}'\text{)}\qquad [\lambda X_2+\mathrm{Exp_p}(\mu X_1)(1+\sum_{\substack{0\le j<m_1\\0<i<p}}\mathrm{T^{-j}}\mu^{\mathrm{i}}(\mathrm{p-i})!\mathrm{P_{j,p-i}(f(T))})]^{\mathrm{p}}$$

$$=(G(T^{-1})+p\mu^p\sum_{0<s<r}A_sT^{-s})(\lambda X_1+1),$$

where $\mathrm{Exp_pX}\overset{\mathrm{def}}{=}1+\mathrm{X}+...+\frac{\mathrm{X^{p-1}}}{\mathrm{(p-1)!}}$ is the truncated exponential, and $\mu\overset{\mathrm{def}}{=}\log_p(\zeta_2)=1-\zeta_2+......+(-1)^{p-1}\frac{\zeta_2^{p-1}}{p-1}$, are as in the proof of Lemma 2.2.1, the polynomial $G\overset{\mathrm{def}}{=}G(\frac{(\lambda X_1+1)^p-1}{\lambda^{\ p}})$ is defined in a similar way as in [Gr-Ma], Lemma 5.4, $P_{j,p-i}\in R[X]$ are primitive polynomials which lift the $p_{j,p-i} \in k[x]$, and $A_s \in R$ lift the $a_s$ (cf. loc. cit). Then $f : Z \to \tilde{X}$ is a Galois cover with a cyclic Galois group (isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$) and $Z$ is smooth over $R$, as follows from the local criterion for good reduction (cf. [Gr-Ma], 3.4), by using Lemma 5.4 in [Gr-Ma] (where among others the degree of $G$ in $T^{-1}$ is computed), and the same argument as in the proof of Theorem 5.5 in loc. cit. The key points here are that $X_1^{-\frac{1}{m_1}}$ is a parameter for the disc $Z'$ and the key Lemma 5.4 in [Gr-Ma] is valid by replacing $G \stackrel{\text{def}}{=} G(T^{-m_1})$ there by $G \stackrel{\text{def}}{=} G(f(T))$ in our case (formally speaking only the degree in $T^{-1}$ of $f(T)$, which is $m_1$, plays a role in loc. cit.). Moreover, we have (by construction) a natural factorisation $f : Z \to Z' \xrightarrow{h} \tilde{X}$. Q.E.D.

**2.3.**

Next, we will introduce the notion of fake liftings of cyclic Galois covers between smooth curves. We will work within the framework of **[Conj-O2-Rev]**.

Let $n \geq 1$ be a positive integer. Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover, where $Y_k$ is a smooth $k$-curve, with group $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$. We denote by $g_k : X_k \to \mathbb{P}^1_k$ the unique subcover of $f_k$ which is Galois with group $H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}$. We have a canonical factorisation

$$f_k : Y_k \xrightarrow{h_k} X_k \xrightarrow{g_k} \mathbb{P}^1_k,$$

where $h_k : Y_k \to X_k$ is a cyclic Galois cover between smooth $k$-curves of degree $p$. We assume that the Galois cover $g_k : X_k \to \mathbb{P}^1_k$ can be lifted to a Galois cover between smooth $R$-curves, i.e. there exists a finite Galois cover $g : \mathcal{X} \to \mathbb{P}^1_R$ with group $H$, where $\mathcal{X}$ is smooth over $R$, $\mathcal{X}_k \stackrel{\text{def}}{=} \mathcal{X} \times_R k$ is isomorphic to $X_k$, and such that the morphism induced by $g$ at the level of special fibres $g_k : \mathcal{X}_k \to \mathbb{P}^1_k$ is isomorphic to the Galois cover $g_k : X_k \to \mathbb{P}^1_k$. There exists a Garuti lifting of the Galois cover $f_k$ which dominates $g$ (cf. [Sa1], Definition 2.5.2. for the definition of Garuti liftings of Galois covers between smooth curves). We assume (for simplicity) that such a Garuti lifting is defined over $R$, i.e. there exists a finite Galois cover $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ with group $G$ and $\mathcal{Y}$ normal, which dominates $g$, i.e. we have a factorisation $\tilde{f} : \mathcal{Y} \xrightarrow{\tilde{h}} \mathcal{X} \xrightarrow{g} \mathbb{P}^1_R$, and such that the morphism $\tilde{f}_k : \mathcal{Y}_k \stackrel{\text{def}}{=} \mathcal{Y} \times_R k \to \mathbb{P}^1_k$ between special fibres is generically étale, Galois with group $G$, dominates $g_k$ (i.e. we have a factorisation $\tilde{f}_k : \mathcal{Y}_k \to X_k \xrightarrow{g_k} \mathbb{P}^1_k$), the normalisation $\mathcal{Y}_k^{\text{nor}}$ of $\mathcal{Y}_k$ is isomorphic to $Y_k$ (in particular, $\mathcal{Y}_k$ is irreducible), and the natural morphism between the normalisations $\mathcal{Y}_k^{\text{nor}} \to \mathbb{P}^1_k$ (which is Galois) is isomorphic to $f_k$ (cf. loc. cit. Theorem 2.5.3). (Note that a smooth lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ is by definition a Garuti lifting). Let $\delta_\eta \stackrel{\text{def}}{=} \delta_{\tilde{f}_K}$ (resp. $\delta_s \stackrel{\text{def}}{=} \delta_{f_k}$) be the degree of the different in the morphism $\tilde{f}_K : \mathcal{Y}_K \stackrel{\text{def}}{=} \mathcal{Y} \times_R K \to \mathbb{P}^1_K$ between generic fibres (resp. in the morphism $f_k : Y_k \to \mathbb{P}^1_k$). It is well known (and easy to verify) that we have the inequality $\delta_\eta \geq \delta_s$. Furthermore, the equality $\delta_\eta = \delta_s$ holds if and only if $\mathcal{Y}$ is smooth over $R$ (which is equivalent to $\mathcal{Y}_k$ being isomorphic to $Y_k$), as follows from the local criterion for good reduction (cf. [Gr-Ma], 3.4). We will consider the following assumption.

2.3.1. **Assumption (A):** Let $n \geq 1$ be a positive integer and $f_k : Y_k \to \mathbb{P}^1_k$ a cyclic Galois cover with group $G \stackrel{\sim}{\to} \mathbb{Z}/p^n\mathbb{Z}$, with $Y_k$ a smooth $k$-curve. Let $g_k : X_k \to \mathbb{P}^1_k$ be the unique Galois subcover of $f_k$ of degree $p^{n-1}$. Assume that $g_k$ has a smooth Galois lifting $g : \mathcal{X} \to \mathbb{P}^1_{R'}$ (over some finite extension $R'/R$). We say that the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ satisfies the assumption **(A)**, with respect to the smooth lifting $g$ of the subcover $g_k$, if for all possible Garuti liftings $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_{R''}$ of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ which dominate $g$ (see preceding discussion) and are defined over a finite extension $R''/R'$, the strict inequality $\delta_\eta \stackrel{\text{def}}{=} \delta_{\tilde{f}_{K''}} > \delta_s \stackrel{\text{def}}{=} \delta_{f_k}$ (where $K'' \stackrel{\text{def}}{=} \text{Fr}(R'')$) holds. In other words the assumption **(A)** is satisfied if there doesn't exist a smooth lifting of $f_k$ which dominates the given smooth lifting $g$ of the subcover $g_k$ of $f_k$. Note that if **[Conj-O2-Rev]** (cf Remark 2.1.2) is true then no Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ as above satisfies the assumption **(A)**.

Next, we introduce the notion of fake liftings of cyclic Galois covers between curves which naturally arise if cyclic Galois covers satisfy the above assumption **(A)**.

**Definition 2.3.2. Fake liftings of cyclic covers between curves:** Assume that the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ satisfies the assumption **(A)** with respect to the smooth lifting $g$ of the subcover $g_k$ (cf. 2.3.1). Let $\delta \stackrel{\text{def}}{=} \min\{\delta_{\tilde{f}_{K''}}\}$, where the minimum is taken among all possible Garuti liftings $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_{R''}$ of $f_k$ as above which dominate the smooth lifting $g : \mathcal{X} \to \mathbb{P}^1_{R'}$ of the subcover $g_k : X_k \to \mathbb{P}^1_k$. Note that $\delta > \delta_s$ by assumption. We call a lifting $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_{R''}$ as above satisfying the equality $\delta_{\tilde{f}_{K''}} = \delta$ a fake lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$, relative to the smooth lifting $g$ of the sub-cover $g_k$. Note that if $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_{R''}$ is a fake lifting of the Galois cover $f_k$ then $\mathcal{Y}$ is (by definition) not smooth over $R''$.

**Remark 2.3.3.** Fake liftings as in Definition 2.3.2 won't exist if **[Conj-O2-Rev]** is true, hence the reason we call them fake. Moreover, in order to prove the (revisited) Oort conjecture it suffices to prove that fake liftings do not exist, as follows from the various definitions above.

**2.4.**

In this subsection we introduce some notations related to the semi-stable geometry of curves which will be used in the next Subsection 2.5, where we investigate the geometry of the (minimal) semi-stable models of fake liftings of cyclic Galois covers between smooth curves.

Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover with group $G \overset{\sim}{\to} \mathbb{Z}/p^n\mathbb{Z}$, $n \geq 1$. Let $G \twoheadrightarrow H \overset{\sim}{\to} \mathbb{Z}/p^{n-1}\mathbb{Z}$ be the (unique) quotient of $G$ with cardinality $p^{n-1}$. Let $g_k : X_k \to \mathbb{P}^1_k$ be the cyclic subcover of $f_k$ with group $H$. Assume that there exists $g : \mathcal{X} \to \mathbb{P}^1_R$ a smooth Galois lifting of $g_k$ over $R$. Let $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ be a fake lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ (with respect to the smooth lifting $g$ of $g_k$), which dominates the smooth lifting $g$ of $g_k$ (cf. Definition 2.3.2). We assume that both $\tilde{f}$ and $g$ are defined over $R$ for simplicity. We have a natural factorisation $\tilde{f} : \mathcal{Y} \overset{h}{\to} \mathcal{X} \overset{g}{\to} \mathbb{P}^1_R$ where $h : \mathcal{Y} \to \mathcal{X}$ is a finite Galois cover of degree $p$ with $\mathcal{Y}$ normal and non smooth over $R$.

It follows from the semi-stable reduction theorem for curves (cf. [De-Mu], and [Ab1]) that $\mathcal{Y}$ admits a semi-stable model after possibly a finite extension of $R$. Next, we assume that $\mathcal{Y}$ admits a semi-stable model over $R$. More precisely, we assume that there exists a birational morphism $\sigma : \mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'$ semi-stable, i.e. the special fibre $\mathcal{Y}'_k \overset{\text{def}}{=} \mathcal{Y}' \times_R k$ of $\mathcal{Y}'$ is reduced, and its only singularities are ordinary double points. We also assume that the ramified points in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ specialise in smooth distinct points of $\mathcal{Y}'_k$. Moreover, we will assume that the birational morphism $\sigma$ is minimal with respect to the above properties. In particular, the action of the group $G \overset{\sim}{\to} \mathbb{Z}/p^n\mathbb{Z}$ on $\mathcal{Y}$ extends to an action of $G$ on $\mathcal{Y}'$. Let $\mathcal{P} \overset{\text{def}}{=} \mathcal{Y}'/G$ be the quotient of $\mathcal{Y}'$ by $G$ and $\tilde{f}' : \mathcal{Y}' \to \mathcal{P}$ the natural morphism which is Galois with group $G$. Let $\tilde{g} : \mathcal{X}' \to \mathcal{P}$ be the unique subcover of $\tilde{f}'$ which is Galois with group $H$ ($\mathcal{X}'$ is the quotient of $\mathcal{Y}'$ by the unique subgroup of $G$ with cardinality $p$). Then $\mathcal{P}$ and $\mathcal{X}'$ are semi-stable $R$-curves (cf. [Ra], appendice), and we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{h} & \mathcal{X} & \xrightarrow{g} & \mathbb{P}^1_R \\ {\scriptstyle\sigma}\uparrow & & \uparrow & & \uparrow \\ \mathcal{Y}' & \xrightarrow{\tilde{h}} & \mathcal{X}' & \xrightarrow{\tilde{g}} & \mathcal{P} \end{array}$$

where the vertical maps are birational morphisms and the horizontal maps are finite morphisms. To the special fibre $\mathcal{Y}'_k \overset{\text{def}}{=} \mathcal{Y}' \times_R k$ of $\mathcal{Y}'$ (which is a semi-stable $k$-curve) one associates a graph $\Gamma$ whose vertices $\text{Ver}(\Gamma) \overset{\text{def}}{=} \{\text{Y}_\text{i}\}_{\text{i}=0}^{\text{n}'}$ are the irreducible components of $\mathcal{Y}'_k$ and edges are the double points $\text{Edg}(\Gamma) \overset{\text{def}}{=} \{\text{y}_\text{j}\}_{\text{j}\in\text{J}}$ of $\mathcal{Y}'_k$. A double point $y_j \in Y_t \cap Y_s$ defines and edge linking the vertices $Y_t$ and $Y_s$. We assume that $Y_0$ is the strict transform of $\mathcal{Y}_k$ (which is irreducible) in $\mathcal{Y}'$. In a similar way one associates to the special fibre $\mathcal{X}'_k \overset{\text{def}}{=} \mathcal{X}' \times_R k$ of $\mathcal{X}'$ a graph $\Gamma'$ whose vertices $\text{Ver}(\Gamma') \overset{\text{def}}{=} \{\text{X}_\text{i}\}_{\text{i}=0}^{\text{m}}$ are the irreducible components of $\mathcal{X}'_k$ and edges are the double points $\text{Edg}(\Gamma') \overset{\text{def}}{=} \{\text{x}_\text{j}\}_{\text{j}\in\text{J}'}$ of $\mathcal{X}'_k$. We assume that $X_0$ is the strict transform of $\mathcal{X}_k \tilde{\to} X_k$ in $\mathcal{X}'$. Then it follows easily (from the fact that $\mathcal{X}$ is smooth) that the graph $\Gamma'$ is a tree and all the irreducible components of $\mathcal{X}'_k$ which are distinct from $X_0$ are isomorphic to $\mathbb{P}^1_k$. We choose an orientation of $\Gamma'$ starting from $X_0$ towards the end vertices of the tree $\Gamma'$. We have a natural morphism of graphs $\Gamma \to \Gamma'$. Similarly one associates to the special fibre $\mathcal{P}_k \overset{\text{def}}{=} \mathcal{P} \times_R k$ of $\mathcal{P}$ a graph $\Gamma''$ whose vertices $\text{Ver}(\Gamma'') \overset{\text{def}}{=} \{\text{P}_\text{i}\}_{\text{i}=0}^{\text{n}}$ are the irreducible components of $\mathcal{P}_k$ and edges are the double points $\text{Edg}(\Gamma'') \overset{\text{def}}{=} \{\tilde{\text{x}}_\text{j}\}_{\text{j}\in\text{J}''}$ of $\mathcal{P}_k$. We assume that $P_0$ is the strict transform of $\mathbb{P}^1_k$ (the special fibre of $\mathbb{P}^1_R$) in $\mathcal{P}$. The graph $\Gamma''$ is a tree and all the irreducible components of $\mathcal{P}_k$ are isomorphic to $\mathbb{P}^1_k$. We choose an orientation of $\Gamma''$ starting from $P_0$ towards the end vertices of the tree $\Gamma''$. We have natural morphisms of graphs

$$\Gamma \to \Gamma' \to \Gamma''.$$

The graph $\Gamma$ (resp. $\Gamma'$) is naturally endowed with an action of the group $G$ (resp. $H$). Moreover, the morphism $\Gamma \to \Gamma''$ (resp. $\Gamma' \to \Gamma''$) is $G$-equivariant (resp. $H$-equivariant).

Let $Y_i$ be a vertex of the graph $\Gamma$. To $Y_i$ one associates two subgroups of the Galois group $G \tilde{\to} \mathbb{Z}/p^n\mathbb{Z}$ of the cover $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$: the decomposition subgroup $D_i \subseteq G$ and the inertia subgroup $I_i \subseteq D_i$ at the generic point of $Y_i$ in the Galois cover $\tilde{f}$. We call the (irreducible component) vertex $Y_i$ of $\Gamma$ an end vertex (or end component) of $\Gamma$ if the graph $\Gamma$ is a tree and if $Y_i$ is an end vertex of this tree. We call $Y_i$ a separable vertex of $\Gamma$ if the inertia subgroup $I_i$ which is associated to $Y_i$ is trivial. Finally, we call the irreducible component $Y_i$ a ramified vertex if there exists a ramified point in the morphism $f_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ which specialises in the component $Y_i$. Similarly let $X_i$ be a vertex of the graph $\Gamma'$. To $X_i$ one associates two subgroups of the Galois group $H \tilde{\to} \mathbb{Z}/p^{n-1}\mathbb{Z}$ of the cover $g : \mathcal{X} \to \mathbb{P}^1_R$: the decomposition subgroup

$\tilde{D}_i \subseteq H$ and the inertia subgroup $\tilde{I}_i \subseteq \tilde{D}_i$ at the generic point of $X_i$ in the Galois cover $g$. We call the vertex $X_i$ of $\Gamma'$ an end vertex of $\Gamma'$ if $X_i$ is an end vertex of the tree $\Gamma'$. We call $X_i$ an internal vertex of $\Gamma'$ if $X_i$ is distinct from $X_0$ and the end vertices of $\Gamma'$. We call $X_i$ a separable vertex of $\Gamma'$ if the inertia subgroup $\tilde{I}_i$ which is associated to $X_i$ is trivial. We call the irreducible component $X_i$ a ramified vertex if there exists a ramified point in the morphism $g_K : \mathcal{X}_K \to \mathbb{P}^1_K$ which specialises in the component $X_i$. Finally, by a geodesic in a finite tree linking two vertices we mean the path, or subtree, with smallest length which links the two vertices.

**2.5.**

In this subsection we first establish in 2.5.1 some properties of the (not necessarily minimal) semi-stable model $\mathcal{X}' \to \mathcal{X}$ of the smooth lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ of the Galois subcover $g_k : X_k \to \mathbb{P}^1_k$ of $f_k : Y_k \to \mathbb{P}^1_k$.

2.5.1. Let $g_k : X_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover with group $H \tilde{\to} \mathbb{Z}/p^{n-1}\mathbb{Z}$ ($n > 1$), and $X_k$ a smooth $k$-curve. Let $g : \mathcal{X} \to \mathbb{P}^1_R$ be a smooth Galois lifting of $g_k$ over $R$ (i.e. $g$ is a Galois cover between smooth $R$-curves which lifts $g_k$). Assume that there exists a birational morphism $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is semi-stable, the action of $H$ on $\mathcal{X}$ extends to an action on $\mathcal{X}'$, and the ramified points in the Galois cover $g_K : \mathcal{X}_K \to \mathbb{P}^1_K$ specialise in smooth distinct points of $\mathcal{X}'_k$. We do not assume that $\mathcal{X}'$ is minimal with respect to the above properties. Let $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{X}'/H$ be the quotient of $\mathcal{X}'$ by $H$. We have a commutative digram:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathbb{P}^1_R \\ \uparrow & & \uparrow \\ \mathcal{X}' & \xrightarrow{\tilde{g}} & \mathcal{P} \end{array}$$

where $\mathcal{P}$ is a semi-stable $R$-curve and the vertical maps are birational morphisms. .Let $\Gamma'$ (resp. $\Gamma''$) be the graph associated to the semi-stable $k$-curve $\mathcal{X}'_k$ (resp. $\mathcal{P}_k$). Let $\mathrm{Ver}(\Gamma') \stackrel{\text{def}}{=} \{\mathrm{X_i}\}_{\mathrm{i=0}}^{\mathrm{m}}$ (resp. $\mathrm{Ver}(\Gamma'') \stackrel{\text{def}}{=} \{\mathrm{P_i}\}_{\mathrm{i=0}}^{\mathrm{n}}$) be the set of vertices of $\Gamma'$ (resp. of $\Gamma''$). We have a natural morphism $\Gamma' \to \Gamma''$ of graphs.

**Lemma 2.5.1 (i).** *The graphs $\Gamma'$ and $\Gamma''$ are trees. Furthermore, each vertex $X_i$ (resp. $P_i$) of $\Gamma'$ (resp. of $\Gamma''$) which is distinct from the strict transform of $\mathcal{X}_k$ is isomorphic to $\mathbb{P}^1_k$.*

*Proof.* Clear and follows immediately from the fact that $\mathcal{X}$ is smooth.
Q.E.D.

Let $X_0$ be the strict transform of $\mathcal{X}_k \tilde{\to} X_k$ in $\mathcal{X}'$. We choose an orientation of the tree $\Gamma'$ starting form $X_0$ towards the end vertices of $\Gamma'$. For a vertex $X_i$ of $\Gamma'$ we will denote by $\tilde{D}_i$ (resp. $\tilde{I}_i \subseteq \tilde{D}_i$) the decomposition (resp. inertia) subgroup of $H$ at the generic point of $X_i$. Then:

**Lemma 2.5.1 (ii).** *$\tilde{D}_0 = H$ and $\tilde{I}_0 = \{1\}$.*

*Proof.* This is also clear since $\mathcal{X}_k$ is irreducible and the natural morphism $\mathcal{X}_k \to \mathbb{P}^1_k$, which is isomorphic to $g_k : X_k \to \mathbb{P}^1_k$, is generically Galois with Galois group $H$. Q.E.D.

**Lemma 2.5.1 (iii).** *Let $X_i$ be an internal vertex of $\Gamma'$, i.e. $X_i$ is distinct from $X_0$ and from the end vertices of $\Gamma'$, and $X_j$ an adjacent vertex to $X_i$ in the direction moving towards the end vertices of $\Gamma'$. Then the following two cases occur:*

*(1) either $\tilde{D}_i = \tilde{I}_i$, in this case $\tilde{D}_j = \tilde{D}_i$,*

*(2) or $\tilde{I}_i \subsetneq \tilde{D}_i$. In this case $\tilde{D}_j = \tilde{I}_i$ and we have an exact sequence*

$$1 \to \tilde{D}_j \to \tilde{D}_i \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

*Furthermore, in the case (2) if $\tilde{X}_i$ denotes the image of $X_i$ in the quotient $\mathcal{X}'/\tilde{I}_i$ of $\mathcal{X}'$ by $\tilde{I}_i$ then the natural morphism $\tilde{X}_i \to P_i$, where $P_i \tilde{\to} \mathbb{P}^1_k$ is the image of $X_i$ in $\Gamma''$, is a Galois cover of degree $p$ ramified above a unique point $\infty \in P_i$ (which is the edge of the geodesic linking $P_i$ to $P_0$, which is linked to $P_i$) with Hasse conductor $m = 1$ at $\infty$ (i.e. given by an Artin–Schreier equation $z^p - z = t^{-1}$ where $t$ is a local parameter at $\infty$). In particular, when we move in the graph $\Gamma'$ starting from $X_0$ towards the end vertices of $\Gamma'$ then the cardinality of the decomposition group $\tilde{D}_i$ (resp. the cardinality of the inertia subgroup $\tilde{I}_i$) of a vertex $X_i$ decreases. More precisely, if when moving from a vertex $X_i$ towards the end vertices of $\Gamma'$ we encounter a vertex $X_j$ then $\tilde{D}_j \subseteq \tilde{D}_i$ and $\tilde{I}_j \subseteq \tilde{I}_i$.*

*Proof.* Let $X_i$ be an internal vertex of $\Gamma'$ and $X_j$ an adjacent vertex to $X_i$ in the direction moving towards the end vertices of $\Gamma'$. Let $P_i$ (resp. $P_j$) be the image of $X_i$ (resp. $X_j$) in $\mathcal{P}$. Assume first that $\tilde{D}_i = \tilde{I}_i$, we will show that $\tilde{D}_j = \tilde{D}_i$ in this case. Let $\mathcal{X}_1 \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{D}_i$ be the quotient of $\mathcal{X}'$ by $\tilde{D}_i$. Then $\mathcal{X}_1$ is a semi-stable $R$-curve and the configuration of the special fibre $(\mathcal{X}_1)_k$ of $\mathcal{X}_1$ is tree-like (cf. Lemma 2.5.1 (i)). The natural morphism $\mathcal{X}_1 \to \mathcal{P}$ is by assumption completely split above the irreducible component $P_i$ of $\mathcal{P}_k$, hence is also a fortiori completely split above $P_j$. This shows that $\tilde{D}_j \subseteq \tilde{D}_i$. Assume that $\tilde{D}_j \subsetneq \tilde{D}_i$. Let $x \stackrel{\text{def}}{=} X_i \cap X_j$ which is a double point of $\mathcal{X}'$ and $x' \stackrel{\text{def}}{=} P_i \cap P_j$ its image in

$\mathcal{P}$. Let $\mathcal{X}'' \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{D}_j$ be the quotient of $\mathcal{X}'$ by $\tilde{D}_j$ and $X_i''$ the image of $X_i$ in $\mathcal{X}''$. The natural morphism $\mathcal{X}'' \to \mathcal{P}$ is by assumption completely split above $P_j$, hence also completely split above the double point $x'$. In particular, the natural morphism $X_i'' \to P_i$ is étale above $x'$ and is generically Galois with Galois group $\tilde{D}_i/\tilde{D}_j$. This contradicts the fact that $\tilde{D}_i = \tilde{I}_i$. Hence $\tilde{D}_j = \tilde{D}_i$ necessarily.

Assume now that $\tilde{I}_i \subsetneq \tilde{D}_i$ and write $D_i' \stackrel{\text{def}}{=} \tilde{D}_i/\tilde{I}_i \neq \{1\}$. Let $\tilde{X}_i$ be the image of $X_i$ in the quotient $\mathcal{X}'/\tilde{I}_i$ of $\mathcal{X}'$ by $\tilde{I}_i$. We have a natural morphism $\tilde{X}_i \to P_i$ which is generically Galois with group $D_i'$. The vertex $P_i \in \operatorname{Ver}(\Gamma'')$ is an internal vertex of the tree $\Gamma''$ (since $X_i$ is an internal vertex of $\Gamma'$), hence is linked to more than one double point of $\Gamma''$. More precisely, $P_i$ is linked to a unique double point $x'$ which links $P_i$ to the geodesic joining $P_i$ and the vertex $P_0$ ($P_0$ is the image of $X_0$ in $\mathcal{P}$), and (at least another) other double points linking $P_i$ to the geodesics joining $P_i$ and some of the end vertices of the graph $\Gamma''$. If the natural morphism $\tilde{X}_i \to P_i$ is unramified above the double point $x'$ then this would introduce loops in the configuration of the tree $\Gamma'$. Thus, the morphism $\tilde{X}_i \to P_i$ must (totally) ramify above the double point $x'$. In particular, this morphism is necessarily unramified above the remaining double points linking $P_i$ to the end vertices of $\Gamma''$. Indeed, for otherwise the genus of $\tilde{X}_i$ (hence that of $X_i$) would be $> 0$ since the degree of this morphism is a power of $p$, as follows easily from the Riemann–Hurwitz genus formula, and this would contradict the second assertion in Lemma 2.5.1 (i). Also the degree of the morphism $\tilde{X}_i \to P_i$ is necessarily equal to $p$, and this morphism is only ramified above the double point $x'$ with Hasse conductor $m = 1$ at $x'$ (i.e. is given by an Artin–Schreier equation $z^p - z = t^{-1}$ where $t$ is a local parameter at $x'$). For otherwise the genus of $\tilde{X}_i$ (hence that of $X_i$) would be $> 0$ for similar reasons as above. This also shows that $\tilde{D}_j \subset \tilde{I}_i$ (indeed, the natural morphism $\mathcal{X}'/\tilde{I}_i \to \mathcal{P}$ is easily seen to be completely split above the component $P_j$ which is the image of $X_j$ in $\mathcal{P}$), and that we have a natural exact sequence

$$1 \to \tilde{I}_i \to \tilde{D}_i \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Now we show that $\tilde{D}_j = \tilde{I}_i$. Assume that $\tilde{D}_j \subsetneq \tilde{I}_i$. Let $\tilde{\mathcal{X}}' \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{D}_j$ (resp. $\tilde{\mathcal{X}}'' \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{I}_i$) be the quotient of $\mathcal{X}'$ by $\tilde{D}_j$ (resp. the quotient of $\mathcal{X}'$ by $\tilde{I}_i$) and $\tilde{X}_i'$ (resp. $\tilde{X}_i''$) the image of $X_i$ in $\tilde{\mathcal{X}}'$ (resp. $\tilde{\mathcal{X}}''$). By assumption the natural morphism $\tilde{X}_i' \to \tilde{X}_i''$ (which is of degree $\geq p$) must be on the one hand a homeomorphism, and on the other hand completely split above the image of the double point $x \stackrel{\text{def}}{=} X_i \cap X_j$. This is a contradiction. Hence we necessarily have the equality $\tilde{I}_i = \tilde{D}_j$. This proves the assertions (1) and (2) in Lemma 2.5.1 (iii). The remaining assertion follows easily from this. Q.E.D.

**Lemma 2.5.1 (iv).** *Let $X_i$ be a separable vertex of $\Gamma'$ (i.e. $\tilde{I}_i = \{1\}$) which is distinct from $X_0$. Then, either $\tilde{D}_i = \mathbb{Z}/p\mathbb{Z}$, in this case $X_i$ is a Galois cover of $\mathbb{P}^1_k$ ramified above a unique point $\infty \in \mathbb{P}^1_k$ with Hasse conductor $m = 1$ at $\infty$, or $\tilde{D}_i = 1$. In both cases if $X_j$ is a vertex adjacent to $X_i$ in the direction moving towards the ends of $\Gamma'$ then $\tilde{D}_j = 1$.*

*Proof.* Follows easily from Lemma 2.5.1 (iii) and the fact that if $C$ is a smooth and connected curve of genus 0, and $f : C \to \mathbb{P}^1_k$ is a generically Galois cover with group a cyclic $p$-group, then $f$ has necessarily degree $p$ and is ramified above a unique point $\infty \in \mathbb{P}^1_k$ with Hasse conductor $m = 1$, as follows easily from the Riemann–Hurwitz genus formula and Artin–Schreier–Witt theory. Q.E.D.

Let $0 < j \leq n - 1$ be an integer. Let $x \in \mathcal{X}_K$ be a ramified point in the morphism $g_K : \mathcal{X}_K \to \mathbb{P}^1_K$. We say that the ramified point $x$ is of type $j$ if the inertia subgroup $\tilde{I}_x \subseteq H$ at $x$ is isomorphic to $\mathbb{Z}/p^j\mathbb{Z}$. A vertex $X_i$ of $\Gamma'$ is called a ramified vertex of type $j$ if there exists a ramified point $x$ of type $j$ in the morphism $g_K : \mathcal{X}_K \to \mathbb{P}^1_K$ which specialises in the component $X_i$.

**Lemma 2.5.1 (v).** *Let $X_i$ be a ramified vertex of $\Gamma'$. Then $X_i$ is of type $j$ for a unique integer $0 < j \leq n - 1$. In other words if $0 < j < j' \leq n-1$ are integers then ramified points of type $j$ (resp. type $j'$) in the morphism $g_K : \mathcal{X}_K \to \mathbb{P}^1_K$ specialise in distinct irreducible components of $\mathcal{X}_k$. More precisely, if $X_i$ is a ramified vertex of type $j$ then the inertia subgroup $\tilde{I}_i$ which is associated to $X_i$ has cardinality $p^j$, i.e. $\tilde{I}_i \xrightarrow{\sim} \mathbb{Z}/p^j\mathbb{Z}$. Furthermore, let $P_i$ be the image of $X_i$ in $\mathcal{P}$. Then the natural morphism $X_i \to P_i$ has the structure of a $\mu_{p^j}$-torsor outside the double points supported by $P_i$ and the specialisation of the branched points in $P_i$ (in this case $\tilde{D}_i = \tilde{I}_i$).*

*Proof.* Let $0 < j \leq n - 1$ be an integer. Let $x \in \mathcal{X}_K$ be a ramified point in the morphism $g_K : \mathcal{X}_K \to \mathbb{P}^1_K$ of type $j$ which specialises in the irreducible component $X_i$ of $\mathcal{X}'_k$. We will show that $\tilde{I}_i = \tilde{I}_x$, where $\tilde{I}_x \xrightarrow{\sim} \mathbb{Z}/p^j\mathbb{Z}$ is the inertia subgroup at $x$. Let $\mathcal{X}_2 \overset{\text{def}}{=} \mathcal{X}'/\tilde{I}_x$ be the quotient of $\mathcal{X}'$ by $\tilde{I}_x$ and $\tilde{X}_i$ the image of $X_i$ in $\mathcal{X}_2$. The natural morphism $X_i \to \tilde{X}_i$ is a radicial morphism as follows from [Sa], Corollary 4.1.2, hence $\tilde{I}_x \subset \tilde{I}_i$. Assume that $\tilde{I}_x \subsetneq \tilde{I}_i$. Let $\mathcal{X}'_2 \overset{\text{def}}{=} \mathcal{X}'/\tilde{I}_i$ and $\tilde{X}'_i$ the image of $X_i$ in $\mathcal{X}'_2$. The natural morphism $\tilde{X}_i \to \tilde{X}'_i$ (which has degree

$> 1$) is by assumption on the one hand radicial, and on the other hand unramified above the image of the specialisation of the ramified point $x$ in $\tilde{X}'_i$, which is a contradiction. Hence we necessarily have $\tilde{I}_x = \tilde{I}_i$. The last assertion in Lemma 2.5.1 (v) follows from Lemma 2.5.5 (see end of §2) and the corresponding assertion in the case where $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ in [Sa], Corollary 4.1.2. Q.E.D.

**Lemma 2.5.1 (vi).** *Let $X_i$ be a ramified vertex of $\Gamma'$ of type $j$. Then when moving in the graph $\Gamma'$ from $X_i$ towards the end vertices of $\Gamma'$ we encounter at most a unique ramified vertex $X_{i'} \neq X_i$. Moreover, in such a component $X_{i'}$ specialises a unique ramified point in the morphism $f_K : \mathcal{X}_K \to \mathbb{P}^1_K$ and the component $X_{i'}$ is necessarily of the same type $j$ as $X_i$. In other words the graph $\Gamma'$ separates the directions of the ramified vertices of $\Gamma'$ which are of distinct types.*

*Proof.* Follows directly from the next Lemma 2.5.2 by passing to the quotient of $\mathcal{X}'$ by the unique subgroup $H'$ of $H$ with cardinality $p$. Q.E.D.

**Lemma 2.5.1 (vii).** *Assume that $\mathcal{X}$ is minimal with respect to its defining properties above. Then the ramified vertices in the graph $\Gamma'$ are the end vertices of the tree $\Gamma'$.*

*Proof.* Assume that $\mathcal{X}$ is minimal with respect to its defining properties. Let $X_i$ be a ramified vertex of the tree $\Gamma'$. We will show that $X_i$ is necessarily an end vertex of $\Gamma'$. Assume that $X_i$ (which is distinct from $X_0$) is an internal vertex of $\Gamma'$. Let $X_{\tilde{i}}$ be an end vertex of $\Gamma'$ which we encounter when moving in $\Gamma'$ from $X_i$ towards the end vertices of $\Gamma'$ and $\gamma$ the geodesic linking $X_i$ and $X_{\tilde{i}}$. All vertices of $\gamma$ are projective lines (cf. Lemma 2.5.1 (i)). In $\gamma$ there exists at most a unique vertex $X_j \neq X_i$ which is a ramified vertex (cf. Lemma 2.5.1 (vi)). All vertices of $\gamma$ which are not ramified vertices can be contracted in $\mathcal{X}$ without destroying the defining properties of $\Gamma'$. Thus, we deduce that $\gamma$ contains a unique vertex which is distinct from $X_i$, namely $X_j$, and the later $X_j = X_{\tilde{i}}$ is an end vertex of $\Gamma$. By Lemma 2.5.1 (vi) the vertex $X_j$ is of the same type as the vertex $X_i$ and there exists a unique ramified point in the morphism $\mathcal{X}_K \to \mathbb{P}^1_K$ which specialises in (a smooth point of) $X_j$. The vertex $X_j$ can also be contracted in a smooth point of $\mathcal{X}'$ which is supported by $X_i$ and in this point will specialise (after contracting $X_j$) a unique ramified point, which doesn't destroy the defining properties of $\mathcal{X}'$. But this would contradict the minimality of $\mathcal{X}'$. Thus, $X_i$ is necessarily a terminal vertex to start with. Q.E.D.

The following lemma is used in the proof of Lemma 2.5.1 (vi).

**Lemma 2.5.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite Galois cover between smooth $R$-curves with group $H' \tilde{\to} \mathbb{Z}/p\mathbb{Z}$, such that the morphism $f_K : \mathcal{X}_K \to \mathcal{Y}_K$ between generic fibres is ramified. Assume that there exists a birational morphism $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is a semi-stable $R$-curve, the action of the group $H'$ on $\mathcal{X}$ extends to an action of $H'$ on $\mathcal{X}'$, and the ramified points in the morphism $f_K : \mathcal{X}_K \to \mathcal{Y}_K$ specialise in smooth distinct points of $\mathcal{X}'_k$. Then the graph $\Gamma'$ associated to the special fibre $\mathcal{X}'_k$ of $\mathcal{X}'$ is a tree. Let $X_0$ be the strict transform of $\mathcal{X}_k$ in $\mathcal{X}'$. Choose an orientation of $\Gamma'$ starting from $X_0$ towards the end vertices of $\Gamma'$. Let $X_i$ be a vertex of $\Gamma'$. Assume that $X_i$ is a ramified vertex of $\Gamma'$, i.e. there exists a ramified point in the morphism $f_K : \mathcal{X}_K \to \mathcal{Y}_K$ which specialises in $X_i$. Then when moving in the graph $\Gamma'$ from $X_i$ towards the end vertices we encounter at most a unique ramified vertex $X_j \neq X_i$. Moreover, in such a component $X_j$ specialises a unique ramified point in the morphism $f_K : \mathcal{X}_K \to \mathcal{Y}_K$.*

*Proof.* We can assume that the birational morphism $\mathcal{X}' \to \mathcal{X}$ is not an isomorphism. The fact that the graph $\Gamma'$ is a tree follows immediately from the fact that $\mathcal{X}$ is smooth over $R$. Let $X_0$ be the strict transform of $\mathcal{X}_k$ in $\Gamma'$. Let $X_i$ be a ramified component of $\mathcal{X}_k$. Then $X_i \neq X_0$ as follows from [Sa], Corollary 4.1.2. Thus, $X_i$ is either an internal or an end component of $\Gamma'$. Assume that $X_i$ is an internal component. Let $X_j$ be an irreducible component of $\Gamma'$ which is a ramified vertex and that we encounter when moving from $X_i$ towards the end vertices of $\Gamma'$. We will show that only a unique ramified point in the morphism $f_K : \mathcal{X}_K \to \mathcal{Y}_K$ specialises in such a component $X_j$, and that such a component is unique. After possibly contracting all the irreducible components which form the vertices of the geodesics of $\Gamma'$ which link $X_i$ to the end vertices of $\Gamma'$ we can assume that $X_i$ is an end vertex of $\Gamma'$. The component $X_j$ then contracts to a smooth point $x$ of $X_i$, which is the specialisation of some ramified points in the morphism $f_K : \mathcal{X}_K \to \mathcal{Y}_K$. Let $P_i$ be the image of $X_i$ in the quotient $\mathcal{Y}' \overset{\text{def}}{=} \mathcal{X}'/H'$ of $\mathcal{X}'$ by $H'$, and $y$ the image of $x$ in $\mathcal{Y}'$, which is a smooth point. The natural morphism $X_i \to P_i$ is a $\mu_p$-torsor (cf. loc. cit). Furthermore, the natural morphism $\hat{\mathcal{O}}_{\mathcal{X},x} \to \hat{\mathcal{O}}_{\mathcal{Y},y}$ between the formal completions at the smooth points $x$ and $y$ has a degeneration on the boundary of the formal completion $\hat{\mathcal{O}}_{\mathcal{Y},y}$ of type $(\mu_p, 0, h)$ (cf. [Sa], Corollary 4.1.2) and there is a unique ramified point which specialises in $x$ (cf. loc. cit). Q.E.D.

2.5.3. The various results in 2.5.1 have the following local analogs, which describe the geometry of a (minimal) semi-stable model of an order $p^n$ automorphism of a $p$-adic open disc (over $K$) without inertia at $\pi$ (cf. [Gr-Ma], 1), and which was proven in [Gr-Ma1] in the case of an order $p$-automorphism.

Let $f : \tilde{\mathcal{X}} \stackrel{\text{def}}{=} \text{SpfA} \to \tilde{\mathcal{Y}} \stackrel{\text{def}}{=} \text{SpfB}$ be a Galois cover between connected formal germs of smooth $R$-curves (i.e. $A \tilde{\to} B \tilde{\to} R[[T]]$) which is Galois with group $G \tilde{\to} \mathbb{Z}/p^n\mathbb{Z}$, $n \geq 1$, and such that the natural morphism $f_k : \tilde{\mathcal{X}}_k \stackrel{\text{def}}{=} \text{SpecA}/\pi\text{A} \to \tilde{\mathcal{Y}}_\text{k} \stackrel{\text{def}}{=} \text{SpecB}/\pi\text{B}$ between special fibres is generically separable. Assume that there exists a birational morphism $\tilde{\mathcal{X}}' \to \tilde{\mathcal{X}}$ with $\tilde{\mathcal{X}}'$ semi-stable such that the ramified points in the morphism $f_K : \tilde{\mathcal{X}}_K \stackrel{\text{def}}{=} \text{Spec}(\text{A} \otimes_\text{R} \text{K}) \to \tilde{\mathcal{Y}}_\text{K} \stackrel{\text{def}}{=} \text{Spec}(\text{B} \otimes_\text{R} \text{K})$ specialise in smooth distinct points of $\tilde{\mathcal{X}}'_k$, and the action of $G$ on $\tilde{\mathcal{X}}$ extends to an action of $G$ on $\tilde{\mathcal{X}}'$. (We do not assume that $\tilde{\mathcal{X}}'$ is minimal with respect to the above properties). Let $\tilde{\mathcal{Y}}' \stackrel{\text{def}}{=} \tilde{\mathcal{X}}'/G$ be the quotient of $\tilde{\mathcal{X}}'$ by $G$. Then $\tilde{\mathcal{Y}}'$ is semi-stable (cf. [Ra], Appendice). We have a commutative digram:

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \xrightarrow{f} & \tilde{\mathcal{Y}} \\ \uparrow & & \uparrow \\ \tilde{\mathcal{X}}' & \xrightarrow{\tilde{f}} & \tilde{\mathcal{Y}}' \end{array}$$

where the vertical maps are birational morphisms. Let $\Gamma'$ (resp. $\Gamma''$) be the graph associated to the special fibre of $\tilde{\mathcal{X}}'$ (resp. of $\tilde{\mathcal{Y}}'$). Let $\text{Ver}(\Gamma') \stackrel{\text{def}}{=} \{\text{X}_\text{i}\}_{\text{i}=0}^{\text{m}}$ (resp. $\text{Ver}(\Gamma'') \stackrel{\text{def}}{=} \{\text{Y}_\text{i}\}_{\text{i}=0}^{\text{n}}$) be the set of vertices of $\Gamma'$ (resp. of $\Gamma''$). We have a natural morphism $\Gamma' \to \Gamma''$ of graphs.

**Lemma 2.5.3 (i).** *The graphs $\Gamma'$ and $\Gamma''$ are trees. Furthermore, each vertex $X_i$ (resp. $Y_i$) of $\Gamma'$ (resp. of $\Gamma''$) which is distinct from the strict transform of the generic point of $\tilde{\mathcal{X}}_k$ in $\tilde{\mathcal{X}}'$ (resp. distinct from the strict transform of the generic point of $\tilde{\mathcal{Y}}_k$ in $\tilde{\mathcal{Y}}'$) is isomorphic to $\mathbb{P}^1_k$.*

Let $X_0$ be the strict transform of the generic point of $\tilde{\mathcal{X}}_k$ in $\tilde{\mathcal{X}}'$. We choose an orientation of the tree $\Gamma'$ starting from $X_0$ towards the end vertices of $\Gamma'$. For a vertex $X_i$ of $\Gamma'$ we will denote by $\tilde{D}_i$ (resp. $\tilde{I}_i \subseteq \tilde{D}_i$) the decomposition (resp. inertia) subgroup of $H$ at the generic point of $X_i$. Then:

**Lemma 2.5.3 (ii).** *$\tilde{D}_0 = H$ and $\tilde{I}_0 = \{1\}$.*

**Lemma 2.5.3 (iii).** *Let $X_i$ be an internal vertex of $\Gamma'$, i.e. $X_i$ is distinct from $X_0$ and from the end vertices of $\Gamma'$, and $X_j$ an adjacent vertex to $X_i$ in the direction moving towards the end vertices of $\Gamma'$. Then the following two cases occur:*

*(1) either $\tilde{D}_i = \tilde{I}_i$, in this case $\tilde{D}_j = \tilde{D}_i$,*

*(2) or $\tilde{I}_i \subsetneq \tilde{D}_i$, in this case $\tilde{D}_j = \tilde{I}_i$ and we have a natural exact sequence*

$$1 \to \tilde{D}_j \to \tilde{D}_i \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

*Furthermore, in the case (2) if $\tilde{X}_i$ denotes the image of $X_i$ in the quotient $\tilde{\mathcal{X}}'/\tilde{I}_i$ of $\tilde{\mathcal{X}}'$ by $\tilde{I}_i$ then the natural morphism $\tilde{X}_i \to P_i$, where $P_i \tilde{\to} \mathbb{P}^1_k$ is the image of $X_i$ in $\Gamma''$, is a Galois cover of degree p ramified above a unique point $\infty \in P_i$ (which is the edge of the geodesic linking $P_i$ to $P_0$, which is linked to $P_i$) with Hasse conductor $m = 1$ at $\infty$ (i.e. given by an Artin–Schreier equation $z^p - z = t^{-1}$ where $t$ is a local parameter at $\infty$). In particular, when we move in the graph $\Gamma'$ starting from $X_0$ towards the end vertices of $\Gamma'$ then the cardinality of the decomposition group $\tilde{D}_i$ (resp. the cardinality of the inertia subgroup $\tilde{I}_i$) of a vertex $X_i$ decreases. More precisely, if when moving from a vertex $X_i$ towards the end vertices of $\Gamma'$ we encounter a vertex $X_j$ then $\tilde{D}_j \subseteq \tilde{D}_i$ and $\tilde{I}_j \subseteq \tilde{I}_i$.*

**Lemma 2.5.3 (iv).** *Let $X_i$ be a separable vertex of $\Gamma'$ (i.e. $\tilde{I}_i = \{1\}$) which is distinct from $X_0$. Then, either $\tilde{D}_i = \mathbb{Z}/p\mathbb{Z}$, in this case $X_i$ is a Galois cover of $\mathbb{P}^1_k$ ramified above a unique point $\infty \in \mathbb{P}^1_k$ with Hasse conductor $m = 1$ at $\infty$, or $\tilde{D}_i = 1$. In both cases if $X_j$ is a vertex adjacent to $X_i$ in the direction moving towards the ends of $\Gamma'$ then $\tilde{D}_j = 1$.*

Let $0 < j \leq n$ be an integer. Let $x \in \tilde{\mathcal{X}}_K$ be a ramified point in the morphism $f_K : \tilde{\mathcal{X}}_K \to \tilde{\mathcal{Y}}_K$. We say that the ramified point $x$ is of type $j$ if the inertia subgroup $\tilde{I}_x \subseteq G$ at $x$ is isomorphic to $\mathbb{Z}/p^j\mathbb{Z}$. A vertex $X_i$ of $\Gamma'$ is called a ramified vertex of type $j$ if there exists a ramified point $x$ of type $j$ in the morphism $f_K : \tilde{\mathcal{X}}_K \to \tilde{\mathcal{Y}}_K$ which specialises in the component $X_i$.

**Lemma 2.5.3 (v).** *Let $X_i$ be a ramified component of $\Gamma'$. Then $X_i$ is of type $j$ for a unique integer $0 < j \leq n$. In other words if $0 < j < j' \leq n$ are integers then ramified points of type $j$ (resp. type $j'$) in the morphism $f_K : \tilde{\mathcal{X}}_K \to \tilde{\mathcal{Y}}_K$ specialise in distinct irreducible components of $\mathcal{X}_k$. More precisely, if $X_i$ is a ramified vertex of type $j$ then the inertia subgroup $\tilde{I}_i$ which is associated to $X_i$ has cardinality $p^j$, i.e. $I_i \tilde{\to} \mathbb{Z}/p^j\mathbb{Z}$. Furthermore, let $Y_i$ be the image of $X_i$ in $\Gamma''$. Then the natural morphism $X_i \to Y_i$ has the structure of a $\mu_{p^j}$-torsor outside the specialisation of the branched points in $Y_i$ and the double points of $\tilde{\mathcal{Y}}'_k$ which are supported by $Y_i$.*

**Lemma 2.5.3 (vi).** *Let $X_i$ be a ramified vertex of $\tilde{\mathcal{X}}'_k$ of type $j$. Then when moving in the graph $\Gamma'$ from $X_i$ towards the end vertices of*

*$\Gamma'$ we encounter at most a unique ramified vertex $X_{i'} \neq X_i$. Moreover, in such a component $X_{i'}$ specialises a unique ramified point in the morphism $f_K : \tilde{\mathcal{X}}_K \to \tilde{\mathcal{Y}}_K$, and the component $X_{i'}$ is necessarily of the same type $j$ as $X_i$.*

**Lemma 2.5.3 (vii).** *Assume that $\tilde{\mathcal{X}}'$ is minimal with respect to its defining properties above. Then the ramified vertices in the graph $\Gamma'$ are the end vertices of the tree $\Gamma'$.*

*Proof.* Similar to the proofs of Lemma 2.5.1 (i), Lemma 2.5.1 (ii), Lemma 2.5.1 (iii), Lemma 2.5.1 (iv), Lemma 2.5.1 (v), Lemma 2.5.1 (vi), and Lemma 2.5.1 (vii). Q.E.D.

2.5.4. Our main results in this section describe the semi-stable reduction of fake liftings of cyclic Galois covers between smooth curves, and show that fake liftings (if they exist) have semi-stable models with some very specific properties which in some sense are reminiscent of the properties of semi-stable models of smooth liftings of cyclic Galois covers between curves (cf. 2.5.1).

Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover with Galois group $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ $(n \geq 1)$ with $Y_k$ a smooth $k$-curve. Let $g_k : X_k \to \mathbb{P}^1_k$ be the (unique) subcover of $f_k$ with group $H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}$. Assume that there exists a smooth Galois lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ of $g_k$ defined over $R$, and that $f_k$ satisfies the assumption **(A)** (with respect to the smooth lifting $g$ of $g_k$) (cf. 2.3.1). Let $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ be a fake lifting (relative to the smooth lifting $g$ of $g_k$) of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ which dominates the smooth lifting $g$ of $g_k$, and which we suppose is defined over $R$ (cf. Definition 2.3.2). Assume that there exists a minimal birational morphism $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'_k \overset{\text{def}}{=} \mathcal{Y}' \times_R k$ semi-stable, and such that the ramified points in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ specialise in smooth distinct points of $\mathcal{Y}'_k$. Let $\Gamma$ be the graph associated to the semi-stable $k$-curve $\mathcal{Y}'_k$. Write $Y_0$ for the vertex of $\Gamma$ which is the strict transform of $\mathcal{Y}_k$ ($\mathcal{Y}_k$ is irreducible) in $\mathcal{Y}'_k$. For a vertex $Y_i$ of $\Gamma$ we denote by $D_i$ (resp. $I_i \subseteq D_i$) the decomposition (resp. inertia) subgroup of $G$ at the generic point of $Y_i$. We will follow the notations in 2.4.

**Theorem 2.5.4 (i).** *The graph $\Gamma$ is a tree.*

*Proof.* In the course of proving Theorem 2.5.4 (i) we will also prove the second assertion in Lemma 2.5.4 (v) below.

Let's move in the graph $\Gamma'$ starting from the origin vertex $X_0$ towards a given end vertex $X_{\tilde{i}}$ of $\Gamma'$ along the geoedesic $\gamma$ of $\Gamma'$ which links $X_0$ and $X_{\tilde{i}}$. Let $X_i$ be a vertex of $\gamma$ which is distinct from both $X_0$ and $X_{\tilde{i}}$. Then $X_i$ is an internal vertex of $\Gamma'$ and the pre-image of $X_i$ in $\Gamma$ via the natural morphism $\Gamma \to \Gamma'$ consists of a unique vertex $Y_i$ (cf. Lemma 2.5.4 (iii) below, more precisely the exact sequence $0 \to H' \to D_i \to \tilde{D}_i \to 0$, where $H'$ is the unique subgroup of $G$ with cardinality $p$). Moreover, the natural morphism $Y_i \to X_i$ is either radicial (this occurs only if $H' \subseteq I_i$), or is a separable morphism in which case $I_i = \tilde{I}_i = \{1\}$ and $X_i$ is adjacent to an end vertex of $\Gamma'$ as follows from Lemma 2.5.4 (iii). In fact we will show below that the latter case can not occur. Note that there is no vicious circle here since the proof of Lemma 2.5.4 (iii) doesn't use Theorem 2.5.4 (i). Let now $Y_{\tilde{i}}$ be the unique vertex of $\Gamma$ which is in the pre-image of the end vertex $X_{\tilde{i}}$ of $\Gamma'$. The following two cases occur. Either the inertia subgroup $I_{\tilde{i}} \neq \{1\}$ which is associated to the vertex $Y_{\tilde{i}}$ is non trivial, in which case we have an exact sequence $0 \to H' \to I_{\tilde{i}} \to \tilde{I}_{\tilde{i}} \to 0$, or the inertia subgroups $I_{\tilde{i}} = \tilde{I}_{\tilde{i}} = \{1\}$ are trivial. In the first case the natural morphism $Y_{\tilde{i}} \to X_{\tilde{i}}$ is radicial, hence a homeomorphism. In summary two cases occur: either for every vertex $X_i$ of the geodesic $\gamma$ which is distinct from $X_0$ (in particular $X_i$ may be equal to $X_{\tilde{i}}$) and its unique pre-image $Y_i$ in $\Gamma$ we have $I_i \neq \{1\}$ (in particular, $H \subseteq I_i$ in this case), or there exists a vertex $X_i$ of $\gamma$ which is distinct from $X_0$ and its unique pre-image $Y_i$ in $\Gamma$ such that $I_i = \tilde{I}_i = \{1\}$.

In the first case the natural morphism $Y_i \to X_i$ is radicial and the natural morphism $\tilde{h}^{-1}(\gamma) \to \gamma$, where $\tilde{h}^{-1}(\gamma)$ is the pre-image of $\gamma$ in $\Gamma$, is a homeomorphism. In particular, $\tilde{h}^{-1}(\gamma)$ is a tree in this case. More precisely, in this case $\tilde{h}^{-1}(\gamma)$ is a geodesic which links $Y_0$ to the unique vertex $Y_{\tilde{i}}$ in the pre-image of $X_{\tilde{i}}$ which is an end vertex of $\Gamma$. Moreover, all vertices of $\tilde{h}^{-1}(\gamma)$ which are distinct from $X_0$ are projective lines in this case and the vertex $Y_{\tilde{i}}$ is necessarily a ramified vertex. For otherwise the component $Y_{\tilde{i}}$ would be a (non ramified) projective line hence can be contracted in the semi-stable model $\mathcal{Y}'$ without destroying the defining properties of $\mathcal{Y}'$, and this would contradict the minimal character of $\mathcal{Y}'$. Now we shall investigate the second case. Assume that the second case above occurs. In order to show that the graph $\Gamma$ is a tree it suffices to show that the pre-image $\tilde{h}^{-1}(\gamma)$ of the geodesic $\gamma$ is also a tree in this case (for every possible choice of $\gamma$). More precisely, we will show that the natural map $\tilde{h}^{-1}(\gamma) \to \gamma$ is a homeomorphism of trees. Let $X_i$ be the first vertex of $\gamma$ that we encounter when moving from $X_0$ towards $X_{\tilde{i}}$, and $Y_i$ the unique pre-image of $X_i$ in $\Gamma$, such that the inertia groups $I_i = \tilde{I}_i = \{1\}$ are trivial. We will show that $X_i = X_{\tilde{i}}$ is necessarily the end vertex of $\gamma$ and that the natural morphism $Y_i \to X_i$, which is generically Galois with group $H'$, is only (totally) ramified above the unique double point $x_i$ of $\mathcal{X}'_k$ which is supported by $X_i$. This will complete the proof of the assertion that $\Gamma$ is a tree, and will also prove the second assertion in Lemma 2.5.4 (v) below.

Assume the contrary that $X_i \neq X_{\tilde{i}}$ is not the end vertex of $\gamma$. Then $X_i$ is an internal vertex of $\Gamma$, which is linked to a unique double point $x_i$ which is an edge of the geodesic which links $X_i$ to $X_0$, and is linked to (at least) another double point $x_{i'}$ which is an edge of the geodesic which links $X_i$ to $X_{\tilde{i}}$ (there may be more double points linked to $X_i$ which are edges of the possible geodesics linking $X_i$ to other end vertices of $\Gamma'$). Moreover, $\tilde{D}_i \tilde{\to} \mathbb{Z}/p\mathbb{Z}$ in this case (cf. Lemma 2.5.1 (iv)) which necessarily implies that $D_i \tilde{\to} \mathbb{Z}/p^2\mathbb{Z}$ and the natural morphism $X_i \to P_i$ (where $P_i$ is the image of $X_i$ in $\mathcal{P}$) is a Galois cover of degree $p$ ramified above a unique point $\infty \in P_i$ (which is the image of the double point $x_i$ in $\mathcal{P}$) with Hasse conductor $m = 1$ at $\infty$ (cf. Lemma 2.5.1 (iii)). In particular, $X_i \tilde{\to} \mathbb{P}^1_k$ is a projective line. The natural morphism $Y_i \to X_i$ is a generically Galois morphism with group $\mathbb{Z}/p\mathbb{Z}$, and is ramified above the double point $x_i$ with Hasse conductor $m_i$ at this point (if $X_j$ is the vertex of $\gamma$ such that $x_i = X_i \cap X_j$ and $Y_j$ its unique pre-image in $\Gamma$ then $I_j \neq \{1\}$ by assumption). Above the double point $x_{i'}$ this morphism is either ramified with Hasse conductor $m_{i'}$ or is unramified. In both cases the double point $x_{i'}$ produces a non trivial contribution to the arithmetic genus of $\mathcal{Y}'_k$. More precisely, in the first case the contribution of $x_{i'}$ to the arithmetic genus is $p-1$ and in the second case it is $\frac{(m_{i'}+1)(p-1)}{2}$.

We will construct, in order to contradict the above assumption, a new Garuti lifting $f_1 : \mathcal{Y}_1 \to \mathbb{P}^1_R$ of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ which dominates the smooth lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ of the Galois subcover $g_k : X_k \to \mathbb{P}^1_R$ of degree $p^{n-1}$, and such that the degree of ramification $\delta_1 \overset{\text{def}}{=} \delta_{f_{1,K}}$ in the morphism $f_{1,K} : \mathcal{Y}_{1,K} \to \mathbb{P}^1_K$ between generic fibres satisfies the inequality $\delta_1 < \delta \overset{\text{def}}{=} \delta_{\tilde{f}_K}$. This would contradict the minimality of $\delta$, i.e. contradicts the fact that $\tilde{f}$ is a fake lifting of $f_k$. To simplify the arguments below we will assume that $G = D_i = \mathbb{Z}/p^2\mathbb{Z}$. The construction of $f_1$ in the general case is done in a similar fashion by using induced covers from $D_i$ to $G$ (cf. the construction of Garuti in [Ga], 3, for similar arguments). Let $X_{1,k}$ be the semi-stable $k$-curve which is obtained from $\mathcal{X}'_k$ by removing the geodesic of the graph $\Gamma'$ which links $X_i$ to the terminal vertex $X_{\tilde{i}}$, with the vertex $X_i$ removed. Thus, $X_{1,k}$ is a semi-stable $k$-curve with the same arithmetic genus as $\mathcal{X}'_k$ (which is the same as that of $X_k$). Moreover, the graph associated to the semi-stable $k$-curve $X_{1,k}$ is a tree with origin vertex $X_0$, and the irreducible component $X_i$ is an end vertex of this tree. Let $P_{1,k}$ be the image of $X_{1,k}$ in $\mathcal{P}$ (here we view $X_{1,k}$ as a closed sub-scheme of $\mathcal{X}_k$), and $Y_{1,k}$ the pre-image of $X_{1,k}$ in $\mathcal{Y}'_k$. We have natural finite morphisms $Y_{1,k} \to X_{1,k} \to P_{1,k}$ between semi-stable $k$-curves.

One can construct a new finite morphism $Y'_{1,k} \to X_{1,k} \to P_{1,k}$ which above $P_{1,k} \setminus P_i$ coincides with the finite cover which is induced by the above cover $Y_{1,k} \to X_{1,k} \to P_{1,k}$, above $P_i$ is a generically separable Galois cover with group $D_i = G$ which is ramified only above the unique double point $\infty$ of $P_{1,k}$ linking $P_i$ to the geodesic of $\Gamma''$ which links $P_i$ and $P_0$ (the point $\infty$ is the image of $x_i$ in $\mathcal{P}$), and which above the formal completion of $P_{1,k}$ at the double point $\infty$ coincides with the cover that is induced by the morphisms $Y_{1,k} \to X_{1,k} \to P_{1,k}$. In other words in this new cover we eliminate all the irreducible components of the geodesic $\gamma$ that we encounter when moving from $X_i$ in the direction of $X_{\tilde{i}}$, and we also eliminate the ramification in the morphism $Y_i \to X_i$ which may arise above points of $X_i$ which are distinct from the double point $x_i$ (cf. above discussion). The finite morphisms $Y'_{1,k} \to X_{1,k} \to P_{1,k}$ can be lifted (uniquely) to finite morphisms $\tilde{\mathcal{Y}}_1 \to \mathcal{X}_1 \to \mathcal{P}_1$, where $\tilde{\mathcal{Y}}_1 \to \mathcal{P}_1$ is a Galois cover with group $G$ which lifts the finite morphism $Y'_{1,k} \to P_{1,k}$, and $\mathcal{X}_1 \to \mathcal{P}_1$ is the unique subcover with group $H$ which lifts the finite morphism $X_{1,k} \to P_{1,k}$, as follows.

First, we have a natural Galois lifting of the finite morphism

$$Y'_{1,k} \setminus Y'_1 \to P_{1,k} \setminus P_i$$

which is the restriction of the finite Galois morphism $\mathcal{Y}' \to \mathcal{P}$ to the formal fibre of $P_{1,k} \setminus P_i$ in $\mathcal{P}$. The restriction of the finite morphism $\mathcal{Y}' \to \mathcal{P}$ to the formal fibre at the double point $\infty$ (above) provides a natural lifting of the cover above the formal completion of $P_{1,k}$ at the double point $\infty$ which is induced by $Y_{1,k} \to X_{1,k} \to P_{1,k}$. Second, the restriction of the finite morphism $Y'_{1,k} \to X_{1,k} \to P_{1,k}$ to the irreducible component $P_i \setminus \{\infty\}$ (which is an étale torsor) can be lifted to an étale torsor of the formal fibre of $P_i \setminus \{\infty\}$ in $\mathcal{P}_1$ with group $G$ by the theorems of liftings of étale covers (cf. [Gr]). Theses liftings can be patched using formal patching techniques to construct the required Galois cover $\tilde{\mathcal{Y}}_1 \to \mathcal{X}_1 \to \mathcal{P}_1$ (cf. Proposition 1.2.2). Let's now contract (in a Galois equivariant fashion) in $\tilde{\mathcal{Y}}_1$ (resp. in $\mathcal{X}_1$) all the irreducible components of the special fibre $\tilde{\mathcal{Y}}_{1,k}$ (resp. $\mathcal{X}_{1,k}$) which are distinct from $Y_0$ (resp. distinct from $X_0$). We then obtain a normal $R$-curve $\mathcal{Y}_1$ (resp. obtain the smooth $R$-curve $\mathcal{X}$). We have natural finite Galois morphisms $f_1 : \mathcal{Y}_1 \to \mathcal{X} \xrightarrow{g} \mathbb{P}^1_R$ and the Galois cover $f_1 : \mathcal{Y}_1 \to \mathbb{P}^1_R$ is by construction a Garuti lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ (the fact that $f_1$ dominates the smooth lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ of $g_k$ is easily verified, and follows from the above construction). Let $\delta_1 \stackrel{\text{def}}{=} \delta_{f_{1,K}}$ be the degree of the different in the cover $f_{1,K} : \mathcal{Y}_{1,K} \to \mathbb{P}^1_K$ between generic fibres. Then (by construction) we have $\delta_1 < \delta$, since the only point of the irreducible component $X_i$ of $\mathcal{X}_{1,k}$ which contributes to the arithmetic genus of $\mathcal{Y}_{1,k}$ is the double point $x_i$ (and this contribution is the same contribution as in the original cover $\mathcal{Y}'_k \to \mathcal{P}_k$ by construction, cf. the above discussion). But this contradicts the minimality of $\delta$ and the fact that $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ is a fake lifting of the Galois cover $f_k$.

This shows that the irreducible component $X_i = X_{\tilde{i}}$ is necessarily an end vertex of the geodesic $\gamma$, hence also an end vertex of the graph $\Gamma'$. A similar argument shows that the natural morphism $Y_i \to X_i$ (which is generically separable) is only ramified above the unique double point $x_i$ of $X_i$. This, in particular, shows that $\tilde{h}^{-1}(\gamma)$ is a tree, and the natural morphism $h^{-1}(\gamma) \to \gamma$ is a homeomorphism of trees. Thus, the graph $\Gamma$ is a tree as claimed. Furthermore, $Y_i$ can not be a ramified component by [Sa], Corollary 4.1.2, which proves the last assertion in Lemma 2.5.4 (v). Q.E.D.

**Lemma 2.5.4 (ii).** *The vertex $Y_0 \in \text{Ver}(\Gamma)$ is a separable vertex, i.e. $I_0 = \{1\}$, and $D_0 = G$.*

*Proof.* Clear since the natural morphism $Y_0 \to \mathbb{P}^1_k$ is generically Galois with group $G$. Q.E.D.

Let $H' \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ be the (unique) subgroup of $G$ with cardinality $p$. Let $\mathcal{X}' \stackrel{\text{def}}{=} \mathcal{Y}'/H'$ and $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$. Then $\mathcal{X}'$ and $\mathcal{P}$ are semi-stable $R$-curves, and we have a commutative diagram where the vertical maps are birational morphisms:

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{h} & \mathcal{X} & \xrightarrow{g} & \mathbb{P}^1_R \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{Y}' & \xrightarrow{\tilde{h}} & \mathcal{X}' & \xrightarrow{\tilde{g}} & \mathcal{P} \end{array}$$

Let $\Gamma'$ (resp. $\Gamma''$) be the graph associated to the semi-stable $k$-curve $\mathcal{X}'_k$ (resp. $\mathcal{P}_k$). Then the graphs $\Gamma'$ and $\Gamma''$ are trees (cf. Lemma 2.5.1 (i)), and we have natural morphisms of graphs (actually these are morphisms of trees by Lemma 2.5.1 (i), and Theorem 2.5.4 (i) above)

$$\Gamma \to \Gamma' \to \Gamma''.$$

Let $Y_i$ be a vertex of $\Gamma$ which is distinct from $Y_0$. Let $X_i$ (resp. $P_i$) be the image of $Y_i$ in $\mathcal{X}'$ (resp. $\mathcal{P}$). Let $\tilde{D}_i$ (resp. $\tilde{I}_i \subseteq \tilde{D}_i$) be the decomposition subgroup (resp. inertia subgroup) of the Galois group

$H \stackrel{\text{def}}{=} G/H'$ which is associated to the generic point of the irreducible component $X_i$.

**Lemma 2.5.4 (iii).** *We have a natural exact sequence*

$$0 \to H' \to D_i \to \tilde{D}_i \to 0.$$

*Furthermore, either we have an exact sequence*

$$0 \to H' \to I_i \to \tilde{I}_i \to 0,$$

*or $I_i = \tilde{I}_i = \{1\}$, and the inertia subgroups $I_i$ and $\tilde{I}_i$ are trivial. See Lemma 2.5.4 (v) below for a more precise statement related to this case.*

*Proof.* Let $Y_i$ be a vertex of $\Gamma$ which is distinct from $Y_0$. Let $X_i$ (resp. $P_i$) be the image of $Y_i$ in $\mathcal{X}'$ (resp. in $\mathcal{P}$). Let $\tilde{D}_i$ (resp. $\tilde{I}_i$) be the decomposition (resp. inertia) subgroup of the Galois group $H \stackrel{\text{def}}{=} G/H'$ which is associated to the generic point of the irreducible component $X_i$. The image of the decomposition group $D_i$ in $G/H$ via the natural morphism $G \twoheadrightarrow G/H$ coincides with $\tilde{D}_i$. Hence we necessarily either have an exact sequence $0 \to H' \to D_i \to \tilde{D}_i \to 0$, since the group $G$ is cyclic, or we have $D_i = \tilde{D}_i = \{1\}$ (if $D_i \cap H' = \{1\}$ then $D_i = \{1\}$ is trivial) in which case the vertex $X_i$ (resp. $Y_i$) is an end vertex of $\Gamma'$ (resp. of $\Gamma$) (cf. Lemma 2.5.1 (iv) and use the minimality of $\mathcal{Y}'$). The latter case can not occur for otherwise the irreducible component $Y_i$ would be a projective line which is an end vertex of $\Gamma$, and is not a ramified vertex of $\Gamma$ as is easily seen since $I_i = \tilde{I}_i = \{1\}$ (cf. [Sa], Proposition 4.1.1), hence can be contracted in the semi-stable model $\mathcal{Y}'$ without destroying the defining properties of $\mathcal{Y}'$ and this would contradict the minimal character of $\mathcal{Y}'$. Also the image of the subgroup $I_i$ in $G/H$ via the natural morphism $G \twoheadrightarrow G/H$ coincides with $\tilde{I}_i$. Hence we either have an exact sequence $0 \to H' \to I_i \to \tilde{I}_i \to 0$, or the inertia groups $I_i = \tilde{I}_i = \{1\}$ are trivial, since the group $G$ is cyclic. Q.E.D.

Let $0 < j \leq n$ be an integer. Let $y \in \mathcal{Y}_K$ be a ramified point in the morphism $f_K : \mathcal{Y}_K \to \mathbb{P}^1_K$. We say that the ramified point $y$ is of type $j$ if the inertia subgroup $I_y \subseteq G$ at $y$ is isomorphic to $\mathbb{Z}/p^j\mathbb{Z}$. A vertex $Y_i$ of $\Gamma$ is called a ramified vertex of type $j$ if there exists a ramified point $y$ of type $j$ in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ which specialises in the component $Y_i$.

**Lemma 2.5.4 (iv).** *Let $Y_i$ be a ramified vertex of $\Gamma$. Then $Y_i$ is of type $j$ for a unique integer $0 < j \leq n$. In other words if $0 < j < j' \leq n$ are integers then ramified points of type $j$ (resp. type $j'$) in the morphism $f_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ specialise in distinct irreducible components of $\mathcal{Y}_k$.*

*Furthermore,* $D_i = I_i \tilde{\to} \mathbb{Z}/p^j\mathbb{Z}$ *in this case, and the natural morphism* $Y_i \to P_i$ *has the structure of a* $\mu_{p^j}$*-torsor outside the specialisation of the branched points in* $P_i$ *and the double points of* $\mathcal{P}_k$ *which are supported by* $P_i$.

*Proof.* Similar to the proof of Lemma 2.5.1 (v). Q.E.D.

**Lemma 2.5.4 (v).** *The set of separable vertices of* $\Gamma$ *which are distinct from* $Y_0$ *is non empty. Furthermore, let* $Y_i$ *be a separable vertex of* $\Gamma$*, i.e.* $I_i = \{1\}$ *is trivial, which is distinct from* $Y_0$*. Then* $Y_i$ *is an end vertex of* $\Gamma$ *and either* $D_i \tilde{\to} \mathbb{Z}/p\mathbb{Z}$ *or* $D_i \tilde{\to} \mathbb{Z}/p^2\mathbb{Z}$*. In the second case the natural morphism* $Y_i \to P_i$ *is Galois with group* $D_i \tilde{\to} \mathbb{Z}/p^2\mathbb{Z}$*,* $X_i \to P_i$ *is its unique Galois subcover of degree p, and* $X_i$ *is ramified above a unique point* $\infty$ *of* $P_i$ *with Hasse conductor* $1$ *at* $\infty$*. In particular,* $X_i$ *has genus* $0$ *in this case. Moreover, the genus of* $Y_i$ *is* $> 0$*. Also no separable vertex of* $\Gamma$ *is a ramified vertex.*

*Proof.* We prove the first assertion in Lemma 2.5.4 (v). Assume that the set of separable vertices of $\Gamma$ which are distinct from $Y_0$ is empty. Let $Y_i$ be a vertex of $\Gamma$ which is distinct from $Y_0$ and $X_i$ its image in $\Gamma'$. The inertia subgroup $I_i \neq \{1\}$ is non trivial by assumption and we have a natural exact sequence $0 \to H' \to I_i \to \tilde{I}_i \to 0$ (cf. Lemma 2.5.4 (iii)). In particular, the natural morphism $Y_i \to X_i$ is radicial hence a homeomorphism. Thus, $Y_i$ is a projective line. Moreover, the natural morphism of graphs $\Gamma \to \Gamma'$ is a homeomorphism in this case and the graph $\Gamma$ is a tree. In particular, the arithmetic genus of the special fibre $\mathcal{Y}'_k$ is equal to the genus of $Y_k$. Hence the genera of $Y_K$ and $Y_k$ are equal. This implies that $Y_K$ has good reduction, which contradicts the fact that $\mathcal{Y}$ is a fake lifting of $f_k$ (more precisely this contradicts the fact that $\mathcal{Y}$ is not smooth over $R$ (cf. Definition 2.3.2)).

The proof of the second assertion follows from the proof of Theorem 2.5.4 (i).

The last assertion is proven in the course of proving Theorem 2.5.4 (i) (cf. loc. cit.) Q.E.D.

**Lemma 2.5.4 (vi).** *When we move in the tree* $\Gamma$ *from a given vertex towards the end vertices of* $\Gamma$ *we encounter either ramified vertices or separable vertices of* $> 0$ *genus (the later are necessarily end components by (v) above). In particular, an end vertex of the graph* $\Gamma$ *(which is a tree by Theorem 2.5.4 (i)) is either a ramified vertex or a separable vertex of* $\Gamma$.

*Proof.* Let $Y_i$ be an internal vertex of $\Gamma$ and $Y_{\tilde{i}}$ and end vertex of $\Gamma$ which we encounter when moving from $Y_i$ towards the end vertices. Let

$\gamma$ be the geodesic of $\Gamma$ which links $Y_i$ and $Y_{\tilde{i}}$. We argue by contradiction. Assume that $Y_{\tilde{i}}$ is neither a ramified component nor a separable component. Then all vertices of $\gamma$ are projective lines (as is easily seen), and can be contracted in $\mathcal{Y}'$ without destroying the defining properties of $\mathcal{Y}'$, which would contradict the minimal character of $\mathcal{Y}'$. But this would imply that $Y_i$ is an end vertex and is not internal. Q.E.D.

The following Lemma 2.5.5 is used in the proof of Lemma 2.5.1 (v), and Lemma 2.5.4 (iv).

**Lemma 2.5.5.** *Let* $\mathcal{X} \overset{def}{=} \mathrm{Spf}\mathrm{A}$ *be a connected smooth R-formal affine scheme. Let* $f : \mathcal{Y} \to \mathcal{X}$ *be a finite Galois cover between smooth R-formal schemes with* $\mathcal{Y}$ *connected, with group* $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ $(n \geq 1)$*, such that the natural morphism* $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ *between generic fibres is étale. Here the generic fibres* $\mathcal{Y}_K$ *and* $\mathcal{X}_K$ *denote the rigid analytic spaces associated to* $\mathcal{Y}$ *and* $\mathcal{X}$ *respectively (cf. [Ab]). Let* $\eta$ *be the generic point of the special fibre of* $\mathcal{X}$ *and* $\delta$ *the degree of the different in the morphism* $f$ *above* $\eta$*. Assume that* $\delta = v_K(p)(1 + p + p^2 + ... + p^{n-1})$*. Then the natural morphism* $f_k : \mathcal{Y}_k \to \mathcal{X}_k \overset{def}{=} \mathrm{Spec}\mathrm{A}/\pi\mathrm{A}$ *between special fibres has the structure of a* $\mu_{p^n}$*-torsor.*

*Proof.* The Galois cover $f$ has a natural factorisation

$$f : \mathcal{Y} = \mathcal{Y}_n \xrightarrow{f_{n-1}} \mathcal{Y}_{n-1} \longrightarrow .... \longrightarrow \mathcal{Y}_2 \xrightarrow{f_1} \mathcal{Y}_1 \overset{\mathrm{def}}{=} \mathcal{X},$$

where $f_i : \mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is a Galois cover of degree $p$. Let $\delta_i$ be the degree of the different in the morphism $f_i$ above the generic point $\eta_i$ of $\mathcal{Y}_i$. Then $\delta_i \leq v_K(p)$ (cf. [Sa], Proposition 2.3). The assumption on $\delta$ implies that $\delta_i = v_K(p)$, $\forall i \in \{1, ..., n-1\}$. Hence $f_i : \mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is a torsor under the group scheme $\mu_{p,R}$ (cf. loc. cit.). In fact this later property is equivalent to $\delta_i = v_K(p)$. This implies in particular that the Galois cover $f$ is given by an equation $Z^{p^n} = u$ where $u \in A$ is a unit whose image $\bar{u}$ in $A/\pi A$ is not a $p$-th power and hence has the structure of a $\mu_{p^n}$-torsor (over $R$). Q.E.D.

## §3. The smoothening process

In this section we introduce the process of smoothening of fake liftings of cyclic Galois covers between smooth curves. The idea of smoothening of fake liftings already germs in the proof of Theorem 2.5.4 (i). The smoothening process ultimately aims to show that fake liftings as introduced in §2 do not exist. This in turn would imply the validity of the (revisited) Oort conjecture (cf. Remark 2.3.3). We use the same notations as in §1, and §2, especially the notations in 1.1.

**3.1.**

Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover with group $G \tilde{\to} \mathbb{Z}/p^n\mathbb{Z}$ ($n \geq 1$) and $Y_k$ smooth over $k$. Let $g_k : X_k \to \mathbb{P}^1_k$ be the (unique) subcover of $f_k$ with group $H \tilde{\to} \mathbb{Z}/p^{n-1}\mathbb{Z}$. Assume that there exists a smooth lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ of $g_k$ defined over $R$. Assume that $f_k$ satisfies the assumption **(A)** in 2.3.1 with respect to the smooth lifting $g$ of $g_k$. Let $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ be a fake lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ (with respect to the smooth lifting $g$ of $g_k$) which dominates the smooth lifting $g$ of $g_k$, and which we suppose defined over $R$ (cf. Definition 2.3.2). We assume that there exists a minimal birational morphism $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$ semi-stable, and such that the ramified points in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ specialise in smooth distinct points of $\mathcal{Y}'_k$. Let $\Gamma$ be the graph associated to the semi-stable curve $\mathcal{Y}'_k$ which is a tree by Theorem 2.5.4 (i). Let $H' \tilde{\to} \mathbb{Z}/p\mathbb{Z}$ be the unique subgroup of $G$ with cardinality $p$. Let $\mathcal{X}' \stackrel{\text{def}}{=} \mathcal{Y}'/H'$ and $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$ be the quotient of $\mathcal{Y}'$ by $H'$, and the quotient of $\mathcal{Y}'$ by $G$, respectively. Then $\mathcal{X}'$ and $\mathcal{P}$ are semi-stable $R$-curves and we have a natural Galois morphism $f' : \mathcal{Y}' \to \mathcal{P}$ with group $G$. We have a commutative diagram where the vertical maps are birational morphisms:

$$\begin{array}{ccccc}
\mathcal{Y} & \xrightarrow{h} & \mathcal{X} & \xrightarrow{g} & \mathbb{P}^1_R \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{Y}' & \xrightarrow{\tilde{h}} & \mathcal{X}' & \xrightarrow{\tilde{g}} & \mathcal{P}
\end{array}$$

Let $\Gamma'$ (resp. $\Gamma''$) be the graph associated to the semi-stable $k$-curve $\mathcal{X}'_k$ (resp. $\mathcal{P}_k$). Then the graphs $\Gamma'$ and $\Gamma''$ are trees (cf. Lemma 2.5.1 (i)) and we have natural morphisms of trees

$$\Gamma \to \Gamma' \to \Gamma''.$$

Let $Y_0$ be the origin vertex of $\Gamma$ (which is the strict transform of $\mathcal{Y}_k$ in $\mathcal{Y}'$), and let $P_0$ be its image in $\Gamma''$ which is the origin vertex of $\Gamma''$.

3.1.1. **The semi-stable curve $\mathcal{P}_i$ associated to an internal vertex $P_i$.** Let $P_i$ be an internal vertex of $\Gamma''$. Let $P_{i,k}$ be the semi-stable $k$-curve of arithmetic genus 0 which is obtained from the semi-stable $k$-curve $\mathcal{P}_k \stackrel{\text{def}}{=} \mathcal{P} \times_R k$ by removing all the geodesics of $\Gamma''$ which link the vertex $P_i$ to the end vertices of $\Gamma''$, excluding the vertex $P_i$. The graph associated to the semi-stable curve $P_{i,k}$ is a tree $\Gamma''_i$ in which the vertex $P_i$ is a terminal vertex. Denote by $\infty$ the unique double point of $P_{i,k}$ which is supported by $P_i$ and which links $P_i$ to the geodesic of $\Gamma''_i$ joining $P_i$ and $P_0$. Let $\mathcal{P}_i$ be the semi-stable $R$-model of $\mathbb{P}^1_R$ which is obtained from the semi-stable $R$-model $\mathcal{P}$ by contracting all the irreducible components of $\mathcal{P}_k \setminus P_{i,k}$ (here we view $P_{i,k}$ as a closed sub-scheme of $\mathcal{P}_k$). Then the special fibre $\mathcal{P}_{i,k} \overset{\text{def}}{=} \mathcal{P}_i \times_R k$ of $\mathcal{P}_i$ equals $P_{i,k}$, and we have natural birational morphisms $\mathcal{P} \to \mathcal{P}_i \to \mathbb{P}^1_R$. Let $\mathcal{P}'_i$ be the formal fibre of $P_i \setminus \{\infty\}$ in $\mathcal{P}_i$. Then $\mathcal{P}'_i \overset{\sim}{\to} \text{SpfR} < \text{S} >$ is a formal closed disc. Let $\mathcal{P}''_i$ be the formal fibre of $P_{i,k} \setminus \{P_i\}$ in $\mathcal{P}_i$ and $\mathcal{P}_{i,\infty}$ the formal fibre of $\mathcal{P}_i$ at $\infty$ which is a formal open annulus, i.e. $\mathcal{P}_{i,\infty} \overset{\sim}{\to} \text{Spf}\frac{\text{R[[S,T]]}}{(\text{ST}-\pi^e)}$ for some integer $e \geq 1$. Note that the semi-stable $R$-curve $\mathcal{P}_i$ is obtained by patching $\mathcal{P}'_i$ and $\mathcal{P}''_i$ along the open annulus $\mathcal{P}_{i,\infty}$. Next, we define the important concept of a removable vertex in Definition 3.1.2, and the smoothening process in Definition 3.1.3.

**Definition 3.1.2.** (**Removable vertex of** $\Gamma''$) We use the same notations and assumptions as above. We say that $P_i$ is a removable vertex of the tree $\Gamma''$ if there exists a finite Galois cover $f'_1 : \mathcal{Y}'_1 \to \mathcal{P}_i$ where $\mathcal{P}_i$ is as in 3.1.1, with group $G$, satisfying the following three conditions.

(i) The restriction of the Galois cover $f'_1$ to $\mathcal{P}''_i$ (resp. to $\mathcal{P}_{i,\infty}$) is isomorphic to the restriction of the Galois cover $f' : \mathcal{Y}' \to \mathcal{P}$ (which is the semi-stable minimal model of the fake lifting $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ of $f_k$) above $\mathcal{P}''_i$ (resp. above $\mathcal{P}_{i,\infty}$).

(ii) Let $g'_1 : \mathcal{X}'_1 \to \mathcal{P}_i$ be the unique Galois subcover of $f'_1$ of degree $p^{n-1}$. Then $g'_1$ is generically isomorphic to the Galois cover $g : \mathcal{X} \to \mathbb{P}^1_R$ which is the given smooth lifting of $g_k$.

(iii) The arithmetic genera $g$ (resp. $g_1$) of the special fibres $\mathcal{Y}'_k$ (resp. $\mathcal{Y}'_{1,k} \overset{\text{def}}{=} \mathcal{Y}'_1 \times_R k$) satisfy the inequality $g_1 < g$.

**Definition 3.1.3.** (**Smoothening of a fake lifting**) We use the same notations and assumptions as above. Assume that $P_i$ is a removable vertex in the sense of Definition 3.1.2. Let $f'_1 : \mathcal{Y}'_1 \to \mathcal{P}_i$ be the corresponding Galois cover with group $G$ (which is given in Definition 3.1.2). Let $\mathcal{Y}_1$ be the normal $R$-curve which is obtained from $\mathcal{Y}'_1$ by contracting all the irreducible components of $\mathcal{Y}'_{1,k}$ which are distinct from $Y_0$. The Galois cover $f'_1$ induces naturally a Galois cover $f_1 : \mathcal{Y}_1 \to \mathbb{P}^1_R$ with group $G$ since the above contraction procedure is Galois equivariant. The inequality $g_1 < g$ implies (in fact is equivalent to the fact) that the degree of the generic different $\delta_1 \overset{\text{def}}{=} \delta_{f_{1,K}}$ in the natural morphism $f_{1,K} : \mathcal{Y}_{1,K} \overset{\text{def}}{=} \mathcal{Y}_1 \otimes_R K \to \mathbb{P}^1_K$ between generic fibres satisfies the inequality $\delta_1 < \delta \overset{\text{def}}{=} \delta_{\tilde{f}_K}$. We call the Galois cover $f_1 : \mathcal{Y}_1 \to \mathbb{P}^1_R$ a smoothening of the fake lifting $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$.

Note that by property (ii) in Definition 3.1.2 the Galois cover $f_1 : \mathcal{Y}_1 \to \mathbb{P}^1_R$ is a Garuti lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$, which dominates the smooth lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ of the Galois sub-cover $g_k : X_k \to \mathbb{P}^1_k$. This last property may be used to define the notion of a smoothening of a fake lifting independently from Definition 3.1.2

**3.2.**

The existence of a removable vertex in the tree $\Gamma''$, which implies (by definition) the existence of a smoothening $f_1 : \mathcal{Y}_1 \to \mathbb{P}^1_R$ of the fake lifting $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ (more precisely, the above inequality $\delta_1 < \delta$) (cf. Definition 3.1.3), contradicts the fact that $\tilde{f}$ is a fake lifting (i.e. contradicts the minimality of the generic different $\delta$ of $\tilde{f}$), hence will prove the (revisited) Oort conjecture for the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ and the smooth lifting $g$ of $g_k$ (cf. Remark 2.3.3). More precisely, we have the following.

**Proposition 3.2.1.** *Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover with group $G \tilde{\to} \mathbb{Z}/p^n\mathbb{Z}$, with $Y_k$ a smooth $k$-curve. Let $g_k : X_k \to \mathbb{P}^1_k$ be the Galois subcover of $f_k$ with group $H \tilde{\to} \mathbb{Z}/p^{n-1}\mathbb{Z}$. Assume that there exists a smooth Galois lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ of $g_k$ defined over $R$. Assume that $f_k$ satisfies the assumption* **(A)** *in 2.3.1 with respect to the smooth lifting $g$ of $g_k$. Let $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ be a fake lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$, which dominates the smooth lifting $g$ of $g_k$, and which we suppose defined over $R$ (cf. Definition 2.3.2). We assume that there exists a minimal birational morphism $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'_k \stackrel{def}{=} \mathcal{Y}' \times_R k$ semi-stable, and such that the ramified points in the morphism $f_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ specialise in smooth distinct points of $\mathcal{Y}'_k$. Let $\mathcal{P} \stackrel{def}{=} \mathcal{Y}'/G$ be the quotient of $\mathcal{Y}'$ by $G$ and $\Gamma''$ the tree which is associated to the special fibre $\mathcal{P}_k$ of $\mathcal{P}$.*

*Then, under these assumptions, no internal vertex $P_i$ of the tree $\Gamma''$ is a removable vertex of $\Gamma''$ in the sense of Definition 3.1.2. Equivalently, suppose that there exists an internal vertex $P_i$ of the tree $\Gamma''$ which is a removable vertex of $\Gamma''$ in the sense of Definition 3.1.2 (which implies the existence of a smoothening $f_1 : \mathcal{Y}_1 \to \mathbb{P}^1_R$ of the fake lifting $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ in the sense of Definition 3.1.3). Then the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ doesn't satisfy assumption* **(A)** *and the (revisited) Oort conjecture* **[Conj-O2-Rev]** *is true for the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ and the smooth lifting $g$ of the Galois sub-cover $g_k$.*

One can show that fake liftings of cyclic Galois covers between smooth curves (assuming they exist) always admit a smoothening in the case of cyclic Galois covers of degree $p$. This provides an alternative proof of the Oort conjecture in the case of a cyclic Galois group $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ of order $p$. This proof doesn't use the equation describing the degeneration of the Kummer equation of degree $p$ to the Artin–Schreier equation (as in [Se-Oo-Su], and [Gr-Ma]), but rather uses the degeneration of the Kummer equation to a radicial equation (see proof of Proposition 3.2.2). More precisely, we have the following.

**Proposition 3.2.2.** *Assume that $R$ contains a primitive $p$-th root of unity. Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover with group $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$, and $Y_k$ a smooth $k$-curve. Assume that $f_k$ satisfies the assumption* **(A)** *in 2.3.1. The assumption* **(A)** *in this case means that $f_k$ admits no smooth lifting, and a fake lifting is a Garuti lifting with minimal generic different. Let $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ be a fake lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$, which we suppose defined over $R$ (cf. Definition 2.3.2). We assume that there exists a minimal birational morphism $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'_k \stackrel{def}{=} \mathcal{Y}' \times_R k$ semi-stable and such that the ramified points in the morphism $f_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ specialise in smooth distinct points of $\mathcal{Y}'_k$. Let $\mathcal{P} \stackrel{def}{=} \mathcal{Y}'/G$ be the quotient of $\mathcal{Y}'$ by $G$ ($\mathcal{P}$ is a semi-stable $R$-model of $\mathbb{P}^1_R$), and $\Gamma'$ the tree which is associated to the special fibre $\mathcal{P}_k$ of $\mathcal{P}$. Then there exists an internal vertex $P_i$ of the tree $\Gamma'$ which is a removable vertex of $\Gamma'$ in the sense of Definition 3.1.2. In particular, assumption* **(A)** *is not satisfied by $f_k : Y_k \to \mathbb{P}^1_k$ and the (revisited) Oort conjecture is true for the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ (cf. Proposition 3.2.1).*

*Proof.* We can assume, without loss of generality, that the morphism $f_k$ is ramified above a unique point $\infty$ of $\mathbb{P}^1_k$, i.e. work within the framework of **[Conj-O3]**. Let $P_0$ be the origin vertex of the tree $\Gamma'$, and $P_1$ the (unique) vertex of $\Gamma'$ which is adjacent to $P_0$. We will show that $P_1$ is a removable vertex of $\Gamma'$. The semi-stable $R$-curve $\mathcal{P}_1$ (cf. 3.1.1) in this case has a special fibre $\mathcal{P}_{1,k}$ which consists of the two irreducible (smooth) components $P_0$ and $P_1$ which meet at the unique double point $\infty$. Let $\mathcal{P}'_1 \xrightarrow{\sim} \mathrm{SpfR} < \frac{1}{\mathrm{T}} >$ be the formal fibre of $P_1 \setminus \{\infty\}$ in $\mathcal{P}_1$, $\mathcal{P}_{1,\infty}$ the formal completion of $\mathcal{P}_1$ at $\infty$, and $\mathcal{P}''_1$ the formal fibre of $\mathcal{P}_{1,k} \setminus P_1$ in $\mathcal{P}_1$. The natural Galois morphism $\mathcal{Y}' \to \mathcal{P}$ restricts to Galois morphisms $\mathcal{Y}''_1 \to \mathcal{P}''_1$, and $\mathcal{Y}'_y \to \mathcal{P}_{1,\infty}$, where $\mathcal{Y}'_y$ is the formal completion of $\mathcal{Y}'$ at the unique double point $y$ above $\infty$.

The degeneration type of the Galois cover $\mathcal{Y}'_y \to \mathcal{P}_{1,\infty}$ on the boundary which is linked to $\mathcal{P}'_1$ is necessarily radicial of type $(\alpha_p, -m, 0)$ where $m > 0$ is an integer prime to $p$ (since $P_1$ is an internal vertex of $\Gamma'$), or of type $(\mu_p, -m, 0)$ where $m$ is as above. We only treat the first case, the second case is treated in a similar way (use [Sa], Proposition 3.3.1, (a2)). In the first case the Galois cover $\mathcal{Y}'_y \to \mathcal{P}_{1,\infty}$ induces a Galois cover on the boundary which is linked to $\mathcal{P}'_1$ given by an equation $X^p = 1 + \pi^{tp}T^m$, for a suitable choice of $T$ as above, and $t < v_K(\lambda)$ (cf. Proposition 1.2.2). Here $\lambda = \zeta_1 - 1$, and $\zeta_1$ is a primitive $p$-th root of 1. Consider the Galois cover $\mathcal{Y}'_1 \to \mathcal{P}'_1$ which is generically given by the equation $X^p = T^{-\alpha}(T^{-m} + \pi^{pt})$ where $\alpha$ is an integer such that $\alpha + m \equiv 0 \mod p$. Then $\mathcal{Y}'_1$ is smooth over $R$, and the natural morphism $\mathcal{Y}'_{1,k} \to \mathcal{P}_{1,k}$ between special fibres is radicial (cf. [Sa], Proposition 3.3.1, (b)). The above coverings can be patched using formal patching techniques to construct a Galois cover $\tilde{\mathcal{Y}}_1 \to \mathcal{P}_1$ with group $G$ between semi-stable $R$-curves (cf. Proposition 1.2.2), and by construction the arithmetic genus $g_1$ of the special fibre $\tilde{\mathcal{Y}}_{1,k}$ (which is in fact equal to that of $Y_k$) satisfies the inequality $g_1 < g$ as required. Q.E.D.

### 3.3.

Next, we will give some sufficient conditions for the existence of removable vertices in the case where the Galois group $G \tilde{\to} \mathbb{Z}/p^2\mathbb{Z}$ has order $p^2$.

**Proposition 3.3.1.** *Assume that $R$ contains a primitive $p^2$-th root of unity. Let $f_k : Y_k \to \mathbb{P}^1_k$ be a finite ramified Galois cover with group $G \tilde{\to} \mathbb{Z}/p^2\mathbb{Z}$, and $Y_k$ a smooth $k$-curve. Let $g_k : X_k \to \mathbb{P}^1_k$ be the Galois subcover of $f_k$ with group $H \tilde{\to} \mathbb{Z}/p\mathbb{Z}$. Assume that there exists a smooth Galois lifting $g : \mathcal{X} \to \mathbb{P}^1_R$ of $g_k$ defined over $R$. Assume that $f_k$ satisfies the assumption* **(A)** *in 2.3.1 with respect to the smooth lifting $g$ of the Galois sub-cover $g_k$. Let $\tilde{f} : \mathcal{Y} \to \mathbb{P}^1_R$ be a fake lifting of the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ which dominates the smooth lifting $g$ of $g_k$, which we suppose defined over $R$ (cf. Definition 2.3.2). We assume that there exists a minimal birational morphism $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'_k \overset{def}{=} \mathcal{Y}' \times_R k$ semi-stable, and such that the ramified points in the morphism $f_K : \mathcal{Y}_K \to \mathbb{P}^1_K$ specialise in smooth distinct points of $\mathcal{Y}'_k$. Let $\mathcal{P} \overset{def}{=} \mathcal{Y}'/G$ be the quotient of $\mathcal{Y}'$ by $G$ (which is a semi-stable $R$-model of $\mathbb{P}^1_R$) and $\Gamma''$ the tree which is associated to the special fibre $\mathcal{P}_k$ of $\mathcal{P}$. Assume that there exists an internal vertex $P_i$ of $\Gamma''$ which satisfies the following properties.*

*(i) The pre-image of $P_i$ in $\Gamma$ contains no ramified vertex.*

*(ii) When moving in the tree $\Gamma''$ from the vertex $P_i$ towards the end vertices of $\Gamma''$ we encounter a vertex (necessarily terminal by Lemma 2.5.4 (v)) whose pre-image in $\Gamma$ contains a separable vertex.*

*(iii) When moving in the tree $\Gamma''$ from the vertex $P_i$ towards the end vertices of $\Gamma''$ we encounter a unique vertex whose pre-image in $\Gamma$ contains (in fact consists of) a ramified vertex of type* 2.

*(iv) When moving in the tree $\Gamma''$ from the vertex $P_i$ towards the end vertices of $\Gamma''$ we encounter no vertex whose pre-image in $\Gamma$ contains a ramified vertex of type* 1.

*Then $P_i$ is a removable vertex of $\Gamma''$ in the sense of Definition 3.1.2, and the revisited Oort conjecture* [**Conj-O2-Rev**] *is true for the Galois cover $f_k : Y_k \to \mathbb{P}^1_k$ and the smooth lifting $g$ of the Galois sub-cover $g_k$.*

*Proof.* Let $Y_i$ be a vertex of the graph $\Gamma$ which is in the pre-image of the vertex $P_i$, and $D_i$ (resp. $I_i$) the decomposition (resp. inertia) subgroup of the Galois group $G$ at the generic point of $Y_i$. Then $I_i \neq \{1\}$ since the vertex $Y_i$ is not terminal (cf. Lemma 2.5.4 (v)). Moreover, $I_i = D_i = G$, for otherwise we will contradict the assumption (iii) satisfied by $P_i$ above. (Indeed, the cardinality of the decomposition and inertia subgroups of the various vertices of $\Gamma$ decrease when we move towards the end vertices of $\Gamma$, compare with Lemma 2.5.3 (iii)). Let $\mathcal{P}_i$, $\mathcal{P}'_i$, $\mathcal{P}''_i$ and $\mathcal{P}_{i,\infty}$ be as in 3.1.1. Let $H' \subset G$ be the unique subgroup of $G$ with cardinality $p$ and $\mathcal{X}' \overset{\text{def}}{=} \mathcal{Y}'/H'$ the quotient of $\mathcal{Y}'$ by $H'$. We have natural morphisms $f' : \mathcal{Y}' \to \mathcal{X}' \to \mathcal{P}$. The Galois cover $\mathcal{X}' \to \mathcal{P}$ induces above the irreducible component $P_i$ of $\mathcal{P}_k$, outside the specialisation of the branched points and the double points of $\mathcal{P}$ supported by $P_i$, an $\mathcal{H}_{pt,R}$-torsor (cf. 1.2.1), where $pt < v_K(\zeta_1 - 1)$ and $\zeta_1$ is a primitive $p$-th root of 1. This torsor is generically given by an equation

$$Z^p = 1 + \pi^{tp^2} g(T),$$

where $1 + \pi^{tp^2} g(T) \in \mathrm{Fr}(R < \frac{1}{T} >)$ has $m+1$ distinct geometric zeros in $\mathcal{P}'_1$, which we may assume without loss of generality specialise in the point $\frac{1}{t} = 0$ at infinity (the later follows from the uniqueness of the ramified vertex of type 2 in the assumption (iii)). We will assume for simplicity that $g(T) = T^m$. The general case is treated in a similar fashion.

The above Galois cover $f' : \mathcal{Y}' \to \mathcal{P}$ induces a cyclic Galois cover $\mathcal{Y}'_\infty \to \mathcal{P}_{i,\infty}$ of degree $p^2$ above the formal open annulus $\mathcal{P}_{i,\infty}$, with $\mathcal{Y}'_\infty$ connected (since $D_i = I_i = G$), which induces a cyclic Galois cover $f'_{\infty,1} : \mathcal{Y}'_{\infty,1} \to \mathcal{X}'_{\infty,1} \to \mathcal{P}_{i,\infty,1} \tilde{\to} \mathrm{Spf R[[T^{-1}]]\{T\}}$ of degree $p^2$ above the formal boundary $\mathcal{P}_{i,\infty,1} \tilde{\to} \mathrm{Spf R[[T^{-1}]]\{T\}}$ of $\mathcal{P}_{i,\infty}$ which is linked to $\mathcal{P}'_i \tilde{\to} \mathrm{Spf R} < \frac{1}{\mathrm{T}} >$. We will give an explicit description of the Galois cover $f'_{\infty,1}$ using the assumptions satisfied by the vertex $P_i$. The Galois cover $\mathcal{X}'_{\infty,1} \to \mathcal{P}_{i,\infty,1}$ is a torsor under the group scheme $\mathcal{H}_{pt,R}$ (where $t$ is as above) which has a degeneration type $(\alpha_p, -m, 0)$ where $m > 1$ is as above (this results from the assumption (iii) satisfied by $P_i$), and is given by an equation

$$(*)\qquad \frac{(\pi^{pt}X_1+1)^p-1}{\pi^{p^2t}}=T^m,$$

where $pt < v_K(\zeta_1 - 1)$ and $\zeta_1$ is a primitive p-th root of 1 as above (in the general case replace $T^m$ by $g(T)$ above). The $\alpha_p$-torsor $\mathcal{X}'_{\infty,1,k} \to \mathcal{P}_{i,\infty,1,k}$ at the level of special fibres is given by the equation $x_1^p = t^m$, where $x_1 = X_1 \mod \pi$ and $t = T \mod \pi$. From the above equation $(*)$ we deduce that in $\mathcal{X}'_{\infty,1}$, we have

$$T=(X_1^{\frac{1}{m}})^p[1+\sum_{k=1}^{p-1}\binom{p}{k}\pi^{pt(k-p)}X_1^{k-p}]^{\frac{1}{m}}.$$

In particular, $\mathcal{X}'_{\infty,1,k} \xrightarrow{\sim} \mathrm{SpfR[[T_i]]\{T_i^{-1}\}}$ and $X_1^{\frac{1}{m}}$ is a parameter of $\mathcal{X}'_{\infty,1,k}$. Moreover, the Galois cover $\mathcal{Y}'_{\infty,1} \to \mathcal{X}'_{\infty,1}$ is given by an equation

$$X_2^p=(1+\pi^{pt}X_1)(1+\pi^{ps}f(T))$$

where $f(T) \in \mathrm{Fr}(R < \frac{1}{T} >)$ is such that $(1 + \pi^{ps}f(T))$ is a unit in $\mathcal{P}'_i$, for otherwise we will contradict the assumption (iv) satisfied by $P_i$. We can assume without loss of generality that $1 + \pi^{pt}f(T) \in R < \frac{1}{T} >$. We will give an explicit description (by equations) of the degeneration of the Galois cover $\mathcal{Y}'_{\infty,1} \to \mathcal{X}'_{\infty,1}$. Assume for simplicity that $f(T) = T^{-m_1}$, with $m_1 > 0$. The general case is treated in a similar way. Thus, the above equation is

$$X_2^p=(1+\pi^{pt}X_1)(1+\pi^{ps}T^{-m_1}).$$

Assume first that $t \leq s$. Then on the level of special fibres the $\alpha_p$-torsor $\mathcal{Y}'_{\infty,1,k} \to \mathcal{X}'_{\infty,1,k}$ is given (in the case where $s = t$ one has to eliminate $p$-powers) by the equation

$$(x'_2)^p=x_1=(x_1^{\frac{1}{m}})^m$$

where $x_1 = X_1 \mod \pi$ ($t^{-1}$ becomes a $p$-power in $\mathcal{X}'_{\infty,1,k}$). Here, $X_2 = (1 + \pi^t X'_2)$ and $x'_2 = X'_2 \mod \pi$. In this case the above cover $\mathcal{Y}'_{\infty,1} \to \mathcal{X}'_{\infty,1}$ is a torsor under the group scheme $\mathcal{H}_{t,R}$, and has a degeneration of type $(\alpha_p, -m, 0)$. Note that $x_1^{\frac{1}{m}}$ is a parameter of $\mathcal{X}'_{\infty,1,k}$. Assume now that $s < t$. Then

$$X_2^p=1+\pi^{ps}T^{-m_1}+\pi^{pt}X_1+\pi^{p(t+s)}X_1T^{-m_1},$$

which is not an integral equation for $\mathcal{Y}'_{\infty,1}$ since $T^{-m_1}$ is a $p$-power mod $\pi$ in $\mathcal{X}'_{\infty,1,k}$. To obtain an integral equation we need first to replace $T^{-m_1}$ by its expression in terms of $X_1$, which is deduced from the above expression of $T$,

$$T^{-m_1} = (X_1^{\frac{1}{m}})^{-m_1 p}[1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{pt(k-p)} X_1^{k-p}]^{\frac{-m_1}{m}}.$$

Thus,

$$X_2^p = 1 + \pi^{ps}(X_1^{\frac{1}{m}})^{-m_1 p} + ..... + \pi^{pt} X_1 + ...,$$

where the remaining terms have coefficients with a valuation which is greater than $ps$. After replacing $1 + \pi^{ps}(X_1^{\frac{1}{m}})^{-m_1 p}$ by

$$(1 + \pi^{s}(X_1^{\frac{1}{m}})^{-m_1})^{p} - .......,$$

and multiplying the above equation by $(1 + \pi^{s}(X_1^{\frac{1}{m}})^{-m_1})^{-p}$, we reduce to an equation

$$(X_2')^{p} = 1 + \pi^{pt}(X_1^{\frac{1}{m}})^{m} + .....,$$

where the remaining terms have coefficients with a valuation which is greater than $pt$. In particular, the Galois cover $\mathcal{Y}'_{\infty,1} \to \mathcal{X}'_{\infty,1}$ is a torsor under the group scheme $\mathcal{H}_{t,R}$ and has a degeneration of type $(\alpha_p, -m, 0)$. More precisely, the $\alpha_p$-torsor $\mathcal{Y}'_{\infty,1,k} \to \mathcal{X}'_{\infty,1,x}$ on the level of special fibres is given by an equation $\tilde{x}_2^p = x_1 = (x_1^{\frac{1}{m}})^m$.

The Galois cover $f' : \mathcal{Y}' \to \mathcal{P}$ restricts to Galois covers $\mathcal{Y}''_1 \to \mathcal{P}''_i$, and $\mathcal{Y}'_{1,\infty} \to \mathcal{P}'_{i,\infty}$, above $\mathcal{P}''_i$, and $\mathcal{P}_{i,\infty}$, respectively. Consider the cyclic Galois cover $\mathcal{Y}_1 \to \mathcal{X}_1 \to \mathcal{P}'_i$ of degree $p^2$ which is generically given by the equations

$$\frac{(\pi^{pt} X_1 + 1)^p - 1}{\pi^{p^2 t}} = T^m,$$

(in the general case replace $T^m$ by $g(T)$ above), and

$$X_2^p = (1 + \pi^{pt} X_1)(1 + \pi^{ps} f(T)),$$

where $t$, $s$, and $f(T)$ are as above. This Galois cover on the generic fibre is ramified only at ramified points of type 2 ($(1 + \pi^{ps} f(T))$ is a unit in $\mathcal{P}'_i$). Furthermore, both $\mathcal{X}_1$ and $\mathcal{Y}_1$ are smooth, and the arithmetic genus of the special fibre $\mathcal{Y}_{1,k}$ is 0. Indeed, $\mathcal{X}_1$ is smooth, and the $\alpha_p$-torsor $\mathcal{Y}_{1,k} \to \mathcal{X}_{1,k}$ is given by an equation $\tilde{x}_2^p = x_1$ by arguments similar to the one above ($x_1^{\frac{1}{m}}$ is a parameter on $\mathcal{X}_{1,k}$). The above coverings can be patched using formal patching techniques to construct a Galois cover $\tilde{\mathcal{Y}}_1 \to \mathcal{P}_i$ with Galois group $G$ between semi-stable $R$-curves (cf. Proposition 1.2.2), and by construction the arithmetic genera $g_1$ and $g$ of the special fibres $\tilde{\mathcal{Y}}_{1,k}$ and $\mathcal{Y}'_k$ satisfy the inequality $g_1 < g$. Indeed, we have eliminated the contribution to the arithmetic genus of $\mathcal{Y}'_k$ which arise from the separable end components of $\Gamma$, that lie above the end components of $\Gamma''$ that we encounter when moving form the vertex $P_i$ towards the ends of $\Gamma''$, and which exist by the assumption (ii) satisfied by $P_i$. This proves that $P_i$ is a removable vertex as claimed. Q.E.D.

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