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Dimensions of moduli spaces of finite flat models

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Abstract.

In this survey paper, we explain a moduli space of finite flat models and a dimensional bound of the moduli spaces. We also explain some variant of the moduli space and a conjecture.

§ Introduction

Let K be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. We assume $p > 2$. Let e be the ramification index of K over $\mathbb{Q}_p$, and k be the residue field of K. We consider a finite-dimensional continuous representation $V_{\mathbb{F}}$ of the absolute Galois group G_K over a finite field $\mathbb{F}$ of characteristic p. By a finite flat model of $V_{\mathbb{F}}$, we mean a pair of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, equipped with a structure of an $\mathbb{F}$-vector space, and an isomorphism $V_{\mathbb{F}} \xrightarrow{\sim} \mathcal{G}(\overline{K})$ that respects the action of G_K and the structure of $\mathbb{F}$-vector space. If $e < p-1$, the finite flat model of $V_{\mathbb{F}}$ is unique by Raynaud's result (Theorem 2.2). In general, there are finitely many finite flat models of $V_{\mathbb{F}}$.

A finite flat model of $V_{\mathbb{F}}$ corresponds to a linear algebraic object, which is called a Kisin module. Using theory of Kisin modules, Kisin constructed a moduli space of finite flat model of $V_{\mathbb{F}}$ and studied it to deduce a theorem comparing a deformation ring and a Hecke ring, which is called $R = T$ theorem, in [Ki]. A study of the connected components of the moduli space is important for the application to $R = T$ theorem. For studies of the connected components, see also [Ge], [Im1] and [He2].

The moduli space of finite flat models itself is an interesting geometric object. Even in some simple case, classical geometric objects such as Schubert varieties appear in this moduli space (cf. [He1, Theorem 3.9. (b)]).

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In this survey paper, we explain a dimensional bound of this moduli space in the case $\dim V_{\mathbb{F}} = 2$. The bounds of the dimensions are given by using ramification index e, and this result gives a dimensional generalization of Raynaud's result [Ra, Theorem 3.3.3] in the case $\dim V_{\mathbb{F}} = 2$.

In [Ca], Caruso consider a variant of the moduli space of finite flat model, and give a dimensional bound in the setting. We also explain his result and conjecture.

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Notation

Throughout this paper, we use the following notation. Let $p > 2$ be a prime number, and k be a finite field of characteristic p. The Witt ring of k is denoted by $W(k)$. Let K_0 be the quotient field of $W(k)$, and K be a totally ramified finite extension of K_0. The ring of integers of K is denoted by $\mathcal{O}_K$. Let $\mathbb{F}$ be a finite field of characteristic p. For a ring A, the formal power series ring of u over A is denoted by $A[[u]]$, and we put $A((u)) = A[[u]](1/u)$. For a field F, we denote the algebraic closure of F by $\overline{F}$, the separable closure of F by F^{sep} and the absolute Galois group of F by G_F. Let v_u be the valuation of $\mathbb{F}((u))$ normalized by $v_u(u) = 1$, and we put $v_u(0) = \infty$. For $x \in \mathbb{R}$, the greatest integer less than or equal to x is denoted by $[x]$. The category of sets is denoted by **Set**. For a positive integer d, the d-dimensional affine space over $\mathbb{F}$ is denoted by $\mathbb{A}^d_{\mathbb{F}}$ and the d-dimensional projective space over $\mathbb{F}$ is denoted by $\mathbb{P}^d_{\mathbb{F}}$. Let $\mathbb{G}_{m,\mathbb{F}}$ be $\mathbb{A}^1_{\mathbb{F}} - \{0\}$.

§1. Kisin module

In this section, we recall a definition and properties of a Kisin module with coefficients in an $\mathbb{F}_p$-algebra.

We put $\mathfrak{S} = W(k)[[u]]$. We define an action of ϕ on $\mathfrak{S}$ by the Frobenius action on $W(k)$ and $u \mapsto u^p$. For an $\mathbb{F}_p$-algebra A, we put $\mathfrak{S}_A = \mathfrak{S} \otimes_{\mathbb{F}_p} A$ and extend the action of ϕ on $\mathfrak{S}$ to an A-linear action on $\mathfrak{S}_A$. We put $e = [K : K_0]$ until the end of the section 3.

Definition 1.1. *A Kisin module with coefficients in an $\mathbb{F}_p$-algebra A is a finite projective $\mathfrak{S}_A$-module $\mathfrak{M}$ with ϕ-semi-linear map $\phi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the induced $\mathfrak{S}$-linear map*

$$\mathfrak{S} \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M};\ s \otimes m \mapsto s\phi_{\mathfrak{M}}(m)$$

is killed by u^e.

The category of the Kisin modules with coefficients in an $\mathbb{F}_p$-algebra A is denoted by $(\mathrm{Mod}/\mathfrak{S})_A$.

Remark 1.2. The definition of a Kisin module with coefficients in an $\mathbb{F}_p$-algebra depends only on the ramification index e of K over K_0 and not on K.

Let $(\mathbb{F}\text{-Gr}/\mathcal{O}_K)$ be the category of finite flat group schemes over $\mathcal{O}_K$ with a structure of an $\mathbb{F}$-vector space.

Theorem 1.3. *There exists an equivalence of categories*

$$\mathrm{Gr} : (\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}} \to (\mathbb{F}\text{-Gr}/\mathcal{O}_K).$$

Proof. This follows from [Br2, Théorème 4.2.1.6] and [Ki, Proposition 1.1.11]. Q.E.D.

Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. There is a p-adically continuous action of ϕ on $\mathcal{O}_{\mathcal{E}}$ determined by the Frobenius action on $W(k)$ and $u \mapsto u^p$. We fix a uniformizer π of $\mathcal{O}_K$, and choose elements $\pi_m \in \overline{K}$ such that $\pi_0 = \pi$ and $\pi_{m+1}^p = \pi_m$ for $m \geq 0$, and put $K_\infty = \bigcup_{m \geq 0} K(\pi_m)$.

Let $\Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}}$ be the category of finite $(\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F})$-modules M equipped with ϕ-semi-linear map $\phi_M : M \to M$ such that the induced $(\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F})$-linear map

$$\mathcal{O}_{\mathcal{E}} \otimes_{\phi, \mathcal{O}_{\mathcal{E}}} M \to M;\ s \otimes m \mapsto s\phi_M(m)$$

is an isomorphism. Let $\mathrm{Rep}_{\mathbb{F}}(G_{K_\infty})$ be the category of finite-dimensional continuous representations of G_{K_∞} over $\mathbb{F}$. By the theory of norm fields, there is an isomorphism $G_{K_\infty} \simeq G_{k((u))}$ (cf. [Br1, 2.1]). Then the functor

$$T : \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}} \to \mathrm{Rep}_{\mathbb{F}}(G_{K_\infty});\ M \mapsto \left(k((u))^{\mathrm{sep}} \otimes_{k((u))} M\right)^{\phi=1}$$

gives an equivalence of abelian categories as in [Ki, (1.1.12)]. Here ϕ acts on $k((u))^{\mathrm{sep}}$ by the p-th power map.

Let $V_{\mathbb{F}}$ be a continuous finite-dimensional representation of G_K over $\mathbb{F}$. We take a ϕ-module $M_{\mathbb{F}} \in \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}}$ such that $T(M_{\mathbb{F}})$ is isomorphic to $V_{\mathbb{F}}(-1)|_{G_{K_\infty}}$. Here (-1) denotes the inverse of the Tate twist.

Proposition 1.4 ([Ki, Proposition 1.1.13])**.** *For an object* $\mathfrak{M}$ *of* $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}}$*, there exists a canonical isomorphism*

$$T(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M})(1) \xrightarrow{\sim} \mathrm{Gr}(\mathfrak{M})(\overline{K})|_{G_{K_\infty}}$$

as G_{K_∞}*-representations. Here* (1) *denotes the Tate twist.*

Remark 1.5. In this paper, we consider only Kisin modules and ϕ-modules that are killed by p. Therefore we do not need to introduce the coefficient rings $\mathfrak{S}$ and $\mathcal{O}_{\mathcal{E}}$. However, we introduce them for consistency with usual notation as in [Ki].

§2. Moduli space

In this section, we explain the moduli spaces of finite flat models constructed by Kisin in [Ki].

First, we recall the definition of a finite flat model.

Definition 2.1. *A finite flat model of $V_{\mathbb{F}}$, is a pair of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, equipped with a structure of an $\mathbb{F}$-vector space, and an isomorphism $V_{\mathbb{F}} \xrightarrow{\sim} \mathcal{G}(\overline{K})$ that respects the action of G_K and the structure of $\mathbb{F}$-vector space.*

Theorem 2.2 (Raynaud). *If $e < p-1$, then $V_{\mathbb{F}}$ has at most one finite flat model.*

Proof. This follows from [Ra, Théorème 3.3.3 and Corollaire 3.3.6]. Q.E.D.

Next, we explain the moduli space of finite flat models. We define a functor

$$G : \{\mathbb{F}\text{-algebra}\} \to \mathbf{Set}$$

by putting $G(A)$ to be the set of finite projective $(k[[u]] \otimes_{\mathbb{F}_p} A)$-submodules $\mathfrak{M}_A$ of $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ satisfying

(1) $\mathfrak{M}_A$ generates $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ over $k((u)) \otimes_{\mathbb{F}_p} A$
(2) $\mathfrak{M}_A$ is stable by $\phi_M \otimes \mathrm{id}_A$ and the action of $\phi_M \otimes \mathrm{id}_A$ makes $\mathfrak{M}_A$ a Kisin module with coefficients in A

for $\mathbb{F}$-algebra A. Then the functor G is represented by a projective scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ over $\mathbb{F}$ by [Ki, Proposition 2.1.7].

The scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ is called a moduli space of finite flat model, because we have the following proposition.

Proposition 2.3. *For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, there is a natural bijection between the set of isomorphism classes of finite flat models of $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $\mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$.*

Proof. This follows from Theorem 3.4 and the construction of the scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$. Q.E.D.

Remark 2.4. Theorem 2.2 for $V_{\mathbb{F}'}$, where $\mathbb{F}'$ ranges over all finite extension of $\mathbb{F}$, says that if $e < p-1$ and $\mathcal{GR}_{V_{\mathbb{F}},0}$ is not empty, then $\mathcal{GR}_{V_{\mathbb{F}},0}$ zero-dimensional and connected.

§3. Bound of dimensions

To fix the notation, we recall the definition of the zeta function of a scheme of finite type over a finite field.

Definition 3.1. *Let X be a scheme of finite type over $\mathbb{F}$. We put $q_{\mathbb{F}} = |\mathbb{F}|$. The zeta function $Z(X;T)$ of X is defined by*

$$Z(X;T) = \exp\left(\sum_{m=1}^{\infty} \frac{\left|X(\mathbb{F}_{q_{\mathbb{F}}^m})\right|}{m} T^m\right),$$

where

$$\exp\bigl(f(T)\bigr) = \sum_{m=0}^{\infty} \frac{1}{m!} f(T)^m \in \mathbb{Q}[[T]]$$

for $f(T) \in T\mathbb{Q}[[T]]$.

Theorem 3.2. *We assume that $\dim_{\mathbb{F}} V_{\mathbb{F}} = 2$. We put $d_{V_{\mathbb{F}}} = \dim \mathscr{GR}_{V_{\mathbb{F}},0}$ and $n = [k : \mathbb{F}_p]$. Then followings are true.*

(1) *After extending the field $\mathbb{F}$ sufficiently, we have*

$$Z(\mathscr{GR}_{V_{\mathbb{F}},0}; T) = \prod_{i=0}^{d_{V_{\mathbb{F}}}} (1 - |\mathbb{F}|^i T)^{-m_i}$$

for some $m_i \in \mathbb{Z}$ such that $m_{d_{V_{\mathbb{F}}}} > 0$.

(2) *If $n = 1$, we have*

$$0 \leq d_{V_{\mathbb{F}}} \leq \left[\frac{e+2}{p+1}\right].$$

If $n \geq 2$, we have

$$0 \leq d_{V_{\mathbb{F}}} \leq \left[\frac{n+1}{2}\right]\left[\frac{e}{p+1}\right] + \left[\frac{n-2}{2}\right]\left[\frac{e+1}{p+1}\right] + \left[\frac{e+2}{p+1}\right].$$

Here, $[x]$ is the greatest integer less than or equal to x for $x \in \mathbb{R}$.

Furthermore, each equality in the above inequalities can happen for any finite extension K of $\mathbb{Q}_p$.

Remark 3.3. If $e < p - 1$, Theorem 3.2 also implies that $\mathscr{GR}_{V_{\mathbb{F}},0}$ is zero-dimensional. Therefore it gives a dimensional generalization of Raynaud's result for two-dimensional Galois representations.

The connectedness of $\mathscr{GR}_{V_{\mathbb{F}},0}$ is completely false in general. For example, we can check that if $K = \mathbb{Q}_p(\zeta_p)$ and $V_{\mathbb{F}}$ is trivial representations then $\mathscr{GR}_{V_{\mathbb{F}},0}$ consists of $\mathbb{P}^1_{\mathbb{F}}$ and two points (c.f. [Ki, Proposition 2.5.15(2)]).

In the remaining of this section, we explain an outline of a proof of Theorem 3.2.

From now on, we assume $\mathbb{F}_q \subset \mathbb{F}$ and fix an embedding $k \hookrightarrow \mathbb{F}$. This assumption does not matter, because we may extend $\mathbb{F}$ to prove Theorem 3.2. We consider the isomorphism

$$\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \xrightarrow{\sim} \prod_{\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)} \mathbb{F}((u)) \ ;$$
$$\Big(\sum a_i u^i\Big) \otimes b \mapsto \Big(\sum \sigma(a_i) b u^i\Big)_\sigma$$

and let $\epsilon_\sigma \in k((u)) \otimes_{\mathbb{F}_p} \mathbb{F}$ be the primitive idempotent corresponding to σ. Take $\sigma_1, \cdots, \sigma_n \in \mathrm{Gal}(k/\mathbb{F}_p)$ such that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$. Here we regard ϕ as the p-th power Frobenius, and use the convention that $\sigma_{n+i} = \sigma_i$. In the following, we often use such conventions. Then we have $\phi(\epsilon_{\sigma_i}) = \epsilon_{\sigma_{i+1}}$ and $\phi : M_{\mathbb{F}} \to M_{\mathbb{F}}$ determines $\phi : \epsilon_{\sigma_i} M_{\mathbb{F}} \to \epsilon_{\sigma_{i+1}} M_{\mathbb{F}}$. For $(A_i)_{1 \le i \le n} \in GL_2\big(\mathbb{F}((u))\big)^n$, we write

$$M_{\mathbb{F}} \sim (A_1, A_2, \ldots, A_n) = (A_i)_i$$

if there is a basis $\{e_1^i, e_2^i\}$ of $\epsilon_{\sigma_i} M_{\mathbb{F}}$ over $\mathbb{F}((u))$ such that $\phi \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} = A_i \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix}$. We use the same notation for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ similarly. Here and in the following, we consider only sublattices that are $(\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathbb{F})$-modules.

Let A be an $\mathbb{F}$-algebra, and $\mathfrak{M}_A$ be a finite free $(k[[u]] \otimes_{\mathbb{F}_p} A)$-submodule of $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ that generates $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ over $k((u)) \otimes_{\mathbb{F}_p} A$. We choose a basis $\{e_1^i, e_2^i\}_i$ of $\mathfrak{M}_A$ over $k[[u]] \otimes_{\mathbb{F}_p} A$. For $B = (B_i)_{1 \le i \le n} \in GL_2\big(\mathbb{F}((u)) \otimes_{\mathbb{F}_p} A\big)^n$, the $(\mathfrak{S} \otimes_{\mathbb{Z}_p} A)$-module generated by the entries of $\left\langle B_i \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} \right\rangle$ for $1 \le i \le n$ with the basis given by these entries is denoted by $B \cdot \mathfrak{M}_A$. Note that $B \cdot \mathfrak{M}_A$ depends on the choice of the basis of $\mathfrak{M}_A$. We can see that if $\mathfrak{M}_{\mathbb{F}} \sim (A_i)_i$ for $(A_i)_{1 \le i \le n} \in GL_2\big(\mathbb{F}((u))\big)^n$ with respect to a given basis, then we have

$$B \cdot \mathfrak{M}_{\mathbb{F}} \sim \big(\phi(B_i) A_i (B_{i+1})^{-1}\big)_i$$

with respect to the induced basis.

Lemma 3.4. *Suppose $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, and $x \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ corresponds to $\mathfrak{M}_{\mathbb{F}'}$. Put $\mathfrak{M}_{j,\mathbb{F}'} = \left(\begin{pmatrix} u^{s_{j,i}} & v_{j,i} \\ 0 & u^{t_{j,i}} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{\mathbb{F}'}$*

for $1 \leq j \leq 2$, $s_{j,i}, t_{j,i} \in \mathbb{Z}$ *and* $v_{j,i} \in \mathbb{F}'((u))$. *Assume* $\mathfrak{M}_{1,\mathbb{F}'}$ *and* $\mathfrak{M}_{2,\mathbb{F}'}$ *correspond to* $x_1, x_2 \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ *respectively. Then* $x_1 = x_2$ *if and only if*

$$s_{1,i} = s_{2,i},\ t_{1,i} = t_{2,i} \text{ and } v_{1,i} - v_{2,i} \in u^{t_{1,i}}\mathbb{F}'[[u]] \text{ for all } i.$$

Proof. The equality $x_1 = x_2$ is equivalent to the existence of $B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}'[[u]])^n$ such that

$$B_i \begin{pmatrix} u^{s_{1,i}} & v_{1,i} \\ 0 & u^{t_{1,i}} \end{pmatrix} = \begin{pmatrix} u^{s_{2,i}} & v_{2,i} \\ 0 & u^{t_{2,i}} \end{pmatrix}$$

for all i. It is further equivalent to the condition that

$$\begin{pmatrix} u^{s_{2,i}-s_{1,i}} & v_{2,i}u^{-t_{1,i}} - u^{s_{2,i}-s_{1,i}-t_{1,i}}v_{1,i} \\ 0 & u^{t_{2,i}-t_{1,i}} \end{pmatrix} \in GL_2(\mathbb{F}'[[u]])$$

for all i. The last condition is equivalent to the desired condition. Q.E.D.

The claim on existence of a zero-dimensional moduli space follows from the following example.

Example 3.5 ([Im2, Proposition 1.3]). *If* $M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^e & u \\ 0 & 1 \end{pmatrix} \right)_i$, *then* $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ *is one point for any finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$.

We explain the outline of a proof of Theorem 3.2 only in the case where $V_{\mathbb{F}}$ is not absolutely irreducible. A proof in the case where $V_{\mathbb{F}}$ is absolutely irreducible is similar, but more complicated.

Extending the field $\mathbb{F}$, we may assume that $V_{\mathbb{F}}$ is reducible. Let $\mathfrak{M}_{0,\mathbb{F}}$ be a lattice of $M_{\mathbb{F}}$ corresponding to a point of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F})$. Then we take and fix a basis of $\mathfrak{M}_{0,\mathbb{F}}$ over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ such that $\mathfrak{M}_{0,\mathbb{F}} \sim \left(\begin{pmatrix} \alpha_i u^{a_{0,i}} & w_{0,i} \\ 0 & \beta_i u^{b_{0,i}} \end{pmatrix} \right)_i$ for $\alpha_i, \beta_i \in \mathbb{F}^\times$, $0 \leq a_{0,i}, b_{0,i} \leq e$ and $w_{0,i} \in \mathbb{F}[[u]]$. For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, we put $\mathfrak{M}_{0,\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $M_{\mathbb{F}'} = M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. By the Iwasawa decomposition, any sublattice of $M_{\mathbb{F}'}$ can be written as $\left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'}$ for $s_i, t_i \in \mathbb{Z}$ and $v_i' \in \mathbb{F}'((u))$.

We put

$$I = \left\{ (\underline{a}, \underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n \;\middle|\; \underline{a} = (a_i)_{1 \leq i \leq n},\ \underline{b} = (b_i)_{1 \leq i \leq n},\ 0 \leq a_i, b_i \leq e \right\},$$

and

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \left\{ \left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \;\middle|\; s_i, t_i \in \mathbb{Z}, v_i' \in \mathbb{F}'((u)),\ a_i = a_{0,i} + ps_i - s_{i+1},\ b_i = b_{0,i} + pt_i - t_{i+1} \right\}$$

for $(\underline{a},\underline{b}) = \big((a_i)_{1\le i\le n}, (b_i)_{1\le i\le n}\big) \in I$. Then we have

$$\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') = \bigcup_{(\underline{a},\underline{b})\in I} \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}'),$$

and this is a disjoint union by Lemma 3.4.

Take $\mathfrak{M}_{\mathbb{F}'} = \left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ with the basis induced from the basis of $\mathfrak{M}_{0,\mathbb{F}'}$, then $\mathfrak{M}_{\mathbb{F}'} \sim \left(\begin{pmatrix} \alpha_i u^{a_i} & w_i \\ 0 & \beta_i u^{b_i} \end{pmatrix} \right)_i$ for some $(w_i)_{1\le i\le n} \in \mathbb{F}'[[u]]^n$. We note that $a_i + b_i - v_u(w_i) \le e$ for all i by the condition $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big)$.

Now, any $\mathfrak{M}'_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ can be written as $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix} \right)_i \cdot \mathfrak{M}_{\mathbb{F}'}$ for some $(v_i)_{1\le i\le n} \in \mathbb{F}'((u))^n$. With the basis induced from $\mathfrak{M}_{\mathbb{F}'}$, we have

$$\begin{aligned} \mathfrak{M}'_{\mathbb{F}'} &\sim \left(\begin{pmatrix} 1 & \phi(v_i) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_i u^{a_i} & w_i \\ 0 & \beta_i u^{b_i} \end{pmatrix} \begin{pmatrix} 1 & -v_{i+1} \\ 0 & 1 \end{pmatrix} \right)_i \\ &= \left(\begin{pmatrix} \alpha_i u^{a_i} & w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i}\phi(v_i) \\ 0 & \beta_i u^{b_i} \end{pmatrix} \right)_i . \end{aligned}$$

We are going to examine the condition for $(v_i)_{1\le i\le n} \in \mathbb{F}'((u))^n$ to give a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ as $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix} \right)_i \cdot \mathfrak{M}_{\mathbb{F}'}$. Extending the field $\mathbb{F}$, we may assume that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}) = \emptyset$ if and only if $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \emptyset$ for each $(\underline{a},\underline{b}) \in I$ and any finite extension $\mathbb{F}'$ of $\mathbb{F}$.

For $(v_i)_{1\le i\le n} \in \mathbb{F}'((u))^n$, we have $\mathfrak{M}'_{\mathbb{F}'} = \left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ if and only if

$$v_u\big(w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i}\phi(v_i)\big) \ge 0 \text{ and}$$
$$v_u(\alpha_i u^{a_i}) + v_u(\beta_i u^{b_i}) - v_u\big(w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i}\phi(v_i)\big) \le e \text{ for all } i,$$

by the condition $u^e\mathfrak{M}'_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}'_{\mathbb{F}'})\big) \subset \mathfrak{M}'_{\mathbb{F}'}$. This is further equivalent to

$$v_u\big(\alpha_i u^{a_i} v_{i+1} - \beta_i u^{b_i}\phi(v_i)\big) \ge \max\{0, a_i + b_i - e\},$$

because $v_u(w_i) \ge \max\{0, a_i + b_i - e\}$.

We define an $\mathbb{F}'$-vector space $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'}$ by

$$\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'} = \big\{(v_1,\dots,v_n) \in \mathbb{F}'((u))^n \;\big|\; v_u\big(\alpha_i u^{a_i} v_{i+1} - \beta_i u^{b_i}\phi(v_i)\big) \ge \max\{0, a_i + b_i - e\} \text{ for all } i\big\}.$$

We note that $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'} \supset \mathbb{F}'[[u]]^n$, and put $N_{\underline{a},\underline{b},\mathbb{F}'} = \widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'}/\mathbb{F}'[[u]]^n$. Then we have a bijection $N_{\underline{a},\underline{b},\mathbb{F}'} \to \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ by Lemma 3.4. We put $d_{\underline{a},\underline{b}} = \dim_{\mathbb{F}'} N_{\underline{a},\underline{b},\mathbb{F}'}$, and note that $\dim_{\mathbb{F}'} N_{\underline{a},\underline{b},\mathbb{F}'}$ is independent of finite extensions $\mathbb{F}'$ of $\mathbb{F}$.

We take a basis $(\mathbf{v}_j)_{1\le j\le d_{\underline{a},\underline{b}}}$ of $N_{\underline{a},\underline{b},\mathbb{F}}$ over $\mathbb{F}$, where

$$\mathbf{v}_j = (v_{j,1},\dots,v_{j,n}) \in \mathbb{F}((u))^n.$$

Then an $(\mathbb{F}[[u]] \otimes_{\mathbb{F}} \mathbb{F}[X_1,\dots,X_{d_{\underline{a},\underline{b}}}])$-module

$$\mathfrak{M}'_{\mathbb{F}[X_1,\dots,X_{d_{\underline{a},\underline{b}}}]} = \left(\begin{pmatrix} 1 & \sum_j v_{j,i}X_j \\ 0 & 1 \end{pmatrix}\right)_i \cdot (\mathfrak{M}_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}[X_1,\dots,X_{d_{\underline{a},\underline{b}}}])$$

gives a morphism $f_{\underline{a},\underline{b}} : \mathbb{A}_{\mathbb{F}}^{d_{\underline{a},\underline{b}}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$ such that $f_{\underline{a},\underline{b}}(\mathbb{F}')$ is injective and the image of $f_{\underline{a},\underline{b}}(\mathbb{F}')$ is $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$. Then we have the claim on the zeta functions and

$$d_{V_{\mathbb{F}}} = \max_{(\underline{a},\underline{b})\in I,\ \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F})\neq\emptyset} \{d_{\underline{a},\underline{b}}\}.$$

We can express $d_{\underline{a},\underline{b}}$ explicitly and bound it by combinatorial arguments.

We can also give an explicit ϕ-module $M_{\mathbb{F}}$ such that $\dim \mathscr{GR}_{V_{\mathbb{F}},0}$ coincide with the upper bound. See the proof of [Im2, Proposition 2.1] for more details.

Remark 3.6. In the case where $V_{\mathbb{F}}$ is not absolutely irreducible, we have decomposed $\mathscr{GR}_{V_{\mathbb{F}},0}$ into affine spaces in the level of rational points after extending the field $\mathbb{F}$. However, in the case where $V_{\mathbb{F}}$ is absolutely irreducible, we have to decompose $\mathscr{GR}_{V_{\mathbb{F}},0}$ into $\mathbb{A}^d_{\mathbb{F}}$, $\mathbb{A}^{d-1}_{\mathbb{F}} \times \mathbb{G}_{m,\mathbb{F}}$ and $\mathbb{A}^{d-2}_{\mathbb{F}} \times \mathbb{G}^2_{m,\mathbb{F}}$ in the level of rational points after extending the field $\mathbb{F}$.

§4. Variant and conjecture

In this section, we explain a variant of the moduli space of finite flat models after Caruso. We also explain his result and conjecture in some case.

We consider a ring homomorphism

$$\phi_{\sigma,b} : k((u)) \to k((u)); \ \sum_i a_i u^i \mapsto \sum_i \sigma(a_i) u^{bi}$$

for $\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)$ and $b \in \mathbb{Z}_{\geq 2}$.

Remark 4.1. If we change k to $\mathbb{F}_p$ and $\mathbb{F}$ to k in the notation in the section 1, then ϕ is equal to $\phi_{\sigma,p}$ for the p-th power map σ.

Let M be a finite $k((u))$-module equipped with $\phi_{\sigma,b}$-semi-linear map $\phi_M : M \to M$ such that the induced $k((u))$-linear map

$$k((u)) \otimes_{\phi,k((u))} M \to M; \ s \otimes m \mapsto s\phi_M(m)$$

is an isomorphism. For any positive integer e, we define a functor

$$F_{\leq e} : \{k\text{-algebra}\} \to \mathbf{Set}$$

by putting $F_{\leq e}(A)$ to be the set of finite projective $(k[[u]] \otimes_k A)$-submodules $\mathfrak{M}_A$ of $M \otimes_k A$ satisfying

(1) $\mathfrak{M}_A$ generates $M \otimes_k A$ over $k((u)) \otimes_k A$
(2) $\mathfrak{M}_A$ is stable by $\phi_M \otimes \mathrm{id}_A$ and the cokernel of the induced $(k[[u]] \otimes_k A)$-linear map

$$k[[u]] \otimes_{\phi_{\sigma,b},k((u))} \mathfrak{M}_A \to \mathfrak{M}_A; \ a \otimes m \mapsto a(\phi_M \otimes \mathrm{id}_A)(m)$$

is killed by u^e

for a k-algebra A. Then the functor $F_{\leq e}$ is represented by a projective scheme $\mathcal{X}_{\leq e}(\phi_M)$ over k.

Theorem 4.2 ([Ca, Théorème 2]). *We assume that M has a basis $\{e_i\}_{1\leq i\leq d}$ over $k((u))$ such that $\phi_M(e_i) = e_i$ for $1 \leq i \leq d$. If $\sigma \neq \mathrm{id}_k$, we have*

$$\left[\frac{d^2}{4}\right] \cdot \left[\frac{e-b+2}{b+1}\right] \leq \dim \mathcal{X}_{\leq e}(\phi_M) \leq \left[\frac{d^2}{4}\right] \cdot \frac{e}{b+1}.$$

If $\sigma = \mathrm{id}_k$, *we have*

$$\left[\frac{d^2}{4}\right] \cdot \left[\frac{e-b+2}{b+1}\right] \leq \dim \mathcal{X}_{\leq e}(\phi_M) \leq \frac{d(d-1)}{2} + \left[\frac{d^2}{4}\right] \cdot \frac{e}{b+1}.$$

In fact, Caruso considered also a more general moduli space and gave some bound of the dimension of the moduli space (cf. [Ca, Théorème 4 and 5]).

Conjecture 4.3 ([Ca, Conjecture 4.5]). *We fix* M *as a* $k((u))$*-module. There are constant real numbers* b_0, c_1 *and* c_2 *satisfies the following:*

If $b \geq b_0$, *then we have*

$$\dim \mathcal{X}_{\leq e}(\phi_M) \leq c_1 + \left[\frac{d^2}{4}\right] \cdot \frac{e}{b+1} \tag{1}$$

for all positive integer e *and*

$$\dim \mathcal{X}_{\leq e}(\phi_M) \geq -c_2 + \left[\frac{d^2}{4}\right] \cdot \frac{e}{b+1} \tag{2}$$

for all sufficiently large integer e.

Remark 4.4. The upper bound in Theorem 3.2 is compatible with (1) in Conjecture 4.3. Furthermore, there is no contradiction between the strict lower bound in Theorem 3.2 and (2) in Conjecture 4.3, because we fix e and change $M_{\mathbb{F}}$ in Theorem 3.2 and, on the other hand, fix M and change ϕ_M and e in Conjecture 4.3.

References

[Br1] C. Breuil, Une application du corps des normes, Compositio Math., **117** (1999), 189–203.

[Br2] C. Breuil, Groupes p-divisibles, groupes finis et modules filtrés, Ann. of Math. (2), **152** (2000), 489–549.

[Ca] X. Caruso, Estimation des dimensions de certaines variétés de Kisin, preprint, arXiv:1005.2394.

[Ge] T. Gee, A modularity lifting theorem for weight two Hilbert modular forms, Math. Res. Lett., **13** (2006), 805–811.

[He1] E. Hellmann, On the structure of some moduli spaces of finite flat group schemes, Mosc. Math. J., **9** (2009), 531–561.

[He2] E. Hellmann, Connectedness of Kisin varieties for GL_2, Adv. Math., **228** (2011), 219–240.

[Im1] N. Imai, On the connected components of moduli spaces of finite flat models, Amer. J. Math., **132** (2010), 1189–1204.

[Im2] N. Imai, Ramification and moduli spaces of finite flat models, Ann. Inst. Fourier, **61** (2011), 1943–1975.

[Ki] M. Kisin, Moduli of finite flat group schemes, and modularity, Ann. of Math. (2), **170** (2009), 1085–1180.

[Ra] M. Raynaud, Schémas en groupes de type $(p, \dots, p)$, Bull. Soc. Math. France, **102** (1974), 241–280.

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