Three sides of the geometric Langlands correspondence for $\mathfrak{gl}_N$ Gaudin model and Bethe vector averaging maps

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Abstract.

   We consider the $\mathfrak{gl}_N$ Gaudin model of a tensor power of the standard vector representation. The geometric Langlands correspondence in the Gaudin model relates the Bethe algebra of the commuting Gaudin Hamiltonians and the algebra of functions on a suitable space of N-th order differential operators. In this paper we introduce a third side of the correspondence: the algebra of functions on the critical set of a master function. We construct isomorphisms of the third algebra and the first two. Our main technical tool is the Bethe vector averaging maps, which is a new object.

§1. Introduction

   We consider the $\mathfrak{gl}_N$ Gaudin model associated with a tensor power of the standard vector representation. The geometric Langlands correspondence identifies the Bethe algebra of the commuting Gaudin Hamiltonians and the algebra of functions on a suitable space of N-th order differential operators. In this paper we introduce a third ingredient of the correspondence: the algebra of functions on the critical set of a master function. We construct isomorphisms of the three algebras.

   Master functions were introduced in [16] to construct hypergeometric integral solutions of the KZ equations,

$$\kappa \frac{\partial I}{\partial z_i} = H_i(z)I(z), \ \ i = 1, \ldots, n \ , \qquad I(z) \ = \ \int \Phi(z,t)^{1/\kappa}\omega(z,t)dt \ ,$$

where $H_i(z)$ are the Gaudin Hamiltonians, $\Phi(z, t)$ is a scalar master functions, $\omega(z, t)$ is a universal weight function, which is a vector-valued function. It was realized almost immediately [2, 14] that the value of the universal weight function at a critical point of the master function is an eigenvector of the Gaudin Hamiltonians. This construction of the eigenvectors is called the Bethe ansatz. The critical point equations for the master function are called the Bethe ansatz equations and the eigenvectors are called the Bethe vectors. The Bethe ansatz gives a relation between the critical points of the master function and the algebra generated by Gaudin Hamiltonians. The algebra of all (in particular, generalized) Gaudin Hamiltonians is called the Bethe algebra. Higher Gaudin Hamiltonians were introduced using different approaches in [6] and [18], see also [9].

In [17, 12], an N-th order differential operator was assigned to every critical point of the master function. The differential operators appearing in that construction form the second component of the geometric Langlands correspondence.

In this paper we introduce the third component of the geometric Langlands correspondence. It is the algebra of functions on the critical set of the master function. We show that all three components of the geometric Langlands correspondence are on equal footing; they are isomorphic.

The main results of the paper are Corollaries 5.4 and 8.6.

The paper is organized as follows. In Section 2 we recall the definition of the Bethe algebra $\mathcal{B}_\mathcal{V}$ of a tensor power of the vector representation of $\mathfrak{gl}_N$ [10]. In Section 3 we introduce the algebra $\mathcal{O}_W$ of functions on a suitable Schubert cell $\mathcal{W}$. Points of $\mathcal{W}$ are some N-dimensional spaces of polynomials in one variable. Such a space X is characterized by a monic N-th order differential operator with kernel X. The algebra $\mathcal{O}_W$ can be considered as the algebra of functions on the space of those differential operators. In Section 4 we recall an isomorphism $\zeta : \mathcal{O}_W \to \mathcal{B}_\mathcal{V}$ constructed in [10]. In Section 5 a master function and its quotient critical set $\mathcal{C}$ are introduced and an isomorphism $\iota^* : \mathcal{O}_W \to \mathcal{O}_\mathcal{C}$ is constructed. Here $\mathcal{O}_\mathcal{C}$ is the algebra of functions on $\mathcal{C}$. Consequently, we obtain a composition isomorphism $\mathcal{B}_\mathcal{V} \xrightarrow{\zeta^{-1}} \mathcal{O}_W \xrightarrow{\iota^*} \mathcal{O}_\mathcal{C}$. In Section 6 we introduce the universal weight function $\omega(z, t)$ and describe the basic facts of the Bethe ansatz. In Section 7 the Bethe vector averaging maps

$$v_F : z \mapsto \frac{1}{l_1! \dots l_{N-1}!} \sum_{(z,p) \in C_z} \frac{F(z, p) \, \omega(z, p)}{\mathrm{Hess}_t \log \Phi(z, p)}$$

are introduced. Here $\Phi(\boldsymbol{z}, \boldsymbol{t})$ is the master function, $\mathcal{C}_{\boldsymbol{z}}$ the critical set of the function $\Phi(\boldsymbol{z}, \cdot)$, $\omega(\boldsymbol{z}, \boldsymbol{t})$ the Bethe vector, $F(\boldsymbol{z}, \boldsymbol{t})$ an auxiliary polynomial function. Theorem 7.1 says that the Bethe vector averaging maps are polynomial maps. This is the main technical result of the paper. Using the Bethe vector averaging maps, we construct in Section 8 a new (direct) isomorphism $\nu : \mathcal{O}_{\mathcal{C}} \to \mathcal{B}_{\mathcal{V}}$. We prove that the throughout composition $\mathcal{B}_{\mathcal{V}} \xrightarrow{\zeta^{-1}} \mathcal{O}_{\mathcal{W}} \xrightarrow{\iota^*} \mathcal{O}_{\mathcal{C}} \xrightarrow{\nu} \mathcal{B}_{\mathcal{V}}$ is the identity map. Section 9 contains the proof of Theorem 7.1.

The paper discusses one example: the Gaudin model on a tensor power of the vector representation of $\mathfrak{gl}_N$. But the picture presented here presumably holds for more general representations and more general Lie algebras. All the ingredients of our considerations (the Bethe algebras, master functions, Bethe vector averaging maps) are available in other situations.

§2. Bethe algebra $\mathcal{B}_{\boldsymbol{\lambda}}$

2.1. Lie algebra $\mathfrak{gl}_N$

Let e_{ij}, $i, j = 1, \ldots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{ij}, e_{sk}] = \delta_{js} e_{ik} - \delta_{ik} e_{sj}$. Let $\mathfrak{h} \subset \mathfrak{gl}_N$ be the Cartan subalgebra generated by e_{ii}, $i = 1, \ldots, N$.

Let M be a $\mathfrak{gl}_N$-module. A vector $v \in M$ has weight $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ if $e_{ii} v = \lambda_i v$ for $i = 1, \ldots, N$. A vector v is singular if $e_{ij} v = 0$ for $1 \leqslant i < j \leqslant N$. Denote by $(M)_{\boldsymbol{\lambda}}$ the subspace of M of weight $\boldsymbol{\lambda}$, by $(M)^{sing}$ the subspace of all singular vectors in M, and by $(M)^{sing}_{\boldsymbol{\lambda}}$ the subspace of all singular vectors of weight $\boldsymbol{\lambda}$.

Denote by $L_{\boldsymbol{\lambda}}$ the irreducible finite-dimensional $\mathfrak{gl}_N$-module with highest weight $\boldsymbol{\lambda}$. The $\mathfrak{gl}_N$-module $L_{(1,0,\ldots,0)}$ is the standard N-dimensional vector representation of $\mathfrak{gl}_N$, denoted below by V. We choose a highest weight vector of V and denote it by v_+.

The Shapovalov form on V is the unique symmetric bilinear form S defined by the conditions $S(v_+, v_+) = 1$, $S(e_{ij} u, v) = S(u, e_{ji} v)$, for all $u, v \in V$ and $1 \leqslant i, j \leqslant N$. For a natural number n, the tensor Shapovalov form on $V^{\otimes n}$ is the tensor product of the Shapovalov forms of factors.

A sequence of integers $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N \geqslant 0$ is called a partition with at most N parts. Denote $|\boldsymbol{\lambda}| = \lambda_1 + \cdots + \lambda_N$.

2.2. Current algebra $\mathfrak{gl}_N[t]$

Let $\mathfrak{gl}_N[t] = \mathfrak{gl}_N \otimes \mathbb{C}[t]$ be the complex Lie algebra of $\mathfrak{gl}_N$-valued polynomials with the pointwise commutator. We identify $\mathfrak{gl}_N$ with the subalgebra $\mathfrak{gl}_N \otimes 1$ of constant polynomials in $\mathfrak{gl}_N[t]$. Hence, any $\mathfrak{gl}_N[t]$-module has a canonical structure of a $\mathfrak{gl}_N$-module.

It is convenient to collect elements of $\mathfrak{gl}_N[t]$ in generating series in a formal variable u. For $g \in \mathfrak{gl}_N$, set

$$g(u) = \sum_{s=0}^{\infty} (g \otimes t^s) u^{-s-1}.$$

For each $a \in \mathbb{C}$, there exists an automorphism ρ_a of $\mathfrak{gl}_N[t]$, $\rho_a : g(u) \mapsto g(u - a)$. Given a $\mathfrak{gl}_N[t]$-module M, we denote by $M(a)$ the pull-back of M through the automorphism ρ_a. As $\mathfrak{gl}_N$-modules, M and $M(a)$ are isomorphic by the identity map, since $\rho_a : g \otimes 1 \mapsto g \otimes 1$.

We have the evaluation homomorphism,

$$ev : \mathfrak{gl}_N[t] \to \mathfrak{gl}_N, \qquad ev : g(u) \mapsto g\, u^{-1},$$

that is, we have $g \otimes 1 \mapsto g \otimes 1$ and $g \otimes t^r \mapsto 0$ for $r > 0$. The restriction of the evaluation homomorphism to the subalgebra $\mathfrak{gl}_N \subset \mathfrak{gl}_N[t]$ is the identity map. For any $\mathfrak{gl}_N$-module M, we denote by the same letter the $\mathfrak{gl}_N[t]$-module, obtained by pulling M back through the evaluation homomorphism.

There is a $\mathbb{Z}_{\geqslant 0}$-grading on $\mathfrak{gl}_N[t]$: for any $g \in \mathfrak{gl}_N$, we have

$$\deg g \otimes t^r = r.$$

2.3. The $\mathfrak{gl}_N[t]$-module $\mathcal{V}^S$

Let n be a positive integer. Let $\mathcal{V}$ be the space of polynomials in $z_1, \ldots, z_n$ with coefficients in $V^{\otimes n}$, $\mathcal{V} = V^{\otimes n} \otimes_{\mathbb{C}} \mathbb{C}[z_1, \ldots, z_n]$. For $v \in V^{\otimes n}$ and $p(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$, we write $p(z_1, \ldots, z_n)\, v$ instead of $v \otimes p(z_1, \ldots, z_n)$.

The symmetric group S_n acts on $\mathbb{C}[z_1, \ldots, z_n]$ by permuting the variables $z_1, \ldots, z_n$ and acts on $\mathcal{V}$ by permuting the factors of $V^{\otimes n}$ and the variables $z_1, \ldots, z_n$ simultaneously,

$$\sigma\big(p(z_1, \ldots, z_n)\, v_1 \otimes \cdots \otimes v_n\big) = p(z_{\sigma(1)}, \ldots, z_{\sigma(n)})\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)},$$

$\sigma \in S_n$. For an S_n-module M denote by M^S the subspace of S_n-invariants of M. The space $\mathcal{V}^S$ is a free $\mathbb{C}[z_1, \ldots, z_n]^S$-module of rank N^n, see [4], cf. [10].

The space $\mathcal{V}$ is a $\mathfrak{gl}_N[t]$-module with a series $g(u)$, $g \in \mathfrak{gl}_N$, acting by

$$g(u) \big(p(z_1, \ldots, z_n)\, v_1 \otimes \cdots \otimes v_n\big) =$$
$$= p(z_1, \ldots, z_n) \sum_{s=1}^{n} \frac{v_1 \otimes \cdots \otimes g v_s \otimes \cdots \otimes v_n}{u - z_s}\,.$$

The $\mathfrak{gl}_N[t]$-action on $\mathcal{V}$ commutes with the S_n-action. Hence, $\mathcal{V}^S \subset \mathcal{V}$ is a $\mathfrak{gl}_N[t]$-submodule.

Define a grading on $\mathbb{C}[z_1, \ldots, z_n]$ by setting $\deg z_i = 1$ for all i. Define a grading on $\mathcal{V}$ by setting $\deg(v \otimes p) = \deg p$ for any $v \in V^{\otimes n}$ and $p \in \mathbb{C}[z_1, \ldots, z_n]$. The grading on $\mathcal{V}$ induces a grading on $\mathcal{V}^S$ and $\mathrm{End}\,(\mathcal{V}^S)$. The $\mathfrak{gl}_N[t]$-action on $\mathcal{V}^S$ is graded [4].

Let $\boldsymbol{\lambda}$ be a partition of n. The space $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ is a free graded $\mathbb{C}[z_1, \ldots, z_n]^S$-module. Its graded character is

$$(2.1) \qquad \mathrm{ch}((\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}) = \frac{\prod_{1 \leqslant i < j \leqslant N}(1 - q^{\lambda_i - \lambda_j + j - i})}{\prod_{i=1}^{N}(q)_{\lambda_i + N - i}}\, q^{\sum_{i=1}^{N}(i-1)\lambda_i},$$

where $(q)_a = \prod_{j=1}^{a}(1 - q^j)$, see [4, 3, 10].

2.4. Bethe algebra

Given an $N \times N$ matrix $A = (a_{ij})$, we define its row determinant to be

$$\mathrm{rdet}\, A = \sum_{\sigma \in S_N} (-1)^{\sigma}\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{N\sigma(N)}\,.$$

Let ∂ be the operator of differentiation in the variable u. Define the *universal differential operator* $\mathcal{D}$ by the formula

$$\mathcal{D} = \mathrm{rdet} \begin{pmatrix} \partial - e_{11}(u) & -e_{21}(u) & \ldots & -e_{N1}(u) \\ -e_{12}(u) & \partial - e_{22}(u) & \ldots & -e_{N2}(u) \\ \ldots & \ldots & \ldots & \ldots \\ -e_{1N}(u) & -e_{2N}(u) & \ldots & \partial - e_{NN}(u) \end{pmatrix}.$$

It is a differential operator in u, whose coefficients are formal power series in u^{-1} with coefficients in $U(\mathfrak{gl}_N[t])$,

$$\mathcal{D} = \partial^N + \sum_{i=1}^{N} B_i(u)\, \partial^{N-i}, \qquad B_i(u) = \sum_{j=i}^{\infty} B_{ij}\, u^{-j}\,,$$

and $B_{ij} \in U(\mathfrak{gl}_N[t])$, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{\geqslant i}$. The unital subalgebra of $U(\mathfrak{gl}_N[t])$ generated by B_{ij}, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{\geqslant 0}$, is called the *Bethe algebra* and denoted by $\mathcal{B}$.

By [18], cf. [9], the algebra $\mathcal{B}$ is commutative, and $\mathcal{B}$ commutes with the subalgebra $U(\mathfrak{gl}_N) \subset U(\mathfrak{gl}_N[t])$.

Let M be a $\mathcal{B}$-module and $v \in M$ an eigenvector of $\mathcal{B}$. For every coefficient $B_i(u)$ we have $B_i(u)v = h_i(u)v$, where $h_i(u)$ is a scalar series. The scalar differential operator

$$(2.2) \qquad \mathcal{D}_v = \partial^N + \sum_{i=1}^{N} h_i(u)\, \partial^{N-i}$$

will be called the differential operator associated with an eigenvector v.

As a subalgebra of $U(\mathfrak{gl}_N[t])$, the algebra $\mathcal{B}$ acts on any $\mathfrak{gl}_N[t]$-module M. Since $\mathcal{B}$ commutes with $U(\mathfrak{gl}_N)$, it preserves the weight subspaces of M and the subspaces $(M)^{sing}_{\boldsymbol{\lambda}}$.

For a $\mathcal{B}$-module M, the image of $\mathcal{B}$ in $\mathrm{End}\,(M)$ is called the Bethe algebra of M.

Let $\boldsymbol{\lambda}$ be a partition of n with at most N parts. The space $(\mathcal{V}^S)^{sing}_{\boldsymbol{\lambda}}$ is a $\mathcal{B}$-module. Set

$$(2.3) \qquad \mathcal{D}^{\mathcal{V}} = \partial^N + \sum_{i=1}^{N} B_i^{\mathcal{V}}(u)\, \partial^{N-i}, \qquad B_i^{\mathcal{V}}(u) = \sum_{j=i}^{\infty} B_{ij}^{\mathcal{V}} u^{-j},$$

where $B_{ij}^{\mathcal{V}}$ is the image of B_{ij} in $\mathrm{End}\,((\mathcal{V}^S)^{sing}_{\boldsymbol{\lambda}})$.

For any (i, j), the element $B_{ij}^{\mathcal{V}}$ is homogeneous of degree $j - i$. For any i the series $B_i^{\mathcal{V}}(u)$ is homogeneous of degree $-i$, see [10].

Denote by $\mathcal{B}_{\mathcal{V}}$ the Bethe algebra of $(\mathcal{V}^S)^{sing}_{\boldsymbol{\lambda}}$. The Bethe algebra $\mathcal{B}_{\mathcal{V}}$ is our first main object.

§3.   Algebra of functions $\mathcal{O}_{\mathcal{W}}$

3.1.   Cell $\mathcal{W}$ and algebra $\mathcal{O}_{\mathcal{W}}$

Let $N, d \in \mathbb{Z}_{>0}$, $N \leqslant d$. Let $\mathbb{C}_d[u]$ be the space of polynomials in u of degree less than d, $\dim \mathbb{C}_d[u] = d$. Let $\mathrm{Gr}(N, d)$ be the Grassmannian of all N-dimensional vector subspaces of $\mathbb{C}_d[u]$.

Given a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ with $\lambda_1 \leqslant d - N$, introduce a sequence

$$P = \{d_1 > d_2 > \cdots > d_N\}, \qquad d_i = \lambda_i + N - i.$$

Denote by $\mathcal{W}$ the subset of $\mathrm{Gr}(N,d)$ consisting of all N-dimensional subspaces $X \subset \mathbb{C}_d[u]$ such that for every $i = 1, \ldots, N$, the subspace X contains a polynomial of degree d_i.

In other words, $\mathcal{W}$ consists of subspaces $X \subset \mathbb{C}_d[u]$ with a basis $f_1(u), \ldots, f_N(u)$ of the form

$$(3.1) \qquad f_i(u) = u^{d_i} + \sum_{j=1,\ d_i - j \notin P}^{d_i} f_{ij} u^{d_i - j}.$$

For a given $X \in \mathcal{W}$, such a basis is unique. The basis $f_1(u), \ldots, f_N(u)$ will be called the flag basis of X.

The set $\mathcal{W}$ is a (Schubert) cell isomorphic to an affine space of dimension $|\boldsymbol{\lambda}|$ with coordinate functions f_{ij}. Let $\mathcal{O}_{\mathcal{W}}$ be the algebra of regular functions on $\mathcal{W}$,

$$\mathcal{O}_{\mathcal{W}} = \mathbb{C}[f_{ij}, \ i = 1, \ldots, N, \ j = 1, \ldots, d_i, \ d_i - j \notin P].$$

We may regard the polynomials $f_i(u)$, $i = 1, \ldots, N$, as generating functions for the generators f_{ij} of the algebra $\mathcal{O}_{\mathcal{W}}$.

The algebra $\mathcal{O}_{\mathcal{W}}$ is graded with $\deg f_{ij} = j$. A polynomial $f_i(u)$ is homogeneous of degree d_i. The graded character of $\mathcal{O}_{\mathcal{W}}$ is

$$(3.2) \qquad \mathrm{ch}(\mathcal{O}_{\mathcal{W}}) = \frac{\prod_{1 \leqslant i < j \leqslant N}(1 - q^{d_i - d_j})}{\prod_{i=1}^{N}(q)_{d_i}}$$

$$= \frac{\prod_{1 \leqslant i < j \leqslant N}(1 - q^{\lambda_i - \lambda_j + j - i})}{\prod_{i=1}^{N}(q)_{\lambda_i + N - i}},$$

see [10].

3.2. New generators of $\mathcal{O}_{\mathcal{W}}$

For $g_1, \ldots, g_N \in \mathbb{C}[u]$, introduce the Wronskian

$$\mathrm{Wr}(g_1(u), \ldots, g_N(u)) = \det \begin{pmatrix} g_1(u) & g_1'(u) & \cdots & g_1^{(N-1)}(u) \\ g_2(u) & g_2'(u) & \cdots & g_2^{(N-1)}(u) \\ \cdots & \cdots & \cdots & \cdots \\ g_N(u) & g_N'(u) & \cdots & g_N^{(N-1)}(u) \end{pmatrix},$$

where an i-th row is formed by derivatives of g_i.

Let $f_i(u)$, $i = 1, \ldots, N$, be the generating functions in (3.1). We have

$$\mathrm{Wr}(f_1(u), \ldots, f_N(u)) = \prod_{1 \leqslant i < j \leqslant N}(d_j - d_i)\left(u^n + \sum_{s=1}^{n}(-1)^s A_s\, u^{n-s}\right),$$

where $n = |\boldsymbol{\lambda}|$ and $A_1, \ldots, A_n$ are elements of $\mathcal{O}_\mathcal{W}$. Define

$$\mathcal{D}^\mathcal{W} = \frac{1}{\mathrm{Wr}(f_1(u), \ldots, f_N(u))} \, \mathrm{rdet} \begin{pmatrix} f_1(u) & f_1'(u) & \ldots & f_1^{(N)}(u) \\ f_2(u) & f_2'(u) & \ldots & f_2^{(N)}(u) \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \partial & \ldots & \partial^N \end{pmatrix}.$$

We have

$$\mathcal{D}^\mathcal{W} = \partial^N + \sum_{i=1}^{N} B_i^\mathcal{W}(u)\, \partial^{N-i}, \qquad B_i^\mathcal{W}(u) = \sum_{j=i}^{\infty} B_{ij}^\mathcal{W} u^{-j},$$

and $B_{ij}^\mathcal{W} \in \mathcal{O}_\mathcal{W}$, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{\geqslant i}$. For any (i, j), the element $B_{ij}^\mathcal{W}$ is homogeneous of degree $j - i$. For any i the series $B_i^\mathcal{W}(u)$ is homogeneous of degree $-i$. The elements $B_{ij}^\mathcal{W} \in \mathcal{O}_\mathcal{W}$, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{\geqslant i}$, generate the algebra $\mathcal{O}_\mathcal{W}$, see [10].

For $X \in \mathcal{W}$, denote by $\mathcal{D}_X$ the monic scalar differential operator of order N with kernel X. We call $\mathcal{D}_X$ the differential operator associated with X. The operator $\mathcal{D}_X$ is obtained from $\mathcal{D}^\mathcal{W}$ by specialization of variables f_{ij} to their values at X.

3.3. Wronski map

Let $X \in \mathcal{W}$. The Wronskian determinant of a basis of the subspace X does not depend on the choice of the basis up to multiplication by a number. The monic polynomial representing the Wronskian determinant of a basis of X is called the Wronskian of X and denoted by $\mathrm{Wr}_X(u)$.

The Wronski map $\mathcal{W} \to \mathbb{C}^n$ sends a point $X \in \mathcal{W}$ to a point $\boldsymbol{a} = (a_1, \ldots, a_n)$, if $\mathrm{Wr}_X(u) = u^n + \sum_{s=1}^{n}(-1)^s a_s u^{n-s}$. The Wronski map has finite degree.

§4. Isomorphism of $\mathcal{B}_\mathcal{V}$ and $\mathcal{O}_\mathcal{W}$

Theorem 4.1 ([10]). *The map*

$$\zeta : \mathcal{O}_\mathcal{W} \to \mathcal{B}_\mathcal{V}, \qquad B_{ij}^\mathcal{W} \mapsto B_{ij}^\mathcal{V},$$

is a well-defined isomorphism of graded algebras.

The degrees of elements of $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ are not less than $\sum_{i=1}^{N}(i-1)\lambda_i$ and the homogeneous component of $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ of degree $\sum_{i=1}^{N}(i-1)\lambda_i$ is one-dimensional, see formula (2.1). Let $v_1 \in (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ be a nonzero vector of degree $\sum_{i=1}^{N}(i-1)\lambda_i$.

Theorem 4.2 ([10]). *The map*

$$\eta : \mathcal{O}_{\mathcal{W}} \to (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing} , \qquad B_{ij}^{\mathcal{W}} \mapsto B_{ij}^{\mathcal{V}} v_1 ,$$

is an isomorphism of degree $\sum_{i=1}^{N}(i-1)\lambda_i$ of graded vector spaces. The maps ζ and η intertwine the action of the multiplication operators on $\mathcal{O}_{\mathcal{W}}$ and the action of the Bethe algebra $\mathcal{B}_{\mathcal{V}}$ on $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$, that is, for any $F, G \in \mathcal{O}_{\mathcal{W}}$, we have

$$\eta(FG) = \zeta(F)\,\eta(G) .$$

§5. Critical points of the master function

5.1. Master function

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ be a partition of n. Set $l_a = \sum_{b=a+1}^{N} \lambda_b$, $a = 0, \ldots, N$, where $l_0 = n$ and $l_N = 0$. Denote $l = l_0 + \cdots + l_{N-1}$, $\boldsymbol{l} = (l_0, \ldots, l_{N-1})$. Consider a set of l variables

$$\boldsymbol{T} = (t_1^{(0)}, \ldots, t_{l_0}^{(0)}, t_1^{(1)}, \ldots, t_{l_1}^{(1)}, \ldots, t_1^{(N-1)}, \ldots, t_{l_{N-1}}^{(N-1)})$$

and its subsets $\boldsymbol{t}^0 = (t_1^{(0)}, \ldots, t_{l_0}^{(0)})$ and $\boldsymbol{t} = (t_1^{(1)}, \ldots, t_{l_1}^{(1)}, \ldots, t_1^{(N-1)}, \ldots, t_{l_{N-1}}^{(N-1)})$. Consider the affine space $\mathbb{C}_{\boldsymbol{T}}^l = \mathbb{C}^l$ with coordinates $\boldsymbol{T} = (\boldsymbol{t}^0, \boldsymbol{t})$. The rational function $\Phi : \mathbb{C}^l \to \mathbb{C}$,

$$\Phi(\boldsymbol{T}) = \prod_{a=1}^{N-1} \prod_{1 \leqslant i < j \leqslant l_a} (t_i^{(a)} - t_j^{(a)})^2 \prod_{a=0}^{N-2} \prod_{i=1}^{l_a} \prod_{j=1}^{l_{a+1}} (t_i^{(a)} - t_j^{(a+1)})^{-1}$$

is called a *master function*. The master functions arise in the hypergeometric solutions of the KZ equations, see [8, 16, 19], and in the Bethe ansatz method for the Gaudin model, see [2, 14].

The product of symmetric groups $S_{\boldsymbol{l}} = S_{l_0} \times \cdots \times S_{l_{N-1}}$ acts on the coordinates $\boldsymbol{T}$ by permuting the coordinates with the same upper index. The master function is $S_{\boldsymbol{l}}$-invariant.

We consider the master function as a function of $\boldsymbol{t}$ depending on the parameters $\boldsymbol{t}^{(0)}$.

A point $\boldsymbol{T} = (\boldsymbol{t}^0, \boldsymbol{t}) \in \mathbb{C}^l$ is called a *critical point* of $\log \Phi(\boldsymbol{t}^0, \cdot)$ if

$$\frac{\partial}{\partial t_i^{(a)}} \log \Phi(\boldsymbol{T}) = 0 , \qquad a = 1, \ldots, N-1, \quad i = 1, \ldots, l_a .$$

That is, a point $\boldsymbol{T}$ is a critical point if the following system of $l - n$ equations is satisfied:

$$(5.1) \quad \sum_{j=1}^{l_{a-1}} \frac{1}{t_i^{(a)} - t_j^{(a-1)}} - \sum_{\substack{j=1 \\ j \neq i}}^{l_a} \frac{2}{t_i^{(a)} - t_j^{(a)}} + \sum_{j=1}^{l_{a+1}} \frac{1}{t_i^{(a)} - t_j^{(a+1)}} = 0,$$

here $a = 1, \ldots, N - 1$, $j = 1, \ldots, l_a$. In this definition we assume that all the denominators in (5.1) are nonzero. In the Gaudin model, equations (5.1) are called the Bethe ansatz equations. For a point $\boldsymbol{T} \in \mathbb{C}^l$, denote

$$\mathrm{Hess}_t \log \Phi(\boldsymbol{T}) = \det \left(\frac{\partial^2}{\partial t_i^{(a)} \partial t_j^{(b)}} \log \Phi(\boldsymbol{T}) \right),$$

where we take the determinant of the $(l - n) \times (l - n)$ matrix of second derivatives of the function $\log \Phi$ with respect to all of the variables $t_i^{(a)}$ with $a > 0$.

For a fixed $\boldsymbol{t}^0$, the function $\log \Phi(\boldsymbol{t}^0, \cdot)$ has finitely many critical points, see [17, 12, 13].

Theorem 5.1 ([17, 13]). *For generic $\boldsymbol{t}^0 \in \mathbb{C}^n$, all critical points of the function $\log \Phi(\boldsymbol{t}^0, \cdot)$ are nondegenerate. The number of the $S_{l_1} \times \cdots \times S_{l_{N-1}}$-orbits of critical points equals $\dim (V^{\otimes n})_{\boldsymbol{\lambda}}^{sing}$.*

Denote by $\widetilde{C} \subset \mathbb{C}_{\boldsymbol{T}}^l$ the union of all critical points of the functions $\log \Phi(\boldsymbol{t}^0, \cdot)$ for all $\boldsymbol{t}^0 \in \mathbb{C}^n$ with distinct coordinates $t_1^{(0)}, \ldots, t_n^{(0)}$. Denote by $C \subset \mathbb{C}_{\boldsymbol{T}}^l$ the Zariski closure of $\widetilde{C}$. The set C is S_l-invariant.

5.2. Factorization by S_l

For $a = 0, \ldots, N - 1$, let $\sigma_1^{(a)}, \ldots, \sigma_{l_a}^{(a)}$ be the elementary symmetric functions of $t_1^{(a)}, \ldots, t_{l_a}^{(a)}$. Denote by $\mathbb{C}_{\boldsymbol{\Sigma}}^l = \mathbb{C}^l$ the affine space with coordinates

$$\boldsymbol{\Sigma} = (\sigma_1^{(0)}, \ldots, \sigma_{l_0}^{(0)}, \sigma_1^{(1)}, \ldots, \sigma_{l_1}^{(1)}, \ldots, \sigma_1^{(N-1)}, \ldots, \sigma_{l_{N-1}}^{(N-1)}).$$

The space $\mathbb{C}_{\boldsymbol{\Sigma}}^l$ is the quotient of $\mathbb{C}_{\boldsymbol{T}}^l$ by the S_l-action.

Denote by $\mathcal{C}$ the image of C under the natural projection $\mathbb{C}_{\boldsymbol{T}}^l \to \mathbb{C}_{\boldsymbol{\Sigma}}^l$. The set $\mathcal{C}$ will be called the *quotient critical set* of the master function. Let $\mathcal{O}_{\mathcal{C}}$ be the algebra of regular functions on $\mathcal{C}$, that is, the restriction of $\mathbb{C}[\boldsymbol{\Sigma}]$ to $\mathcal{C}$.

The algebra $\mathbb{C}[\boldsymbol{T}]$ is a graded algebra with $\deg t_i^{(a)} = 1$ for all (a, i). The algebra $\mathbb{C}[\boldsymbol{\Sigma}]$ is a graded algebra with $\deg \sigma_i^{(a)} = i$ for all (a, i).

Equations (5.1) are homogeneous. Hence, $\mathcal{C}$ is a quasi-homogeneous algebraic set and the algebra $\mathcal{O}_{\mathcal{C}}$ has a grading with $\deg(\sigma_i^{(a)}|_{\mathcal{C}}) = i$.

5.3. A map $\theta : \mathcal{W} \to \mathbb{C}_{\Sigma}^l$

For $X \in \mathcal{W}$, let $f_{1,X}(u), \ldots, f_{N,X}(u)$ be the flag basis of X. Introduce the polynomials $y_{0,X}(u), y_{1,X}(u), \ldots, y_{N-1,X}(u)$ by the formula

$$y_{a,X}(u) \prod_{a < i < j \leqslant N} (d_i - d_j) = \mathrm{Wr}(f_{a+1,X}(u), \ldots, f_{N,X}(u)),$$

$a = 0, \ldots, N - 1$. For each a, the polynomial $y_{a,X}(u)$ is a monic polynomial of degree l_a, $y_{a,X}(u) = u^{l_a} + \sum_{i=1}^{l_a} (-1)^i \sigma_{i,X}^{(a)} u^{l_a - i}$. Denote by $t_{1,X}^{(a)}, \ldots, t_{l_a,X}^{(a)}$ the roots of $y_{a,X}(u)$. Then $\sigma_{1,X}^{(a)}, \ldots, \sigma_{l_a,X}^{(a)}$ are the elementary symmetric functions of $t_{1,X}^{(a)}, \ldots, t_{l_a,X}^{(a)}$. The sequence

$$\boldsymbol{T}_X = (t_{1,X}^{(0)}, \ldots, t_{l_0,X}^{(0)}, \ldots, t_{1,X}^{(N-1)}, \ldots, t_{l_{N-1},X}^{(N-1)})$$

will be called the *root coordinates* of X. For every a, the numbers $t_{1,X}^{(a)}, \ldots, t_{l_a,X}^{(a)}$ are determined up to a permutation. Let $\boldsymbol{\Sigma}_X$ be the image of $\boldsymbol{T}_X$ in $\mathbb{C}_{\boldsymbol{\Sigma}}^l$.

A point $X \in \mathcal{W}$ will be called *nice* if all roots of the polynomials $y_{0,X}(u), y_{1,X}(u), \ldots, y_{N-1,X}(u)$ are simple and for each $a = 1, \ldots, N-1$, the polynomials $y_{a-1,X}(u)$ and $y_{a,X}(u)$ do not have common roots. Nice points form a Zariski open subset of $\mathcal{W}$, see [11]. If X is nice, then the root coordinates $\boldsymbol{T}_X$ satisfy the critical point equations (5.1), see [12].

Define a polynomial map $\theta : \mathcal{W} \to \mathbb{C}_{\boldsymbol{\Sigma}}^l$, $X \mapsto \boldsymbol{\Sigma}_X$. This map induces a graded algebra homomorphism $\mathbb{C}[\boldsymbol{\Sigma}] \to \mathcal{O}_{\mathcal{W}}$.

Lemma 5.2. *We have $\theta(\mathcal{W}) \subset \mathcal{C}$.*

Proof. The lemma follows from the fact that the nice points of $\mathcal{W}$ are mapped to $\mathcal{C}$. Q.E.D.

5.4. Differential operator $\mathcal{D}^{\mathcal{T}}$ and a map $\iota : \mathcal{C} \to \mathcal{W}$

Set

$$\chi^a(u, \boldsymbol{T}) = \sum_{j=1}^{l_{a-1}} \frac{1}{u - t_j^{(a-1)}} - \sum_{i=1}^{l_a} \frac{1}{u - t_j^{(a)}}, \qquad a = 1, \ldots, N,$$

and

$$\mathcal{D}^{\mathcal{T}} = (\partial - \chi^1(u, \boldsymbol{T})) \ldots (\partial - \chi^N(u, \boldsymbol{T})).$$

We have

$$\mathcal{D}^{\mathcal{T}} = \partial^N + \sum_{i=1}^{N} B_i^{\mathcal{T}}(u)\, \partial^{N-i}, \qquad B_i^{\mathcal{T}}(u) = \sum_{j=i}^{\infty} B_{ij}^{\mathcal{T}}\, u^{-j},$$

and $B_{ij}^{\mathcal{T}} \in \mathbb{C}[\boldsymbol{T}]^{S_l} = \mathbb{C}[\boldsymbol{\Sigma}]$, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{\geqslant i}$. For a point $\boldsymbol{T} \in \mathbb{C}^l$, denote by $\mathcal{D}_{\boldsymbol{T}}$ the specialization of $\mathcal{D}^{\mathcal{T}}$ at $\boldsymbol{T}$. We call $\mathcal{D}_{\boldsymbol{T}}$ the differential operator associated with a point $\boldsymbol{T}$.

If $\boldsymbol{T} = (\boldsymbol{t}^0, \boldsymbol{t}) \in \mathbb{C}^l$ is a critical point of $\log \Phi(\boldsymbol{t}^0, \cdot)$, then the kernel $X_{\boldsymbol{T}}$ of $\mathcal{D}_{\boldsymbol{T}}$ consists of polynomials; moreover, $X_{\boldsymbol{T}}$ is a point of $\mathcal{W}$, see [12]. The correspondence $\boldsymbol{T} \mapsto X_{\boldsymbol{T}}$ defines a rational map $\iota : \mathcal{C} \to \mathcal{W}$.

5.5. Quotient critical set is a nonsingular subvariety

Theorem 5.3. *The quotient critical set $\mathcal{C} \subset \mathbb{C}_{\boldsymbol{\Sigma}}^l$ is a nonsingular subvariety. The map $\theta : \mathcal{W} \to \mathbb{C}_{\boldsymbol{\Sigma}}^l$ is an embedding with $\theta(\mathcal{W}) = \mathcal{C}$. The map $\iota : \mathcal{C} \to \mathcal{W}$ is an isomorphism and $\iota\theta = \mathrm{id}_{\mathcal{W}}$.*

Proof. The map θ, considered as a map from $\mathcal{W}$ to $\theta(\mathcal{W})$ is finite. The set $\theta(\mathcal{W})$ is Zariski closed since $\mathcal{W}$ is Zariski closed. We know from [12] that $\theta(\mathcal{W})$ contains the subset $\widetilde{\mathcal{C}} \subset \mathcal{C}$, the image of nondegenerate critical points. We have $\theta(\mathcal{W}) = \mathcal{C}$, since $\mathcal{C}$ is the Zariski closure of $\widetilde{\mathcal{C}}$ and $\theta(\mathcal{W})$ is Zariski closed.

The fact that $\iota\theta = \mathrm{id}_{\mathcal{W}}$ at generic points of $\mathcal{W}$ is proved in [12]. Therefore, $\iota\theta = \mathrm{id}_{\mathcal{W}}$ for all points of $\mathcal{W}$.

Consider the algebra homomorphism $\iota^* : \mathcal{O}_{\mathcal{W}} \to \mathcal{O}_{\mathcal{C}}$ induced by ι. Under the map ι^* the elements $B_{ij}^{\mathcal{W}}$ are mapped to the polynomials $B_{ij}^{\mathcal{T}} \in \mathbb{C}[\boldsymbol{\Sigma}]$ restricted to $\mathcal{C}$. Since the elements $B_{ij}^{\mathcal{W}}$ generate $\mathcal{O}_{\mathcal{W}}$, the map $\theta : \mathcal{W} \to \mathbb{C}_{\boldsymbol{\Sigma}}^l$ is an embedding and the map $\iota : \mathcal{C} \to \mathcal{W}$ is an isomorphism.   Q.E.D.

Corollary 5.4. *The map $\iota^* : \mathcal{O}_{\mathcal{W}} \to \mathcal{O}_{\mathcal{C}}$, $B_{ij}^{\mathcal{W}} \mapsto B_{ij}^{\mathcal{T}}|_{\mathcal{C}}$, is an isomorphism of graded algebras. In particular, the elements $B_{ij}^{\mathcal{T}}|_{\mathcal{C}}$ generate $\mathcal{O}_{\mathcal{C}}$.*

§6. Universal weight function and Bethe vectors

We remind a construction of a rational map $\omega : \mathbb{C}^l \to (V^{\otimes n})_{\boldsymbol{\lambda}}$, called the *universal weight function*, see [8, 16], cf. [15].

A basis of $V^{\otimes n}$ is formed by the vectors $e_J v = e_{j_1,1} v_+ \otimes \cdots \otimes e_{j_n,1} v_+$, where $J = (j_1, \ldots, j_n)$ and $1 \leqslant j_a \leqslant N$ for $a = 1, \ldots, N$. A basis of $(V^{\otimes n})_{\boldsymbol{\lambda}}$ is formed by the vectors $e_J v$ such that $\#\{a \mid j_a > i\} = l_i$ for every $i = 1, \ldots, N-1$. Such a multi-index J will be called admissible.

The universal weight function has the form $\omega(\boldsymbol{T}) = \sum_J \omega_J(\boldsymbol{T}) e_J v$ where the sum is over the set of all admissible J, and the functions $\omega_J(\boldsymbol{T})$ are defined below.

For an admissible J and $i = 1, \ldots, N-1$, define $A_i(J) = \{ a \mid 1 \leqslant a \leqslant n, \; 1 \leqslant i < j_a \}$. Then $|A_i(J)| = l_i$.

Let $\Gamma(J)$ be the set of sequences $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{N-1})$ of bijections $\gamma_i : A_i(J) \to \{1, \ldots, l_i\}$, $i = 1, \ldots, N-1$. Then $|\Gamma(J)| = \prod_{i=1}^{N-1} l_i!$.

For $a \in A_1(J)$ and $\boldsymbol{\gamma} \in \Gamma(J)$, introduce a rational function

$$\omega_{a,\gamma}(\boldsymbol{T}) = \frac{1}{t^{(1)}_{\gamma_1(a)} - t^{(0)}_a} \prod_{i=2}^{j_a-1} \frac{1}{t^{(i)}_{\gamma_i(a)} - t^{(i-1)}_{\gamma_{i-1}(a)}} \;.$$

Define

$$\omega_J(\boldsymbol{T}) = \sum_{\boldsymbol{\gamma} \in \Gamma(J)} \prod_{a \in A_1(J)} \omega_{a,\gamma} \;.$$

Theorem 6.1. *Let $\boldsymbol{T} = (\boldsymbol{z}, \boldsymbol{p})$ be a nondegenerate critical point of the function $\log \Phi(\boldsymbol{z}, \cdot)$, here $\boldsymbol{z} = (z_1, \ldots, z_n)$ lies in $\mathbb{C}^n$ and $\boldsymbol{p}$ lies in $\mathbb{C}^{l-n}$ with coordinates $t^{(i)}_j$, $i > 0$. Consider the value $\omega(\boldsymbol{z}, \boldsymbol{p})$ of the universal weight function $\omega : \mathbb{C}^l \to (V^{\otimes n})_{\boldsymbol{\lambda}}$ at $(\boldsymbol{z}, \boldsymbol{p})$. Consider $V^{\otimes n}$ as the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^n V(z_s)$. Then*

(i) *The vector $\omega(\boldsymbol{z}, \boldsymbol{p})$ belongs to $(V^{\otimes n})^{sing}_{\boldsymbol{\lambda}}$.*

(ii) *The vector $\omega(\boldsymbol{z}, \boldsymbol{p})$ is an eigenvector of the Bethe algebra $\mathcal{B}$, acting on $\otimes_{s=1}^n V(z_s)$. Moreover, $\mathcal{D}_{\omega(\boldsymbol{z},\boldsymbol{p})} = \mathcal{D}_{(\boldsymbol{z},\boldsymbol{p})}$, where $\mathcal{D}_{\omega(\boldsymbol{z},\boldsymbol{p})}$ and $\mathcal{D}_{(\boldsymbol{z},\boldsymbol{p})}$ are the differential operators associated with the eigenvector $\omega(\boldsymbol{z}, \boldsymbol{p})$ and the point $(\boldsymbol{z}, \boldsymbol{p}) \in \mathbb{C}^l$, respectively, see (2.2) and 5.4.*

(iii) *Let S be the tensor Shapovalov form on $V^{\otimes n}$, then*

$$S(\omega(\boldsymbol{z}, \boldsymbol{p}), \omega(\boldsymbol{z}, \boldsymbol{p})) = \text{Hess}_{\boldsymbol{t}} \log \Phi(\boldsymbol{z}, \boldsymbol{p}) \;.$$

(iv) *If $(\boldsymbol{z}, \boldsymbol{p})$ and $(\boldsymbol{z}, \boldsymbol{p}')$ are two nondegenerate critical points of the function $\log \Phi(\boldsymbol{z}, \cdot)$, which lie in different $S_{l_1} \times \cdots \times S_{l_{N-1}}$-orbits, then $S(\omega(\boldsymbol{z}, \boldsymbol{p}), \omega(\boldsymbol{z}, \boldsymbol{p}')) = 0$.*

Part (i) is proved in [2] and [14]. Part (i) also follows directly from Theorems 6.16.2 in [16]. Part (ii) is proved in [9]. Part (iii) is proved in [13, 20]. Part (iv) is proved in [20] and also follows from [10].

The vector $\omega(\boldsymbol{z}, \boldsymbol{p})$ is called the *Bethe vector* corresponding to a critical point $(\boldsymbol{z}, \boldsymbol{p})$.

§7. Bethe vector averaging maps

Consider $\mathbb{C}^l$ with coordinates $\boldsymbol{T} = (\boldsymbol{t}^0, \boldsymbol{t})$. We denote the variables $\boldsymbol{t}^0 = (t_1^{(0)}, \dots, t_n^{(0)})$ also by $\boldsymbol{z} = (z_1, \dots, z_n)$.

Let $\boldsymbol{z}$ be a generic point of $\mathbb{C}^n$ with distinct coordinates and such that all the critical points of the function $\log \Phi(\boldsymbol{z}, \cdot)$ are nondegenerate. The critical set $C_{\boldsymbol{z}}$ of $\log \Phi(\boldsymbol{z}, \cdot)$ consists of $\dim (V^{\otimes n})_{\boldsymbol{\lambda}}^{sing}$ $S_{l_1} \times \cdots \times S_{l_{N-1}}$-orbits. Each orbit has $l_1! \cdots l_{N-1}!$ points. For any $F \in \mathbb{C}[\boldsymbol{T}]^{S_l} = \mathbb{C}[\boldsymbol{\Sigma}]$, let us define

$$(7.1) \qquad v_F(\boldsymbol{z}) \; = \; \frac{1}{l_1! \dots l_{N-1}!} \sum_{(\boldsymbol{z}, \boldsymbol{p}) \in C_{\boldsymbol{z}}} \frac{F(\boldsymbol{z}, \boldsymbol{p}) \, \omega(\boldsymbol{z}, \boldsymbol{p})}{\mathrm{Hess}_{\boldsymbol{t}} \log \Phi(\boldsymbol{z}, \boldsymbol{p})} \; .$$

The term of this sum corresponding to a critical point $(\boldsymbol{z}, \boldsymbol{p})$ can be written as the following integral, see Chapter 5 of [7]. Choose a small neighborhood U of $\boldsymbol{p}$ in $\mathbb{C}^{l-n}$. Define a torus $\Gamma_{\boldsymbol{z}, \boldsymbol{p}}$ in U by $l - n$ equations $|\Phi_j^a(\boldsymbol{z}, \boldsymbol{t})| = \epsilon_j^a$ where Φ_j^a are derivatives of $\log \Phi(\boldsymbol{z}, \cdot)$ with respect to the variables $t_j^{(a)}$, $a > 0$, and where ϵ_i^a are small positive numbers. Then

$$(7.2) \qquad \frac{F(\boldsymbol{z}, \boldsymbol{p}) \, \omega(\boldsymbol{z}, \boldsymbol{p})}{\mathrm{Hess}_{\boldsymbol{t}} \log \Phi(\boldsymbol{z}, \boldsymbol{p})} \; = \; \frac{1}{(2\pi i)^{l-n}} \int_{\Gamma_{\boldsymbol{z}, \boldsymbol{t}}} \frac{F(\boldsymbol{z}, \boldsymbol{t}) \, \omega(\boldsymbol{z}, \boldsymbol{t}) \, d\boldsymbol{t}}{\prod_{a,j} \Phi_j^a(\boldsymbol{z}, \boldsymbol{t})} \; .$$

The $l_1! \cdots l_{N-1}!$ terms of the sum in (7.1) corresponding to a single $S_{l_1} \times \cdots \times S_{l_{N-1}}$-orbit are all equal due to the $S_{l_1} \times \cdots \times S_{l_{N-1}}$-invariance of Φ, ω and F.

The correspondence $\boldsymbol{z} \mapsto v_F(\boldsymbol{z})$ defines a map $v_F : \mathbb{C}^n \to (V^{\otimes n})_{\boldsymbol{\lambda}}^{sing}$ which will be called *a Bethe vector averaging map*.

The map v_F is a rational map. Indeed, the map is well defined on a Zariski open subset of $\mathbb{C}^n$ and has bounded growth as the argument approached the possible singular points or infinity.

Theorem 7.1. *For any $F \in \mathbb{C}[\boldsymbol{\Sigma}]$, the Bethe vector averaging map v_F is a polynomial map.*

Theorem 7.1 is proved in Section 9.

§8. Quotient critical set and Bethe algebra

8.1. Construction of isomorphisms

Recall that $\mathbb{C}[\boldsymbol{\Sigma}]$ is graded by $\deg \sigma_j^{(a)} = j$. For any $F \in \mathbb{C}[\boldsymbol{\Sigma}]$, consider the Bethe vector averaging map $v_F : \mathbb{C}^n \to (V^{\otimes n})_{\boldsymbol{\lambda}}^{sing}$.

Lemma 8.1. *If F is quasi-homogeneous and $\deg F = d$, then v_F is homogeneous and $\deg v_F = d + l - n = d + \sum_{i=1}^{N} (i-1)\lambda_i$.*

It is clear that the map v_F is an element of $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$. Thus, the correspondence $F \mapsto v_F$ defines a graded linear map $\mu : \mathbb{C}[\boldsymbol{\Sigma}] \to (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$.

Theorem 8.2. *The kernel of $\mu : \mathbb{C}[\boldsymbol{\Sigma}] \to (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ is the defining ideal $I_{\mathcal{C}} \subset \mathbb{C}[\boldsymbol{\Sigma}]$ of $\mathcal{C}$. The map μ induces a graded linear isomorphism $\mathcal{O}_{\mathcal{C}} \to (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ of degree $\sum_{i=1}^{N}(i-1)\lambda_i$.*

We shall denote this isomorphism by the same letter μ.

Proof. If $F \in I_{\mathcal{C}}$, then $v_F = 0$ for generic $\boldsymbol{z}$. Hence, $v_F = 0$ as an element of $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$. If $v_F = 0$ as an element of $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$, then $F = 0$ on a Zariski open subset of $\mathcal{C}$. Hence, $F \in I_{\mathcal{C}}$. Therefore, $\ker \mu = I_{\mathcal{C}}$.

The graded character of $\mathcal{O}_{\mathcal{C}}$ equals the graded character of $\mathcal{O}_{\mathcal{W}}$ by Corollary 5.4. The graded character of $\mathcal{O}_{\mathcal{W}}$ is given by (3.2). The graded character of $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ is given by (2.1). Comparing the characters and using Lemma 8.1, we conclude that the induced map $\mu : \mathcal{O}_{\mathcal{C}} \to (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ is an isomorphism. Q.E.D.

Corollary 8.3. *Consider the element $v_1 \in (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$, corresponding to $F = 1$ under the isomorphism μ. Then v_1 is a generator of the one-dimensional graded component of $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ of degree $\sum_{i=1}^{N}(i-1)\lambda_i$ (compare this v_1 with the element v_1 in Theorem 4.2).*

Given an element $F \in \mathcal{O}_{\mathcal{C}}$, define a linear map $\nu(F) : (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing} \to (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$, $v_G \mapsto v_{FG}$. By Theorem 8.2, this map is well-defined.

Consider the generators $B_{ij}^{\mathcal{T}}|_{\mathcal{C}}$ of $\mathcal{O}_{\mathcal{C}}$ and generators $B_{ij}^{\mathcal{V}}$ of $\mathcal{B}_{\mathcal{V}}$, see Corollary 5.4 and formula (2.3).

Lemma 8.4. *For any (i,j), the linear map $\nu(B_{ij}^{\mathcal{T}}|_{\mathcal{C}}) : (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing} \to (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$, $v_F \mapsto v_{B_{ij}^{\mathcal{T}}F}$, coincides with the map $B_{ij}^{\mathcal{V}}$.*

Proof. The lemma follows from part (ii) of Theorem 6.1. Q.E.D.

Corollary 8.5. *The map $F \mapsto \nu(F)$ is an algebra isomorphism $\nu : \mathcal{O}_{\mathcal{C}} \to \mathcal{B}_{\mathcal{V}}$.*

Corollary 8.6. *The maps $\mu : \mathcal{O}_{\mathcal{C}} \to (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$ and $\nu : \mathcal{O}_{\mathcal{C}} \to \mathcal{B}_{\mathcal{V}}$ intertwine the action of the multiplication operators on $\mathcal{O}_{\mathcal{C}}$ and the action of the Bethe algebra $\mathcal{B}_{\mathcal{V}}$ on $(\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$, that is, for any $F, G \in \mathcal{O}_{\mathcal{C}}$, we have*
$$\mu(FG) = \nu(F)\,\mu(G).$$

Corollary 8.7. *Consider the element $v_1 \in (\mathcal{V}^S)_{\boldsymbol{\lambda}}^{sing}$, corresponding to $F = 1$ under the isomorphism μ. Let us use this element in the*

definition of the isomorphism η of Theorem 4.2. Then the throughout compositions

$$\mathcal{O}_\mathcal{C} \xrightarrow{\mu} (\mathcal{V}^S)^{sing}_{\boldsymbol\lambda} \xrightarrow{\eta^{-1}} \mathcal{O}_\mathcal{W} \xrightarrow{\iota^*} \mathcal{O}_\mathcal{C} \, , \qquad \mathcal{O}_\mathcal{C} \xrightarrow{\nu} \mathcal{B}_\mathcal{V} \xrightarrow{\zeta^{-1}} \mathcal{O}_\mathcal{W} \xrightarrow{\iota^*} \mathcal{O}_\mathcal{C}$$

are the identity maps.

8.2. Inverse map to $\nu : \mathcal{O}_\mathcal{C} \to (\mathcal{V}^S)^{sing}_{\boldsymbol\lambda}$

For $v \in (\mathcal{V}^S)^{sing}_{\boldsymbol\lambda}$, define a function f_v on a Zariski open subset of $\mathcal{C}$ as follows. For a generic point $\boldsymbol\Sigma \in \mathcal{C}$, let $\boldsymbol{T} = (\boldsymbol{z}, \boldsymbol{t})$ be a point of the critical set $C \subset \mathbb{C}^l$ which projects to $\boldsymbol\Sigma$. Let $\omega(\boldsymbol{z}, \boldsymbol{t})$ be the Bethe vector corresponding the point $(\boldsymbol{z}, \boldsymbol{t})$. Set

$$f_v(\boldsymbol\Sigma) \; = \; S(v(\boldsymbol{z}), \omega(\boldsymbol{z}, \boldsymbol{t})) \, ,$$

where S is the tensor Shapovalov form on $V^{\otimes n}$, cf. Theorem 6.1.

Theorem 8.8. *For any $v \in (\mathcal{V}^S)^{sing}_{\boldsymbol\lambda}$, the scalar function f_v is the restriction to $\mathcal{C}$ of a polynomial. Moreover, the map $(\mathcal{V}^S)^{sing}_{\boldsymbol\lambda} \to \mathcal{O}_\mathcal{C}$, $v \mapsto f_v$, is the inverse map to the isomorphism $\nu : \mathcal{O}_\mathcal{C} \to (\mathcal{V}^S)^{sing}_{\boldsymbol\lambda}$.*

Proof. Any element of $(\mathcal{V}^S)^{sing}_{\boldsymbol\lambda}$ has the form of v_F for a suitable $F \in \mathbb{C}[\boldsymbol\Sigma]$, see (7.1). In that case,

$$f_{v_F}(\boldsymbol{z}, \boldsymbol{t}) = S \left(\frac{1}{l_1! \dots l_{N-1}!} \sum_{(\boldsymbol{z}, \boldsymbol{p}) \in C_{\boldsymbol{z}}} \frac{F(\boldsymbol{z}, \boldsymbol{p}) \, \omega(\boldsymbol{z}, \boldsymbol{p})}{\mathrm{Hess}_{\boldsymbol{t}} \log \Phi(\boldsymbol{z}, \boldsymbol{p})} , \omega(\boldsymbol{z}, \boldsymbol{t})) \right)$$

$$= F(\boldsymbol{z}, \boldsymbol{t})$$

by Theorem 6.1. This identity proves the theorem.   Q.E.D.

§9. Proof of Theorem 7.1

9.1. The Shapovalov form and asymptotics of v_F

Let $\boldsymbol{T}^0$ be a point of the critical set $C \subset \mathbb{C}^l$, see Section 5.1.

Consider the germ at $0 \in \mathbb{C}$ of a generic analytic curve $\mathbb{C} \to \mathbb{C}^l$, $s \mapsto \boldsymbol{T}(s) = (\boldsymbol{z}(s), \boldsymbol{t}(s))$, with $\boldsymbol{T}(0) = \boldsymbol{T}^0$ such that for any small nonzero s, the point $(\boldsymbol{z}(s), \boldsymbol{t}(s))$ is a nondegenerate critical point of $\log \Phi(\boldsymbol{z}(s), \cdot)$, and $\boldsymbol{z}(s)$ has distinct coordinates. The corresponding Bethe vector has the form, $\omega(\boldsymbol{T}(s)) = w_\alpha s^\alpha + o(s^\alpha)$, where α is a rational number and $w_\alpha \in (V^{\otimes n})^{sing}_{\boldsymbol\lambda}$ is a nonzero vector.

Let X^0 denote the point of $\mathcal{W}$ corresponding to $\boldsymbol{T}^0$. Namely, we take the image $\boldsymbol\Sigma^0$ of $\boldsymbol{T}^0$ in $\mathcal{C}$ under the factorization by the $S_{\boldsymbol{l}}$-action and then set $X^0 = \iota(\boldsymbol\Sigma^0)$.

Lemma 9.1. *Assume that X^0 is not a critical point of the Wronski map $\mathcal{W} \to \mathbb{C}^n$. Then $S(w_\alpha, w_\alpha)$ is a nonzero number, where S is the tensor Shapovalov form.*

Proof. For a small nonzero s, the Bethe vectors corresponding to $S_{l_1} \times \cdots \times S_{l_{N-1}}$-orbits of the critical points of $\log \Phi(\boldsymbol{z}(s), \,\cdot\,)$ form a basis of $(V^{\otimes n})^{sing}_{\boldsymbol{\lambda}}$, see [12]. That basis is orthogonal with respect to the Shapovalov form. The Shapovalov form is nondegenerate on $(V^{\otimes n})^{sing}_{\boldsymbol{\lambda}}$. By assumptions of the lemma, the limit of the direction of the Bethe vector $\omega(\boldsymbol{z}(s), \boldsymbol{t}(s))$ as $s \to 0$ is different from the limits of the directions of the other Bethe vectors of the basis. These remarks imply the lemma.
Q.E.D.

Corollary 9.2. *If $\alpha \leqslant 0$, then the ratio $\omega(\boldsymbol{T}(s))/\mathrm{Hess}_{\boldsymbol{t}} \log \Phi(\boldsymbol{T}(s))$ has well-defined limit as $s \to 0$.*

Proof. We have

$$\mathrm{Hess}_{\boldsymbol{t}} \log \Phi(\boldsymbol{T}(s)) = S(\omega(\boldsymbol{T}(s)), \omega(\boldsymbol{T}(s)))$$

$$= s^{-2\alpha} S(\omega_\alpha, \omega_\alpha) + o(s^{-2\alpha}),$$

so the ratio $\omega(\boldsymbol{T}(s))/\mathrm{Hess}_{\boldsymbol{t}} \log \Phi(\boldsymbol{T}(s))$ has order $s^{-\alpha}$ as $s \to 0$. Q.E.D.

9.2. Possible places of irregularity of v_F

To prove Theorem 7.1, we need to show that v_F is regular outside of at most a codimension-two algebraic subset of $\mathbb{C}^n$. There are three possible codimension-one irregularity places of v_F:

(9.1) A pole of v_F may occur at a place where $\boldsymbol{z}$ has equal coordinates.

(9.2) A pole of v_F may occur at a place where $\boldsymbol{z}$ has distinct coordinates and the function $\log \Phi(\boldsymbol{z}, \,\cdot\,)$ has a degenerate critical point.

(9.3) A pole of v_F may occur at a place where $\boldsymbol{z}$ has distinct coordinates and there is a critical point which moved to a position with $t_i^{(1)} = z_j$ for some pair (i,j), or to a position with $t_i^{(a)} = t_j^{(a)}$ for some triple (a,i,j), $a > 0$, $i \neq j$, or to a position with $t_i^{(a)} = t_j^{(a+1)}$ for some triple (a,i,j), $a > 0$.

Problem (9.1) is treated in [13]. By Lemmas 4.3 and 4.4 of [13], the map v_F is regular at generic points of the hyperplanes $z_i = z_j$. (In fact, it is shown in Lemmas 4.3 and 4.4 of [13], that the number α of Corollary 9.2 is negative at generic points of possible irregularity corresponding to such hyperplanes, see [13].)

Problem (9.2) of possible irregularity of v_F at the places, where $\log \Phi(\boldsymbol{z}, \cdot)$ has a degenerate critical point, is treated in a standard way using integral representation (7.2). One replaces the sum in (7.1) by an integral over a cycle which can serve all $\boldsymbol{z}$ that are close to a given one, and then observes that the integral is holomorphic in $\boldsymbol{z}$; see, for example, Sections 5.13, 5.17, 5.18 in [1].

Thus, to prove Theorem 7.1 we need to show that generic points of type (9.3) correspond to the points of $\mathcal{W}$ which are noncritical for the Wronski map and which have $\alpha \leqslant 0$.

9.3. Flag exponents

A point $X \in \mathcal{W}$ is an N-dimensional space of polynomials with a basis $g_1(u), \ldots, g_N(u)$ such that $\deg g_i = \lambda_i + N - i$. Each polynomial g_i is defined up to multiplication by a number and addition of a linear combination of $g_{i+1}, \ldots, g_N$.

For any $a \in \mathbb{C}$ define distinct integers $\boldsymbol{d}_{X,a} = (d_1, \ldots, d_N)$ called the flag exponents of X as follows. Choose a basis $g_1, \ldots, g_N$ of X (not changing the degrees of these polynomials) so that $g_1, \ldots, g_N$ have different orders at $u = a$ and set d_i to be the order of g_i at $u = a$.

We say that X is of type $\boldsymbol{d}$ if there exists $a \in \mathbb{C}$ such that $\boldsymbol{d}_{X,a} = \boldsymbol{d}$. For every $\boldsymbol{d}$, denote by $\mathcal{W}_{\boldsymbol{d}} \subset \mathcal{W}$ the closure of the subset of points of type $\boldsymbol{d}$. We are interested in the subsets $\mathcal{W}_{\boldsymbol{d}} \subset \mathcal{W}$ which are of codimension one and whose points correspond to Problem (9.3). Such subsets will be called *essential*.

For example, for $N = 2$, the subset $\mathcal{W}_{(0,2)}$ is the only essential subset. For $N = 3$, the only essential subsets are $\mathcal{W}_{(1,3,0)}$, $\mathcal{W}_{(1,0,2)}$ and $\mathcal{W}_{(0,2,1)}$.

Lemma 9.3. *For given N, if $\mathcal{W}_{\boldsymbol{d}}$ is essential, then $\boldsymbol{d}$ is one of the following $2N - 3$ indices,*

$$\boldsymbol{d}_{1+} = (N - 2, N, N - 3, N - 4, \ldots, 1, 0),$$

$$\boldsymbol{d}_{i+} = (N - 1, N - 2, \ldots, N - i + 1,$$
$$N - i - 1, N - i - 2, N - i, N - i - 3, \ldots, 1, 0),$$

$$\boldsymbol{d}_{i-} = (N - 1, N - 2, \ldots, N - i + 1,$$
$$N - i - 2, N - i, N - i - 1, N - i - 3, \ldots, 1, 0),$$

for $i = 2, \ldots, N - 1$.

Proof. The lemma is proved by straightforward counting of codimensions.   Q.E.D.

If X is a point of $\mathcal{W}_{\boldsymbol{d}_{1+}}$, then for a suitable ordering of its root coordinates we have $z_1 = t_1^{(1)} = t_2^{(1)}$. If X is a point of $\mathcal{W}_{\boldsymbol{d}_{i+}}$, $i > 1$, then for a suitable ordering of its root coordinates we have $t_1^{(i-1)} = t_1^{(i)} = t_2^{(i)}$. If X is a point of $\mathcal{W}_{\boldsymbol{d}_{i-}}$, then for a suitable ordering of its root coordinates we have $t_1^{(i-1)} = t_2^{(i-1)} = t_1^{(i)}$. Each of these properties is a problem of type (9.3).

Lemma 9.4. *Each essential subset is irreducible.*

Proof. It is easy to see that an essential subset is the image of an affine space under a suitable map. Q.E.D.

Lemma 9.5. *Generic points of every essential subset are not critical for the Wronski map.*

Proof. The proof is similar to the proof in Proposition 8 of [5] of the fact that the Jacobian $\det \Delta_{\boldsymbol{q}}$ is nonzero. Q.E.D.

9.4. Proof of Theorem 7.1

9.4.1. Let $\mathcal{W}_{\boldsymbol{d}}$ be an arbitrary essential subset. We fix a certain positive integer q. Then for any numbers $\boldsymbol{r} = (r_0, r_1, r_2, \ldots, r_q)$, such that $r_0 \in \mathbb{C}$, $r_i \in \mathbb{R}$ for $i > 0$, $0 < r_1 < r_2 < \cdots < r_q$, we choose a point $X_{\boldsymbol{r}}(\epsilon, s) \in \mathcal{W}$ depending on two parameters ϵ, s so that $X_{\boldsymbol{r}}(\epsilon, 0) \in \mathcal{W}_{\boldsymbol{d}}$ and the point $X_{\boldsymbol{r}}(\epsilon, s)$ is nice for small nonzero s. The dependence of $X_{\boldsymbol{r}}(\epsilon, s)$ on $\boldsymbol{r}$ in our construction is generic in the following sense. For any hypersurface $\mathcal{Z} \subset \mathcal{W}_{\boldsymbol{d}}$ we can fix $\boldsymbol{r}$ so that the curve $X_{\boldsymbol{r}}(\epsilon, 0)$ does not lie in $\mathcal{Z}$.

For any fixed $\boldsymbol{r}$, we choose ordered root coordinates $\boldsymbol{T}_{\boldsymbol{r}}(\epsilon, s)$ of $X_{\boldsymbol{r}}(\epsilon, s)$ and consider the corresponding Bethe vector $\omega(\boldsymbol{T}_{\boldsymbol{r}}(\epsilon, s))$. We choose a suitable coordinate $\omega_J(\boldsymbol{T}_{\boldsymbol{r}}(\epsilon, s))$ of the Bethe vector and show that for small ϵ the coordinate $\omega_J(\boldsymbol{T}_{\boldsymbol{r}}(\epsilon, s))$ has nonzero limit as $s \to 0$. That statement and Corollary 9.2 show that the corresponding summand in (7.1) is regular at $\mathcal{W}_{\boldsymbol{d}}$.

The proof that $\omega_J(\boldsymbol{T}_{\boldsymbol{r}}(\epsilon, s))$ has nonzero limit is lengthy. We present it for the Lie algebras $\mathfrak{gl}_2$ and $\mathfrak{gl}_3$. The proof for the Lie algebra $\mathfrak{gl}_N$ with $N > 3$ is similar, see also the end of Section 9.4.5.

9.4.2. *Proof for $N = 2$.* A point $X \in \mathcal{W}$ is a two-dimensional space of polynomials. The only essential subset is $\mathcal{W}_{(0,2)}$. This essential subset corresponds to the problem $z_{\lambda_1 + \lambda_2} = t_{\lambda_2-1}^{(1)} = t_{\lambda_2}^{(1)}$ of type (9.3) (after relabeling the root coordinates).

For any numbers $\boldsymbol{r} = (r_0, r_1, r_2, \ldots, r_{\lambda_2+\lambda_1-1})$, such that $r_0 \in \mathbb{C}$, $r_i \in \mathbb{R}$ for $i > 0$, $0 < r_1 < r_2 < \cdots < r_{\lambda_2+\lambda_1-1}$, we choose $X_{\boldsymbol{r}}(\epsilon, s)$ to be

the two-dimensional space of polynomials spanned by

$$g_1(u) = (u - r_0)^{\lambda_1 + 1} + \sum_{i=0}^{\lambda_1} b_i(u - r_0)^i \, ,$$

$$g_2(u) = (u - r_0)^{\lambda_2} + \sum_{i=2}^{\lambda_2 - 1} a_i(u - r_0)^i - s^2 a_2 \, ,$$

where $a_{\lambda_2 - 1} = \epsilon^{r_1}$, $a_{\lambda_2 - i}/a_{\lambda_2 - i + 1} = \epsilon^{r_i}$, $i = 2, \ldots, \lambda_2 - 2$, $b_{\lambda_1} = \epsilon^{r_{\lambda_2 - 1}}$, $b_{\lambda_1 - i}/b_{\lambda_1 - i + 1} = \epsilon^{r_{\lambda_2 + i - 1}}$, $i = 1, \ldots, \lambda_1$. We have $X_{\boldsymbol{r}}(\epsilon, 0) \in \mathcal{W}_{(0,2)}$. Clearly, the dependence of $X_{\boldsymbol{r}}(\epsilon, s)$ on $\boldsymbol{r}$ is generic in the sense defined in Section 9.4.1.

We consider the asymptotic zone $1 \gg |\epsilon| \gg |s| > 0$ and describe the asymptotics in that zone of the roots of g_2 and Wronskian $\mathrm{Wr}(g_1, g_2)$. The leading terms of asymptotics are obtained by the Newton polygon method. If the leading term of some root is at least of order s^2, we shall write that this root equals zero.

The roots of g_2 have the form:

$$t_1^{(1)} \sim r_0 - \epsilon^{r_1}, \quad t_2^{(1)} \sim r_0 - \epsilon^{r_2}, \quad \ldots \; , \quad t_{\lambda_2 - 2}^{(1)} \sim r_0 - \epsilon^{r_{\lambda_2 - 2}},$$

$$t_{\lambda_2 - 1}^{(1)} \sim r_0 + s \, , \quad t_{\lambda_2}^{(1)} \sim r_0 - s \, .$$

The Wronskian is a polynomial in u, ϵ. Below we present only the monomials corresponding to the line segments of the Newton polygon important for the leading asymptotics of the roots,

$$\mathrm{Wr}(g_1, g_2) = (\lambda_1 + 1 - \lambda_2)(u - r_0)^{\lambda_2 + \lambda_1} +$$

$$+ \sum_{i=2}^{\lambda_2 - 1} (\lambda_1 + 1 - i) \, a_i(u - r_0)^{\lambda_1 + i} +$$

$$+ a_2 \sum_{i=0}^{\lambda_1} (i - 2) \, b_i(u - r_0)^{i+1} + \ldots \, .$$

It follows from this formula that the roots of $\mathrm{Wr}(g_1, g_2)$ have the form:

$$z_1 \sim r_0 - \frac{\lambda_1 - \lambda_2 + 2}{\lambda_1 - \lambda_2 + 1} \epsilon^{r_1}, \quad z_2 \sim r_0 - \frac{\lambda_1 - \lambda_2 + 3}{\lambda_1 - \lambda_2 + 2} \epsilon^{r_2}, \quad \ldots \, ,$$

$$z_{\lambda_2 - 2} \sim r_0 - \frac{\lambda_1 - 1}{\lambda_1 - 2} \epsilon^{r_{\lambda_2 - 2}}, \quad z_{\lambda_2 - 1} \sim r_0 - \frac{\lambda_1 - 2}{\lambda_1 - 1} \epsilon^{r_{\lambda_2 - 1}}, \quad \ldots \, ,$$

$$z_{\lambda_2+\lambda_1-4} \sim r_0 - \frac{1}{2}\,\epsilon^{r_{\lambda_2+\lambda_1-4}}, \quad z_{\lambda_2+\lambda_1-3} \sim r_0 + \epsilon^{(r_{\lambda_2+\lambda_1-3}+r_{\lambda_2+\lambda_1-2})/2},$$

$$z_{\lambda_2+\lambda_1-2} \sim r_0 - \epsilon^{(r_{\lambda_2+\lambda_1-3}+r_{\lambda_2+\lambda_1-2})/2}, \quad z_{\lambda_2+\lambda_1-1} \sim r_0 - 2\epsilon^{r_{\lambda_2+\lambda_1-1}},$$

$$z_{\lambda_2+\lambda_1} \sim r_0 \ .$$

The point $\boldsymbol{T_r}(\epsilon, s) = (z_1, \dots, z_{\lambda_2+\lambda_1}, t_1^{(1)}, \dots, t_{\lambda_2}^{(1)})$ is a point of root coordinates of $X_{\boldsymbol{r}}(\epsilon, s)$.

Let us call the root coordinates $t_{\lambda_2-1}^{(1)}$, $t_{\lambda_2}^{(1)}$, $z_{\lambda_2+\lambda_1}$ exceptional, and the remaining root coordinates regular. For each regular root coordinate y the leading term of asymptotics of $y - r_0$ as $\epsilon \to 0$ has the form $A\epsilon^B$ for suitable numbers $A \neq 0$, B.

Lemma 9.6. *The pairs (A, B) are different for different regular root coordinates.*

Proof. A proof is by inspection of the list. Q.E.D.

For each exceptional coordinate y the absolute value of the difference $y - r_0$ is much smaller as $\epsilon \to 0$ than for any regular coordinate.

The Bethe vector is the vector $\omega(\boldsymbol{T_r}(\epsilon, s)) = \sum_J \omega_J(\boldsymbol{T_r}(\epsilon, s))\,e_J v$, where the sum is over all admissible J, see Section 6. An admissible $J = (j_1, \dots, j_{\lambda_1+\lambda_2})$ consists of ones and twos with exactly λ_2 twos. Choose J with $j_i = 2$ for $i = 1, \dots, \lambda_2$. Then

$$(9.4) \qquad \omega_J(\boldsymbol{T_r}(\epsilon, s)) = \sum_{\sigma \in S_{\lambda_2}} \prod_{i=1}^{\lambda_2} \frac{1}{t_{\sigma(i)}^{(1)} - z_i} \ .$$

Lemma 9.7. *For small ϵ, the function $\omega_J(\boldsymbol{T_r}(\epsilon, s))$ has well-defined limit as $s \to 0$.*

Proof. By Lemma 9.6, each summand in (9.4) has well-defined limit. Q.E.D.

Our goal is to show that $\bar{\omega}_J(\epsilon) = \lim_{s\to 0} \omega_J(\boldsymbol{T_r}(\epsilon, s))$ is nonzero for small ϵ.

If f is a function of ϵ and $f \sim A(f)\epsilon^{B(f)}$ for some numbers $A(f) \neq 0$, $B(f)$ as $\epsilon \to 0$, then we call f acceptable, $B(f)$ the order of f and $A(f)$ the leading coefficient of f. If the absolute value of f is smaller than any positive power of ϵ or is the zero function, then we set $B(f) = \infty$.

For every σ, the limit $q_\sigma = \lim_{s\to 0}(\prod_{i=1}^{\lambda_2} \frac{1}{t_{\sigma(i)}^{(1)}-z_i})$ is an acceptable function of order $B(q_\sigma) = -\sum_{i=1}^{\lambda_2} \min(B(t_{\sigma(i)}^{(1)} - r_0), B(z_i - r_0))$. In particular, $B(q_\sigma) \geqslant -B(z_{\lambda_2-1} - r_0) - B(z_{\lambda_2} - r_0) - \sum_{i=1}^{\lambda_2-2} B(t_i^{(1)} - r_0)$.

Lemma 9.8. *The function $\bar{\omega}_J(\epsilon)$ is acceptable. Its order and leading coefficient are given by the formulas*

$$B(\bar{\omega}_J(\epsilon)) = -B(z_{\lambda_2-1} - r_0) - B(z_{\lambda_2} - r_0) - \sum_{i=1}^{\lambda_2-2} B(t_i^{(1)} - r_0) \, ,$$

$$A(\bar{\omega}_J(\epsilon)) = \frac{2}{A(z_{\lambda_2-1} - r_0)A(z_{\lambda_2} - r_0)} \prod_{i=1}^{\lambda_2-2} \frac{1}{A(t_i^{(1)} - r_0) - A(z_i - r_0)} \, .$$

By Lemma 9.8, $\bar{\omega}_J(\epsilon)$ is nonzero for small ϵ and therefore, $\alpha \leqslant 0$ for generic points of $\mathcal{W}_{(0,2)}$. Theorem 7.1 is proved for $N = 2$.

9.4.3. *Proof for $N = 3$ and $\mathcal{W}_{(1,3,0)}$.* A point $X \in \mathcal{W}$ is a three-dimensional space of polynomials. We study the problem $z_{\lambda_1+\lambda_2+\lambda_3} = t_{\lambda_2+\lambda_3-1}^{(1)} = t_{\lambda_2+\lambda_3}^{(1)}$ of type (9.3) (after relabeling the root coordinates).

For any numbers $\boldsymbol{r} = (r_0, r_1, r_2, \ldots, r_{\lambda_3+\lambda_2+\lambda_1-1})$, such that $r_0 \in \mathbb{C}$, $r_i \in \mathbb{R}$ for $i > 0$, $0 < r_1 < r_2 < \cdots < r_{\lambda_3+\lambda_2+\lambda_1-1}$, we choose $X_{\boldsymbol{r}}(\epsilon, s)$ to be the three-dimensional space of polynomials spanned by

$$g_1(u) = (u - r_0)^{\lambda_1+2} + \sum_{i=1}^{\lambda_1+1} c_i(u - r_0)^i \, ,$$

$$g_2(u) = (u - r_0)^{\lambda_2+1} + \sum_{i=3}^{\lambda_2} b_i(u - r_0)^i - 3s^2 b_3(u - r_0) \, ,$$

$$g_3(u) = (u - r_0)^{\lambda_3} + \sum_{i=0}^{\lambda_3-1} a_i(u - r_0)^i \, ,$$

where

$$
\begin{aligned}
&a_{\lambda_3-1} = \epsilon^{r_1}, &&a_{\lambda_3-i}/a_{\lambda_3-i+1} = \epsilon^{r_i}, &&i = 2, \ldots, \lambda_3 \, , \\
&b_{\lambda_2} = \epsilon^{r_{\lambda_3}+1}, &&b_{\lambda_2-i}/b_{\lambda_2-i+1} = \epsilon^{r_{\lambda_3}+i+1}, &&i = 1, \ldots, \lambda_2 - 3 \, , \\
&c_{\lambda_1+1} = \epsilon^{r_{\lambda_3}+\lambda_2-1}, &&c_{\lambda_1-i}/c_{\lambda_1-i+1} = \epsilon^{r_{\lambda_3}+\lambda_2+i}, &&i = 0, \ldots, \lambda_1 - 1 \, .
\end{aligned}
$$

We have $X_{\boldsymbol{r}}(\epsilon, 0) \in \mathcal{W}_{(1,3,0)}$. Clearly, the dependence of $X_{\boldsymbol{r}}(\epsilon, s)$ on $\boldsymbol{r}$ is generic in the sense defined in Section 9.4.1.

We consider the asymptotic zone $1 \gg |\epsilon| \gg |s| > 0$ and describe the asymptotics in that zone of the roots of the polynomials g_3, $\mathrm{Wr}(g_2, g_3)$, $\mathrm{Wr}(g_1, g_2, g_3)$. We obtain the leading terms of asymptotics by the Newton polygon method. If the leading term of some root is at least of order s^2, we shall write that this root equals zero.

The roots of g_3 are of the form:

$$t_1^{(2)} \sim r_0 - \epsilon^{r_1}, \quad t_2^{(2)} \sim r_0 - \epsilon^{r_2}, \quad \ldots \quad , \quad t_{\lambda_3}^{(2)} \sim r_0 - \epsilon^{r_{\lambda_3}} \; .$$

We have

$$\mathrm{Wr}(g_2, g_3) = (\lambda_2 + 1 - \lambda_3)(u - r_0)^{\lambda_2 + \lambda_3} +$$

$$+ \sum_{i=0}^{\lambda_3 - 1} (\lambda_2 + 1 - i)\, a_i (u - r_0)^{\lambda_2 + i - 1} +$$

$$+ a_0 \sum_{i=3}^{\lambda_2} i b_i (u - r_0)^{i-1} - 3s^2 a_0 b_3 + \ldots \; ,$$

where the dots denote the monomials which are not important for the leading asymptotics of the roots. The roots of $\mathrm{Wr}(g_2, g_3)$ are of the form

$$t_1^{(1)} \sim r_0 - \frac{\lambda_2 - \lambda_3 + 2}{\lambda_2 - \lambda_3 + 1}\, \epsilon^{r_1}, \quad \ldots \quad , \quad t_{\lambda_3}^{(1)} \sim r_0 - \frac{\lambda_2 + 1}{\lambda_2}\, \epsilon^{r_{\lambda_3}},$$

$$t_{\lambda_3 + 1}^{(1)} \sim r_0 - \frac{\lambda_2}{\lambda_2 + 1}\, \epsilon^{r_{\lambda_3 + 1}}, \quad \ldots \quad , \quad t_{\lambda_3 + \lambda_2 - 2}^{(1)} \sim r_0 - \frac{3}{4}\, \epsilon^{r_{\lambda_3 + \lambda_2 - 2}},$$

$$t_{\lambda_3 + \lambda_2 - 1}^{(1)} \sim r_0 + s, \quad t_{\lambda_3 + \lambda_2}^{(1)} \sim r_0 - s.$$

We have

$$\mathrm{Wr}\,(g_1, g_2, g_3) =$$

$$= (\lambda_1 + 1 - \lambda_2)(\lambda_1 + 2 - \lambda_3)(\lambda_2 + 1 - \lambda_3)(u - r_0)^{\lambda_3 + \lambda_2 + \lambda_1} +$$

$$+ \sum_{i=0}^{\lambda_3 - 1} (\lambda_1 + 1 - \lambda_2)(\lambda_1 + 2 - i)(\lambda_2 + 1 - i)\, a_i (u - r_0)^{i + \lambda_2 + \lambda_1} +$$

$$+ a_0 \sum_{i=3}^{\lambda_2} (\lambda_1 + 2 - i)(\lambda_1 + 2)\, i\, b_i (u - r_0)^{\lambda_1 + i - 1} +$$

$$+ a_0 b_3 \sum_{i=1}^{\lambda_1 + 1} 3i(i - 3)\, c_i (u - r_0)^i + \ldots \; .$$

The roots of $\mathrm{Wr}(g_1, g_2, g_3)$ are of the form

$$z_1 \sim r_0 - \frac{(\lambda_1 + 3 - \lambda_3)(\lambda_2 + 2 - \lambda_3)}{(\lambda_1 + 2 - \lambda_3)(\lambda_2 + 1 - \lambda_3)}\, \epsilon^{r_1}, \quad \ldots \quad ,$$

$$z_{\lambda_3} \sim r_0 - \frac{(\lambda_1 + 2)(\lambda_2 + 1)}{(\lambda_1 + 1)\lambda_2}\, \epsilon^{r_{\lambda_3}},$$

$$z_{\lambda_3+1} \sim r_0 - \frac{(\lambda_1 + 2 - \lambda_2)\lambda_2}{(\lambda_1 + 1 - \lambda_2)(\lambda_2 + 1)}\,\epsilon^{r_{\lambda_3+1}},\ \ \dots\ ,$$

$$z_{\lambda_3+\lambda_2-2} \sim r_0 - \frac{3(\lambda_1 - 1)}{4(\lambda_1 - 2)}\,\epsilon^{r_{\lambda_3+\lambda_2-2}},$$

$$z_{\lambda_3+\lambda_2-1} \sim r_0 - \frac{(\lambda_1 + 1)(\lambda_1 - 2)}{(\lambda_1 + 2)(\lambda_1 - 1)}\,\epsilon^{r_{\lambda_3+\lambda_2-1}},\ \ \dots\ ,$$

$$z_{\lambda_3+\lambda_2+\lambda_1-4} \sim r_0 - \frac{2}{5}\,\epsilon^{r_{\lambda_3+\lambda_2+\lambda_1-4}},$$

$$z_{\lambda_3+\lambda_2+\lambda_1-3} \sim r_0 + \frac{1}{\sqrt{2}}\,\epsilon^{(r_{\lambda_3+\lambda_2+\lambda_1-3}+r_{\lambda_3+\lambda_2+\lambda_1-2})/2},$$

$$z_{\lambda_3+\lambda_2+\lambda_1-2} \sim r_0 - \frac{1}{\sqrt{2}}\,\epsilon^{(r_{\lambda_3+\lambda_2+\lambda_1-3}+r_{\lambda_3+\lambda_2+\lambda_1-2})/2},$$

$$z_{\lambda_3+\lambda_2+\lambda_1-1} \sim r_0 - \epsilon^{r_{\lambda_3+\lambda_2+\lambda_1-1}},\quad z_{\lambda_3+\lambda_2+\lambda_1} \sim r_0\,.$$

The point $\boldsymbol{T_r}(\epsilon, s) = (z_1, \dots, z_{\lambda_3+\lambda_2+\lambda_1}, t_1^{(1)}, \dots, t_{\lambda_3+\lambda_2}^{(1)}, t_1^{(2)}, \dots, t_{\lambda_3}^{(2)})$ is a point of root coordinates of $X_{\boldsymbol{r}}(\epsilon, s)$.

Let us call the root coordinates $t_{\lambda_3+\lambda_2-1}^{(1)}$, $t_{\lambda_3+\lambda_2}^{(1)}$, $z_{\lambda_3+\lambda_2+\lambda_1}$ exceptional, and the remaining root coordinates regular. For each regular root coordinate y the leading term of asymptotics of $y - r_0$ as $\epsilon \to 0$ has the form $A\epsilon^B$ for suitable numbers $A \neq 0$, B.

Lemma 9.9. *The pairs (A, B) are different for different regular root coordinates.*

Proof.   A proof is by inspection of the list.   Q.E.D.

For each exceptional coordinate y the absolute value of the difference $y - r_0$ is much smaller as $\epsilon \to 0$ than for any regular coordinate.

The Bethe vector has the form $\omega(\boldsymbol{T_r}(\epsilon, s)) = \sum_J \omega_J(\boldsymbol{T_r}(\epsilon, s))\,e_J v$, where the sum is over all admissible J, see Section 6. An admissible $J = (j_1, \dots, j_{\lambda_3+\lambda_2+\lambda_1})$ consists of ones, twos and threes with exactly λ_3 threes and λ_2 twos. Choose J with $j_i = 3$ for $i = 1, \dots, \lambda_3$ and $j_i = 2$ for $i = \lambda_3 + \lambda_2 - 1, \lambda_3 + \lambda_2, \dots, \lambda_3 + 2\lambda_2 - 2$. Then $\omega_J(\boldsymbol{T_r}(\epsilon, s))$ is given by the formula

$$(9.5)\quad \omega_J(\boldsymbol{T_r}(\epsilon, s)) = \sum_{\sigma \in S_{\lambda_3+\lambda_2}} \sum_{\tau \in S_{\lambda_3}} \prod_{i=1}^{\lambda_3} \frac{1}{(t_{\tau(i)}^{(2)} - t_{\sigma(i)}^{(1)})(t_{\sigma(i)}^{(1)} - z_i)} \times$$

$$\times \prod_{i=\lambda_3+1}^{\lambda_3+\lambda_2} \frac{1}{t_{\sigma(i)}^{(1)} - z_{\lambda_2+i-2}}\,.$$

Lemma 9.10. *For small ϵ, the function $\omega_J(\boldsymbol{T_r}(\epsilon, s))$ has well-defined limit as $s \to 0$.*

Proof. By Lemma 9.9, each summand in (9.5) has well-defined limit. Q.E.D.

Our goal is to show that $\bar{\omega}_J(\epsilon) = \lim_{s\to 0} \omega_J(\boldsymbol{T_r}(\epsilon, s))$ is nonzero for small ϵ.

For every σ the second product in (9.5), has well-defined limit

$$q_\sigma = \lim_{s\to 0} \left(\prod_{i=\lambda_3+1}^{\lambda_3+\lambda_2} \frac{1}{t_{\sigma(i)}^{(1)} - z_{\lambda_2+i-2}} \right).$$

That limit is an acceptable function of order

$$B(q_\sigma) = -\sum_{i=\lambda_3+1}^{\lambda_3+\lambda_2} \min(B(t_{\sigma(i)}^{(1)} - r_0), B(z_{\lambda_2+i-2} - r_0)).$$

In particular,

$$B(q_\sigma) \geqslant -B(z_{\lambda_3+2\lambda_2-3} - r_0) - B(z_{\lambda_3+2\lambda_2-2} - r_0) - \sum_{i=\lambda_3+1}^{\lambda_3+\lambda_2-2} B(t_i^{(1)} - r_0).$$

The largest second products are those with

$$B(q_\sigma) = -B(z_{\lambda_3+2\lambda_2-3} - r_0) - B(z_{\lambda_3+2\lambda_2-2} - r_0) - \sum_{i=\lambda_3+1}^{\lambda_3+\lambda_2-2} B(t_i^{(1)} - r_0).$$

For every σ, τ, the first product in (9.5) has well-defined limit

$$p_{\sigma\tau} = \lim_{s\to 0} \left(\prod_{i=1}^{\lambda_3} \frac{1}{(t_{\tau(i)}^{(2)} - t_{\sigma(i)}^{(1)})(t_{\sigma(i)}^{(1)} - z_i)} \right).$$

That limit is an acceptable function of order

$$B(p_{\sigma\tau}) = -\sum_{i=1}^{\lambda_3} \Big(\min(B(t_{\tau(i)}^{(2)} - r_0), B(t_{\sigma(i)}^{(1)} - r_0)) +$$
$$+ \min(B(t_{\sigma(i)}^{(1)} - r_0), B(z_i - r_0)) \Big).$$

In particular, $B(p_{\sigma\tau}) \geqslant -\sum_{i=1}^{\lambda_3}(B(t_i^{(2)} - r_0) + B(z_i - r_0))$. The largest first products are those with

$$B(p_{\sigma\tau}) = -\sum_{i=1}^{\lambda_3}(B(t_i^{(2)} - r_0) + B(z_i - r_0)).$$

Lemma 9.11. *The function* $\bar{\omega}_J(\epsilon)$ *is acceptable. Its order and leading coefficient are given by the formulas*

$$B(\bar{\omega}_J(\epsilon)) = -\sum_{i=1}^{\lambda_3}(B(t_i^{(2)} - r_0) + B(z_i - r_0)) - B(z_{\lambda_3+2\lambda_2-3} - r_0) -$$

$$- B(z_{\lambda_3+2\lambda_2-2} - r_0) - \sum_{i=\lambda_3+1}^{\lambda_3+\lambda_2-2} B(t_i^{(1)} - r_0).$$

$$A(\bar{\omega}_J(\epsilon)) = 2\,(\lambda_2 - 2)!\,\frac{1}{A(z_{\lambda_3+2\lambda_2-3} - r_0)A(z_{\lambda_3+2\lambda_2-2} - r_0)} \times$$

$$\times \prod_{i=1}^{\lambda_3} \frac{1}{(A(t_i^{(2)} - r_0) - A(t_i^{(1)} - r_0))(A(t_i^{(1)} - r_0) - A(z_i - r_0))} \times$$

$$\times \prod_{i=\lambda_3+1}^{\lambda_3+\lambda_2-2} \frac{1}{A(t_i^{(1)} - r_0)}.$$

Proof. It is easy to see that if σ, τ are such that the second product in (9.5) has order

$$-B(z_{\lambda_3+2\lambda_2-3} - r_0) - B(z_{\lambda_3+2\lambda_2-2} - r_0) - \sum_{i=\lambda_3+1}^{\lambda_3+\lambda_2-2} B(t_i^{(1)} - r_0),$$

then the first product has order $-\sum_{i=1}^{\lambda_3}(B(t_i^{(2)} - r_0) + B(z_i - r_0))$ only if it equals $\prod_{i=1}^{\lambda_3}(t_i^{(2)} - t_i^{(1)})^{-1}(t_i^{(1)} - z_i)^{-1}$. This implies the lemma.
Q.E.D.

9.4.4. *Proof for $N = 3$ and $\mathcal{W}_{(0,2,1)}$.* We study the problem $t_{\lambda_2+\lambda_3}^{(1)} = t_{\lambda_2+\lambda_3-1}^{(1)} = t_{\lambda_3}^{(2)}$ of type (9.3) (after relabeling the root coordinates).

For any numbers $\boldsymbol{r} = (r_0, r_1, r_2, \ldots, r_{\lambda_3+\lambda_2+\lambda_1})$, such that $r_0 \in \mathbb{C}$, $r_i \in \mathbb{R}$ for $i > 0$, $0 < r_1 < r_2 < \cdots < r_{\lambda_3+\lambda_2+\lambda_1}$, we choose $X_{\boldsymbol{r}}(\epsilon, s)$ to

be the three-dimensional space of polynomials spanned by

$$g_1(u) = (u - r_0)^{\lambda_1+2} + \sum_{i=0}^{\lambda_1+1} c_i (u - r_0)^i \, ,$$

$$g_2(u) = (u - r_0)^{\lambda_2+1} + \sum_{i=2}^{\lambda_2} b_i (u - r_0)^i + s^2 b_2 \, ,$$

$$g_3(u) = (u - r_0)^{\lambda_3} + \sum_{i=1}^{\lambda_3-1} a_i (u - r_0)^i \, ,$$

where

$$a_{\lambda_3-1} = \epsilon^{r_1}, \qquad a_{\lambda_3-i}/a_{\lambda_3-i+1} = \epsilon^{r_i}, \qquad i = 2, \ldots, \lambda_3 - 1 \, ,$$

$$b_{\lambda_2} = \epsilon^{r_{\lambda_3}}, \qquad b_{\lambda_2-i}/b_{\lambda_2-i+1} = \epsilon^{r_{\lambda_3+i}}, \qquad i = 1, \ldots, \lambda_2 - 2 \, ,$$

$$c_{\lambda_1+1} = \epsilon^{r_{\lambda_3+\lambda_2-1}}, \quad c_{\lambda_1-i}/c_{\lambda_1-i+1} = \epsilon^{r_{\lambda_3+\lambda_2+i}} \qquad i = 0, \ldots, \lambda_1 \, .$$

We have $X_{\boldsymbol{r}}(\epsilon, 0) \in \mathcal{W}_{(0,2,1)}$. Clearly, the dependence of $X_{\boldsymbol{r}}(\epsilon, s)$ on $\boldsymbol{r}$ is generic in the sense defined in Section 9.4.1.

We consider the same asymptotic zone $1 \gg |\epsilon| \gg |s| > 0$. The roots of g_3 are of the form:

$$t_1^{(2)} \sim r_0 - \epsilon^{r_1}, \;\; t_2^{(2)} \sim r_0 - \epsilon^{r_2}, \;\; \ldots \;, \;\; t_{\lambda_3-1}^{(2)} \sim r_0 - \epsilon^{r_{\lambda_3-1}}, \;\; t_{\lambda_3}^{(2)} = r_0 \, .$$

We have

$$\mathrm{Wr}(g_2, g_3) = (\lambda_2 + 1 - \lambda_3)(u - r_0)^{\lambda_3+\lambda_2} +$$

$$+ \sum_{i=1}^{\lambda_3-1} (\lambda_2 + 1 - i) \, a_i (u - r_0)^{\lambda_2+i} +$$

$$+ a_1 \sum_{i=2}^{\lambda_2} (i - 1) \, b_i (u - r_0)^i - s^2 a_1 b_2 + \ldots \, .$$

The roots of $\mathrm{Wr}(g_2, g_3)$ are of the form

$$t_1^{(2)} \sim r_0 - \frac{\lambda_2 - \lambda_3 + 2}{\lambda_2 - \lambda_3 + 1} \epsilon^{r_1}, \;\; \ldots \;, \;\; t_{\lambda_3-1}^{(2)} \sim r_0 - \frac{\lambda_2}{\lambda_2 - 1} \epsilon^{r_{\lambda_3-1}},$$

$$t_{\lambda_3}^{(2)} \sim r_0 - \frac{\lambda_2 - 1}{\lambda_2} \epsilon^{r_{\lambda_3}}, \;\; \ldots \;, \;\; t_{\lambda_3+\lambda_2-2}^{(2)} \sim r_0 - \frac{1}{2} \epsilon^{r_{\lambda_3+\lambda_2-2}},$$

$$t_{\lambda_3+\lambda_2-1}^{(2)} \sim r_0 + s \, , \;\; t_{\lambda_3+\lambda_2}^{(2)} \sim r_0 - s \, .$$

We have

$$\mathrm{Wr}\,(g_1, g_2, g_3) =$$

$$= (\lambda_1 + 1 - \lambda_2)(\lambda_1 + 2 - \lambda_3)(\lambda_2 + 1 - \lambda_3)(u - r_0)^{\lambda_3 + \lambda_2 + \lambda_1} +$$

$$+ \sum_{i=1}^{\lambda_3 - 1} (\lambda_1 + 1 - \lambda_2)(\lambda_1 + 2 - i)(\lambda_2 + 1 - i)\,a_i(u - r_0)^{i + \lambda_2 + \lambda_1} +$$

$$+ a_1 \sum_{i=2}^{\lambda_2} (\lambda_1 + 2 - i)(\lambda_1 + 1)(i - 1)\,b_i(u - r_0)^{\lambda_1 + i} +$$

$$+ a_1 b_2 \sum_{i=0}^{\lambda_1 + 1} (i - 2)(i - 1)\,c_i(u - r_0)^i + \ldots \ .$$

The roots of $\mathrm{Wr}(g_1, g_2, g_3)$ are of the form

$$z_1 \sim r_0 - \frac{(\lambda_1 + 3 - \lambda_3)(\lambda_2 + 2 - \lambda_3)}{(\lambda_1 + 2 - \lambda_3)(\lambda_2 + 1 - \lambda_3)}\,\epsilon^{r_1}, \ \ldots \ ,$$

$$z_{\lambda_3 - 1} \sim r_0 - \frac{(\lambda_1 + 1)\lambda_2}{\lambda_1(\lambda_2 - 1)}\,\epsilon^{r_{\lambda_3 - 1}},$$

$$z_{\lambda_3} \sim r_0 - \frac{(\lambda_1 + 2 - \lambda_2)(\lambda_2 - 1)}{(\lambda_1 + 1 - \lambda_2)\lambda_2}\,\epsilon^{r_{\lambda_3}}, \ \ldots \ ,$$

$$z_{\lambda_3 + \lambda_2 - 3} \sim r_0 - \frac{\lambda_1}{2(\lambda_1 + 1)}\,\epsilon^{r_{\lambda_3 + \lambda_2 - 3}},$$

$$z_{\lambda_3 + \lambda_2 - 2} \sim r_0 - \frac{(\lambda_1 - 1)}{(\lambda_1 + 1)}\,\epsilon^{r_{\lambda_3 + \lambda_2 - 2}}, \ \ldots \ ,$$

$$z_{\lambda_3 + \lambda_2 + \lambda_1 - 3} \sim r_0 - \frac{1}{3}\,\epsilon^{r_{\lambda_3 + \lambda_2 + \lambda_1 - 3}},$$

$$z_{\lambda_3 + \lambda_2 + \lambda_1 - 2} \sim r_0 + x_1\epsilon^m, \quad z_{\lambda_3 + \lambda_2 + \lambda_1 - 1} \sim r_0 + x_2\epsilon^m,$$

$$z_{\lambda_3 + \lambda_2 + \lambda_1} \sim r_0 + x_3\epsilon^m,$$

where x_1, x_2, x_3 are distinct roots of the equation $x^3 + 1 = 0$ and

$$m = \frac{1}{3}\left(r_{\lambda_3 + \lambda_2 + \lambda_1 - 2} + r_{\lambda_3 + \lambda_2 + \lambda_1 - 1} + r_{\lambda_3 + \lambda_2 + \lambda_1}\right).$$

The point $\boldsymbol{T_r}(\epsilon, s) = (z_1, \ldots, z_{\lambda_3 + \lambda_2 + \lambda_1}, t_1^{(1)}, \ldots, t_{\lambda_3 + \lambda_2}^{(1)}, t_1^{(2)}, \ldots, t_{\lambda_3}^{(2)})$ is a point of root coordinates of $X_{\boldsymbol{r}}(\epsilon, s)$.

Let us call the root coordinates $t^{(2)}_{\lambda_3}$, $t^{(1)}_{\lambda_3+\lambda_2-1}$, $t^{(1)}_{\lambda_3+\lambda_2}$ exceptional, and the remaining root coordinates regular. For each regular root coordinate y the leading term of asymptotics of $y - r_0$ as $\epsilon \to 0$ has the form $A\epsilon^B$ for suitable numbers $A \neq 0$, B.

Lemma 9.12. *The pairs (A, B) are different for different regular root coordinates.*

Proof. A proof is by inspection of the list. Q.E.D.

For each exceptional coordinate y the absolute value of the difference $y - r_0$ is much smaller as $\epsilon \to 0$ than for any regular coordinate.

The Bethe vector has the form $\omega(\boldsymbol{T_r}(\epsilon, s)) = \sum_J \omega_J(\boldsymbol{T_r}(\epsilon, s)) e_J v$, where the sum is over all admissible J, see Section 6. An admissible $J = (j_1, \ldots, j_{\lambda_3+\lambda_2+\lambda_1})$ consists of ones, twos and threes with exactly λ_3 threes and λ_2 twos. Choose J with $j_i = 3$ for $i = 1, 2, \ldots, \lambda_3 - 1$, $\lambda_3 + \lambda_2 - 2$ and $j_i = 2$ for $i = \lambda_3, \lambda_3 + 1, \ldots, \lambda_3 + \lambda_2 - 3, \lambda_3 + \lambda_2 - 1$, $\lambda_3 + \lambda_2$. Then $\omega_J(\boldsymbol{T_r}(\epsilon, s))$ is given by the formula

$$(9.6) \quad \omega_J(\boldsymbol{T_r}(\epsilon, s)) = \sum_{\sigma \in S_{\lambda_3+\lambda_2}} \sum_{\tau \in S_{\lambda_3}} \prod_{i=1}^{\lambda_3-1} \frac{1}{(t^{(2)}_{\tau(i)} - t^{(1)}_{\sigma(i)})(t^{(1)}_{\sigma(i)} - z_i)} \times$$

$$\times \prod_{i=\lambda_3+1}^{\lambda_3+\lambda_2-2} \frac{1}{t^{(1)}_{\sigma(i)} - z_{i-1}} \times$$

$$\times \frac{1}{(t^{(2)}_{\tau(\lambda_3)} - t^{(1)}_{\sigma(\lambda_3)})(t^{(1)}_{\sigma(\lambda_3)} - z_{\lambda_3+\lambda_2-2})} \prod_{i=\lambda_3+\lambda_2-1}^{\lambda_3+\lambda_2} \frac{1}{t^{(1)}_{\sigma(i)} - z_i}.$$

Lemma 9.13. *For small ϵ, the function $\omega_J(\boldsymbol{T_r}(\epsilon, s))$ has well-defined limit as $s \to 0$.*

Proof. Divergent summands in (9.6) are the summands with factors $(t^{(2)}_{\lambda_3} - t^{(1)}_{\lambda_3+\lambda_2-1})^{-1}$ or $(t^{(2)}_{\lambda_3} - t^{(1)}_{\lambda_3+\lambda_2})^{-1}$. The divergent summands come in pairs. There are two types of such pairs. The first type has the form

$$p_1^{Ckij} = \frac{C}{(t^{(2)}_{\lambda_3} - t^{(1)}_{\lambda_3+\lambda_2-1})(t^{(1)}_{\lambda_3+\lambda_2-1} - z_i)(t^{(2)}_k - t^{(1)}_{\lambda_3+\lambda_2})(t^{(1)}_{\lambda_3+\lambda_2} - z_j)},$$

$$p_2^{Ckij} = \frac{C}{(t^{(2)}_{\lambda_3} - t^{(1)}_{\lambda_3+\lambda_2})(t^{(1)}_{\lambda_3+\lambda_2} - z_i)(t^{(2)}_k - t^{(1)}_{\lambda_3+\lambda_2-1})(t^{(1)}_{\lambda_3+\lambda_2-1} - z_j)},$$

where C is a common factor, regular as $s \to 0$. The second type has the form

$$q_1^{Cij} = \frac{C}{(t_{\lambda_3}^{(2)} - t_{\lambda_3+\lambda_2-1}^{(1)})(t_{\lambda_3+\lambda_2-1}^{(1)} - z_i)(t_{\lambda_3+\lambda_2}^{(1)} - z_j)} \,,$$

$$q_2^{Cij} = \frac{C}{(t_{\lambda_3}^{(2)} - t_{\lambda_3+\lambda_2}^{(1)})(t_{\lambda_3+\lambda_2}^{(1)} - z_i)(t_{\lambda_3+\lambda_2-1}^{(1)} - z_j)} \,,$$

where C is a common factor, regular as $s \to 0$. Each pair has well-defined limit as $s \to 0$,

$$\lim_{s \to 0} \left(p_1^{Ckij} + p_2^{Ckij} \right) =$$

$$= \lim_{s \to 0} \frac{C}{(t_k^{(2)} - r_0)(z_i - r_0)(z_j - r_0)} \left(\frac{2}{t_k^{(2)} - r_0} + \frac{2}{z_j - r_0} - \frac{2}{z_i - r_0} \right),$$

$$\lim_{s \to 0} \left(q_1^{Cij} + q_2^{Cij} \right) =$$

$$= \lim_{s \to 0} \frac{C}{(z_i - r_0)(z_j - r_0)} \left(\frac{2}{z_j - r_0} - \frac{2}{z_i - r_0} \right).$$

These limits will be called resonant pairs.   Q.E.D.

It is easy to see that $\bar{\omega}_J(\epsilon) = \lim_{s \to 0} \omega_J(\boldsymbol{T_r}(\epsilon, s))$ is an acceptable function and its order equals

$$b = -B(z_{\lambda_3+\lambda_2} - r_0) - \sum_{i=1}^{\lambda_3-1} B(t_i^{(2)} - r_0) - \sum_{i=1}^{\lambda_3+\lambda_2} B(z_i - r_0) \,.$$

Indeed, the order of the limit of any convergent summand in (9.6) is greater than b. The order of any resonant pair is not less than b. There is exactly one resonant pair of order b. That pair is of the second type and corresponds to

$$q_1 = \prod_{i=1}^{\lambda_3-1} \frac{1}{(t_i^{(2)} - t_i^{(1)})(t_i^{(1)} - z_i)} \prod_{i=\lambda_3}^{\lambda_3+\lambda_2-3} \frac{1}{t_i^{(1)} - z_i} \times$$

$$\times \frac{1}{(0 - s)(s - z_{\lambda_3+\lambda_2-2})(t_{\lambda_3+\lambda_2-2}^{(1)} - z_{\lambda_3+\lambda_2-1})(-s - z_{\lambda_3+\lambda_2})} \,,$$

$$q_2 = \prod_{i=1}^{\lambda_3-1} \frac{1}{(t_i^{(2)} - t_i^{(1)})(t_i^{(1)} - z_i)} \prod_{i=\lambda_3}^{\lambda_3+\lambda_2-3} \frac{1}{t_i^{(1)} - z_i} \times$$

$$\times \frac{1}{(0+s)(-s-z_{\lambda_3+\lambda_2-2})(t_{\lambda_3+\lambda_2-2}^{(1)} - z_{\lambda_3+\lambda_2-1})(s-z_{\lambda_3+\lambda_2})} .$$

Thus, $\bar{\omega}_J(\epsilon)$ is nonzero for small ϵ.

9.4.5. *Proof for $N=3$ and $\mathcal{W}_{(1,0,2)}$.* We study the problem $t_{\lambda_2+\lambda_3}^{(1)} = t_{\lambda_3-1}^{(2)} = t_{\lambda_3}^{(2)}$ of type (9.3) (after relabeling the root coordinates).

For any numbers $\boldsymbol{r} = (r_0, r_1, r_2, \ldots, r_{\lambda_3+\lambda_2+\lambda_1})$, such that $r_0 \in \mathbb{C}$, $r_i \in \mathbb{R}$ for $i > 0$, $0 < r_1 < r_2 < \cdots < r_{\lambda_3+\lambda_2+\lambda_1}$, we choose $X_{\boldsymbol{r}}(\epsilon, s) \in \mathcal{W}$ to be the three-dimensional space of polynomials spanned by

$$g_1(u) = (u - r_0)^{\lambda_1+2} + \sum_{i=1}^{\lambda_1+1} c_i(u - r_0)^i ,$$

$$g_2(u) = (u - r_0)^{\lambda_2+1} + \sum_{i=0}^{\lambda_2} b_i(u - r_0)^i ,$$

$$g_3(u) = (u - r_0)^{\lambda_3} + \sum_{i=2}^{\lambda_3-1} a_i(u - r_0)^i - a_2 s^2 ,$$

where

$$a_{\lambda_3-1} = \epsilon^{r_1}, \qquad a_{\lambda_3-i}/a_{\lambda_3-i+1} = \epsilon^{r_i}, \qquad\qquad i = 2, \ldots, \lambda_3 - 2 ,$$
$$b_{\lambda_2} = \epsilon^{r_{\lambda_3}-1}, \qquad b_{\lambda_2-i}/b_{\lambda_2-i+1} = \epsilon^{r_{\lambda_3+i-1}}, \qquad i = 1, \ldots, \lambda_2 ,$$
$$c_{\lambda_1+1} = \epsilon^{r_{\lambda_3+\lambda_2}}, \quad c_{\lambda_1-i}/c_{\lambda_1-i+1} = \epsilon^{r_{\lambda_3+\lambda_2+i+1}}, \qquad i = 0, \ldots, \lambda_1 - 1 .$$

We have $X_{\boldsymbol{r}}(\epsilon, 0) \in \mathcal{W}_{(1,0,2)}$. Clearly, the dependence of $X_{\boldsymbol{r}}(\epsilon, s)$ on $\boldsymbol{r}$ is such that the corresponding curve $X_{\boldsymbol{r}}(\epsilon, 0)$ is generic in $\mathcal{W}_{\boldsymbol{d}}$ in the sense defined in Section 9.4.1.

We consider the same asymptotic zone $1 \gg |\epsilon| \gg |s| > 0$. The roots of g_3 are of the form:

$$t_1^{(2)} \sim r_0 - \epsilon^{r_1}, \ \ldots\ , \ t_{\lambda_3-2}^{(2)} \sim r_0 - \epsilon^{r_{\lambda_3-2}}, \ \ t_{\lambda_3-1}^{(2)} \sim r_0 + s, \ \ t_{\lambda_3}^{(1)} \sim r_0 - s .$$

We have

$$\mathrm{Wr}(g_2, g_3) = (\lambda_2 + 1 - \lambda_3)(u - r_0)^{\lambda_3 + \lambda_2} +$$

$$+ \sum_{i=2}^{\lambda_3 - 1} (\lambda_2 + 1 - i)\, a_i (u - r_0)^{\lambda_3 + i - 1} +$$

$$+ a_2 \sum_{i=0}^{\lambda_2} (i - 2)\, b_i (u - r_0)^{i+1} + \dots \,.$$

The roots of $\mathrm{Wr}(g_2, g_3)$ are of the form

$$t_1^{(1)} \sim r_0 - \frac{\lambda_2 - \lambda_3 + 2}{\lambda_2 - \lambda_3 + 1}\, \epsilon^{r_1}, \ \dots \ , \ t_{\lambda_3 - 2}^{(1)} \sim r_0 - \frac{\lambda_2 - 1}{\lambda_2 - 2}\, \epsilon^{r_{\lambda_3 - 2}},$$

$$t_{\lambda_3 - 1}^{(1)} \sim r_0 - \frac{\lambda_2 - 2}{\lambda_2 - 1}\, \epsilon^{r_{\lambda_3 - 1}}, \ \dots \ , \ t_{\lambda_3 + \lambda_2 - 4}^{(1)} \sim r_0 - \frac{1}{2}\, \epsilon^{r_{\lambda_3 + \lambda_2 - 4}},$$

$$t_{\lambda_3 + \lambda_2 - 3}^{(1)} \sim \epsilon^{(r_{\lambda_3 + \lambda_2 - 3} + r_{\lambda_3 + \lambda_2 - 2})/2}, \ \ t_{\lambda_3 + \lambda_2 - 2}^{(1)} \sim -\epsilon^{(r_{\lambda_3 + \lambda_2 - 3} + r_{\lambda_3 + \lambda_2 - 2})/2},$$

$$t_{\lambda_3 + \lambda_2 - 1}^{(1)} \sim r_0 - 2\epsilon^{r_{\lambda_3 + \lambda_2 - 1}}, \ \ t_{\lambda_3 + \lambda_2}^{(1)} \sim r_0\,.$$

We have

$$\mathrm{Wr}\,(g_1, g_2, g_3) =$$

$$+\ (\lambda_1 + 1 - \lambda_2)(\lambda_1 + 2 - \lambda_3)(\lambda_2 + 1 - \lambda_3)(u - r_0)^{\lambda_3 + \lambda_2 + \lambda_1} +$$

$$+\ \sum_{i=2}^{\lambda_3 - 1} (\lambda_1 + 1 - \lambda_2)(\lambda_1 + 2 - i)(\lambda_2 + 1 - i)\, a_i (u - r_0)^{i + \lambda_2 + \lambda_1} +$$

$$+\ a_2 \sum_{i=0}^{\lambda_2} (\lambda_1 + 2 - i)\lambda_1 (i - 2)\, b_i (u - r_0)^{\lambda_1 + i + 1} -$$

$$-\ a_2 b_0 \sum_{i=1}^{\lambda_1 + 1} 2i\, (i - 2)\, c_i (u - r_0)^{i-1} + \dots \,.$$

The roots of $\mathrm{Wr}(g_1, g_2, g_3)$ are of the form

$$z_1 \sim r_0 - \frac{(\lambda_1 + 3 - \lambda_3)(\lambda_2 + 2 - \lambda_3)}{(\lambda_1 + 2 - \lambda_3)(\lambda_2 + 1 - \lambda_3)}\, \epsilon^{r_1}, \ \dots \ ,$$

$$z_{\lambda_3 - 2} \sim r_0 - \frac{\lambda_1(\lambda_2 - 1)}{(\lambda_1 - 1)(\lambda_2 - 2)}\, \epsilon^{r_{\lambda_3 - 2}},$$

$$z_{\lambda_3-1} \sim r_0 - \frac{(\lambda_1 + 2 - \lambda_2)(\lambda_2 - 2)}{(\lambda_1 + 1 - \lambda_2)(\lambda_2 - 1)}\, \epsilon^{r_{\lambda_3+1}}, \ \ldots \ ,$$

$$z_{\lambda_3+\lambda_2-4} \sim r_0 - \frac{\lambda_1 - 1}{2(\lambda_1 - 2)}\, \epsilon^{r_{\lambda_3+\lambda_2-4}},$$

$$z_{\lambda_3+\lambda_2-3} \sim r_0 + \sqrt{\frac{\lambda_1 + 1}{\lambda_1 - 1}}\, \epsilon^{(r_{\lambda_3+\lambda_2-3}+r_{\lambda_3+\lambda_2-2})/2},$$

$$z_{\lambda_3+\lambda_2-2} \sim r_0 - \sqrt{\frac{\lambda_1 + 1}{\lambda_1 - 1}}\, \epsilon^{(r_{\lambda_3+\lambda_2-3}+r_{\lambda_3+\lambda_2-2})/2},$$

$$z_{\lambda_3+\lambda_2-1} \sim r_0 - \frac{2(\lambda_1 + 2)}{\lambda_1 + 1}\, \epsilon^{r_{\lambda_3+\lambda_2-1}},$$

$$z_{\lambda_3+\lambda_2} \sim r_0 - \frac{(\lambda_1 + 1)(\lambda_1 - 1)}{(\lambda_1 + 2)\lambda_1}\, \epsilon^{r_{\lambda_3+\lambda_2}}, \ \ldots \ ,$$

$$z_{\lambda_3+\lambda_2+\lambda_1-2} \sim r_0 - \frac{3}{8}\, \epsilon^{r_{\lambda_3+\lambda_2+\lambda_1-2}},$$

$$z_{\lambda_3+\lambda_2+\lambda_1-1} \sim r_0 + \frac{1}{\sqrt{3}}\, \epsilon^{(r_{\lambda_3+\lambda_2+\lambda_1-1}+r_{\lambda_3+\lambda_2+\lambda_1})/2},$$

$$z_{\lambda_3+\lambda_2+\lambda_1} \sim r_0 - \frac{1}{\sqrt{3}}\, \epsilon^{(r_{\lambda_3+\lambda_2+\lambda_1-1}+r_{\lambda_3+\lambda_2+\lambda_1})/2}.$$

The point $\boldsymbol{T_r}(\epsilon, s) = (z_1, \ldots, z_{\lambda_3+\lambda_2+\lambda_1}, t_1^{(1)}, \ldots, t_{\lambda_3+\lambda_2}^{(1)}, t_1^{(2)}, \ldots, t_{\lambda_3}^{(2)})$ is a point of root coordinates of $X_r(\epsilon, s)$.

Let us call the root coordinates $t_{\lambda_3-1}^{(2)}, t_{\lambda_3}^{(2)}, t_{\lambda_3+\lambda_2}^{(1)}$ exceptional, and the remaining root coordinates regular. For each regular root coordinate y the leading term of asymptotics of $y - r_0$ as $\epsilon \to 0$ has the form $A\epsilon^B$ for suitable numbers $A \neq 0$, B.

Lemma 9.14. *The pairs (A, B) are different for different regular root coordinates.*

Proof. A proof is by inspection of the list. Q.E.D.

For each exceptional coordinate y the absolute value of the difference $y - r_0$ is much smaller as $\epsilon \to 0$ than for any regular coordinate.

The Bethe vector has the form $\omega(\boldsymbol{T_r}(\epsilon, s)) = \sum_J \omega_J(\boldsymbol{T_r}(\epsilon, s))\, e_J v$, where the sum is over all admissible J, see Section 6. An admissible $J = (j_1, \ldots, j_{\lambda_3+\lambda_2+\lambda_1})$ consists of ones, twos and threes with exactly λ_3 threes and λ_2 twos. Choose J with $j_i = 3$ for $i = 1, 2, \ldots, \lambda_3 - 3, \lambda_3 - 2, \lambda_3 + \lambda_2 - 1, \lambda_3 + \lambda_2$ and $j_i = 2$ for $i = \lambda_3 - 1, \lambda_3, \ldots, \lambda_3 + \lambda_2 - 2$. Then

$\omega_J(\boldsymbol{T_r}(\epsilon, s))$ is given by the formula

$$(9.7) \quad \omega_J(\boldsymbol{T_r}(\epsilon, s)) = \sum_{\sigma \in S_{\lambda_3+\lambda_2}} \sum_{\tau \in S_{\lambda_3}} \prod_{i=1}^{\lambda_3-2} \frac{1}{(t^{(2)}_{\tau(i)} - t^{(1)}_{\sigma(i)})(t^{(1)}_{\sigma(i)} - z_i)} \times$$

$$\times \prod_{i=\lambda_3-1}^{\lambda_3} \frac{1}{(t^{(2)}_{\tau(i)} - t^{(1)}_{\sigma(i)})(t^{(1)}_{\sigma(i)} - z_{\lambda_2+i})} \prod_{i=\lambda_3+1}^{\lambda_3+\lambda_2} \frac{1}{t^{(1)}_{\sigma(i)} - z_{i-2}}.$$

It is easy to see that $\bar{\omega}_J(\epsilon) = \lim_{s\to 0} \omega_J(\boldsymbol{T_r}(\epsilon, s))$ is an acceptable function and its order equals

$$b = -\sum_{i=1}^{\lambda_3-2} B(t^{(2)}_i - r_0) - \sum_{i=1}^{\lambda_3+\lambda_2} B(z_i - r_0) - 2B(t^{(1)}_{\lambda_3+\lambda_2-1} - r_0).$$

Namely, consider the following four summands in (9.7):

$$q = \prod_{i=1}^{\lambda_3-2} \frac{1}{t^{(2)}_i - t^{(1)}_i} \prod_{i=1}^{\lambda_3+\lambda_2-4} \frac{1}{t^{(1)}_i - z_i} \prod_{\lambda_3+\lambda_2-1}^{\lambda_3+\lambda_2} \frac{1}{t^{(1)}_i - z_i} \times$$

$$\times \left(\frac{1}{(t^{(2)}_{\lambda_3-1} - t^{(1)}_{\lambda_3+\lambda_2-1})(t^{(2)}_{\lambda_3} - t^{(1)}_{\lambda_3+\lambda_2})} + \right.$$

$$\left. + \frac{1}{(t^{(2)}_{\lambda_3-1} - t^{(1)}_{\lambda_3+\lambda_2})(t^{(2)}_{\lambda_3} - t^{(1)}_{\lambda_3+\lambda_2-1})} \right) \times$$

$$\times \left(\frac{1}{(t^{(1)}_{\lambda_3+\lambda_2-3} - z_{\lambda_3+\lambda_2-3})(t^{(1)}_{\lambda_3+\lambda_2-2} - z_{\lambda_3+\lambda_2-2})} + \right.$$

$$\left. + \frac{1}{(t^{(1)}_{\lambda_3+\lambda_2-3} - z_{\lambda_3+\lambda_2-2})(t^{(1)}_{\lambda_3+\lambda_2-2} - z_{\lambda_3+\lambda_2-3})} \right).$$

Then the order of $\lim_{s\to 0} q$ equals b and the order of

$$\lim_{s\to 0}(\omega_J((\boldsymbol{T_r}(\epsilon, s)) - q))$$

is greater than b. Therefore, $\lim_{s\to 0} \omega_J(\boldsymbol{T_r}(\epsilon, s))$ is nonzero for small ϵ.

For $N = 3$ and every essential subset $\mathcal{W_d}$, we proved that the Bethe vector is nonzero at generic points of $\mathcal{W_d}$ and, hence, the number α of Corollary 9.2 is nonpositive. Thus, Theorem 7.1 is proved for $N = 3$.

For $N > 3$, Theorem 7.1 is proved similarly. Namely, let $\mathcal{W_d}$, $\boldsymbol{d} = (d_1, \ldots, d_N)$, be an arbitrary essential subset. A point $X \in \mathcal{W}$ is an N-dimensional space of polynomials. For any numbers $\boldsymbol{r} = (r_0, r_1, r_2, \ldots)$,

such that $r_0 \in \mathbb{C}$, $r_i \in \mathbb{R}$ for $i > 0$, $0 < r_1 < r_2 < \dots$, we choose $X_{\boldsymbol{r}}(\epsilon, s)$ to be the N-dimensional space of polynomials spanned by the polynomials $g_1, \dots, g_N$, where

$$g_j(u) = (u - r_0)^{\lambda_j + N - j} + \sum_{i=d_j}^{\lambda_j + N - j - 1} a_{j,i}(u - r_0)^i \,,$$

if $j = 1$ or $j > 1$ and $d_j - d_{j-1} \neq 2$, and

$$g_j(u) = (u - r_0)^{\lambda_j + N - j} + \sum_{i=d_j}^{\lambda_j + N - j - 1} a_{j,i}(u - r_0)^i - s^2 a_{j,d_j}(u - r_0)^{d_j - 2} \,,$$

if $j > 1$ and $d_j - d_{j-1} = 2$. We order the coefficients $a_{j,i}$ by defining $(j, i) > (j', i')$ if either $j > j'$, or $j = j'$ and $i > i'$. According to that order we define the coefficients $a_{j,i}$ by setting

$$a_{N,\lambda_N + N - 2} = \epsilon^{r_1}, \quad a_{N,\lambda_N + N - 3}/a_{N,\lambda_N + N - 2} = \epsilon^{r_2}, \quad \dots \quad,$$

see the cases $N = 2, 3$. We have $X_{\boldsymbol{r}}(\epsilon, 0) \in \mathcal{W}_{\boldsymbol{d}}$ and the point $X_{\boldsymbol{r}}(\epsilon, s)$ is nice for small nonzero s.

We consider the asymptotic zone $1 \gg |\epsilon| \gg |s| > 0$ and determine the asymptotics of the roots of the Wronskians $\mathrm{Wr}(g_N, g_{N-1}, \dots, g_i)$. That gives the asymptotics of coordinates of the corresponding critical point of the master function. Among the coordinates there will be exactly three exceptional coordinates with the same ϵ-asymptotics, see examples for $N = 2, 3$. Depending on the indices of the exceptional coordinates we choose a coordinate of the associated Bethe vector and show that the chosen coordinate has a well-defined limit as $s \to 0$. That statement, proved for any exceptional subset $\mathcal{W}_{\boldsymbol{d}}$, implies Theorem 7.1.

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