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Resonance varieties of arrangement complements

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§1. Introduction

This paper is a survey based on the lecture given by the author in the conference "Arrangements of Hyperplanes -Sapporo, 2009" at the University of Hokkaido. The main topic of the survey is the resonance varieties of the complement of an arrangement of several linear complex hyperplanes. These varieties can be defined for an arbitrary topological space M as the jumping loci for the kind of 'secondary cohomology' of M. More precisely this is the cohomology of the graded commutative algebra $H^*(M)$ provided with the differential given by the multiplication by an element of $H^1(M)$. This cohomology has appeared first in topology as the first sheet of the Farber–Novikov spectral system (see, for example [25]) which converges to cohomology with local coefficients for compact manifolds. For arrangement complements first results for this cohomology were vanishing theorems from [30] and [35] and comparison theorems [14]. The jumping loci for this cohomology were first considered explicitly in [15] and called resonance varieties. In [1], this cohomology was considered for an arbitrary graded commutative algebra as the measure of its complexity. For initial results about resonance varieties over arbitrary fields see [16].

At the beginning, the resonance varieties were mainly considered due to their connections with the jumping loci for the cohomology with local coefficients; the most recent results about these connections can be found in [10, 11]. Now resonance varieties appear in many areas of arrangement theory. The most recent appearance is in [5, 6] where these varieties have been used for results on the Milnor fiber cohomology and roots of b-functions. There are also several recent papers (see, for example

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[12, 33]) where some properties of resonance varieties for arrangement complements were analyzed from the point of view which properties of the spaces imply them. This has allowed the authors to generalize them to other topological spaces. See also the survey by A. Suciu in the Proceedings.

The outline of the paper is as follows. In Section 2, we give the main definitions and list the properties of the resonance varieties with just hints of proofs. Then in Section 3, we discuss the structure of the first resonance variety, that is the jumping locus for the first cohomology of the algebra $H^*(M)$. The components of this variety can be characterized equivalently in terms of classical projective geometry and in terms of pencils of plane curves. This theorem has been proved in [17] but here we give a proof that fills a small gap and uses some new ideas and shortcuts. In Section 3, we survey the results on the dimension of the first resonance variety. In Section 4, we list several open questions.

§2. Resonance varieties and their properties

Definition 2.1. *Let M be a topological space and $A = H^*(M)$ its graded cohomology algebra (the coefficients can be apriori from an arbitrary field but for later use we assume they are in $\mathbb{C}$). We write $A^p = H^p(M)$. For every $x \in A^1$ we have $x^2 = 0$ since A is graded-commutative. Thus the multiplication by x defines the differential $A \to A$ of degree +1, i.e., converts A into a cochain complex that we denote by (A, x).*

The main object of this paper is the *cohomology* $H^p(A, x)$ as a function of x.

For the next definition we assume that the linear spaces A^p are finite dimensional.

Definition 2.2. *The p-th resonance variety $R^p = R^p(M)$ is the (determinantal) subvariety of A^1 defined as $R^p = \{x \in A^1 | H^p(A, x) \neq 0\}$.*

The rest of the paper will be devoted to the case where M is the complement in a finite dimensional linear space V ($V \approx \mathbb{C}^\ell$) to an arrangement $\mathcal{A}$ of several (linear) hyperplanes. We will always assume that $\mathcal{A}$ is *essential*, i.e., $\bigcap_{H\in\mathcal{A}} H = 0$. If we consider several arrangements at the same time we will use the symbol $A(\mathcal{A})$ for A.

The cohomology of such an M is determined by theorems of Arnold, Brieskorn and Orlik–Solomon, [3, 4, 26]. For each hyperplane $H \in \mathcal{A}$ fix a linear form α_H with $\ker \alpha_H = H$. Then A can be identified with the subalgbera of the algebra of all the (holomorphic) differential forms on M generated by the logarithmic forms $\frac{d\alpha_H}{\alpha_H}$ $(H \in \mathcal{A})$. The classes e_H of these forms form a canonical basis of A^1 whence for every $x \in A^1$ we have $x = \sum_{H \in \mathcal{A}} x_H e_H$ for some $x_H \in \mathbb{C}$. Relations for the generators are well-known and can be found in [27].

Properties of R^p.

There are several properties of varieties R^p for arrangement complements that hold for all p, $0 \leq p \leq \ell$.

(i) (linearity of components) The irreducible components of R^p are linear subspaces of A^1.

(ii) (sufficient condition for vanishing) If $\sum_{H \in \mathcal{A}} x_H \neq 0$ then $R^p = 0$ for all p (we will write $R^* = 0$).

Remark 2.3. *If $\mathcal{A}_i$ $(i = 1, 2)$ are arrangements in linear spaces V_i then their product is the arrangement $\mathcal{A}_1 \times \mathcal{A}_2$ in $V_1 \oplus V_2$ of the hyperplanes $H \oplus V_2$ $(H \in \mathcal{A}_1)$ and $V_1 \oplus H$ $(H \in \mathcal{A}_2)$, see* [27]. *One easily checks that $A(\mathcal{A}_1 \times \mathcal{A}_2) = A(\mathcal{A}_1) \otimes_{\mathbb{C}} A(\mathcal{A}_2)$. An non-empty arrangement $\mathcal{A}$ is irreducible if it is not isomorphic to the product of non-empty arrangements. Every non-empty arrangement $\mathcal{A}$ is the product of several uniquely defined irreducible arrangements, irreducible components of $\mathcal{A}$.*

(iii) (equivalent condition for vanishing) $R^* \neq 0$ if and only if $\sum_{H \in \mathcal{A}_j} x_H = 0$ for every $j = 1, \dots, r$ where $\mathcal{A}_j$ is an irreducible component of $\mathcal{A}$.

Remark 2.4. *The subset $sing(\mathcal{A}) = sing(A)$ of A^1 consisting of all x satisfying $H^*(A, x) \neq 0$ is called the singular variety of A. It is usually viewed as a module over the exterior algebra E generated in degree 1 by $\{e_H | H \in \mathcal{A}\}$. See* [1] *for another approach to it and its applications.*

(iv)(propagation of cohomology) If $H^p(A, x) \neq 0$ for some p then $H^q(A, x) \neq 0$ (i.e., $R^p \subset R^q$) for every q, $p \leq q \leq \ell$ (i.e., $R^* \neq 0$ is equivalent to $R^\ell \neq 0$).

The known proofs of the properties are of different difficulties. The easiest property is (ii). It is a consequence of the existence of a differential $\partial : A \to A$ of degree -1 that satisfies the signed Leibniz formula and is normed by the condition $\partial(e_H) = 1$ for every $H \in \mathcal{A}$. Then $\frac{1}{\sum_H x_H}\partial$ is a contracting homotopy for (A, x).

If $\mathcal{A}$ is not a product of non-empty arrangements than the converse of (ii) follows from the non-vanishing of the Euler characteristic of $\ker \partial$

(cf. [9]). This and the fact $sing(\mathcal{A}_1 \times \mathcal{A}_2) = sing(\mathcal{A}_1) \times sing(\mathcal{A}_2)$ implies (iii).

Property (i) has been proved for an arbitrary p in [7] and [22]. The idea is to view the cohomology of (A, x) as a linearization of cohomology of another complex.

Finally, property (iv) has been proved in [13] as a corollary of the linearity of the minimal injective resolution for A viewed as an E-module.

A direct linear algebra proof of the propagation for $p = 1$ has been given by J. V. Pereira and goes as follows.

Proof. Assume that $a \in A^1$ satisfies $H^1(A, a) \neq 0$ and $H^p(A, a) = 0$, for some $1 < p \leq \ell$. Let $b \in A^1 \setminus \mathbb{C}a$ with $ab = 0$. Since a and b can be identified with some rational one-forms on $\mathbb{C}^\ell$, there exists a rational function h such that $b = ha$. This immediately implies that the cocycle spaces for a and b in all degrees r coincide: $Z_a^r = Z_b^r$. In particular, this is true for $r = p - 1$ and $r = p$. Since by assumption $B_a^p = Z_a^p$, where B_a^p is the space of coboundaries of degree p for a, we obtain that $\dim B_b^p = \dim B_a^p = \dim Z_a^p = \dim Z_b^p$ whence $B_b^p = B_a^p$.

Now consider the isomorphisms $\phi_a, \phi_b \colon C = A^{p-1}/Z_a^{p-1} \to B_a^p$, given by multiplication by a and b, respectively, and the automorphism $\phi = \phi_a^{-1}\phi_b$ of C. If $c \in A^{p-1}$ is such that its projection to C is an eigenvector of ϕ, then we have $bc = \lambda ac$ for some $\lambda \in \mathbb{C}^*$. But since also $bc = hac$, we have $h = \lambda$, which contradicts the choice of b. Q.E.D.

No proof of this kind is known for arbitrary p.

§3. The first resonance variety

For the first resonance variety R^1 the situation can be simplified.

First, we projectivize the linear space and study an arrangement of projective hyperplanes in the complex projective space. The cohomology algebra of the projectivized complement (that we still denote by A in this section) is the graded subalgebra of the cohomology algebra of M generated by $\bar{e}_{ij} = e_i - e_j$ for $i \neq j$. For these generators the basic relations are $\prod_{j=2,3,\ldots,k} \bar{e}_{i_1 i_j}$ where $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}$ runs through all minimal linearly dependent sets. Second, it suffices to consider arrangements of lines in $\mathbb{P}^2$. Indeed intersect arbitrary $\mathcal{A}$ with a generic plane and either apply the weak Lefschetz theorem or the description of A in terms of the intersection lattice of $\mathcal{A}$. An arrangement of projective lines will be denoted by L.

3.1. Local components

Here we describe the class of the simplest components of R^1 (in some sense trivial).

Suppose $P \in \mathbb{P}^2$ is a point where k lines of L intersect, $k \geq 3$. Define the linear subspace

$$V_P = \{x = \sum_{P \in \ell} x_\ell e_\ell | \sum x_\ell = 0\}$$

of A^1. Using the relations above it is easy to show that $xy = 0$ for every $x, y \in V_P$ and for every $x \in V_P$ the equality $xy = 0$ implies $y \in V_P$. Thus V_P is a component (of dimension $k-1$) of R^1. This component is called a *local component of* R^1.

Notice that there is no uniform upper bound for the dimension of local components of arrangements. As we will see later, this makes local components different from all others.

3.2. Nets in $\mathbb{P}^2$

Now we want to study the non-local components. Our goal is to state the theorem that gives at least two different characterizations of them. For that we need to define the terms to be used.

First we discuss some special configurations of lines and points in $\mathbb{P}^2$.

Definition 3.1. *A finite set L of lines partitioned in k ($k > 2$) blocks $L = \cup_{j=1}^k L_j$ is a k-net if for every point P which is the intersection of lines from different blocks there is a precisely one line from each block passing through P.*

The following numerical equalities for a k-net are almost obvious. In order to state them we denote by $\mathcal{X}$ the set of all points of intersection of lines from different blocks.

(i) For arbitrary $1 \leq i, j \leq k$ and $\ell \in L$ we have $|L_i| = |L_j| = |\mathcal{X} \cap \ell|$. The latter integer is denoted by d and the net is called (k, d)*-net.*
(ii) $|L| = kd$.
(iii) $|\mathcal{X}| = d^2$.

Nets can be defined purely combinatorially using an incidence relation. Then after identifying two blocks of a (k, d)-net with each other, every other block gives a Latin square of size d and these $k-2$ squares are pairwise orthogonal. If $k = 3$ identifying all blocks gives a multiplication table of a quasi-group.

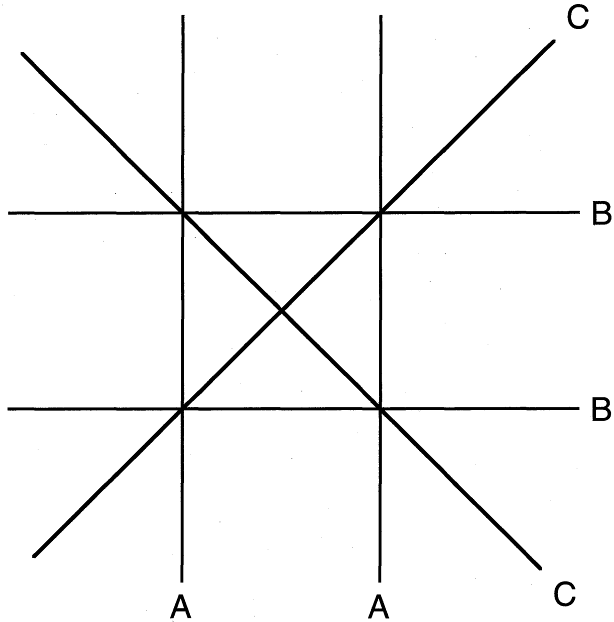

Fig. 1. The (3,2) net

A $(k,1)$-net consists of k lines passing through a point with each block consisting of one line. Clearly maximal $(k,1)$ nets in an arrangements correspond to local components of R^1. Thus from now on we will assume that $d > 1$.

The combinatorial nets that can be realized in $\mathbb{P}^2$ form a very restricted class (e.g., see the restrictions on k below). However there are plenty of examples of 3-nets. The simplest nontrivial one is given by all the reflection lines of the Coxeter group of type A_3. In appropriate coordinates the blocks can be described by the equations

$$(x^2 - y^2) = (x^2 - z^2) = (y^2 - z^2) = 0,$$

see Fig. 1 for the picture of it in the affine plane $z = 1$. This is the only example of a $(3,2)$-net up to a projective isomorphism. As a classical example of a $(3,3)$-net one can use the generic Pappus configuration taking for $\mathcal{X}$ all the triple points.

The first attempt to study 3-nets in $\mathbb{P}^2$ was taken in [36]. In particular, a general way to construct examples of $(3,d)$-nets for every d was found there.

If H is a group of order d then we say that a $(3,d)$-net *realizes* H if there is a way to identify all the blocks of the net to obtain the multiplication table of H.

Theorem 3.2. *Let H be a finite subgroup of a two dimensional torus. Then there exists a 3-net in $\mathbb{P}^2$ realizing H.*

Proof. Fix a non-singular plane cubic curve C and define the additive group operation on it (converting it to the two dimensional torus) by choosing one of the flexes of C as the neutral element. Then three points p_1, p_2, and p_3 from C are collinear if and only if $p_1 + p_2 + p_3 = 0$. We can identify H with a subset of C and put $d = |H|$. Now choose $\alpha, \beta \in C/H$ such that α, β, and $-\alpha - \beta$ are distinct. The union of these three cosets is a set of $3d$ points partitioned in three blocks. It is clear that the lines in the projective plane dual to these points form a 3-net and this net realizes H. Q.E.D.

The construction used in that proof can be extended to reducible cubics. We say that a 3-net is *algebraic* if there exists a cubic curve $C \subset (P^2)^*$ containing the points dual to the lines of the net in the set $\mathbb{C}^0$ of its regular points. Then the following partial converse of the previous theorem has been proved in [36].

Theorem 3.3. *Let H be a finite Abelian group that is either cyclic or it has at least one element of order greater than 9. Then every realization of H by a 3-net in $\mathbb{P}^2$ is algebraic.*

Remark 3.4. *It was also proved in* [36] *that under the extra condition that all the lines in one class of the net are concurrent then* 6 *instead of* 9 *suffices for the conclusion of Theorem. In the case where each of three classes consists of concurrent lines,* 2 *can be substituted for* 9. *In fact, the conclusion of the theorem holds for $\mathbb{Z}_2$ and is false for $\mathbb{Z}_2^k$ for $k > 2$. The group $\mathbb{Z}_2^2$ is an unknown case.*

Further examples of nets and attempts to classify some of them can be found in [31, 34]. In particular, J. Stipins found in [31] example of a $(3,5)$-net that does not represent a group whence is not algebraic.

3.3. Multinets in $\mathbb{P}^2$.

The example of the B_3-arrangement (i.e., the arrangement of all reflection hyperplanes of the Coxeter group of type B_3) shows that there are arrangements that do not support nets but would support very similar configurations if provided with some multiplicities.

Definition 3.5. *(1) A multi-arrangement of lines in $\mathbb{P}^2$ is an arrangement L of lines together with a multiplicity function $m : L \to \mathbb{Z}_{>o}$. For every $\ell \in L$ the integer $m(\ell)$ is called the multiplicity of ℓ.*

(2) Let k be an integer, $k \geq 3$. A k-multinet is a multi-arrangement (L, m) with L partitioned into k blocks $L_1, \dots, L_k$ subject to the following conditions:

(i) As for the nets, let $\mathcal{X}$ be the set of the intersections of lines from different blocks; then for each $P \in \mathcal{X}$ the number $n(P) = \sum_{\ell \in L_i, P \in \ell} m(\ell)$ is independent of i;

This number is called the multiplicity of P;

(ii) For every $1 \leq i \leq k$ and $\ell, \ell' \in L_i$ there exists a sequence $\ell_0 = \ell, \ell_1, \dots, \ell_r = \ell'$ such that $\ell_{j-1} \cap \ell_j \notin \mathcal{X}$, for all $j = 1, 2, \dots, r$.

Remark 3.6. *(1) The axiom (i) is a direct generalization of the definition of nets.*

(2) The more technical axiom (ii) guarantees that the partition is the finest for given (L, m) and $\mathcal{X}$. Due to this axiom, the blocks L_i are the equivalent classes of the transitive closure of the relation $\ell \sim \ell'$ if $\ell \cap \ell' \notin \mathcal{X}$.

(3) If $n(P) = 1$ for all $P \in \mathcal{X}$ then clearly $m(\ell) = 1$ for all $\ell \in L$ and we have a k-net. The converse is false—there are multinets with $m(\ell) = 1$ for all lines ℓ but $n(P) \neq 1$ for some P (see Fig. 3).

Multinets satisfy the numerical equalities which generalize those for nets.

(1) $\sum_{\ell \in L_i} m(\ell)$ does not depend on i;

This integer is denoted by d and the multinet is called (k, d)-multinet.

(2) $\sum_{\ell \in L} m(\ell) = dk$;

(3) $\sum_{P \in \mathcal{X} \cap \ell} n(P) = d$ for every $\ell \in L$;

(4) $\sum_{P \in \mathcal{X}} n(P)^2 = d^2$.

Also dividing all $m(\ell)$ by their common divisor does not spoil the properties. Thus we can and always will assume that the numbers $\{m(\ell) | \ell \in L\}$ are mutually relatively prime.

As stated above, the simplest and motivating example of a multinet that is not a net is supported by the Coxeter arrangement of the type B_3, see Fig. 2. It can be given (up to projective isomorphism) by the

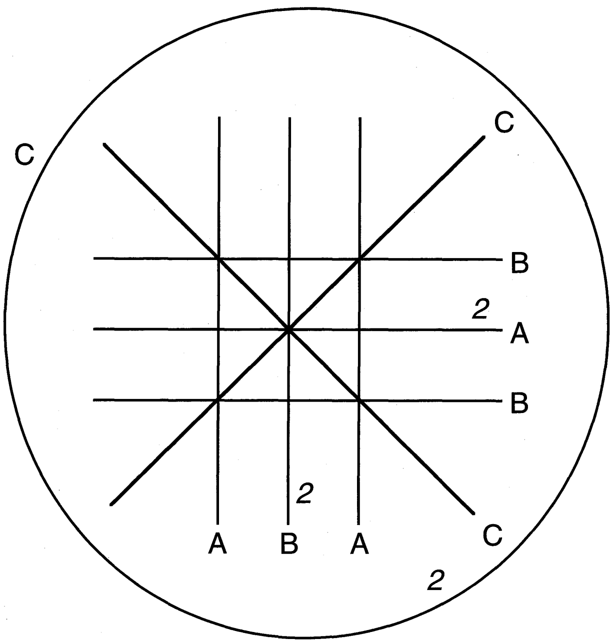

Fig. 2. A (3,4) multinet

equation

$$[x^2(y^2-z^2)][y^2(x^2-y^2)][z^2(x^2-y^2)]=0$$

where the exponents signify that the coordinate axes are taken each with multiplicity 2. The brackets show the partition into 3 classes. It is a $(3,4)$-multinet with 9 lines (12 counting with multiplicities) and 7 points (16 counting with squares of multiplicities).

3.4. Pencils of plane algebraic curves

Here we briefly discuss the second ingredient for the characterization of R^1.

We will identify homogeneous polynomials in three variables that differ by a non-zero constant multiplier. Thus we do not make a distinction between a homogeneous polynomial and the projective plane curve (perhaps non-reduced) defined by it, and treat it either as a polynomial or as a curve. A *pencil* of plane curves is a line in the projective space of homogeneous polynomials of some fixed degree. Thus any two distinct curves of the same degree generate a pencil, and conversely a pencil is determined by any two of its curves C_1, C_2. An arbitrary curve in the

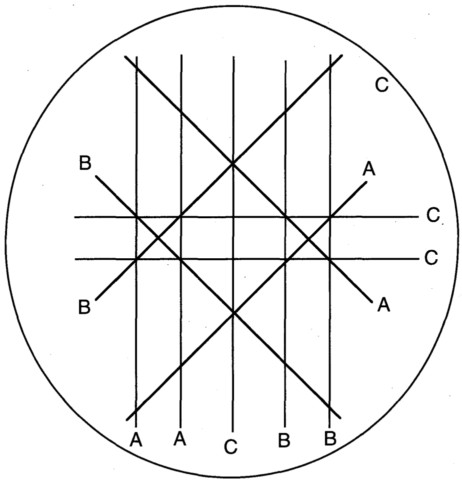

Fig. 3. A multinet with $m(\ell) = 1$ for all $\ell \in L$, which is not a net

pencil (called a *fiber*) is then $aC_1 + bC_2$, $[a : b] \in \mathbb{P}^1$. A pencil π with generators C_1 and C_2 can be equivalently defined as a rational morphism $\mathbb{P}^2 \to \mathbb{P}^1$ via $x \mapsto [C_2(x) : -C_1(x)]$. We denote this morphism also by π.

Every two fibers in a pencil intersect in the same set of points $\mathcal{X} = C_1 \cap C_2$, called the *base* of the pencil. If fibers do not have a common component then the base is a finite set of points. The base $\mathcal{X}$ coincides with the indeterminacy locus of the rational morphism π.

We call a pencil *connected* if the proper transform of each fiber after the blowup at all the points of the base stays connected. Notice that if generic fibers of a pencil π are irreducible then π is connected. A curve of the form $\prod_{i=1}^{q} \alpha_i^{m_i}$, where α_i are different linear forms and m_i are positive integer for $1 \leq i \leq q$ will be called *completely reducible*. We are interested in connected pencils with at least three completely reducible fibers. Following [17], we say that such a pencil is of *Ceva type*.

3.5. Characterizations of R^1

Now we can give a characterization of the resonance variety R^1 by characterizing its components. Let $\mathcal{V}$ be a component of R^1. The *support U of $\mathcal{V}$* is the smallest subarrangement $U = supp\mathcal{V}$ of L such that for every $a = \sum_{\ell \in L} a_\ell e_\ell \in \mathcal{V}$ if $a_\ell \neq 0$ then $\ell \in U$. If $supp\mathcal{V}$ is a set of several concurrent lines from L then $\mathcal{V}$ is local. For every component $\mathcal{V}$

we can restrict our consideration to $supp\mathcal{V}$ whence we can always assume that $\mathcal{V}$ is supported on the whole L.

Theorem 3.7. *Let (L, m) be a multi-arrangement of lines in $\mathbb{P}^2$ not passing all through a point. The following conditions are equivalent:*

(i) There exists a partition $\pi = \{L_1, L_2, \dots, L_k\}$ $(k \geq 3)$ of (L, m) that forms a (k, d)-multinet $(d > 1)$;

(ii) The curves $C_i = \prod_{\ell \in L_i} \alpha_\ell^{m(\ell)}$ are all the completely reducible fibers of a pencil of Ceva type of curves of degree $d > 1$;

(iii) There is an irreducible component of R^1 of dimension $k-1$ supported on the whole L. More precisely, the cohomology classes in A^1 of the logarithmic forms $\frac{dC_i}{C_i} - \frac{dC_1}{C_1}$, $i = 2, 3, \dots, k$, form a basis of the component.

Partial results in this direction have been obtained in [24] in a completely different way.

Proof. The implication (ii)$\Longrightarrow$(iii) can be checked straightforwardly.

The implications (i)$\Longrightarrow$(ii) and (iii)$\Longrightarrow$(i) have been proved in [17]. Here we give the proofs that are somewhat different with several new ideas and short cuts. Also the proof of the former implication fills a small gap in [17].

Proof of implication (i)$\Longrightarrow$(ii). It suffices to prove that the pencil π generated by C_1 and C_2 is connected and C_i is a fiber of π for every $i > 2$. Fix $i > 2$ and put $D = D_i = \prod_{\ell \in L_i} \alpha_\ell$.

(a) **Claim 1**. There exists a fiber $F = F_i$ of π such that D divides F. Indeed by the part (ii) of the definition of multinets we can order $\ell_1, \ell_2, \dots, \ell_k$ of L_i so that $\ell_j \cap \ell_{j+1} \notin \mathcal{X}$ for all j, $1 \leq j \leq k-1$. Choose a point $P \in \ell_1 \setminus \mathcal{X}$ and put $F = C_2(P)C_1 - C_1(P)C_2$. Then by Bèzout's Theorem ℓ_1 is a component of F. Now substituting points $\ell_j \cap \ell_{j+1}$ for P, claim 1 follows by induction on j.

(b) For each $\ell \in L_i$ denote by $k(\ell)$ the largest integer such that $\alpha_\ell^{k(\ell)}$ divides F and put $\bar{D} = \prod_{\ell \in L_i} \alpha_\ell^{k(\ell)}$. Notice that $1 \leq k(\ell) \leq m(\ell)$. We have $F = \bar{D}E$ where E is a polynomial relatively prime with D of degree

$$\deg E = d - \sum_{\ell \in L_i} k(\ell) = \sum_{\ell \in L_i} (m(\ell) - k(\ell)). \tag{1}$$

In particular, using the same idea as in (a) we have $E \cap D \subset \mathcal{X}$. Thus, due again to Bėzout's Theorem, we have for every $\ell_0 \in L_i$

$$\deg E = \sum_{\ell \in L_i'} (m(\ell) - k(\ell)) + s(m(\ell_0) - k(\ell_0)) \tag{2}$$

where $L_i' = \{\ell \in L_i | \ell \cap \ell_0 \in \mathcal{X}\}$ and $s = |L_i'|$. Also by properties of multinets we have

$$sm(\ell_0) = \sum_{\ell \in L_i \setminus L_i'} m(\ell) \tag{3}$$

for every $\ell_0 \in L_i$.

Comparing (1) and (2) and using (3) we obtain

$$sk(\ell_0) = \sum_{\ell \in L_i \setminus L_i'} k(\ell) \tag{4}$$

for every $\ell_0 \in L_i$.

(c) **Claim 2.** The function $\rho : L_i \to \mathbb{Q}$ given by $\ell \mapsto \frac{m(\ell)}{k(\ell)}$ is constant.

Indeed choose $\ell_0 \in L_i$ such that $\rho(\ell_0)$ is the greatest value on L_i. Then multiplying the equality (3) by $\frac{1}{\rho(\ell_0)}$ we have

$$sk(\ell_0) = \sum_{\ell \in L_i \setminus L_i'} m(\ell)\frac{k(\ell_0)}{m(\ell_0)} \leq \sum_{\ell \in L_i \setminus L_i'} k(\ell) \tag{5}$$

which has to be an equality. Thus $\rho(\ell) = \rho(\ell_0)$ for every $\ell \in L_i \setminus L_i'$, i.e., ρ is constant on $L_i \setminus L_i'$. Applying again part (ii) of the definition of multinets, ρ is constant on L_i.

(d) Here we conclude the proof using Stein Factorization Theorem (SFT) ([19, Cor. III.11.5], see also [18, p. 556]).

As it proved in (c), for every $i \geq 1$ there is a constant ρ_i such that $m(\ell)/k(\ell) = \rho_i$ for all $\ell \in L_i$. Suppose π is not connected which is definitely the case if at least one $\rho_i \neq 1$. The rational map $\pi : \mathbb{P}^2 \to \mathbb{P}^1$ lifts to a regular map $\tilde{\pi} : \mathbb{S} \to \mathbb{P}^1$ where $\varphi : \mathbb{S} \to \mathbb{P}^2$ is the blowup at $\mathcal{X}$. By SFT we can write $\tilde{\pi} = f \circ \tilde{\pi}_0$ where $\tilde{\pi}_0 : \mathbb{S} \to \mathcal{C}$ is a regular map with connected fibers to a curve $\mathcal{C}$, and $f : \mathcal{C} \to \mathbb{P}^1$ is a finite regular map. Since φ is birational, $\tilde{\pi}_0$ can be pushed down to a rational map $\pi_0 : \mathbb{P}^2 \to \mathcal{C}$. Since π is not connected the degree e of $f : \mathcal{C} \to \mathbb{P}^1$ is greater than 1. The fibers of π_0 are curves of degree d' where $d = ed'$ for some integer $e > 1$.

Now, again by condition (ii) of definition of multinets, the proper transform $\tilde{D}_i$ of $\bar{D}_i$ is connected. Since D_i and E_i are disjoint away from the base locus, $\tilde{D}_i$ is in fact a connected component of the proper transform of $F_i = \bar{D}_i E_i$. Then $\bar{D}_i$ is a (perhaps multiple) fiber of π_0 whence there exists a curve D_{0i} such that $D_{0i}^{\mu} = \bar{D}_i$ for an integer μ and $D_{0i}^e = C_i$. Since the proper transforms of C_1 and C_2 are also connected by definition of multinets the respective fibers of f are points of multiplicity e each. This implies that all the multiplicities $m(\ell)$ ($\ell \in L$) are divisible by e contradicting the condition that the line multiplicities $m(\ell)$ are mutually relatively prime. We conclude that $e = 1$, π is connected, and C_i is its fiber for every i.

Remark 3.8. *Notice that unlike in the proof of the same implication in* [17] *the condition (ii) from definition of multinets is used for all C_i. In fact, the Example 3.9 from* [17] *shows that if this condition fails even for one C_i then the implication may not hold.*

Proof of the implication (iii)$\Longrightarrow$(i).

For the proof of this implication [17] refers to [23]. Here we give details of a simplified version of the proof in [23].

Fix an irreducible component $\mathcal{V}$ of R^1 supported on a the whole L. For every $a \in \mathcal{V}$ put $Z(a) = \{b \in A^1 | ab = 0\}$. Recall from the end of section 2 that for every $b \in Z(a)$ we have $Z(b) = Z(a)$. This implies that $Z(c) \cap Z(a) = \{0\}$ for each $c \in R^1 \setminus Z(a)$. Since $\mathcal{V}$ is an irreducible component of R^1 it coincides with $Z(a)$ and in particular is a linear subspace of A^1. Now without any loss we can assume that a is in general position in $\mathcal{V}$ in the sense that $a_\ell \neq 0$ for every $\ell \in L$.

Now we define the subset $\mathcal{X}$ of the set of all multiple points of intersections of lines from L: $\mathcal{X} = \{P | \sum_{P \in \ell} a_\ell = 0\}$. Denote by J the incidence matrix (of the size $|\mathcal{X}| \times |L|$) of points from $\mathcal{X}$ and lines from L and define the square $|L| \times |L|$-matrix $Q = J^T J - E$ where J^T is the transpose of J and E is the matrix with all entries equal to 1. The importance of Q for the component $\mathcal{V}$ is that

$$\mathcal{V} = Z(a) = \ker J \cap \ker E = \ker Q \cap \ker E.$$

This follows from the description of local components. Besides Q is a symmetric integer matrix with the diagonal entries equal to $t_\ell - 1$, where t_ℓ is the number of points in $\mathcal{X}$ which are contained in ℓ. The assumption that the support of $\mathcal{V}$ is equal to the whole L implies that $Q_{\ell,\ell} > 0$ for $\ell \in L$. An off-diagonal entry $Q_{\ell,\ell'}$ is equal to zero when $\ell \cap \ell' \in \mathcal{X}$, and is equal to -1 otherwise. In particular, Q is a *(generalized) Cartan matrix.* These matrices were classified by E. Vinberg, see [20].

In order to apply Vinberg's classification (and construct the partition for the multinet) we represent Q as the direct sum of indecomposable matrices, as follows. Define the equivalence relation on L as the transitive hull of the relation $\ell \approx \ell'$ if $\ell \cap \ell' \notin \mathcal{X}$ and denote by $\{L_1, L_2, \ldots, L_k\}$ the partition of L into the equivalence classes. Due to properties of Q we have a direct sum decomposition $Q = \oplus_{i=1}^k Q_i$ where the matrices Q_i are again generalized Cartan and besides are indecomposable.

According to Vinberg's classification, each of these matrices is of one of three possible types. We denote by u elements of the linear space over $\mathbb{Q}$ with L as a basis and write $u > 0$ if all coordinates $u(\ell)$ are positive.

(1) Q_i is *finite* if it is positive definite; equivalently $Q_i u > 0$ for every $u > 0$;
(2) Q_i is *affine* if it is positive semidefinite (i.e., $Q_i u \geq 0$ for every $u > 0$) and $\ker(Q_i) = \mathbb{Q}u_i$ for some $u_i > 0$;
(3) Q_i is *indefinite* if $Q_i u < 0$ for some $u > 0$.

Since $\mathcal{V}$ is supported on the whole L the partition that we defined does not have finite blocks. Since matrix $J^T J$ is positive Q can have at most one indefinite block and in that case no affine blocks, in particular $k = 1$. Indeed otherwise it would be easy to find a positive vector u with $\sum_{\ell \in L} u(\ell) = 0$ and $(J^T Ju, u) = (Qu, u) < 0$. On the other hand, by definition of the partition, the restriction of $\mathcal{V}$ to every L_i is at most one-dmensional and $\dim \mathcal{V} \geq 2$. Thus $k \neq 1$ whence all the blocks are affine. We see that $\mathcal{V} = \ker Q$ is generated by the vectors $u_i - u_j$ where u_i is a positive integer vector generating $\ker Q_i$ such that $\sum_{\ell \in L_i} u_i(\ell)$ does not depend on i. Now we choose for each $i = 1, 2, \ldots, k$ such vectors u_i and put $m(\ell) = u_i(\ell)$ if $\ell \in L_i$. Then it follows from the construction that the partition of the multiarrangement (L, m) defines a (k, d) multinet. Q.E.D.

Examples

We start with the example of a multinet of type B_3 and change the exponent 2 to an arbitrary $d \geq 2$. What we obtain are examples of $(3, d)$-multinets that we call N_d ($N_2 = B_3$). This can be seen easier if we consider first the respective pencil of Ceva type. Indeed the curves $z^d(x^d - y^d)$, and $x^d(y^d - z^d)$, generate the pencil (the *Fermat pencil*) of Ceva type with the third completely rerducible fiber $y^d(x^d - z^d)$.

The other classical example of a pencil of Ceva type is the *Hesse pencil* of cubics generated by $x^3 + y^3 + z^3$ and xyz. This pencil has four completely reducible fibers, each of which is the product of three distinct lines. The resulting $(4, 3)$-net has 12 lines and $|\mathcal{X}| = 9$. In this example each of the four blocks is in general position (i.e., lines of each block intersect at three double points). The set $\mathcal{X}$ can be realized as the the set of all inflection points of a smooth irreducible cubic. It is the only known example of a $(4, d)$-net for any d (see below).

§4. Upper bound on k

Theorem 4.1. *A Ceva pencil of degree $d > 1$ cannot have more than four completely reducible fibers.*

Due to the main theorem this result can be also formulated in at least two other equivalent ways.

(i) For a (k, d)-multinet in $\mathbb{P}^2$ if $d > 1$ then $k < 5$.

(ii) Every non-local irreducible resonance component has dimension either two or three.

(iii) $\dim H^1(A, x) \leq 2$ if x is not supported on several lines passing through a point.

These inequalities have resulted from work by different authors. The first result $k \leq 5$ was proved in [23] for nets (not explicitly defined there) using the pencil corresponding to a components of R^1 and computing the Euler characteristic of the blowup at the base in two different ways. This proof was generalized in [17] to the multinets with all $m(\ell) = 1$. Then this inequality was proved in general case in [29] with a help of foliations of $\mathbb{P}^2$ generated by pencils of curves. The stronger inequality $k \leq 4$ was proved in the dissertation of J. Stipins [31] for nets. He used the pencil of Hessians of the fibers of π. Finally this method was generalized in [37] to the general case.

Let us recall that while there are plenty of examples with $k = 3$, the Hesse pencil is the only example of a Ceva pencil with 4 completely reducible fibers whence the only known $(4, d)$-multinet. $(4, d)$-multinets that are not nets do not exist (see [37]). The only possible examples would be nets with $d \geq 7$.

Open problems

1. Give a direct proof of the propagation property of the resonance varieties R^p.

2. Give a more constructive description of irreducible components of R^p for $p > 1$. In particular, how are they related to linear systems of subsurfaces?

3. Describe all the groups that can be represented by a 3-net in $\mathbb{P}^2$.

Conjecture 4.2. *These are all the finite subgroups of $PGL(2, \mathbb{C})$.*

4. Find other 4-nets besides the Hesse configuration.

Conjecture 4.3. *They do not exist.*

The conjecture has been proved for d=4,5,6.

5. Is every multinet the limit of a family of nets (in $\mathbb{P}^2$)?

For instance, $(3, 2d)$-multinet N_d ($d = 2, 3, \ldots$) considered above has this property. The easiest way to see this is to consider an arrangement of planes in $\mathbb{P}^3$ given by the polynomial

$$(x_0^d - x_1^d)(x_2^d - x_3^d)(x_0^d - x_2^d)(x_1^d - x_3^d)(x_0^d - x_3^d)(x_1^d - x_2^d).$$

Intersection of this arrangement with a generic (projective) plane gives a $(3, 2d)$-net in $\mathbb{P}^2$. Moving this plane to $x_3 = 0$ makes the intersection approach the multinet N_d.

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