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Hyperplane arrangements: computations and conjectures

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Abstract.

This paper provides an overview of selected results and open problems in the theory of hyperplane arrangements, with an emphasis on computations and examples. We give an introduction to many of the essential tools used in the area, such as Koszul and Lie algebra methods, homological techniques, and the Bernstein–Gelfand–Gelfand correspondence, all illustrated with concrete calculations. We also explore connections of arrangements to other areas, such as De Concini–Procesi wonderful models, the Feichtner–Yuzvinsky algebra of an atomic lattice, fatpoints and blowups of projective space, and plane curve singularities.

Contents

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§1. Introduction and algebraic preliminaries

There are a number of wonderful sources available on hyperplane arrangements, most notably Orlik–Terao's landmark 1992 text [58]. In the last decade alone several excellent surveys have appeared: Suciu's paper on aspects of the fundamental group [84], Yuzvinsky's paper on Orlik–Solomon algebras and local systems cohomology [97], and several monographs devoted to connections to areas such as hypergeometric integrals [59], mathematical physics [91], as well as proceedings from conferences at Sapporo [38], Northeastern [16] and Istanbul [30].

The aim of this note is to provide an overview of some recent results and open problems, with a special emphasis on connections to computation. The paper also gives a concrete and example driven introduction for non-specialists, but there is enough breadth here that even experts should find something new. There are few proofs, but rather pointers to original source material. We also explore connections of arrangements to other areas, such as De Concini–Procesi wonderful models, the Feichtner–Yuzvinsky algebra of an atomic lattice, the Orlik–Terao algebra and blowups, and plane curve singularities. All computations in this survey can be performed using Macaulay2 [45], available at: `http://www.math.uiuc.edu/Macaulay2/`, and the arrangements package by Denham and Smith [22].

Let $V = \mathbb{K}^\ell$, and let S be the symmetric algebra on V^*: $S = \bigoplus_{i\in\mathbb{Z}} S_i$ is a $\mathbb{Z}$-graded ring, which means that if $s_i \in S_i$ and $s_j \in S_j$, then $s_i \cdot s_j \in S_{i+j}$. A graded S-module M is defined in similar fashion. Of special interest is the case where S_0 is a field $\mathbb{K}$, so that each M_i is a $\mathbb{K}$–vector space. The free S module with generator in degree i is written $S(-i)$, and in general $M(i)_j = M_{i+j}$.

Definition 1. *The Hilbert function* $HF(M,i) = \dim_{\mathbb{K}} M_i$.

Definition 2. *The Hilbert series* $HS(M,i) = \sum_{\mathbb{Z}} \dim_{\mathbb{K}} M_i t^i$.

Example 3. $S = \mathbb{K}[x,y]$, $M = S/\langle x^2, xy\rangle$. Then

i	M_i	$M(-2)_i$
0	1	0
1	x,y	0
2	y^2	1
3	y^3	x,y
4	y^4	y^2
n	y^n	y^{n-2}

The respective Hilbert series are

$$HS(M,i) = \frac{1-2t^2+t^3}{(1-t)^2} \text{ and } HS(M(-2),i) = \frac{t^2(1-2t^2+t^3)}{(1-t)^2}.$$

An induction shows that $HS(S(-i),t) = t^i/(1-t)^\ell$; this makes it easy to compute the Hilbert series of an arbitrary graded module from a free resolution. For $S/\langle x^2, xy\rangle$, a minimal free resolution is

$$0 \longrightarrow S(-3) \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} S(-2)^2 \xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} S \longrightarrow S/I \longrightarrow 0.$$

$$\text{The map } [x^2, xy] \text{ sends } \begin{array}{l} e_1 \mapsto x^2 \\ e_2 \mapsto xy, \end{array}$$

so in order to have a map of graded modules, the basis elements of the source must have degree two, explaining the shifts in the free resolution. Taking the alternating sum of the Hilbert series yields

$$HS(M,i) = \frac{t^3-2t^2+1}{(1-t)^2}$$

which agrees with the previous computation.

Example 4. The 2×2 minors of $\begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}$ define the twisted cubic $I \subseteq S = \mathbb{K}[x,y,z,w]$.

$$0 \longrightarrow S(-3)^2 \xrightarrow{\begin{bmatrix} -z & w \\ y & -z \\ -x & y \end{bmatrix}} S(-2)^3 \xrightarrow{\begin{bmatrix} y^2-xz & yz-xw & z^2-yw \end{bmatrix}} S \longrightarrow S/I$$

The numerical information in a free resolution may be compactly displayed as a *betti table*:

$$b_{ij} = \dim_{\mathbb{K}} \mathrm{Tor}_i^S(M,\mathbb{K})_{i+j}.$$

total	1	3	2
0	1	–	–
1	–	3	2

In particular, the indexing begins at position $(0,0)$ and is read over and down. So for the twisted cubic, $b_{21}(S/I) = \dim_{\mathbb{K}} \mathrm{Tor}_2^S(S/I,\mathbb{K})_3 = 2$.

We now give a quick review of arrangements. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of complex hyperplanes in $\mathbb{C}^\ell$. We assume $\mathcal{A}$ is *central* and *essential*: the ℓ_i with $H_i = V(\ell_i)$ are homogeneous, and the common zero locus $V(\ell_1, \ldots, \ell_n) = 0 \in \mathbb{C}^\ell$. The central condition means that $\mathcal{A}$ also defines an arrangement in $\mathbb{P}^{\ell-1}$. The main combinatorial object associated to $\mathcal{A}$ is the intersection lattice $L_\mathcal{A}$, which consists of the intersections of elements of $\mathcal{A}$, ordered by reverse inclusion. $\mathbb{C}^n$ is the lattice element $\hat{0}$ and the rank one elements of $L_\mathcal{A}$ are the hyperplanes.

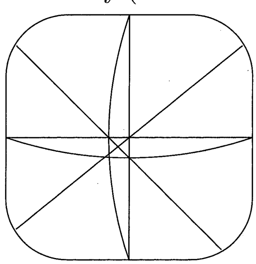
Definition 5. *The Möbius function $\mu : L_\mathcal{A} \longrightarrow \mathbb{Z}$ is defined by*

$$\begin{array}{rcl} \mu(\hat{0}) & = & 1 \\ \mu(t) & = & -\sum\limits_{s<t} \mu(s), \text{ if } \hat{0} < t. \end{array}$$

The Poincaré and characteristic polynomials of $\mathcal{A}$ are defined as

$$\pi(\mathcal{A}, t) = \sum_{x \in L(\mathcal{A})} \mu(x) \cdot (-t)^{\mathrm{rank}(x)}, \text{ and } \chi(\mathcal{A}, t) = t^{\mathrm{rk}(\mathcal{A})} \pi(\mathcal{A}, \frac{-1}{t}).$$

Example 6. The A_3 arrangement is $\bigcup_{1 \leq i < j \leq 4} V(x_i - x_j) \subseteq \mathbb{C}^4$. Projecting along $(1,1,1,1)$ gives a central arrangement in $\mathbb{C}^3$, hence a configuration of lines in $\mathbb{P}^2$. This configuration corresponds to the figure below, but with the line at infinity (which bounds the figure) omitted.



For the 7 rank two elements of $L(A_3)$, the four corresponding to triple points have $\mu = 2$, and the three normal crossings have $\mu = 1$. Thus, $\pi(A_3, t) = 1 + 6t + 11t^2 + 6t^3$. Adding the bounding line gives the non-Fano arrangement NF, with $\pi(NF, t) = 1 + 7t + 15t^2 + 9t^3$.

In [57], Orlik and Solomon showed that the cohomology ring of the complement $M_\mathcal{A} = \mathbb{C}^n \setminus \bigcup_{i=1}^d H_i$ has presentation $H^*(M_\mathcal{A}, \mathbb{Z}) = \bigwedge(\mathbb{Z}^n)/I$, with generators $e_1, \ldots, e_n$ in degree 1 and

$$I = \langle \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r} \mid \operatorname{codim} H_{i_1} \cap \cdots \cap H_{i_r} < r \rangle.$$

For additional background on arrangements, see [58].

§2. $D(\mathcal{A})$ and freeness

Let $\mathcal{A} = \bigcup_{i=1}^{n} H_i \subseteq V = \mathbb{C}^\ell$ be a central arrangement. For each i, fix $V(l_i) = H_i \in \mathcal{A}$, and define $Q_\mathcal{A} = \prod_{i=1}^n l_i \in S = \mathbb{C}[x_1, \ldots, x_\ell]$.

Definition 7. *The module of $\mathcal{A}$-derivations (or Terao module) is the submodule of $Der_\mathbb{C}(S)$ consisting of vector fields tangent to $\mathcal{A}$:*

$$D(\mathcal{A}) = \{\theta \in Der_\mathbb{C}(S) | \theta(l_i) \in \langle l_i \rangle \text{ for all } l_i \text{ with } V(l_i) \in \mathcal{A}\}.$$

An arrangement is free when $D(\mathcal{A})$ is a free S-module. In this case, the degrees of the generators of $D(\mathcal{A})$ are called the exponents of the arrangement. Note that $D(\mathcal{A})$ is always nonzero, since the Euler derivation $\theta_E = \sum_{i=1}^{\ell} x_i \partial/\partial x_i \in D(\mathcal{A})$. It is easy to show that

$$D(\mathcal{A}) \simeq S \cdot \theta_E \oplus syz(J_\mathcal{A}),$$

where $J_\mathcal{A}$ is the Jacobian ideal of $Q_\mathcal{A}$, and syz denotes the module of syzygies on $J_\mathcal{A}$: polynomial relations on the generators of $J_\mathcal{A}$.

Theorem 8 (Saito [72]). *$\mathcal{A}$ is free iff there exist ℓ elements*

$$\theta_i = \sum_{j=1}^{\ell} f_{ij} \frac{\partial}{\partial x_j} \in D(\mathcal{A}),$$

such that $\det([f_{ij}]) = c \cdot Q_\mathcal{A}$, *for some* $c \neq 0$.

Example 9. For Example 6, a computation shows that

$$\begin{array}{rcl} D(A_3) & \simeq & S(-1) \oplus S(-2) \oplus S(-3) \\ D(NF) & \simeq & S(-1) \oplus S(-3) \oplus S(-3). \end{array}$$

Interestingly, the respective Poincaré polynomials factor, as

$$\pi(A_3, t) = (1+t)(1+2t)(1+3t), \text{ and } \pi(NF, t) = (1+t)(1+3t)^2.$$

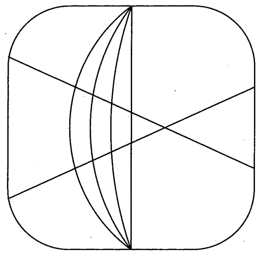
This suggests the possibility of a connection between the exponents of a free arrangement and the Poincaré polynomial.

A landmark result in arrangements is:

Theorem 10 (Terao [86]). *If* $D(\mathcal{A}) \simeq \bigoplus_{i=1}^{\ell} S(-a_i)$, *then*

$$\pi(\mathcal{A}, t) = \prod (1 + a_i t) = \sum \dim_\mathbb{C} H^i(\mathbb{C}^\ell \setminus \mathcal{A}) t^i.$$

Example 11. [Stanley] For $\mathcal{A}$ below, $\pi(\mathcal{A},t) = (1+t)(1+3t)^2$.

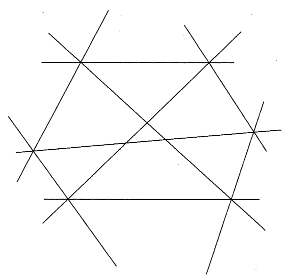


A computation shows that $\mathcal{A}$ is not free, so factorization of $\pi(\mathcal{A},t)$ is a necessary but not sufficient for freeness of $\mathcal{A}$.

A famous open conjecture in the field of arrangements is:

Conjecture 12 (Terao). *If* $char(\mathbb{K}) = 0$, *then freeness of* $D(\mathcal{A})$ *depends only on* $L_{\mathcal{A}}$.

Example 13. [Ziegler's pair [101]] Let $\mathcal{A}$ be an arrangement of 9 lines in $\mathbb{P}^2$, as below.



Then $D(\mathcal{A})$ depends on nonlinear geometry: if the six triple points lie on a smooth conic, we compute:

$$0 \longrightarrow S(-7) \oplus S(-8) \longrightarrow S(-5) \oplus S^3(-6) \longrightarrow syz(J_{\mathcal{A}}) \longrightarrow 0\ ,$$

while if six triple points are not on a smooth conic, the resolution is:

$$0 \longrightarrow S^4(-7) \longrightarrow S^6(-6) \longrightarrow syz(J_{\mathcal{A}}) \longrightarrow 0\ .$$

A version of Terao's theorem applies to any arrangement:

Definition 14. $D^p(\mathcal{A}) \subseteq \Lambda^p(Der_{\mathbb{K}}(S))$ *consists of* θ *such that*

$$\theta(l_i, f_2, \ldots, f_p) \in \langle l_i \rangle, \forall\ V(l_i) \in \mathcal{A}, f_i \in S.$$

Theorem 15 (Solomon–Terao, [82]).

$$\chi(\mathcal{A},t) = (-1)^\ell \lim_{x\to 1} \sum_{p\geq 0} HS(D^p(\mathcal{A});x)(t(x-1)-1)^p.$$

Problem 16. Relate the modules $D^p(\mathcal{A})$, for $p \geq 2$, to $L_\mathcal{A}$.

A closed subarrangement $\hat{\mathcal{A}} \subseteq A$ is a subarrangement such that $\hat{\mathcal{A}} = \mathcal{A}_X$ for some flat X. The best result relating $D(\mathcal{A})$ to $L_\mathcal{A}$ is:

Theorem 17 (Terao, [89]). *If* $\hat{\mathcal{A}} \subset \mathcal{A}$ *is a closed subarrangement, then* $\operatorname{pdim} D(\mathcal{A}) \geq \operatorname{pdim} D(\hat{\mathcal{A}})$.

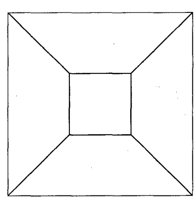
Problem 18. Find bounds on $\operatorname{pdim} D(\mathcal{A})$ depending on $L_\mathcal{A}$.

A particularly interesting class of arrangements are graphic arrangements, which are subarrangements of A_n. Given a simple (no loops or multiple edges) graph G, with ℓ vertices and edge set E, we define $\mathcal{A}_G = \{z_i - z_j = 0 \mid (i,j) \in \mathsf{E} \subseteq \mathbb{C}^\ell\}$.

Theorem 19 (Stanley [83]). $\mathcal{A}_G$ *is supersolvable iff* G *is chordal.*

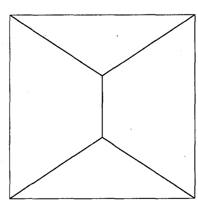
Theorem 20 (Kung–Schenck [52]). *If* $\mathcal{A}_G$ *has an induced* k*-cycle, then* $\operatorname{pdim} D(\mathcal{A}_G) \geq k-3$.

Example 21. The largest induced cycle of G below is a 6-cycle.



A computation shows $\operatorname{pdim}(D(\mathcal{A})) = 3$.

Example 22. The largest induced cycle of G below is a 4-cycle.



A computation shows $\operatorname{pdim}(D(\mathcal{A})) = 2$.

Problem 23. Find a formula for $\operatorname{pdim} D(\mathcal{A}_G)$.

Definition 24. *A triple* $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ *of arrangements consists of a choice of* $H \in \mathcal{A}$, *with* $\mathcal{A}' = \mathcal{A} \setminus H, \mathcal{A}'' = \mathcal{A}|_H$.

A main tool for proving freeness is Terao's addition-deletion theorem.

Theorem 25 (Terao [87]). *For a triple, any two of the following imply the third*

(1) $D(\mathcal{A}) \simeq \oplus_{i=1}^{n} S(-b_i)$.
(2) $D(\mathcal{A}') \simeq S(-b_n+1) \oplus_{i=1}^{n-1} S(-b_i)$.
(3) $D(\mathcal{A}'') \simeq \oplus_{i=1}^{n-1} S/L(-b_i)$.

Example 26. In Example 6, the A_3 arrangement is free with exponents $\{1,2,3\}$. Let H be the line at infinity, which meets A_3 in four points. Then $D(\mathcal{A}'')$ is free, with exponents $\{1,2\}$, so the non-Fano arrangement is free with exponents $\{1,3,3\}$, which agrees with our earlier computation. Example 4.59 of [58] gives a free arrangement for which the addition-deletion theorem does not apply.

As a corollary of Theorem 25, Terao showed that supersolvable arrangements are free.

Definition 27. *An element* X *of a lattice is modular if for all* $Y \in L$ *and all* $Z < Y$, $Z \vee (X \wedge Y) = (Z \vee X) \wedge Y$. *A central arrangement* $\mathcal{A}$ *is supersolvable if there exists a maximal chain* $\hat{0} = X_0 < X_1 < \cdots < X_n = \hat{1}$ *of modular elements in* $L(\mathcal{A})$.

For line configurations in $\mathbb{P}^2$, the supersolvability condition simply means there is a singular point $p \in \mathcal{A}$ such that every other singularity of $\mathcal{A}$ lies on a line of $\mathcal{A}$ which passes through p. For example, the A_3 arrangement is supersolvable, since any triple point is such a singularity. For arrangements in $\mathbb{P}^2$, there is a beautiful characterization of freeness involving multiarrangements.

Definition 28. *A multiarrangement* $(\mathcal{A}, \mathbf{m})$ *consists of an arrangement* $\mathcal{A}$, *along with a multiplicity* $m_i \in \mathbb{N}$ *for each* $H \in \mathcal{A}$.

$$D(\mathcal{A}, \mathbf{m}) = \{\theta \mid \theta(l_i) \in \langle l_i^{m_i} \rangle\}.$$

Theorem 29. $\mathcal{A} \subseteq \mathbb{P}^2$ *is free if and only if*

(1) $\pi(\mathcal{A}, t) = (1+t)(1+at)(1+bt)$ *and*
(2) $D(\mathcal{A}|_H, \mathbf{m}) \simeq S/L(-a) \oplus S/L(-b)$,

where (2) *holds for all* $H = V(L) \in \mathcal{A}$, *with* $\mathbf{m}(H_i) = \mu_{\mathcal{A}}(H \cap H_i)$.

The necessity of these conditions was shown by Ziegler in [100], and sufficiency was proved by Yoshinaga in [94]. In [93], Yoshinaga gives a generalization to higher dimensions.

§3. Multiarrangements

The exponents of free multiarrangements are not combinatorial:

Example 30. [Ziegler, [100]] Consider the two multiarrangements in $\mathbb{P}^1$, with underlying arrangements defined by

$$\begin{aligned}\mathcal{A}_1 &= V(x \cdot y \cdot (x+y) \cdot (x-y)) \\ \mathcal{A}_2 &= V(x \cdot y \cdot (x+y) \cdot (x-ay)),\end{aligned}$$

with $a \neq 1$. To compute $D(\mathcal{A}_1, (1,1,3,3))$, we must find all

$$\theta = f_1(x,y)\partial/\partial x + f_2\partial/\partial y$$

such that

$$\begin{aligned}\theta(x) \in \langle x \rangle,\ \theta(x+y) \in \langle x+y \rangle^3 \\ \theta(y) \in \langle y \rangle,\ \theta(x-y) \in \langle x-y \rangle^3.\end{aligned}$$

Thus, $D(\mathcal{A}_1, (1,1,3,3))$ is the kernel of the matrix

$$\begin{bmatrix} 1 & 0 & x & 0 & 0 & 0 \\ 0 & 1 & 0 & y & 0 & 0 \\ 1 & 1 & 0 & 0 & (x+y)^3 & 0 \\ 1 & -1 & 0 & 0 & 0 & (x-y)^3 \end{bmatrix}.$$

Computations show that $D(\mathcal{A}_1, (1,1,3,3))$ has exponents $\{3,5\}$, and $D(\mathcal{A}_2, (1,1,3,3))$ has exponents $\{4,4\}$.

There is an analog of Theorem 15 for multiarrangements.

Definition 31. *$D^p(\mathcal{A}, \mathbf{m}) \subseteq \Lambda^p(Der_{\mathbb{K}}(S))$ consists of θ such that*

$$\theta(l_i, f_2, \ldots, f_p) \in \langle l_i \rangle^{m(l_i)}, \forall\ V(l_i) \in \mathcal{A}, f_i \in S.$$

Theorem 32 (Abe–Terao–Wakefield [2]). *Define*

$$\begin{aligned}\Psi(\mathcal{A}, \mathbf{m}, t, x) &= \sum_{p=0}^{\ell} HS(D^p(\mathcal{A}, \mathbf{m}), x)(t(x-1)-1)^p. \\ \chi((\mathcal{A}, \mathbf{m}), t) &= (-1)^{\ell} \lim_{x \to 1} \Psi(\mathcal{A}, \mathbf{m}, t, 1).\end{aligned}$$

If $D^1(\mathcal{A}, \mathbf{m}) \simeq \oplus S(-d_i)$, then $\chi((\mathcal{A}, \mathbf{m}), t) = \prod_{i=1}^{\ell}(1 + d_i t)$.

In [1], Abe–Terao–Wakefield prove an addition-deletion theorem for multiarrangements by introducing *Euler multiplicity* for the restriction. It follows from the Hilbert–Burch theorem that any $(\mathcal{A}, \mathbf{m}) \subseteq \mathbb{P}^1$ is free, which leads to the question of whether there exist other arrangements which are free for any $\mathbf{m}$. In [3], Abe–Terao–Yoshinaga prove that any such arrangement is a product of one- and two-dimensional arrangements. Nevertheless, several natural questions arise:

Problem 33. Characterize the projective dimension of $D(\mathcal{A}, \mathbf{m})$.

Problem 34. Define supersolvability for multiarrangements.

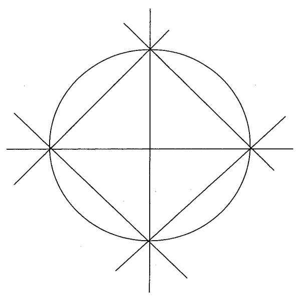
§4. Arrangements of plane curves

For a collection of hypersurfaces

$$\mathcal{C} = \bigcup_i V(f_i) \subseteq \mathbb{P}^n,$$

the module of derivations $D(\mathcal{C})$ is obtained by substituting f_i for l_i in Definition 7. It is not hard to prove that Saito's criterion still applies. Are there other freeness theorems?

Example 35. For the arrangement $\mathcal{C} \subseteq \mathbb{P}^2$ depicted below



we compute that $D(\mathcal{C}) \simeq S(-1) \oplus S(-2) \oplus S(-5)$.

This example can be explained by an addition-deletion theorem [79], but there is subtle behavior related to singular points. For the remainder of this section, $C = \cup_i V(f_i) \subseteq \mathbb{C}^2$ is reduced plane curve, and if $p \in C$ is a singular point, translate so $p = (0,0)$.

Definition 36. *A plane curve singularity is quasihomogeneous if and only if there exists a holomorphic change of variables so that $f(x,y) = \sum c_{ij}x^iy^j$ is weighted homogeneous: there exists $\alpha, \beta \in \mathbb{Q}$ such that $\sum c_{ij}x^{i\cdot\alpha}y^{j\cdot\beta}$ is homogeneous.*

Definition 37. *The Milnor number at $(0,0)$ is*

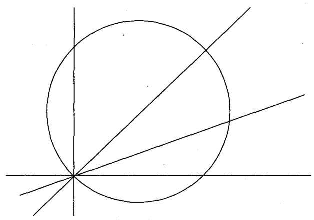
$$\mu_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle.$$

The Tjurina number at $(0,0)$ is

$$\tau_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f\rangle.$$

For a projective plane curve $V(Q) \subseteq \mathbb{P}^2$, it is easy to see that the degree of $Jac(Q) = \sum_{p \in sing(V(Q))} \tau_p$.

Example 38. Let $\mathcal{C}$ be as below:

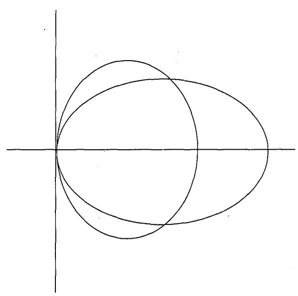


If p is an ordinary singularity with k distinct branches, then $\mu_p(C) = (k-1)^2$, so the sum of the Milnor numbers is 20. However, a computation shows that $\deg(J_{\mathcal{C}}) = 19$. All singularities are ordinary, but the singularity at the origin is not quasihomogeneous.

Theorem 39 (Saito [71]). *If $C = V(f)$ has an isolated singularity at the origin, then $f \in Jac(f)$ iff f is quasihomogeneous.*

For arrangements of lines and conics such that every singular point is quasihomogeneous, [79] proves an addition/deletion theorem; [78] generalizes the result to curves of higher genus.

Example 40. Let $\mathcal{C}$ be as below:



$D(\mathcal{C})$ has exponents $\{1, 2, 3\}$, which can be shown using the aforementioned addition-deletion theorem. Change $\mathcal{C}$ to $\mathcal{C}'$ via:

$$y = 0 \longrightarrow x - 13y = 0.$$

A computation shows that $D(\mathcal{C}')$ is not free. Thus, for line-conic arrangements, freeness is not combinatorial.

Problem 41. Define supersolvability for hypersurface arrangements.

Problem 42. Give combinatorial bounds on $\operatorname{pdim} D(\mathcal{C})$.

Problem 43. Analyze associated primes and Ext modules of $D(\mathcal{C})$.

§5. The Orlik–Terao algebra and blowups

The Orlik–Terao algebra is a symmetric analog of the Orlik–Solomon algebra. While the Orlik–Solomon algebra records the existence of dependencies among sets of hyperplanes, the Orlik–Terao algebra records the actual dependencies. If $\operatorname{codim} \cap_{j=1}^{m} H_{i_j} < m$, then there exist c_{i_j} with

$$\sum_{j=1}^{m} c_{i_j} \cdot l_{i_j} = 0 \text{ a dependency.}$$

Definition 44. *The Orlik–Terao ideal*

$$I_{\mathcal{A}} = \langle \sum_{j=1}^{m} c_{i_j}(y_{i_1} \cdots \hat{y}_{i_j} \cdots y_{i_m}) \mid \textit{ over all dependencies} \rangle$$

The Orlik–Terao algebra is $C(\mathcal{A}) = \mathbb{K}[x_1, \ldots, x_n]/I_{\mathcal{A}}$.

Example 45. $\mathcal{A} = V(x_1 \cdot x_2 \cdot x_3 \cdot (x_1 + x_2 + x_3))$, the only dependency is $l_1 + l_2 + l_3 - l_4 = 0$, so $I_{\mathcal{A}} = \langle y_2y_3y_4 + y_1y_3y_4 + y_1y_2y_4 - y_1y_2y_3 \rangle$.

In [60], Orlik and Terao answer a question of Aomoto by considering the quotient AOT of $C(\mathcal{A})$ by $\langle x_1^2, \ldots, x_n^2 \rangle$. They prove:

Theorem 46 (Orlik–Terao [60]). *$HS(AOT, t) = \pi(\mathcal{A}, t)$.*

Theorem 47 (Terao [90]).

$$HS(C(\mathcal{A}), t) = \pi\Big(\mathcal{A}, \frac{t}{1-t}\Big).$$

It is not hard to show that

$$0 \to I_{\mathcal{A}} \to \mathbb{K}[x_1, \ldots, x_n] \xrightarrow{\phi} \mathbb{K}\left[\frac{1}{l_1}, \ldots, \frac{1}{l_n}\right] \to 0$$

is exact, so $V(I_{\mathcal{A}}) \subseteq \mathbb{P}^{n-1}$ is irreducible and rational. In any situation where weights of dependencies play a role, the Orlik–Terao algebra is the natural candidate to study. One such situation involves 2-formality:

Definition 48. *$\mathcal{A}$ is 2-formal if all dependencies are generated by dependencies among three hyperplanes.*

Theorem 49 (Falk–Randell [37]). *If $\mathcal{A}$ is $K(\pi, 1)$, $\mathcal{A}$ is 2-formal.*

Theorem 50 (Yuzvinsky [95]). *If $\mathcal{A}$ is free, $\mathcal{A}$ is 2-formal.*

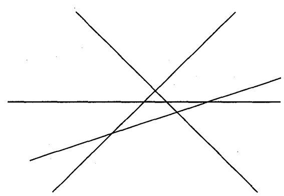
One reason that formality is interesting is that it is not a combinatorial invariant: in Example 13, the arrangement for which the six triple points lie on a smooth conic is not 2-formal, and the arrangement for which the points do not lie on a smooth conic is 2-formal.

Theorem 51 ([79]). *$\mathcal{A}$ is 2-formal iff* $\operatorname{codim}(I_{\mathcal{A}})_2 = n - \ell$.

In [7], Brandt and Terao generalized the notion of 2-formality to k-formality: $\mathcal{A}$ is k-formal if all dependencies are generated by dependencies among $k+1$ or fewer hyperplanes. Brandt and Terao prove that every free arrangement is k-formal.

Problem 52. Find an analog of Theorem 51 for k-formality.

Example 53. The configuration of Example 45 consists of four generic lines:



The Orlik–Terao ideal defines a cubic surface in $\mathbb{P}^3$, and a computation shows that $V(I_{\mathcal{A}})$ has four singular points.

This can be interpreted in terms of a rational map. Let $\alpha_i = Q_{\mathcal{A}}/l_i$, and define $\phi_{\mathcal{A}} = [\alpha_1, \dots, \alpha_n]$.

$$\mathbb{P}^{\ell-1} \xrightarrow{\phi_{\mathcal{A}}} \mathbb{P}^{n-1},$$

Restrict to the case $\mathcal{A} \subseteq \mathbb{P}^2$, and let $X_{\mathcal{A}} \xrightarrow{\pi} \mathbb{P}^2$ denote the blowup of $\mathbb{P}^2$ at the singular points of $\mathcal{A}$, with E_0 denoting the pullback to $X_{\mathcal{A}}$ of the class of a line on $\mathbb{P}^2$, and E_i the exceptional divisors over singular points of $\mathcal{A}$. Let

$$D_{\mathcal{A}} = (n-1)E_0 - \sum_{p_i \in L_2(\mathcal{A})} \mu(p_i)E_i.$$

Utilizing results of Proudfoot–Speyer [67] showing that $C(\mathcal{A})$ is Cohen–Macaulay and the Riemann–Roch theorem, [75] shows that the map $\phi_{\mathcal{A}}$ is determined by the global sections of $D_{\mathcal{A}}$, and that $\phi_{\mathcal{A}}$

(1) is an isomorphism on $\pi^*(\mathbb{P}^2 \setminus \mathcal{A})$
(2) contracts the lines of $\mathcal{A}$ to points
(3) blows up the singularities of $\mathcal{A}$.

Definition 54. *A graded S-module N has Castelnuovo–Mumford regularity d if* $\mathrm{Ext}^j(N,S)_n = 0$ *for all j and all $n \leq -d-j-1$.*

In terms of the betti table, the regularity of N is the label of the last non-zero row, so in Example 4, S/I has Castelnuovo–Mumford regularity one. The regularity of $C(\mathcal{A})$ is determined in [75]:

Theorem 55 ([75]). *For $\mathcal{A} \subseteq \mathbb{P}^{\ell-1}$, $C(\mathcal{A})$ is $\ell-1$-regular.*

To see this, note that since $C(\mathcal{A})$ is Cohen–Macaulay, quotienting $C(\mathcal{A})$ by ℓ generic linear forms yields an Artinian ring whose Hilbert series is the numerator of the Hilbert series of $C(\mathcal{A})$. The regularity of an Artinian module is equal to the length of the module, so the result follows from Theorem 47.

A main motivation for studying $C(\mathcal{A})$ is a surprising connection to nets and resonance varieties, which are the subject of §9. First, the definition of a net:

Definition 56. *Let $3 \leq k \in \mathbb{Z}$. A k-net in $\mathbb{P}^2$ is a partition of the lines of an arrangement $\mathcal{A}$ into k subsets $\mathcal{A}_i$, together with a choice of points $Z \subseteq \mathcal{A}$, such that:*

(1) *for every $i \neq j$ and every $L \in \mathcal{A}_i$, $L' \in \mathcal{A}_j$, $L \cap L' \in Z$.*
(2) *$\forall\ p \in Z$ and every $i \in \{1,\ldots,k\}$, there exists a unique $L \in A_i$ with $Z \in L$.*

In [53], Libgober and Yuzvinsky show that nets are related to the first resonance variety $R^1(\mathcal{A})$. The definition of a net forces each subset $\mathcal{A}_i$ to have the same cardinality, and if $m = |\mathcal{A}_i|$, the net is called a (k,m)-net. Using work of [53] and [39], it is shown in [75] that

Theorem 57. *Existence of a (k,m) net implies that there is a decomposition $D_{\mathcal{A}} = A + B$ with $h^0(A) = 2$ and $h^0(B) = km - \binom{m+1}{2}$.*

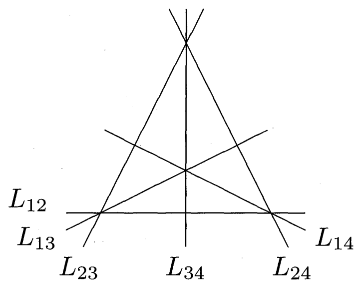
Definition 58. *A matrix of linear forms is 1-generic if it has no zero entry, and cannot be transformed by row and column operations to have a zero entry.*

In [26], Eisenbud shows that if a divisor D on a smooth curve X factors as $D \simeq A+B$, with A having m-sections and B having n-sections, then the ideal of the image of X under the map defined by the global sections of D will contain the 2×2 minors of a 1-generic matrix. Using this result and Theorem 57, it can be shown that $I_{\mathcal{A}}$ contains the ideal $I_2(M)$ of 2×2 minors of a 1-generic $2 \times \left(km - \binom{m+1}{2}\right)$ matrix M. So if $G = S(-1)^{km-\binom{m+1}{2}}$, the Eagon–Northcott complex [27]

$$\cdots \to S_2(S^2)^* \otimes \Lambda^4 G \to (S^2)^* \otimes \Lambda^3 G \to \Lambda^2 G \to \Lambda^2 S^2 \to S/I_2(M) \to 0$$

is a subcomplex of resolution of $S/I_{\mathcal{A}}$. The geometric content of Theorem 57 is that it implies $V(I_{\mathcal{A}})$ lies on a *scroll* [27].

Example 59. For the A_3 arrangement, the set of triple points Z gives a $(3,2)$ net, where the A_i correspond to normal crossing points: $A_1 = 12 \mid 34$, $A_2 = 13 \mid 24$, $A_3 = 14 \mid 23$.



$$\text{Let } A = 2E_0 - \sum_{\{p \mid \mu(p)=2\}} E_p \text{ and } B = 3E_0 - \sum_{p \in L_2(\mathcal{A})} E_p.$$

So $n - \binom{m+1}{2} = 6-3 = 3$ and I contains the 2×2 minors of a 2×3 matrix, whose resolution appears in Example 4. The graded betti diagram for $\mathbb{C}[x_0,\ldots,x_5]/I_A$ is

total	1	4	5	2
0	1	–	–	–
1	–	4	2	–
2	–	–	3	2

From this, it follows that the free resolution of S/I_A is a mapping cone resolution [27]. The geometric meaning is that $X_{\mathcal{A}}$ is the intersection of a generic quadric hypersurface with the scroll.

Since $D_{\mathcal{A}}$ contracts proper transforms of lines to points, it is not very ample. However, it follows from [75] that $D_{\mathcal{A}} + E_0$ is very ample, and gives a De Concini–Procesi wonderful model (see next section) for the blowup.

Problem 60. Determine the graded betti numbers of $C(\mathcal{A})$.

Problem 61. Relate $R^k(\mathcal{A})$ to the graded betti numbers of $C(\mathcal{A})$.

§6. Compactifications

In [44], Fulton–MacPherson provide a compactification $F(X,n)$ for the configuration space of n marked points on an algebraic variety X. The construction is quite involved, but the combinatorial data is that of A_n. In a related vein, in [18], De Concini–Procesi construct a wonderful model X for a subspace complement $M_{\mathcal{A}} = \mathbb{C}^\ell \setminus \mathcal{A}$: a smooth, compact X such that $X \setminus M_{\mathcal{A}}$ is a normal crossing divisor. Here it is the combinatorics which are complex. A key object in their construction is

$$M_{\mathcal{A}} \longrightarrow \mathbb{C}^\ell \times \prod_{D \in G} \mathbb{P}(\mathbb{C}^\ell / D),$$

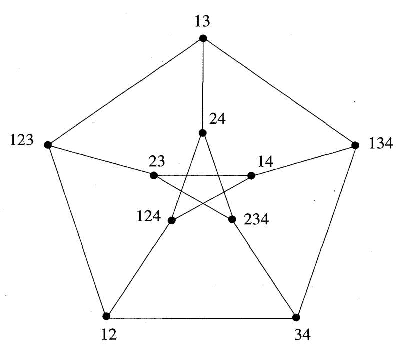
where G is a building set. In [41], Feichtner–Kozlov generalize the construction of [18] to a purely lattice-theoretic setting. See [40] for additional background on this section.

Definition 62. *For a lattice L, a building set G is a subset of L, such that for all $x \in L$, $\max\{G_{\le x}\} = \{x_1, \ldots, x_m\}$ satisfies $[\hat{0}, x] \simeq \prod_{j=1}^m [\hat{0}, x_j]$. A building set contains all irreducible $x \in L$.*

Definition 63. *A subset N of a building set G is nested if for any set of incomparable $\{x_1, \ldots, x_p\} \subseteq N$ with $p \ge 2$, $x_1 \vee x_2 \vee \cdots \vee x_p$ exists in L, but is not in G.*

Nested sets form a simplicial complex $N(G)$, with vertices the elements of G (which are vacuously nested).

Example 64. The minimal building set for A_3 consists of the hyperplanes themselves, the triple intersections in L_2, and the element $\hat{1}$. Since $\hat{1}$ is a member of every face of $N(G)$, the nested set complex $N(G)$ is the cone over



There is an edge $\overline{(12), (123)}$ because there are no incomparable subsets with at least two elements, while $\overline{(12), (34)}$ is an edge because $(12) \vee (34)$ exists in L (it is a normal crossing), but is not in G.

Suppose L is an atomic lattice, and G a building set in L. In [42], Feichtner and Yuzvinsky study a certain algebra associated to the pair L, G:

$$D(L,G) = \mathbb{Z}[x_g | g \in G]/I, \text{ with } x_g \text{ of degree 2.}$$

where I is generated by

$$\prod_{\{g_1,\ldots,g_n\} \notin N(G)\}} x_{g_i} \text{ and } \sum_{g_i \geq H \in L_1} x_{g_i}$$

Theorem 65 (Feichtner–Yuzvinsky [42]). *If $\mathcal{A}$ is a hyperplane arrangement and G a building set containing $\hat{1}$, then*

$$D(L,G) \simeq H^*(Y_{\mathcal{A},G}, \mathbb{Z}),$$

where $Y_{\mathcal{A},G}$ is the wonderful model arising from the building set G.

The importance of this is the relation to the Knudson–Mumford compactification $\overline{M_{0,n}}$ of the moduli space of n marked points on $\mathbb{P}^1$.

Theorem 66 (De Concini–Procesi [19]).

$$\overline{M_{0,n}} \simeq Y_{A_{n-2},G},$$

where G is the minimal building set for A_{n-2}.

A presentation for the cohomology ring of $\overline{M_{0,n}}$ was first described by Keel in [49]; the description which follows from [42] is very economic.

Example 67. By Theorem 65 and [19],

$$H^*(\overline{M_{0,5}}, \mathbb{Z}) \simeq D(L(A_3), G_{min}).$$

The nested set complex for A_3 and G_{min} appears in Example 64, so that $D(L(A_3), G_{min})$ is the quotient of a polynomial ring S with eleven generators by an ideal consisting of 6 linear forms (one form for each hyperplane) and 19 quadrics. To see that there are 19 quadrics, note that the space of quadrics in 11-variables has dimension 45, and $N(G_{min})$ has $15+11 = 26$ edges (recall that $\hat{1}$ is not pictured). A computation shows that

$$D(L(A_3), G_{min}) \simeq \mathbb{Z}[x_1, \ldots, x_5]/I,$$

where I consists of all but one quadric of S (and includes all squares of variables). This meshes with the intuitive picture: to obtain a wonderful model, simply blow up the four triple points, so that $\overline{M_{0,5}}$ is the corresponding Del Pezzo surface X_4, which has $\sum h^i(X_4, \mathbb{Z})t^i = 1 + 5t^2 + t^4$, agreeing with the computation.

Problem 68. Analyze $D(L,G)$ for other lattices.

§7. Associated Lie algebra of π_1 and LCS ranks

Let G be a finitely-generated group, with normal subgroups,

$$G = G_1 \geq G_2 \geq G_3 \geq \cdots,$$

defined inductively by $G_k = [G_{k-1}, G]$. We obtain an associated Lie algebra

$$gr(G) \otimes \mathbb{Q} := \bigoplus_{k=1}^{\infty} G_k/G_{k+1} \otimes \mathbb{Q},$$

with Lie bracket induced by the commutator map. Let $\phi_k = \phi_k(G)$ denote the rank of the k-th quotient. Presentations for $\pi_1(M_{\mathcal{A}})$ are given by Randell [68], Salvetti [73], Arvola [5], and Cohen–Suciu [13]. For computations, the braid monodromy presentation of [13] is easiest to implement. For a detailed survey of $\pi_1(M_{\mathcal{A}})$, see Suciu's survey [84]. The fundamental group is quite delicate, and in this section, we investigate properties of $\pi_1(M_{\mathcal{A}})$ via the associated graded Lie algebra

$$\mathfrak{g} = gr(\pi_1(M_{\mathcal{A}})) \otimes \mathbb{Q}.$$

The Lefschetz-type theorem of Hamm–Le [46] implies that taking a generic two dimensional slice gives an isomorphism on π_1. Thus, to study $\pi_1(M_{\mathcal{A}})$, we may assume $\mathcal{A} \subseteq \mathbb{C}^2$ or $\mathbb{P}^2$. As shown by Rybnikov [70], $\pi_1(M_{\mathcal{A}})$ is not determined by $L_{\mathcal{A}}$; whereas the Orlik–Solomon algebra $H^*(M_{\mathcal{A}}, \mathbb{Z})$ is determined by $L_{\mathcal{A}}$.

Example 69. In Example 6, we saw that the Hilbert series for A_3 is $1 + 6t + 11t^2 + 6t^3$. A computation shows that the LCS ranks begin

$$6 \quad 4 \quad 10 \quad 21 \quad 54 \quad \cdots.$$

For higher k, $\phi_k(\pi_1(A_3)) = w_k(2) + w_k(3)$, where w_k is a Witt number. In general, we may encode the LCS ranks via

$$\prod_{k=1}^{\infty} \frac{1}{(1-t^k)^{\phi_k}}.$$

For A_3, this is

$$\frac{1}{(1-t)^6} \frac{1}{(1-t^2)^4} \frac{1}{(1-t^3)^{10}} \frac{1}{(1-t^4)^{21}} \frac{1}{(1-t^5)^{54}} \cdots.$$

Expanding this and writing out the first few terms yields

$$1 + 6t + 25t^2 + 90t^3 + 301t^4 + 966t^5 + 3025t^6 + \cdots.$$

If we multiply this with

$$\pi(A_3, -t) = 1 - 6t + 11t^2 - 6t^3,$$

the result is 1, and is part of a general pattern.

Theorem 70 (Kohno's LCS formula [51]). *For the arrangement A_{n-1} (graphic arrangement of K_n)*

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = \prod_{i=1}^{n-1}(1-it).$$

This explains the computation of Example 69. We now compute the free resolution of the residue field A/m as an A-module, where $m = \langle E_1 \rangle$. Let

$$b_{ij} = \dim_{\mathbb{Q}} Tor_i^A(\mathbb{Q}, \mathbb{Q})_j.$$

Example 71. For A_3, we compute $b_{ij} = 0$ if $i \neq j$, and

$$\sum_i b_{ii}t^i = 1 + 6t + 25t^2 + 90t^3 + 301t^4 + 966t^5 + 3025t^6 + \cdots.$$

The b_{ii} are the coefficients of the formal power series in Example 69!

Kohno's result was the first of a long line of results on LCS formulas for certain special families of arrangements

(1) Braid arrangements: Kohno [51]
(2) Fiber type arrangements: Falk–Randell [36]
(3) Supersolvable arrangements: Terao [88]
(4) Lower bound for ϕ_k: Falk [33]
(5) Koszul arrangements: Shelton–Yuzvinsky [81]
(6) Hypersolvable arrangements: Jambu–Papadima [48]
(7) Rational $K(\pi, 1)$ arrangements: Papadima–Yuzvinsky [64]
(8) MLS arrangements: Papadima–Suciu [61]
(9) Graphic arrangements: Lima-Filho–Schenck [54]
(10) No such formula in general: Peeva [65]

Let $\mathbb{L}(H_1(M_{\mathcal{A}}, \mathbb{K}))$ denote the free Lie algebra on $H_1(M_{\mathcal{A}}, \mathbb{K})$. Dualizing the cup product gives a map

$$H_2(M_{\mathcal{A}}, \mathbb{Q}) \xrightarrow{c} H_1(M_{\mathcal{A}}, \mathbb{Q}) \wedge H_1(M_{\mathcal{A}}, \mathbb{Q}) \longrightarrow \mathbb{L}(H_1(M_{\mathcal{A}}, \mathbb{Q})).$$

Following Chen [10], define the holonomy Lie algebra

$$\mathfrak{h}_{\mathcal{A}} = \mathbb{L}(H_1(M_{\mathcal{A}}, \mathbb{K}))/I_{\mathcal{A}},$$

where $I_{\mathcal{A}}$ is the Lie ideal generated by $\mathrm{Im}(c)$. As noted by Kohno in [50], taking transpose of cup product shows that the image of c is generated by

$$[x_j, \sum_{i=1}^{k} x_i],$$

where x_i is a generator of $\mathbb{L}(H_1(X, \mathbb{K}))$ corresponding to H_i, and the set $\{H_1, \ldots, H_k\}$ is a maximal dependent set of codimension two, so corresponds to an element of $L_2(\mathcal{A})$. The upshot is that

$$\prod_{k=1}^{\infty} \frac{1}{(1-t^k)^{\phi_k}} = \sum_{i=0}^{\infty} \dim_{\mathbb{Q}} Tor_i^A(\mathbb{Q}, \mathbb{Q})_i t^i.$$

This was first made explicit by Peeva in [65]; the proof runs as follows. First, Brieskorn [8] showed that $M_{\mathcal{A}}$ is formal, in the sense of [85]. Using Sullivan's work and an analysis of the bigrading on Hirsch extensions, Kohno proved

Theorem 72 (Kohno). $\phi_k(\mathfrak{g}) = \phi_k(\mathfrak{h}_{\mathcal{A}})$.

Thus

(1) $\prod_{k=1}^{\infty} \frac{1}{(1-t^k)^{\phi_k}} = HS(U(\mathfrak{h}_{\mathcal{A}}, t))$, which follows from Kohnos work and Poincaré–Birkhoff–Witt.
(2) Shelton–Yuzvinsky show in [81] that $U(\mathfrak{h}_{\mathcal{A}}) = \overline{A}^!$ is the quadratic dual of the quadratic Orlik–Solomon algebra.
(3) Results of Priddy–Löfwall show that the quadratic dual is related to diagonal Yoneda Ext-algebra via

$$\overline{A}^! \cong \bigoplus_i \mathrm{Ext}^i_{\overline{A}}(\mathbb{Q}, \mathbb{Q})_i.$$

Results of Peeva [65] and Roos [69] show that in general there does not exist a standard graded algebra R such that $\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = HS(R, -t)$. For any quotient of a free Lie algebra, can we:

Problem 73. Find spaces for which there is a simple generating function for ϕ_k.

Problem 74. Relate $\mathfrak{h}_{\mathcal{A}}$ to $\bigoplus_{X \in L_2} \mathfrak{h}_{\mathcal{A}_X}$, as in [61].

As Shelton–Yuzvinsky proved in [81], the natural class of arrangements for which an LCS formula holds are arrangements for which A is a Koszul algebra, which we tackle next.

§8. Koszul algebras

Let $T(V)$ denote the tensor algebra on V.

Definition 75. *A quadratic algebra is $T(V)/I$, with $I \subseteq V \otimes V$.*

A quadratic algebra A has a quadratic dual $A^{\perp} = T(V^*)/I^{\perp}$:

$$\langle \alpha \otimes \beta \mid \alpha(a) \cdot \beta(b) = 0 \mid \forall a \otimes b \in I \rangle = I^{\perp} \subseteq V^* \otimes V^*.$$

Definition 76. *A is Koszul if $Tor_i^A(\mathbb{K}, \mathbb{K})_j = 0$, $i \neq j$.*

A quadratic algebra A is Koszul iff the minimal free resolution of the residue field over A has matrices with only linear entries.

Example 77. The Hilbert series of $S = T(\mathbb{K}^n)/\langle x_i \otimes x_j - x_j \otimes x_i \rangle$ is $1/(1-t)^n$, and a computation shows that the minimal free resolution of $\mathbb{K}$ over S is the Koszul complex, so $\dim_{\mathbb{K}} Tor_i^S(\mathbb{K}, \mathbb{K})_i = \binom{n}{i}$. Since

$$I^{\perp} = \langle x_i \otimes x_j + x_j \otimes x_i \rangle,$$

we see that $S^! = E$. The Hilbert series of E is $(1+t)^n = \sum_{i=0}^{n} \binom{n}{i} t^i$. A computation shows that $\dim_{\mathbb{K}} Tor_i^E(\mathbb{K}, \mathbb{K})_i = \binom{n-1+i}{i}$, which are the coefficients in an expansion of $1/(1-t)^n$.

Fröberg [43] proved that if I is a quadratic monomial ideal then S/I is Koszul. By uppersemicontinuity [47], this means S/I is Koszul if I has a quadratic Grobner basis (QGB). See Example 81 below for a Koszul algebra having no QGB. Both S and E are Koszul, and the relation between their Hilbert series is explained by:

Theorem 78. *If A is Koszul, so is $A^!$, and*

$$HS(A, t) \cdot HS(A^!, -t) = 1.$$

Theorem 79 (Björner–Ziegler [6]). *The Orlik–Solomon algebra has a QGB iff $\mathcal{A}$ is supersolvable.*

Example 80. A computation shows that the Orlik–Solomon algebra of A_3 has a quadratic Grobner basis, so is Koszul. For the non-Fano arrangement, $\dim_{\mathbb{K}} Tor_3^A(\mathbb{K}, \mathbb{K})_4 = 1$, so A is not Koszul.

Example 81. [Caviglia [9]] Map $R = \mathbb{K}[a_1, \ldots a_9] \xrightarrow{\phi} \mathbb{K}[x, y, z]$ using all cubic monomials of $\mathbb{K}[x, y, z]$ except xyz, and let $I = \ker(\phi)$. Then R/I is Koszul, but has no quadratic Grobner basis.

Problem 82. For Orlik–Solomon algebras, does Koszul imply supersolvable? In the case of graphic arrangements, it does [76].

Problem 83. Find a combinatorial description of $Tor_i^A(\mathbb{K}, \mathbb{K})_j$.

§9. Resonance varieties

Let A be the Orlik–Solomon algebra of $M_{\mathcal{A}}$, with $|\mathcal{A}| = n$. For each $a = \sum a_i e_i \in A_1$, we consider the Aomoto complex (A, a), whose i^{th} term is A_i, and differential is $\wedge a$:

$$(A, a) \colon\; 0 \longrightarrow A_0 \xrightarrow{a} A_1 \xrightarrow{a} A_2 \xrightarrow{a} \cdots \xrightarrow{a} A_\ell \longrightarrow 0\,.$$

This complex arose in Aomoto's work [4] on hypergeometric functions, as well as in the study of cohomology with local system coefficients [31], [74]. In [96], Yuzvinsky showed that for a generic a, the Aomoto complex is exact; the resonance varieties of $\mathcal{A}$ are the loci of points $a = \sum_{i=1}^{n} a_i e_i \leftrightarrow (a_1 : \cdots : a_n) \in \mathbb{P}^{n-1}$ for which (A, a) fails to be exact, that is:

Definition 84. *For each $k \geq 1$,*

$$R^k(\mathcal{A}) = \{a \in \mathbb{P}^{n-1} \mid H^k(A, a) \neq 0\}.$$

In [34], Falk gave necessary and sufficient conditions for $a \in R^1(\mathcal{A})$.

Definition 85. *A partition Π of $\mathcal{A}$ is neighborly if for all $Y \in L_2(\mathcal{A})$ and π a block of Π,*

$$\mu(Y) \leq |Y \cap \pi| \Longrightarrow Y \subseteq \pi.$$

Falk proved that components of $R^1(\mathcal{A})$ arise from neighborly partitions; he conjectured that $R^1(\mathcal{A})$ is a union of linear components. This was established (essentially simultaneously) by Cohen–Suciu [14] and Libgober–Yuzvinsky [53]. Libgober and Yuzvinsky also showed that $R^1(\mathcal{A})$ is a disjoint union of positive dimensional subspaces in $\mathbb{P}(E_1)$, and Cohen–Orlik [12] show that $R^{k\geq 2}(\mathcal{A})$ is also a subspace arrangement.

On the other hand, as shown by Falk in [35], in positive characteristic, the components of $R^1(\mathcal{A})$ can meet, and need not be linear. The approach of Libgober–Yuzvinsky involves connecting $R^1(\mathcal{A})$ to pencils/nets/webs and there is much recent work in the area, e.g. [39], [64] [99]. Of special interest is the following conjecture relating $R^1(\mathcal{A})$ and the LCS ranks ϕ_k:

Conjecture 86 (Suciu [84]). *If $\phi_4 = \theta_4$, then*

$$\prod_{k\geq 1}(1 - t^k)^{\phi_k} = \prod_{L_i \in R^1(\mathcal{A})} (1 - (dim(L_i)t)),$$

where θ_4 is the fourth Chen rank (Definition 88).

Example 87. Let $\mathcal{A} = V(xy(x-y)z) \subseteq \mathbb{P}^2$, and $E = \Lambda(\mathbb{K}^4)$, with generators $e_1, \ldots, e_4$, so that

$$A = E/\langle \partial(e_1e_2e_3), \partial(e_1e_2e_3e_4) \rangle.$$

Since $\partial(e_1e_2e_3e_4) = e_1 \wedge \partial(e_1e_2e_3) - e_4\partial(e_1e_2e_3)$, the second relation is unnecessary. To compute $R^1(\mathcal{A})$, we need only the first two differentials in the Aomoto complex. Using $e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$ as a basis for A_2, we find that $e_1 \mapsto e_1 \wedge (\sum_{i=1}^{4} a_ie_i) = a_2e_{12} + a_3e_{13} + a_4e_{14}$. Since

$$\partial(e_1e_2e_3) = e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3,$$

in A, $e_{12} = e_{13} - e_{23}$, so that

$$a_2e_{12} = a_2(e_{13} - e_{23}).$$

This means $e_1 \mapsto (a_2 + a_3)e_{13} + a_4e_{14} - a_2e_{23}$; similar computations for the other e_i show that the Aomoto complex is

$$0 \longrightarrow \mathbb{K}^1 \xrightarrow{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}} \mathbb{K}^4 \xrightarrow{\begin{bmatrix} a_2+a_3 & -a_1 & -a_1 & 0 \\ a_4 & 0 & 0 & -a_1 \\ -a_2 & a_1+a_3 & -a_2 & 0 \\ 0 & a_4 & 0 & -a_2 \\ 0 & 0 & a_4 & -a_3 \end{bmatrix}} \mathbb{K}^5.$$

The rank of the first map is always one, $R^1(A) \subseteq \mathbb{P}^3$ is the locus where the second matrix has kernel of dimension at least two, so the 3×3 minors must vanish. A computation shows this locus is $\langle a_4, a_1 + a_2 + a_3 \rangle$.

Letting $a = \sum_{i=1}^{n} a_ie_i$, observe that $a \in R^1(\mathcal{A})$ iff there exists a $b \in E_1$ so that $a \wedge b \in I_2$, so that $R^1(\mathcal{A})$ is the locus of decomposable 2-tensors in I_2. Since I_2 is determined by the intersection lattice $L(\mathcal{A})$ in rank ≤ 2, to study $R^1(\mathcal{A})$, it can be assumed that $\mathcal{A} \subseteq \mathbb{P}^2$.

While the first resonance variety is conjecturally connected (under certain conditions) to the LCS ranks, $R^1(A)$ is *always* connected to the Chen ranks introduced by Chen in [10].

Definition 88. *The Chen ranks of G are the LCS ranks of the maximal metabelian quotient of G:*

$$\theta_k(G) := \phi_k(G/G''),$$

where $G' = [G, G]$.

Conjecture 89 (Suciu [84]). *Let $G = G(\mathcal{A})$ be an arrangement group, and let h_r be the number of components of $R^1(\mathcal{A})$ of dimension r. Then, for $k \gg 0$:*

$$\theta_k(G) = (k-1)\sum_{r\geq 1} h_r \binom{r+k-1}{k}.$$

For the previous example, $R^1(\mathcal{A}) = V(a_4, a_1 + a_2 + a_3) \simeq \mathbb{P}^1$, so

$$\theta_k(G) = (k-1).$$

To discuss the Chen ranks, we need some background. The Alexander invariant G'/G'' is a module over $\mathbb{Z}[G/G']$. For arrangements, $\mathbb{Z}[G/G'] =$ Laurent polynomials in n-variables. In [55], Massey showed that

$$\sum_{k\geq 0} \theta_{k+2}\, t^k = HS(\text{gr } G'/G'' \otimes \mathbb{Q}, t).$$

It turns out to be easier to work with the linearized Alexander invariant B introduced by Cohen–Suciu in [15]

$$(A_2 \oplus E_3) \otimes S \xrightarrow{\Delta} E_2 \otimes S \longrightarrow B \longrightarrow 0,$$

where Δ is built from the Koszul differential and $(E_2 \to A_2)^t$.

Theorem 90 (Cohen–Suciu [15]).

$$V(ann\ B) = R^1(\mathcal{A}).$$

Theorem 91 (Papadima–Suciu [62]). *For $k \geq 2$,*

$$\sum_{k\geq 2} \theta_k\, t^k = HS(B, t).$$

This shows that the Chen ranks are combinatorially determined, and depend only on $L(\mathcal{A})$ in rank ≤ 2.

Example 92. For the A_3 arrangement depicted in Example 59, write $e_0 = L_{12}, e_1 = L_{13}, e_2 = L_{23}, e_3 = L_{24}, e_4 = L_{14}, e_5 = L_{34}$. With this labelling

$$I_2 = \langle \partial(e_1e_4e_5), \partial(e_0e_1e_2), \partial(e_2e_3e_5), \partial(e_0e_3e_4) \rangle,$$

from which a presentation for B can be computed:

$$S^{14} \to S^4 \to B \to 0.$$

A computation shows that $R^1(A_3)$ is

$$\begin{gathered}V(x_1+x_4+x_5,x_0,x_2,x_3)\sqcup\\ V(x_2+x_3+x_5,x_0,x_1,x_4)\sqcup\\ V(x_0+x_3+x_4,x_1,x_2,x_4)\sqcup\\ V(x_0+x_1+x_2,x_3,x_4,x_5)\sqcup\\ V(x_0+x_1+x_2,x_0-x_5,x_1-x_3,x_2-x_4),\end{gathered}$$

and that the Hilbert Series of B is:

$$(4t^2+2t^3-t^4)/(1-t)^2 = 4t^2+10t^3+15t^4+20t^5+\cdots.$$

On the other hand, the graded betti numbers $Tor_i^E(A_3,\mathbb{K})_j$ are

total	1	4	10	21	45	91
0	1	–	–	–	–	–
1	–	4	10	15	20	25
2	–	–	–	6	25	66

So the Hilbert series for B encodes the ranks of $Tor_i^E(A_3,\mathbb{K})_{i+1}$. This suggests a connection between $R^1(A)$ and $Tor_i^E(A_3,\mathbb{K})_{i+1}$, which we tackle in the next section.

Besides the connection to resonance varieties, there is a second reason to study $Tor_i^E(A,\mathbb{K})$: the numbers $b_{ij}=\dim_{\mathbb{K}} Tor_i^A(\mathbb{K},\mathbb{K})_j$ studied in §7 grow very fast, while the numbers $b'_{ij}=\dim_{\mathbb{K}} Tor_i^E(A,\mathbb{K})_j$ grow at a much slower rate.

Example 93. For the non-Fano arrangement of Example 6

total	1	7	23	63	165	387
0	1	–	–	–	–	–
1	–	6	17	27	36	45
2	–	1	6	36	129	342
3	–	–	–	–	–	–

$$b'_{ij},$$

total	1	7	35	156	664	2773	*
0	1	7	34	143	560	2108	*
1	–	–	1	13	103	646	*
2	–	–	–	–	1	19	*
3	–	–	–	–	–	–	1

$$b_{ij}.$$

The spaces $Tor_i^E(A, \mathbb{K})$ and $Tor_i^A(\mathbb{K}, \mathbb{K})$ are related via the change of rings spectral sequence:

$$\mathrm{Tor}_i^A\left(\mathrm{Tor}_j^E(A, \mathbb{K}), \mathbb{K}\right) \Longrightarrow \mathrm{Tor}_{i+j}^E(\mathbb{K}, \mathbb{K}).$$

For arrangements, details of this relationship are investigated in [76].

Problem 94. Find a combinatorial description of $Tor_i^E(A, \mathbb{K})_j$.

Problem 95. If A is Koszul, does this provide data on $Tor_i^E(A, \mathbb{K})_j$?

§10. Linear syzygies

It is fairly easy to see that there is a connection between $R^1(A)$ and linear syzygies, that is, to the module $Tor_2^E(A, \mathbb{K})_3$. Since

$$a \wedge b \in I_2 \longrightarrow a \wedge b = \sum c_i f_i, \ c_i \in \mathbb{K}, f_i \in I_2,$$

the relations $a \wedge a \wedge b = 0 = b \wedge a \wedge b$ yield linear syzygies on I_2:

$$\sum a c_i f_i = 0 = \sum b c_i f_i.$$

Example 87, continued. Since $\partial(e_1 e_2 e_3) = (e_1 - e_2) \wedge (e_2 - e_3)$, both $(e_1 - e_2)$ and $(e_2 - e_3)$ are in $R^1(A)$, as is the line connecting them:

$$s(e_1 - e_2) + t(e_2 - e_3) \subseteq R^1(\mathcal{A}) \subseteq \mathbb{P}(E_1).$$

Parametrically, this may be written

$$(s : t - s : -t : 0) = V(a_4, a_1 + a_2 + a_3),$$

so $s(e_1 - e_2) + t(e_2 - e_3) \wedge \partial(e_1 e_2 e_3) = 0$ gives a family of linear syzygies on I_2, parameterized by $\mathbb{P}^1$. $\diamond$

To make the connection between linear syzygies and the module B precise, we need the following result:

Theorem 96 (Eisenbud–Popescu–Yuzvinsky [29]). *For an arrangement $\mathcal{A}$, the Aomoto complex is exact, as a sequence of S-modules:*

$$0 \longrightarrow A_0 \otimes S \xrightarrow{\cdot a} A_1 \otimes S \xrightarrow{\cdot a} \cdots \xrightarrow{\cdot a} A_\ell \otimes S \longrightarrow F(A) \longrightarrow 0 .$$

Theorem 97 (Schenck–Suciu [77]). *The linearized Alexander invariant B is determined by $F(A)$:*

$$B \cong \mathrm{Ext}_S^{\ell-1}(F(A), S).$$

Furthermore, for $k \geq 2$, $\dim_{\mathbb{K}} B_k = \dim_{\mathbb{K}} \mathrm{Tor}_{k-1}^E(A, \mathbb{K})_k$.

Using this, it is possible to prove one direction of Conjecture 89.

Theorem 98 (Schenck–Suciu [77]). *For $k \gg 0$,*

$$\theta_k(G) \geq (k-1) \sum_{L_i \in R^1(\mathcal{A})} \binom{\dim L_i + k - 1}{k}.$$

Problem 99. Prove the remaining direction of Conjecture 89.

What makes all this work is the Bernstein–Gelfand–Gelfand correspondence, which is our final topic.

§11. Bernstein–Gelfand–Gelfand correspondence

Let $S = Sym(V^*)$ and $E = \bigwedge(V)$. The Bernstein–Gelfand–Gelfand correspondence is an isomorphism between derived categories of bounded complexes of coherent sheaves on $\mathbb{P}(V^*)$ and bounded complexes of finitely generated, graded E-modules. Although this sounds exotic, from this it is possible to extract functors

R: finitely generated, graded S-modules $\longrightarrow$ linear free E-complexes.
L: finitely generated, graded E-modules $\longrightarrow$ linear free S-complexes.

The point is that problems can be translated to a (possibly) simpler setting. For example, BGG yields a very fast way to compute sheaf cohomology, using Tate resolutions.

Definition 100. *Let P be a finitely generated, graded E-module. Then $\mathbf{L}(P)$ is the complex*

$$\cdots \longrightarrow S \otimes P_{i+1} \xrightarrow{\cdot a} S \otimes P_i \xrightarrow{\cdot a} S \otimes P_{i-1} \xrightarrow{\cdot a} \cdots,$$

where $a = \sum_{i=1}^{n} x_i \otimes e_i$, so that $1 \otimes p \mapsto \sum x_i \otimes e_i \wedge p$.

Note that elements of V^* have degree $= 1$, and elements of V have degree $= -1$.

Example 101. $P = E = \bigwedge \mathbb{K}^3$. Then we have

$$0 \longrightarrow S \otimes E_0 \longrightarrow S \otimes E_1 \longrightarrow S \otimes E_2 \longrightarrow S \otimes E_3 \longrightarrow 0.$$

Clearly $1 \mapsto \sum_1^3 x_i \otimes e_i$. For d_1

$$\begin{aligned} e_1 &\mapsto -x_2 e_{12} - x_3 e_{13} \\ e_2 &\mapsto x_1 e_{12} - x_3 e_{23} \\ e_3 &\mapsto x_1 e_{13} + x_2 e_{23} \end{aligned}$$

$$d_2: \ e_{12} \mapsto x_3 e_{123}, \ e_{13} \mapsto -x_2 e_{123} \ e_{23} \mapsto x_1 e_{123}.$$

Thus, $\mathbf{L}(E)$ is

$$S^1 \xrightarrow{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} S^3 \xrightarrow{\begin{bmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{bmatrix}} S^3 \xrightarrow{\begin{bmatrix} x_3 & -x_2 & x_1 \end{bmatrix}} S^1.$$

This is simply the Koszul complex.

If M is a finitely generated, graded S-module, then $\mathbf{R}(M)$ is the complex

$$\cdots \longrightarrow \hat{E} \otimes M_{i-1} \xrightarrow{\cdot a} \hat{E} \otimes M_i \xrightarrow{\cdot a} \hat{E} \otimes M_{i+1} \xrightarrow{\cdot a} \cdots,$$

where $a = \sum_{i=1}^{n} e_i \otimes x_i$, so $1 \otimes m \mapsto \sum e_i \otimes x_i \cdot m$, and $\hat{E}$ is $\mathbb{K}$-dual to E:

$$\hat{E} \simeq E(n) = Hom_{\mathbb{K}}(E, \mathbb{K}).$$

Just as $\mathbf{L}(P) = S \otimes_{\mathbb{K}} P$, $\mathbf{R}(M) = Hom_{\mathbb{K}}(E, M)$.

Example 3, continued. If $M = \mathbb{K}[x_0, x_1]/\langle x_0x_1, x_0^2 \rangle$, then

$$\begin{gathered} 1 \mapsto e_0 \otimes x_0 + e_1 \otimes x_1 \\ x_0 \mapsto e_0 \otimes x_0^2 + e_1 \otimes x_0x_1 \\ x_1 \mapsto e_0 \otimes x_0x_1 + e_1 \otimes x_1^2 \\ x_1^n \mapsto e_0 \otimes x_0x_1^n + e_1 \otimes x_1^{n+1}. \end{gathered}$$

Thus, $\mathbf{R}(M)$ is

$$E(2)^1 \xrightarrow{\begin{bmatrix} e_0 \\ e_1 \end{bmatrix}} E(3)^2 \xrightarrow{\begin{bmatrix} 0 & e_1 \end{bmatrix}} E(4)^1 \xrightarrow{\begin{bmatrix} e_1 \end{bmatrix}} E(5)^1 \xrightarrow{\begin{bmatrix} e_1 \end{bmatrix}} \cdots.$$

This complex is exact, except at the second step. The kernel of

$$\begin{bmatrix} 0 & e_1 \end{bmatrix}$$

is generated by $\alpha = [1, 0]$ and $\beta = [0, e_1]$, with relations $im(d_1) = \beta + e_0\alpha = 0, e_1\beta = 0$, so that

$$H^1(\mathbf{R}(M)) \simeq E(3)/e_0 \wedge e_1.$$

The betti table for M is:

total	1	2	1
0	1	–	–
1	–	2	1

Note that in this example, M is 1-regular.

Theorem 102 (Eisenbud–Fløystad–Schreyer [28]).

$$H^j(\mathbf{R}(M))_{i+j} = Tor_i^S(M, \mathbb{K})_{i+j}.$$

Corollary 103. *The Castelnuovo–Mumford regularity of M is $\leq d$ iff $H^i(\mathbf{R}(M)) = 0$ for all $i > d$.*

What can be said about higher resonance varieties? In [12], Cohen–Orlik prove that for $k \geq 2$,

$$R^k(\mathcal{A}) = \bigcup L_i \text{ linear.}$$

As observed by Suciu, in general the union need not be disjoint.

Theorem 104 (Eisenbud–Popescu–Yuzvinsky [29]). $R^k(\mathcal{A}) \subseteq R^{k+1}(\mathcal{A})$.

The key point is that

$$H^k(A, a) \neq 0 \text{ iff } Tor^S_{\ell-k}(F(A), S/I(p)) \neq 0.$$

The result follows from interpreting this in terms of Koszul cohomology.

Theorem 105 (Denham–Schenck [21]). *Higher resonance may be interpreted via* Ext*:*

$$R^k(\mathcal{A}) = \bigcup_{k' \leq k} V(\operatorname{ann} \operatorname{Ext}^{\ell-k'}(F(A), S)).$$

Furthermore, the differentials in free resolution of A over E can be analyzed using BGG and the Grothendieck spectral sequence.

For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^d$, there is a finitely generated, graded *saturated* S-module M whose sheafification is $\mathcal{F}$. If $\mathcal{F}$ has Castelnuovo–Mumford regularity r, then the Tate resolution of $\mathcal{F}$ is obtained by splicing the complex $\mathbf{R}(M_{\geq r})$:

$$0 \longrightarrow \hat{E} \otimes M_r \xrightarrow{d^r} \hat{E} \otimes M_{r+1} \longrightarrow \hat{E} \otimes M_{r+2} \longrightarrow \cdots,$$

with a free resolution $P_\bullet$ for the kernel of d^r:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[rr] \arrow[dr] & & \hat{E} \otimes M_r \arrow[r] & \hat{E} \otimes M_{r+1} \arrow[r] & \cdots \\
& & & \ker(d^r) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0. & &
\end{tikzcd}$$

By Corollary 103, $\mathbf{R}(M_{\geq r})$ is exact except at the first step, so this yields an exact complex of free E-modules.

Example 106. Since $M = S$ has regularity zero, we obtain Cartan resolutions in both directions, and the splice map $E \to \widehat{E} = E(d+1)$ is multiplication by $e_0 \wedge e_1 \wedge \cdots \wedge e_d = \ker \left[\, e_0, \quad \cdots \quad , e_d \,\right]^t$.

Theorem 107 (Eisenbud–Fløystad–Schreyer [28]). *The i^{th} free module T^i in a Tate resolution for $\mathcal{F}$ satisfies*

$$T^i = \bigoplus_j \widehat{E} \otimes H^j(\mathcal{F}(i-j)).$$

Example 4, continued. The betti table for the twisted cubic shows that S/I has regularity one, which provides us the information needed to compute the Tate resolution. Plugging the resulting numbers into Theorem 107 shows that

i	-3	-2	-1	0	1	2
$h^1(\mathcal{F}(i))$	8	5	2	0	0	0
$h^0(\mathcal{F}(i))$	0	0	0	1	4	7

Does this make sense? Since $\mathcal{F} = \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}(3)$,

$$h^1(\mathcal{F}(i)) = h^1(\mathcal{O}_{\mathbb{P}^1}(3i)) = h^0(\mathcal{O}_{\mathbb{P}^1}(-3i-2))$$

and

$$h^0(\mathcal{F}(i)) = h^0(\mathcal{O}_{\mathbb{P}^1}(3i)) = 3i+1, \ i \geq 0,$$

which agrees with our earlier computation. $\diamond$

Problem 108. Investigate the Tate resolution for $D(\mathcal{A})$ and $C(\mathcal{A})$.

Conclusion. In this note we have surveyed a number of open problems in arrangements. The beauty of the area is that these problems are all interconnected. Perhaps the most central objects are the resonance varieties, which are related to both the LCS ranks studied in §7 and §8 using Koszul and Lie algebras, and to the Chen ranks. The results of §5 tie resonance to the Orlik–Terao algebra, and [75] notes that $J_{\mathcal{A}} \subseteq H^0(D_{\mathcal{A}})$, so the Orlik–Terao algebra is also linked to $D(\mathcal{A})$ and freeness. But freeness ties in to multiarrangements, and can be generalized to hypersurface arrangements, the topics of §3 and §4. To complete the circle, recent work of Cohen–Denham–Falk–Varchenko [11] relates freeness to $R^1(\mathcal{A})$. In short, everything is connected!

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noted in the introduction, the calculations carried out in this survey may be performed using the hyperplane arrangements package of Denham and Smith [22] in Macaulay2 [45]. Code for the individual examples is available at `http://www.math.uiuc.edu/~schenck/`.

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