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# Vanishing products of one-forms and critical points of master functions

*in memory of V. I. Arnol'd*

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**Abstract.**

Let $\mathcal{A}$ be an affine hyperplane arrangement in $\mathbb{C}^\ell$ with complement $U$. Let $f_1, \ldots, f_n$ be linear polynomials defining the hyperplanes of $\mathcal{A}$, and $A^{\cdot}$ the algebra of differential forms generated by the one-forms $\mathrm{d}\log f_1, \ldots, \mathrm{d}\log f_n$. To each $\lambda \in \mathbb{C}^n$ we associate the master function $\Phi = \prod_{i=1}^n f_i^{\lambda_i}$ on $U$ and the closed logarithmic one-form $\omega = \mathrm{d}\log \Phi$. We assume $\omega$ is a general element of a rational linear subspace $D$ of $A^1$ of dimension $q > 1$ such that the map $\bigwedge^k(D) \to A^k$ given by multiplication in $A^{\cdot}$ is zero for all $p < k \leq q$, and is nonzero for $k = p$. With this assumption, we prove the critical locus $\operatorname{crit}(\Phi)$ of $\Phi$ has components of codimension at most $p$, and these are intersections of level sets of $p$ rational master functions. We give conditions that guarantee $\operatorname{crit}(\Phi)$ is nonempty and every component has codimension equal to $p$, in terms of syzygies among polynomial master functions.

If $\mathcal{A}$ is $p$-generic, then $D$ is contained in the degree $p$ resonance variety $\mathcal{R}^p(\mathcal{A})$—in this sense the present work complements previous work on resonance and critical loci of master functions. Any arrangement is 1-generic; we give a precise description of $\operatorname{crit}(\Phi_\lambda)$ in case $\lambda$ lies in an isotropic subspace $D$ of $A^1$, using the multinet structure on $\mathcal{A}$ corresponding to $D \subseteq \mathcal{R}^1(\mathcal{A})$. This is carried out in detail for the Hessian arrangement. Finally, for arbitrary $p$ and $\mathcal{A}$, we establish necessary and sufficient conditions for a set of integral one-forms to span such a subspace, in terms of nested sets of $\mathcal{A}$, using tropical implicitization.

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## §1. Introduction

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of distinct affine hyperplanes in $\mathbb{C}^\ell$, with complement $U = \mathbb{C}^\ell - \bigcup_{i=1}^n H_i$. Choose a linear polynomial $f_i$ with zero locus $H_i$, for each $i$. Let $\lambda = (\lambda_1 \ldots, \lambda_n) \in \mathbb{C}^n$, and consider the *master function*

$$\Phi_\lambda = \prod_{i=1}^n f_i^{\lambda_i}.$$

The multi-valued function $\Phi_\lambda$ has a well-defined critical locus

$$\mathrm{crit}(\Phi_\lambda) = \{x \in U \mid \mathrm{d}\Phi_\lambda(x) = 0\}.$$

Indeed, $\mathrm{crit}(\Phi_\lambda)$ coincides with the zero locus $V(\omega_\lambda)$ of the single-valued closed logarithmic one-form

$$\omega_\lambda = \mathrm{d}\log(\Phi_\lambda) = \sum_{i=1}^n \lambda_i \mathrm{d}\log(f_i).$$

In particular, $\mathrm{crit}(\Phi_\lambda)$ is unchanged if $\lambda$ is multiplied by a non-zero scalar. We are interested in the relation between $\mathrm{crit}(\Phi_\lambda)$ and algebraic properties of the cohomology class represented by $\omega_\lambda$ in $H^1(U, \mathbb{C})$.

For certain arrangements $\mathcal{A}$ and weights $\lambda$, the critical points of $\Phi_\lambda$ yield a complete system of eigenfunctions for the commuting hamilitonians of the $\mathfrak{sl}_n(\mathbb{C})$-Gaudin model, via the Bethe Ansatz [23, 25, 17]. That application was the origin of the term master function, introduced in [28]. Much of that theory depends only on combinatorial properties of arrangements, and can be formulated in that general setting—see [29].

Let $A^{\cdot}$ denote the graded $\mathbb{C}$-algebra of holomorphic differential forms on $U$ generated by $\{\mathrm{d}\log(f_i) \mid 1 \leq i \leq n\}$. By a well-known result of Brieskorn, the inclusion of $A^{\cdot}$ into the de Rham complex of $U$ induces an isomorphism in cohomology, and thus $A^{\cdot} \cong H^{\cdot}(U, \mathbb{C})$, see [1, 4]. In particular $A^1 \cong \mathbb{C}^n$. Since $\omega_\lambda \wedge \omega_\lambda = 0$, left-multiplication by $\omega_\lambda$ makes $A^{\cdot}$ into a cochain complex $(A^{\cdot}, \omega_\lambda)$. For generic $\lambda$, $H^p(A^{\cdot}, \omega_\lambda) = 0$ for $p < \ell$, and $\dim H^\ell(A^{\cdot}, \omega_\lambda) = |\chi(U)|$, see [24, 30]. At the same time, for generic $\lambda$, $\Phi_\lambda$ has $|\chi(U)|$ isolated, nondegenerate critical points [27, 20, 26]. Here we are concerned with $H^p(A^{\cdot}, \omega_\lambda)$ and $\mathrm{crit}(\Phi_\lambda)$ when $\lambda$ is not generic.

For $p < \ell$, the $\omega$ for which $H^p(A^{\cdot}, \omega) \neq 0$ comprise the $p^{\text{th}}$ *resonance variety* $\mathcal{R}^p(\mathcal{A})$ of $\mathcal{A}$, a well-studied invariant of $A^{\cdot}$. On the other hand, precise conditions on $\lambda$ guaranteeing that $\Phi_\lambda$ has $|\chi(U)|$ isolated critical points are not known. There are also examples where the critical points of $\Phi_\lambda$ are isolated but degenerate.

In some cases $\operatorname{crit}(\Phi_\lambda)$ is positive-dimensional. If $\mathcal{A}$ is a discriminantal arrangement, in the sense of [24], then for certain choices of integral weights $\lambda$ arising from a simple Lie algebra $\mathfrak{g}$, $\operatorname{crit}(\Phi_\lambda)$ has components of the same positive dimension [25, 16, 18]. In the particular case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ of this situation, the codimension of $\operatorname{crit}(\Phi_\lambda)$ is $\ell - 1$. In this case, it was shown in [6] that $\omega_\lambda \in \mathcal{R}^{\ell-1}(\mathcal{A})$ for these $\lambda$, with the rank of the skew-symmetric part of $H^{\ell-1}(A^{\cdot}, \omega_\lambda)$ equal to the number of components of $\operatorname{crit}(\Phi_\lambda)$.

Our work in [5] provides a weak generalization of these results. There we study the *universal critical set*, the set $\Sigma$ of pairs $(x, a)$ such that $x \in V(\omega_a)$. For fixed $\lambda$, $\operatorname{crit}(\Phi_\lambda)$ is the $a = \lambda$ slice of $\Sigma$. Let $\overline{\Sigma}$ be the Zariski closure of $\Sigma$ in $\mathbb{C}^\ell \times \mathbb{C}^n$, and $\overline{\Sigma}_\lambda$ the $a = \lambda$ slice of $\overline{\Sigma}$. In [5] we show, if $\omega_\lambda \in \mathcal{R}^p(\mathcal{A})$, then $\overline{\Sigma}_\lambda$ has codimension at most $p$, provided $\mathcal{A}$ is tame and either $p \leq 2$ or $\mathcal{A}$ is free. See [5] for definitions of free and tame arrangements; any affine arrangement in $\mathbb{C}^2$ is tame. It is not true in general that $\overline{\Sigma}_\lambda$ is the closure of $\Sigma_\lambda$. Indeed, $\Sigma_\lambda$ may be empty under the given hypotheses—that is, $\overline{\Sigma}_\lambda \subseteq \mathbb{C}^\ell \times \mathbb{C}^n$ may lie over $\bigcup_{i=1}^n H_i$.

In this paper, we obtain somewhat more precise information on $\operatorname{crit}(\Phi_\lambda)$, for more general arrangements, but impose a different hypothesis on $\omega_\lambda$. Namely, we assume that $\omega_\lambda$ has a *decomposable cocycle*, that is, there exists $\psi \in A^p$ such that $\omega_\lambda \wedge \psi = 0$, and $\psi$ is a product of $p$ elements of $A^1$, whose linear span does not include $\omega_\lambda$.

We say a subspace $D$ of $A^1$ is *singular* if the multiplication map $\bigwedge^q(D) \to A^q$ is zero, where $q = \dim D$. Let $p$ be maximal such that $\bigwedge^p(D) \to A^p$ is not the zero map. If $\omega = \operatorname{d}\log(\Phi) \in D - \{0\}$, then $\omega$ can be included in a basis of $D$, and any $p$-fold product of the other basis elements is a decomposable $p$-cocycle for $\omega$.

We assume that $D$ is a rational subspace of $A^1$, that is, $D$ has a basis $\Lambda = \{\omega_{\xi_1}, \ldots, \omega_{\xi_q}\}$ with each $\omega_{\xi_j}$ an integer linear combination of $\operatorname{d}\log(f_1), \ldots, \operatorname{d}\log(f_n)$. Associated to $\Lambda$ is a rational mapping $\Phi_\Lambda = (\Phi_{\xi_1}, \ldots, \Phi_{\xi_q})\colon \mathbb{C}^\ell \rightarrowtail \mathbb{C}^q$, whose image is a quasi-affine subvariety $Y = Y_\Lambda$ of $\mathbb{C}^q$. The dimension of $Y$ is $p$. If $\omega = \operatorname{d}\log(\Phi) \in D$, then the critical locus $\operatorname{crit}(\Phi)$ is consists of fibers of $\Phi_\Lambda$ and singular points of $\Phi_\Lambda$. In particular, for generic $\omega \in D$, $\operatorname{crit}(\Phi)$ has codimension at most $\dim(Y) = p$. We obtain more precise conclusions in case the projective closure $\overline{Y}$ is a curve ($p = 1$) or a hypersurface ($p = q - 1$), or $\overline{Y}$ is nonsingular and meets the coordinate hyperplanes transversely. If $\overline{Y}$ is linear, of any codimension, with $Y = \overline{Y} \cap (\mathbb{C}^*)^q$ and $\Phi_\Lambda$ nonsingular, we get a complete description of the $\operatorname{crit}(\Phi)$ for $\operatorname{d}\log(\Phi) \in D$, in terms of critical loci of master functions on the complement of the rank-$p$ arrangement cut out on $\overline{Y}$ by the coordinate hyperplanes.

Every component of $\mathcal{R}^1(\mathcal{A})$ is a rationally-defined and isotropic linear subspace [14], and every element of $\mathcal{R}^1(\mathcal{A})$ has a decomposable cocycle. Moreover, by the theory of multinets and Čeva pencils [10], we can choose $\Lambda$ so that the variety $Y_\Lambda$ corresponding to a component of $\mathcal{R}^1(\mathcal{A})$ is linear. We carry out the entire analysis in detail in this case, with special attention to the Hessian arrangement, the one case we know for which $Y_\Lambda$ is not a hypersurface.

Our approach lends itself to tropicalization, using the main result of [7]. Using the nested set subdivison of the Bergman fan [11], we derive a rank condition for a product of integral one-forms $\omega_{\xi_1} \wedge \cdots \wedge \omega_{\xi_q}$ to vanish. The rank condition can be used in case $\mathcal{A}$ is $p$-generic to give a combinatorial description of the $(p+1)$-tuples of integral forms in $A^1$ whose product vanishes, analogous to the description of $\mathcal{R}^1(\mathcal{A})$ in terms of neighborly partitions—see [2].

The outline of this paper is as follows. In Section 2 we introduce Orlik–Solomon algebras and resonance varieties, prove a general result about zero loci of differential forms, and compute critical loci directly for some examples, including the Hessian arrangement. In Section 3 we consider logarithmic one-forms with decomposable $p$-cocycles satisfying the rationality criterion above, obtaining a precise description of their zero loci, especially in case $\Phi_\Lambda$ is nonsingular and $\overline{Y}_\Lambda$ is a hypersurface meeting the coordinate hyperplanes transversely. We revisit the examples from Section 2. In Section 4 we treat the case where $\overline{Y}_\Lambda$ is linear, returning to the example of the Hessian arrangement. In Section 5 we formulate a test for existence of decomposable cocycles using tropical implicitization.

## §2. Resonance, vanishing products, and zeros of one-forms

It will be more convenient for us to consider arrangements of projective hyperplanes in complex projective space $\mathbb{P}^\ell$. Let $[x_0 : \cdots : x_\ell]$ be homogeneous coordinates on $\mathbb{P}^\ell$, and let $\alpha_i \colon \mathbb{C}^{\ell+1} \to \mathbb{C}$ be a nonzero homogeneous linear form, for $0 \leq i \leq n$. Assume without loss that $\alpha_0(x) = x_0$. Let $H_i = \ker(\alpha_i)$, considered as a projective hyperplane in $\mathbb{P}^\ell$, and let $\mathcal{A} = \{H_0, \ldots, H_n\}$. We will denote the corresponding linear hyperplanes in $\mathbb{C}^{\ell+1}$ by $cH_i$, comprising the central arrangement $c\mathcal{A} = \{cH_0, \ldots, cH_n\}$. Let $U = \mathbb{P}^\ell - \bigcup_{i=0}^n H_i$. We identify $[1 : x_1 : \cdots : x_\ell] \in \mathbb{P}^\ell - H_0$ with $(x_1, \ldots, x_\ell) \in \mathbb{C}^\ell$, and set

$$f_i(x_1, \ldots, x_\ell) = \alpha_i(1, x_1, \ldots, x_\ell)$$

for $1 \leq i \leq n$. Then we recover the affine arrangement $\mathcal{A}$ of the Introduction, with the same complement $U$. In $\mathbb{P}^\ell$, $U$ is the complement of the singular projective hypersurface defined by

$$Q = \prod_{i=0}^{n} \alpha_i,$$

the (homogeneous) *defining polynomial* of $\mathcal{A}$.

### 2.1. The projective Orlik–Solomon algebra

Let $\Omega^{\cdot}(U)$ be the complex of holomorphic differential forms on $U$. The *Orlik–Solomon algebra* of $\mathcal{A}$ is the subalgebra $A^{\cdot}(\mathcal{A})$ of $\Omega^{\cdot}(U)$ generated by $\mathrm{d}\log(f_i)$, $1 \le i \le n$, as in the Introduction.

We will also study $A^{\cdot}(\mathcal{A})$ in homogeneous coordinates. Let $\omega_i = \mathrm{d}\log(\alpha_i)$ for $0 \le i \le n$, and let $A^{\cdot}(c\mathcal{A})$ be the algebra of holomorphic forms on $\mathbb{C}^{\ell+1}$ generated by $\omega_0, \ldots, \omega_n$. Define $\partial\colon A^{\cdot}(c\mathcal{A}) \to A^{\cdot}(c\mathcal{A})$ by

$$\partial(\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}) = \sum_{j=1}^{k} (-1)^{j-1} \omega_{i_1} \wedge \cdots \wedge \widehat{\omega}_{i_j} \wedge \cdots \wedge \omega_{i_k}$$

and extending linearly. Then $\partial$ is a graded derivation of degree -1, and

$$\partial\Big(\sum_{i=0}^{n} c_i\omega_i\Big) = \sum_{i=0}^{n} c_i.$$

In general, a holomorphic $p$-form on $\mathbb{C}^{\ell+1} - \{0\}$ descends to a well-defined form on $\mathbb{P}^{\ell}$ if and only if it is $\mathbb{C}^*$-invariant and its contraction along the Euler vector field $\sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$ vanishes, see [8]. This contraction, on $A^{\cdot}(c\mathcal{A})$, is given by $\partial$, and $A^{\cdot}(c\mathcal{A})$ consists of $\mathbb{C}^*$-invariant forms. Then we may identify $A^{\cdot}(\mathcal{A})$ with the subalgebra $\ker(\partial)$ of $A^{\cdot}(c\mathcal{A})$. This is easily seen to coincide with the subalgebra of $A^{\cdot}(c\mathcal{A})$ generated by $\ker(\partial) \cap A^1(c\mathcal{A})$.

With our choice of coordinates, $\mathrm{d}\log(f_i) = \omega_i - \omega_0$ under this identification. $\{\omega_1 - \omega_0, \ldots, \omega_n - \omega_0\}$ generates $A^{\cdot}$ by the remark above. Also, $(A^{\cdot}(c\mathcal{A}), \partial)$ is an exact complex, so that $\mathrm{im}(\partial) = \ker(\partial) = A^{\cdot}(\mathcal{A})$, see [19].

There is a well-known presentation of $A^{\cdot}(c\mathcal{A})$ as a quotient of the exterior algebra $E^{\cdot} = \bigwedge^{\cdot}(e_0, \ldots, e_n)$. For $C = \{i_1, \ldots, i_k\} \subseteq \{0, \ldots, n\}$, write $e_C = e_{i_1} \cdots e_{i_k} \in E^k$. Say $C$ is a *circuit* of $c\mathcal{A}$ if $C$ is minimal with the property that

$$\operatorname{codim} \bigcap_{j \in C} H_j < |C|.$$

Then $A^{\cdot}(c\mathcal{A})$ is isomorphic to $E^{\cdot}/I$, where

$$I = \big(\partial e_C \mid C \text{ is a circuit of } c\mathcal{A}\big).$$

### 2.2. Resonance varieties

Let $\omega = \sum_{i=0}^{n} \lambda_i \omega_i$, where $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{C}^{n+1}$. Assume that $\partial\omega = \sum_{i=0}^{m} \lambda_i = 0$. Since $\mathrm{d}\log(f_i) = \omega_i - \omega_0$, $\omega = \sum_{i=1}^{n} \lambda_i \,\mathrm{d}\log(f_i)$.

Then $\omega \in A^1$, and $\omega \wedge \omega = 0$, so we obtain a cochain complex

$$0 \longrightarrow A^0 \xrightarrow{\omega\wedge -} \cdots \xrightarrow{\omega\wedge -} A^p \xrightarrow{\omega\wedge -} \cdots \xrightarrow{\omega\wedge -} A^\ell \longrightarrow 0.$$

Let

$$\begin{aligned} Z^p(\omega) &= \{\psi \in A^p \mid \omega \wedge \psi = 0\}, \\ B^p(\omega) &= \{\psi \in A^p \mid \psi = \omega \wedge \varphi \text{ for some } \varphi \in A^{p-1}\}, \text{ and} \\ H^p(A^{\cdot}, \omega) &= Z^p(\omega)/B^p(\omega). \end{aligned}$$

Then

$$\mathcal{R}^p(\mathcal{A}) = \{\omega \in A^1 \mid H^p(A^{\cdot}, \omega) \neq 0\}$$

is, by definition, the $p^{\text{th}}$ *resonance variety* of $\mathcal{A}$.

As observed in the Introduction,

$$\omega = \mathrm{d}\log(\Phi) = \frac{\mathrm{d}\Phi}{\Phi},$$

where $\Phi = \prod_{j=1}^{n} f_j^{\lambda_j}$, and $\mathrm{crit}(\Phi)$ coincides with the zero locus of $\omega$. In homogeneous coordinates, $\Phi$ is given by $\prod_{j=0}^{n} \alpha_j^{\lambda_j}$.

### 2.3. Zeros of forms

We start with an elementary observation about products and zeros of differential forms. If $\psi \in \Omega^k(U)$, for some $k$, $0 \leq k \leq \ell$, let

$$V(\psi) = \{x \in U \mid \psi(x) = 0\},$$

a quasi-affine subvariety of $\mathbb{C}^\ell$. Let $U(\psi) = U - V(\psi)$.

**Proposition 2.1.** *Suppose $\omega \in \Omega^1(U)$ and $\psi \in \Omega^p(U)$ satisfy $\omega \wedge \psi = 0$. Then every component of $V(\omega) - V(\psi)$ has codimension less than or equal to $p$.*

*Proof.* We may write $\omega = \sum_{i=1}^{\ell} b_i \mathrm{d}x_i$ for some holomorphic functions $b_1, \dots b_\ell$ on $U$. Then

$$V(\omega) = \bigcap_{i=1}^{\ell} V(b_i).$$

Similarly, $\psi = \sum_I A_I \mathrm{d}x_I$ for some holomorphic functions $A_I$, where $I$ ranges over all subsets $I = \{i_1, \ldots, i_p\}_<$ of $\{1, 2, \ldots, \ell\}$, and $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$. (The subscript "$<$" is meant to indicate that the elements of $I$ are listed in increasing order.) Set $U_I = U(A_I)$, and let $S_I$ denote the coordinate ring of $U_I$, i.e., $S_I$ is $\mathbb{C}[x_1, \ldots, x_\ell]$, localized at $A_I$. Then

$$U(\psi) = \bigcup_I U_I.$$

The equation $\omega \wedge \psi = 0$ says, for each subset $J$ of $\{1, \ldots, \ell\}$ of size $p+1$,

$$\sum_{i \in J} \sigma(i, J) b_i A_{J - \{i\}} = 0. \tag{2.1}$$

Here $\sigma(i, J) = \pm 1$ depending on the position of $i$ in $J$.

We have

$$V(\omega) \cap U(\psi) = \bigcup_I V(\omega) \cap U_I.$$

Fix $I = \{i_1, \cdots, i_p\}_<$. For each $i \notin I$, set $J = I \cup \{i\}$ in equation (2.1). Since $A_I \neq 0$ on $U_I$, one can solve for $b_i$ in terms of $b_{i_1}, \ldots, b_{i_p}$. This means $b_i$ lies in the ideal $(b_{i_1}, \ldots, b_{i_p})$ of $S_I$. Since this holds for every $i \notin I$, the defining ideal of $V(\omega) \cap U_I$ in $S_I$ is contained in $(b_{i_1}, \ldots, b_{i_p})$. Then each irreducible component of $V(\omega) \cap U_I$ has codimension less than or equal to the codimension of $(b_{i_1}, \ldots, b_{i_p})$, which is at most $p$. Since the $U_I$ cover $U(\psi)$, the result follows. Q.E.D.

**Corollary 2.2.** *If*

$$\bigcap \{V(\psi) \mid \psi \in \Omega^p(U), \omega \wedge \psi = 0\} = \emptyset,$$

*then every component $V(\omega)$ has codimension less than or equal to $p$.*

**Corollary 2.3.** *Suppose $X$ is a component of $V(\omega)$ of codimension $c$. If $\psi$ is a $p$-form satisfying $\omega \wedge \psi = 0$ and $p < c$, then $X \subseteq V(\psi)$.*

**Remark 2.4.** The preceding results go through without change for any smooth complex analytic variety $U$, interpreting $x_1, \ldots, x_\ell$ as local holomorphic coordinates on $U$.

A $p$-form $\psi$ satisfying $\omega \wedge \psi = 0$ will be called a *$p$-cocycle* for $\omega$. We say $\psi$ is *trivial* if $\psi = \omega \wedge \varphi$ for some $\varphi \in \Omega^{p-1}(U)$. If $\psi$ is a trivial cocycle for $\omega$, then $V(\omega) \subseteq V(\psi)$.

The trivial cocycle condition $\psi = \omega \wedge \varphi$ is generally difficult to characterize. We propose the following conjecture, the converse to the observation above.

**Conjecture 2.5.** *If $\psi \in A^p$, then $\psi = \omega \wedge \varphi$ for some $\varphi \in A^{p-1}$ if and only if $\omega \wedge \psi = 0$ and $V(\omega) \subseteq V(\psi)$.*

The conjecture is not hard to prove directly in case $p = 1$, and the statement for any $p$ follows from results of [29] if $\omega \in A^1$ is generic.

### 2.4. Examples

Our first example is linearly equivalent to the rank-three braid arrangement.

**Example 2.6.** Let $\mathcal{A} = \{H_0, \ldots, H_5\}$ be the arrangement with defining polynomial

$$Q = xyz(x-y)(x-z)(y-z),$$

with the hyperplanes labelled according to the order of factors in $Q$.

For $a, b, c \in \mathbb{C}$, not all zero, with $a + b + c = 0$, let

$$\Phi_{abc} = [x(y-z)]^a [y(x-z)]^b [z(x-y)]^c.$$

Then $\omega_{abc} := \mathrm{d}\log(\Phi_{abc}) = a(\omega_0 + \omega_5) + b(\omega_1 + \omega_4) + c(\omega_2 + \omega_3)$.

One computes

$$\begin{aligned}
\omega_{abc} &= [\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\
&= [\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z] \begin{bmatrix} 1/x & 1/(x-z) & 1/(x-y) \\ 1/(y-z) & 1/y & 1/(y-x) \\ 1/(z-y) & 1/(z-x) & 1/z \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
\end{aligned}$$

The zero locus $V(\omega_{abc})$ is given by the vanishing of $b_1, b_2$, and $b_3$. The kernel of the matrix is spanned by $(x(y-z), y(z-x), z(x-y))$, so $[x : y : z] \in V(\omega_{abc})$ if and only if $[x(y-z) : y(z-x) : z(x-y)] = [a : b : c]$. Since $a + b + c = 0$, this is equivalent to $[x(y-z) : y(z-x)] = [a : b]$, i.e.,

$$\frac{x(y-z)}{y(z-x)} = \frac{a}{b}$$

or, more symmetrically,

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0.$$

If any of $a$, $b$, or $c$ are zero, then $\mathrm{crit}(\Phi_{abc}) = V(\omega_{abc})$ is empty. Otherwise $\mathrm{crit}(\Phi_{abc})$ has codimension 1. Moreover, $\mathrm{crit}(\Phi_\lambda)$ is a level set of the master function $\frac{x(y-z)}{y(z-x)}$.

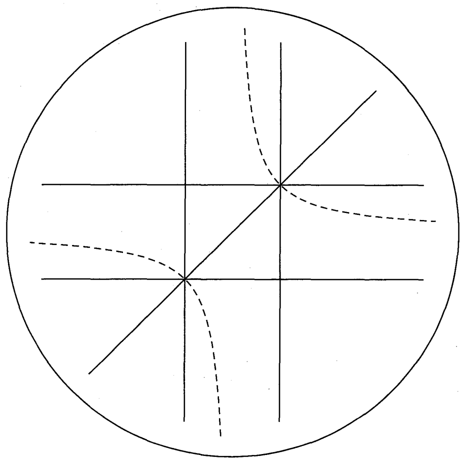

Fig. 1. The braid arrangement with $\mathrm{crit}(\Phi_{1,1,-2})$

Let $D = \{\omega_{abc} \mid a+b+c=0\}$. Then $D \subseteq A^1$ and, for any $\omega \in D$, $Z^1(\omega) = D$. Thus $\psi \in Z^1(\omega_{abc})$ if and only if $\psi = \mathrm{d}\log \Phi_{a'b'c'}$ with $a'+b'+c'=0$. Then $\psi$ has zero locus given by

$$\frac{a'}{x} + \frac{b'}{y} + \frac{c'}{z} = 0,$$

which one can see is disjoint from $V(\omega_{abc})$ if and only if $\psi \notin B^1(\omega_{abc})$. The Zariski closure of every nonempty critical set contains the four points $[0:0:1]$, $[1:1:1]$, $[1:0:0]$, and $[0:1:0]$—see Figure 1.

Here is a rank-four example.

**Example 2.7.** Let $\mathcal{A} = \{H_0, \ldots, H_7\}$ be the arrangement of eight planes in $\mathbb{P}^3$ with defining polynomial

$$Q = xyzw(x+y+z)(x+y+w)(x+z+w)(y+z+w).$$

The dual point configuration consists of the four vertices and four face-centers of the 3-simplex. Fix $a, b, c, d \in \mathbb{C}$ and let

$$\Phi = \left(\frac{x}{y+z+w}\right)^a \left(\frac{y}{x+z+w}\right)^b \left(\frac{z}{x+y+w}\right)^c \left(\frac{w}{x+y+z}\right)^d.$$

The one-form $\omega = \mathrm{d}\log\Phi$ belongs to a 4-dimensional component of $\mathcal{R}^2(\mathcal{A})$. If none of $a, b, c, d$ are zero, then $H^1(A, \omega) = 0$ and $H^2(A, \omega) \cong \mathbb{C}$. A nontrivial 2-cocycle for $\omega$ is given by

$$\psi = b \cdot \partial(\omega_{167}) + c \cdot \partial(\omega_{257}) + d \cdot \partial(\omega_{347}).$$

One sees that $\omega$ is equal to the product

$$[\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w]\begin{bmatrix} \frac{1}{x} & \frac{-1}{x+z+w} & \frac{-1}{x+y+w} & \frac{-1}{x+y+z} \\ \frac{-1}{y+z+w} & \frac{1}{y} & \frac{-1}{x+y+w} & \frac{-1}{x+y+z} \\ \frac{-1}{y+z+w} & \frac{-1}{x+z+w} & \frac{1}{z} & \frac{-1}{x+y+z} \\ \frac{-1}{y+z+w} & \frac{-1}{x+z+w} & \frac{-1}{x+y+w} & \frac{1}{w} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Computing the kernel of the matrix, we see that $[x:y:z:w] \in V(\omega)$ if and only if the vector $(x(y+z+w), y(x+z+w), z(x+y+w), w(x+y+z))$ is proportional to $(a,b,c,d)$, giving the equations

$$\frac{x(y+z+w)}{w(x+y+z)} = \frac{a}{d}, \quad \frac{y(x+z+w)}{w(x+y+z)} = \frac{b}{d}, \quad \frac{z(x+y+w)}{w(x+y+z)} = \frac{c}{d}.$$

$V(\omega)$ has a component of codimension two, given by

$$x+y+z+w = 0, \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w} = 0.$$

For general $a,b,c,d$, the remaining components of $V(\omega)$ consist of four additional points in $\mathbb{P}^3$. It follows from Corollary 2.3 that the cocycle $\psi$ vanishes on the isolated points of $V(\omega)$. This can also be verified here by direct computation.

**Example 2.8.** The Hessian arrangement consists of the 12 lines through the inflection points of a nonsingular cubic in $\mathbb{P}^2$. It is the only known arrangement of rank greater than two that supports a global component of $\mathcal{R}^1(\mathcal{A})$ of dimension greater than two. That is, there is an element $\omega \in A^1$ which has poles along every hyperplane of $\mathcal{A}$, and satisfies $\dim H^1(A^{\cdot},\omega) > 1$.

Any nonsingular cubic is equivalent to

$$x^3+y^3+z^3-3txyz = 0 \tag{2.2}$$

up to projective transformation, for some $t \in \mathbb{C}$. These cubics have the same inflection points. Then, up to projective transformation, the Hessian arrangement $\mathcal{A}$ is defined by

$$\begin{aligned} Q = xyz(x+y+z)(x+y+\zeta z)(x+y+\zeta^2 z)(x+\zeta y+z)(x+\zeta y+\zeta z) \\ \cdot (x+\zeta y+\zeta^2 z)(x+\zeta^2 y+z)(x+\zeta^2 y+\zeta z)(x+\zeta^2 y+\zeta^2 z), \end{aligned}$$

where $\zeta = e^{\frac{2\pi i}{3}}$. The 12 lines of $\mathcal{A}$ are the irreducible components of the four singular cubics in the family 2.2, corresponding to $t = \infty, 1, \zeta$, and $\zeta^2$. See [19, Example 6.30].

Numbering the hyperplanes in order, these singular cubics are given by

$$\begin{aligned} P_0 &= \alpha_0\alpha_1\alpha_2 &&= xyz, \\ P_1 &= \alpha_3\alpha_8\alpha_{10} &&= x^3+y^3+z^3-3xyz, \\ P_2 &= \alpha_4\alpha_6\alpha_{11} &&= x^3+y^3+z^3-3\zeta xyz, \text{ and} \\ P_3 &= \alpha_5\alpha_7\alpha_9 &&= x^3+y^3+z^3-3\zeta^2 xyz. \end{aligned}$$

For $(a_1, a_2, a_3) \in \mathbb{C}^3$ let $\omega = \omega_{a_1a_2a_3} = \mathrm{d}\log(\Phi)$, where

$$\Phi = \Phi_{a_1a_2a_3} = \left(\frac{P_1}{P_0}\right)^{a_1}\left(\frac{P_2}{P_0}\right)^{a_2}\left(\frac{P_3}{P_0}\right)^{a_3}.$$

Let $D \subseteq A^1$ be the space of all such forms. Then $H^1(A^\cdot, \omega) \cong D/\mathbb{C}\omega$ has dimension two.

As in the previous examples, we write

$$\omega = \begin{bmatrix}\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\end{bmatrix} M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

where $M$ is a $3 \times 3$ matrix of rational functions, the Jacobian of

$$(\log(P_1/P_0), \log(P_2/P_0), \log(P_3/P_0)).$$

Then $\omega(x) = 0$ for $x \in U$ if and only if $a = (a_1, a_2, a_3)$ lies in the kernel of $M(x)$.

The matrix $M$ has rank 1. In fact one finds that $M = vw^T$ where

$$v = \begin{bmatrix} \frac{2x^3-y^3-z^3}{x} \\ \frac{x^3-2y^3+z^3}{y} \\ \frac{x^3+y^3-2z^3}{z} \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} \frac{1}{P_1} \\ \frac{1}{P_2} \\ \frac{1}{P_3} \end{bmatrix}.$$

The critical equation becomes $vw^Ta = 0$, which is satisfied if and only if $w^Ta = 0$ or all components of $v$ vanish. The latter occurs at the points given by $x^3 = y^3 = z^3$, which are the common inflection points of the cubics (2.2). In particular those points do not lie in the complement $U$. Thus $\mathrm{crit}(\Phi)$ is defined by the single equation

$$\frac{a_1}{P_1} + \frac{a_2}{P_2} + \frac{a_3}{P_3} = 0. \tag{2.3}$$

Then $\mathrm{crit}(\Phi)$ is empty or has codimension one and degree six. Working in the torus $xyz \neq 0$, set

$$T = \frac{x^3+y^3+z^3}{xyz}.$$

Then equation (2.3) is equivalent to

$$\frac{a_1}{T-3} + \frac{a_2}{T-3\zeta} + \frac{a_3}{T-3\zeta^2} = 0, \tag{2.4}$$

which becomes a quadratic $AT^2+BT+C=0$ in $T$.

Then, if $a=(a_1,a_2,a_3)$ is generic, $\mathrm{crit}(\Phi)$ has two irreducible components. Each component is the intersection with $U$ of a level set $T=3t$ of $T$. These are nonsingular fibers in the pencil (2.2), meeting each other and $\mathcal{A}$ at their nine common inflection points. So $\mathrm{crit}(\Phi) \subseteq U$ has two connected components. Every pair of nonsingular fibers appears as $\mathrm{crit}(\Phi)$ for some $a$. The discriminant $B^2-4AC$ defines a hypersurface in $(a_1,a_2,a_3)$-space for which the corresponding critical locus $\mathrm{crit}(\Phi)$ has a single nonreduced component, and every nonsingular fiber can appear.

For some values of $a$, $\mathrm{crit}(\Phi)$ is empty or has only one reduced component. This is easiest to see by clearing fractions in (2.3), to obtain

$$a_1P_2P_3 + a_2P_1P_3 + a_3P_1P_2 = 0. \tag{2.5}$$

When one component of the variety defined by (2.5) is $P_i=0$ or $P_0 = xyz = 0$, then $\mathrm{crit}(\Phi)$ has one reduced component. This occurs if $a$ is a generic point on $a_i = 0$ or $a_1+a_2+a_3=0$. If $a_i=0$ for some $i$ and $a_1+a_2+a_3=0$, or if $a_i \neq 0$ for only one $i$, then (2.5) becomes $P_iP_j=0$, and $\mathrm{crit}(\Phi)=\emptyset$. If $a$ is a cyclic permutation of $(0,\zeta,1)$ or equals $(1,\zeta,\zeta^2)$, up to scalar multiple, then (2.5) becomes $P_i^2=0$, and $\mathrm{crit}(\Phi)=\emptyset$.

Each cocycle of $\omega=\omega_{a_1a_2a_3}$ has the form $\psi=\omega_{b_1b_2b_3}$ for some $b_1,b_2,b_3$. So we see that $V(\psi)$ and $V(\omega)$ can have a component in common, but if $\omega$ and $\psi$ are not proportional, then $V(\omega)-V(\psi)$ is nonempty and has codimension one.

## §3. Decomposable cocycles

We will now assume $\psi$ is a cocycle for $\omega \in A^1$ and $\psi$ is a product of logarithmic one-forms. Then $\omega$ is a factor in a vanishing product of $p+1$ one-forms in $A^{\cdot}$. To carry out our geometric analysis it is necessary to work initially over the integers, although eventually the results extend to $\mathbb{C}$-linear combinations of the original integral weights. For that reason we state the result in terms of subspaces of $A^1$.

We will be dealing with rational functions parametrizing affine or projective algebraic varieties. For that reason we formulate and prove our results algebraically. For any quasi-projective variety $X$, write $\mathbb{C}[X]$ for the ring of regular functions on $X$, and $\mathbb{C}(X)$ for the field of rational functions on $X$. If $X$ is a subvariety of projective space, then elements of $\mathbb{C}[X]$ are represented by homogeneous polynomials and elements of $\mathbb{C}(X)$ by homogeneous rational functions of degree zero. If $\varphi\colon R \to S$ is a homomorphism of $\mathbb{C}$-algebras, denote by $\Omega_{S|R}$ the $S$-module of Kähler differentials of $S$ over $R$. Write $\Omega[X] = \Omega_{\mathbb{C}[X]|\mathbb{C}}$ and $\Omega(X) = \Omega_{\mathbb{C}(X)|\mathbb{C}}$. Elements of $\Omega[X]$ (resp. $\Omega(X)$) are polynomial (resp. rational) one-forms on $X$.

### 3.1. Singular subspaces of $A^1$

Let $D$ be a subspace of $A^1$. We call $D$ a *singular subspace* if the multiplication map $\bigwedge^q(D) \to A^q$ is the zero map, where $q = \dim D$. The *rank* of $D$ is the largest $p$ such that $\bigwedge^p(D) \to A^p$ is not trivial. We say $D$ is *rational* if $D$ has a basis $\{\omega_{\xi_1}, \dots, \omega_{\xi_q}\}$, with $\xi_i = (\xi_{i0}, \dots, \xi_{in}) \in \mathbb{Z}^{n+1}$ for $1 \le i \le q$. Then $\Phi_{\xi_i} = \prod_{j=1}^n f_j^{\xi_{ij}} = \prod_{j=0}^n \alpha_j^{\xi_{ij}}$ is a single-valued rational function on $\mathbb{P}^\ell$, regular on $U$. (Recall $\sum_{j=0}^n \xi_{ij} = 0$.)

We apply the following general result. It is an easy consequence of the implicit function theorem, but we give an algebraic proof that holds over any algebraically-closed field of characteristic zero. See [13] and [9] for the relevant background on Kähler differentials.

**Proposition 3.1.** *Suppose $F_1, \dots, F_q$ are rational functions on $\mathbb{C}^\ell$, and*

$$F = (F_1, \dots, F_q)\colon \mathbb{C}^\ell \rightarrowtail \mathbb{C}^q.$$

*Then the image of $F$ has dimension less than $k$ if and only if*

$$\mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k} = 0$$

*for all $1 \le i_1 < \cdots < i_k \le q$.*

*Proof.* The image of $F$ is a quasi-affine variety, whose function field is isomorphic to $\mathbb{C}(F_1, \dots, F_q)$. Then the dimension $p$ of $\mathrm{im}(F)$ is equal to the transcendence degree of $\mathbb{C}(F_1, \dots, F_q)$ over $\mathbb{C}$. Without loss of generality, suppose $\{F_1, \dots, F_p\}$ is a transcendence base for $\mathbb{C}(F_1, \dots, F_q)$ over $\mathbb{C}$. Then the set $\{\mathrm{d}F_1, \dots, \mathrm{d}F_p\}$ forms a basis for $\Omega(\mathbb{C}(F_1, \dots, F_q))$, a vector space over $\mathbb{C}(F_1, \dots, F_q)$, see [9, Theorem 16.14]. Then

$$\mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_p \neq 0.$$

If $\{F_{i_1}, \dots, F_{i_k}\} \subseteq \{F_1, \dots, F_q\}$ with $k > p$, then $\{\mathrm{d}F_{i_1}, \dots, \mathrm{d}F_{i_k}\}$ is linearly dependent over $\mathbb{C}(F_1, \dots, F_q)$, hence

$$\mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k} = 0.$$

Q.E.D.

If $\Lambda = (\xi_1, \dots, \xi_q)$ with $\xi_i \in \mathbb{Z}^{n+1}$ satisfying $\sum_{j=0}^{n} \xi_{ij} = 0$, set

$$\Phi_\Lambda := (\Phi_{\xi_1}, \dots, \Phi_{\xi_q}) \colon \mathbb{P}^\ell \rightarrowtail \mathbb{C}^q.$$

Proposition 3.1 applies as follows.

**Corollary 3.2.** *Suppose $D$ is a rational singular subspace of $A^1$, and $\Phi_\Lambda$ is the rational mapping associated to an ordered integral basis $\Lambda$ of $D$. Then* $\dim \Phi_\Lambda(U)$ *is equal to the rank of $D$. In particular, $\Phi_\Lambda(U)$ has positive codimension in $\mathbb{C}^q$.*

Let $[z_0 : \cdots : z_q]$ be homogeneous coordinates on $\mathbb{P}^q$, and identify $\mathbb{C}^q$ with the $\mathbb{P}^q - \{z_0 = 0\}$ as above. Let $\overline{Y}$ be the Zariski closure of $Y = \Phi_\Lambda(U)$ in $\mathbb{P}^q$. In homogeneous coordinates, $\Phi_\Lambda$ is given by

$$\Phi_\Lambda = [1 : \Phi_{\xi_1} : \cdots : \Phi_{\xi_q}] \colon \mathbb{P}^\ell \rightarrowtail \mathbb{P}^q.$$

Clearing fractions, we have

$$\Phi_\Lambda = [\Phi_{\nu_0} : \Phi_{\nu_1} : \cdots : \Phi_{\nu_q}]$$

where the master functions $\Phi_{\nu_i}$ are homogeneous polynomials of the same degree $d$. Moreover we may assume the $\Phi_{\nu_i}$ have no common factors. Equivalently, the weights $\nu_i = (\nu_{i0}, \dots, \nu_{in})$ are non-negative integer vectors with $\sum_{i=0}^{n} \nu_{ij} = d$ for $0 \le i \le q$, whose supports have empty intersection. Then

$$\Phi_{\xi_i} = \frac{\Phi_{\nu_i}}{\Phi_{\nu_0}}, \quad \xi_i = \nu_i - \nu_0, \quad \text{and} \quad \omega_{\xi_i} = \omega_{\nu_i} - \omega_{\nu_0}.$$

**Definition 3.3.** $\Lambda$ is *essential* if every hyperplane $H \in \mathcal{A}$ appears as a component of $V(\Phi_{\nu_i})$ for some $i$, $0 \le i \le q$.

Henceforth we will tacitly assume $\Lambda$ is essential. We have $Y \subseteq \overline{Y} \cap (\mathbb{C}^*)^q$ in any case. If $\Lambda$ is not essential, the inclusion is proper, and may be proper otherwise—see Example 3.16.

The defining ideal $I = I_\Lambda$ of $\overline{Y}$ is generated by homogeneous polynomials $P(z_0, \dots, z_q)$ for which $P(\Phi_{\nu_0}, \dots, \Phi_{\nu_q})$ vanishes identically on $U$, or, equivalently, $P(1, \Phi_{\xi_1}, \dots, \Phi_{\xi_q}) = 0$. We will sometimes refer to $I_\Lambda$

as the *syzygy ideal* of $\Lambda$. The mapping $z_i \mapsto \Phi_{\nu_i}$ induces an isomorphism of rings

$$\mathbb{C}[\overline{Y}] = \mathbb{C}[z_0, \dots, z_q]/I \longrightarrow \mathbb{C}[\Phi_{\nu_0}, \dots, \Phi_{\nu_q}].$$

In particular $\overline{Y}$ is irreducible. Identifying $\mathbb{C}[\overline{Y}]$ with $\mathbb{C}[\Phi_{\nu_0}, \dots, \Phi_{\nu_q}]$, the dominant rational mapping $\Phi_\Lambda \colon \mathbb{P}^\ell \rightarrowtail \overline{Y}$ corresponds to the field extension

$$\mathbb{C}(\Phi_{\xi_1}, \dots, \Phi_{\xi_q}) \subseteq \mathbb{C}(x_1, \dots, x_\ell).$$

The affine ring $\mathbb{C}[Y]$ is isomorphic to a localization of the ring of Laurent polynomials $\mathbb{C}[\Phi_{\nu_0}^{\pm 1}, \dots, \Phi_{\nu_q}^{\pm 1}]$.

If $\omega \in D - \{0\}$, we write $\omega = \omega_a = \sum_{i=0}^q a_i \omega_{\nu_i} = \sum_{i=1}^n a_i \omega_{\xi_i}$ with $a = (a_0, \dots a_q) \in \mathbb{C}^{q+1} - \{0\}$, satisfying $\sum_{i=0}^q a_i = 0$. Note that any $p$-fold wedge product $\psi = \omega_{\xi_{i_1}} \wedge \cdots \wedge \omega_{\xi_{i_p}}$ is a cocycle for $\omega$.

The one-form $\sum_{i=0}^q a_i \mathrm{d}\log(z_i) \in \Omega(\mathbb{C}^{q+1})$ is $\mathbb{C}^*$-invariant, and contracts trivially along the Euler vector field, so it descends to a well-defined rational one-form on $\mathbb{P}^q$. This form restricts to a one-form in $\Omega(\overline{Y})$ which we denote by $\tau_a$. Since $Y \subseteq (\mathbb{C}^*)^q$, $\tau_a$ is regular on $Y$. Note that $\sum_{i=0}^q a_i \mathrm{d}\log(z_i) = \mathrm{d}\log \mu_a$, where $\mu_a = \prod_{i=0}^q z_i^{a_i}$ is a master function for the arrangement of coordinate hyperplanes in $\mathbb{P}^q$.

We show that the zeros of $\tau_a$ pull back to zeros of $\omega_a$.

**Lemma 3.4.** *Let $x \in U$ and $y = \Phi_\Lambda(x) \in Y$. Then $\omega_a(x) = 0$ if and only if $\tau_a(y) \in \ker(\Phi_\Lambda^*)_y$.*

*Proof.* We have

$$\omega_a = \sum_{i=0}^q a_i \mathrm{d}\log \Phi_{\nu_i} = \mathrm{d}\log \prod_{i=0}^q \Phi_{\nu_i}^{a_i}, = \Phi_\Lambda^*(\mathrm{d}\log \prod_{i=0}^q z_i^{a_i}) = \Phi_\Lambda^*(\tau_a).$$

The result follows upon localization at $y$. Q.E.D.

Note that

$$\Phi_\Lambda^*(\tau_a) = \sum_{i=0}^q \sum_{j=0}^\ell \frac{a_i}{\Phi_{\nu_i}} \frac{\partial \Phi_{\nu_i}}{\partial x_j} \mathrm{d}x_j.$$

Then $\tau_a(y) \in \ker(\Phi_\Lambda^*)_y$ if and only if $\begin{bmatrix} \frac{a_0}{y_0} & \cdots & \frac{a_q}{y_q} \end{bmatrix}$ lies in the left null space of the Jacobian of $\Phi_\Lambda$.

Let $\mathrm{Sing}(\Phi_\Lambda)$ denote the singular locus of $\Phi_\Lambda$, and $\mathrm{Sing}(\overline{Y})$ the singular locus of $\overline{Y}$. Let $S_\Lambda = \mathrm{Sing}(\Phi_\Lambda) \cup \Phi_\Lambda^{-1}(\mathrm{Sing}(\overline{Y})) \subseteq U$.

**Theorem 3.5.** *$V(\omega_a)$ contains $\Phi_\Lambda^{-1}(V(\tau_a))$, and $V(\omega_a) - \Phi^{-1}(V(\tau_a))$ is a subset of $S_\Lambda$.*

*Proof.* The first statement is immediate from Lemma 3.4. If $\Phi_\Lambda$ is nonsingular at $x \in U$ then the Jacobian of $\Phi_\Lambda$ attains its maximal rank $p = \dim(Y)$ at $x$. If in addition $\overline{Y}$ is nonsingular at $y = \Phi_\Lambda(x)$, then $\dim(\Omega(Y)_y) = p$. Then $\ker(\Phi_\Lambda^*)_y = 0$. The second statement then follows from Lemma 3.4. Q.E.D.

**Corollary 3.6.** *Suppose $D$ is a rational singular subspace of $A^1$, with integral basis $\Lambda$. Let $p$ be the rank of $D$. If* $\mathrm{d}\log(\Phi) \in D$ *then* $\mathrm{crit}(\Phi) \subseteq S_\Lambda$ *or* $\mathrm{codim}(\mathrm{crit}(\Phi)) \leq p$.

*Proof.* Write $\omega = \mathrm{d}\log(\Phi) = \sum_{i=1}^q a_i \omega_{\xi_i}$. The hypothesis implies $\dim Y = p$, so $V(\tau_a)$ is empty or has codimension at most $p$ in $Y$. In the first case $V(\omega) \subseteq S_\Lambda$ by the preceding theorem. Otherwise, $V(\omega) \supseteq \Phi_\Lambda^{-1}(V(\tau_a))$ has codimension at most $p$. Q.E.D.

**Corollary 3.7.** *Suppose $D$ is a rational singular subspace of $A^1$, with integral basis $\Lambda$. If* $\mathrm{d}\log(\Phi) \in D$*, then* $\mathrm{crit}(\Phi) - S_\Lambda$ *is a union of fibers of $\Phi_\Lambda$.*

The fibers of $\Phi_\Lambda$ are intersections of level sets of the rational master functions $\Phi_{\xi_i}$, for $1 \leq i \leq q$.

We have not used the assumption that $\{\xi_1, \dots, \xi_q\}$ is linearly independent, i.e., that the dimension of $D$ is strictly greater than $p$. This hypothesis rules out a trivial case.

**Proposition 3.8.** *Suppose $a \neq 0$. Then $\tau_a$ is not identically zero on $Y$.*

*Proof.* If $\tau_a$ is zero on $Y$, then

$$\Phi_\Lambda^*(\tau_a) = \sum_{i=0}^{q} a_i \mathrm{d}\log(\Phi_{\nu_i}) = \sum_{i=0}^{q} a_i \omega_{\nu_i} = \sum_{i=1}^{q} a_i \omega_{\xi_i}$$

is zero on $U$. This contradicts the assumption that $\{\omega_{\xi_1}, \dots, \omega_{\xi_q}\}$ is a basis for $D$. Q.E.D.

A singular subspace of rank 1 is called an *isotropic* subspace of $A^1$.

**Corollary 3.9.** *Suppose $\Lambda$ is an integral basis of an isotropic subspace of $A^1$. Then*

(i) *if* $\omega = \mathrm{d}\log(\Phi) \in D$ *then* $\mathrm{Sing}(\Phi_\Lambda) \subseteq \mathrm{crit}(\Phi)$.
(ii) *if* $0 \neq \omega = \mathrm{d}\log(\Phi) \in D$*, then the components of* $\mathrm{crit}(\Phi) - S_\Lambda$ *are disjoint hypersurfaces in $U$.*

*Proof.* Since $\dim(Y) = 1$, the Jacobian of $\Phi_\Lambda$ vanishes identically at points of $\mathrm{Sing}(\Phi_\Lambda)$. Then $\mathrm{Sing}(\Phi_\Lambda) \subseteq V(\omega) = \mathrm{crit}(\Phi)$ by Lemma 3.4.

If $a \neq 0$, then $\tau_a$ doesn't vanish identically on $Y$ by Proposition 3.8. Then $V(\tau_a)$ is zero-dimensional. Assertion (ii) follows from Theorem 3.5.
Q.E.D.

Corollary 3.9(i) can be used to locate singular fibers in Čeva pencils [10, Def. 4.5]—see Example 3.17.

### 3.2. Zeros of $\tau_a$

We apply the method of Lagrange multipliers to find the zeros of $\tau_a$. The argument applies even if $\overline{Y}$ is singular. Fix a set of homogeneous generators $\{P_1, \dots, P_r\}$ of the defining ideal $I = I_\Lambda \subseteq S$ of $\overline{Y}$. Write $\partial_j$ for $\frac{\partial}{\partial z_j}$. Let

$$J_\Lambda = \left[\partial_j P_i\right]$$

be the Jacobian of $(P_1, \dots, P_r)$. The rank of $J_\Lambda$ at a nonsingular point $y \in \overline{Y}$ is equal to $q - p$, the codimension of $\overline{Y}$.

**Lemma 3.10.** *Let $y \in Y$. Then $y \in V(\tau_a)$ if and only if $\left[\frac{a_0}{y_0} \cdots \frac{a_q}{y_q}\right]$ is an element of the row space of $J_\Lambda(y)$.*

*Proof.* The one-form $d\log\mu_a = \sum_{i=0}^q a_i d\log(z_i) \in \Omega(\mathbb{P}^q)$ restricts to $\tau_a \in \Omega(Y)$. There is an exact sequence of $\mathbb{C}[\overline{Y}]$-modules

$$I/I^2 \xrightarrow{d} \mathbb{C}[\overline{Y}] \otimes_{\mathbb{C}[\mathbb{P}^q]} \Omega[\mathbb{P}^q] \to \Omega[\overline{Y}] \to 0,$$

where $d$ is given by right multiplication by $J_\Lambda$ [9, Sec. 16.1]. Localization at the maximal ideal corresponding to $y$ preserves exactness of this sequence, so $\tau_a(y) \in \Omega(Y)_y$ vanishes if and only if $\tau_a(y)$ is in the image of $d$.
Q.E.D.

For each $i \geq 0$, let $\mathrm{Fitt}_i(a, I)$ be the variety in $\mathbb{P}^q$ defined by the $(q+1-i) \times (q+1-i)$ minors of the $(r+1) \times (q+1)$ matrix

$$\begin{bmatrix} a_0/z_0 & \cdots & a_q/z_q \\ \partial_0 P_1 & \cdots & \partial_q P_1 \\ \vdots & \ddots & \vdots \\ \partial_0 P_r & \cdots & \partial_q P_r \end{bmatrix}. \tag{3.1}$$

The ideal $\mathrm{Fitt}_i(a, I)$ is independent of the choice of generating set for $I$. Similarly, let $\mathrm{Fitt}_i(J_\Lambda)$ denote the variety in $\mathbb{P}^q$ defined by the $(q+1-i) \times (q+1-i)$ minors of $J_\Lambda$. Let $\overline{Y}_{\mathrm{reg}} = \overline{Y} - \mathrm{Sing}\,\overline{Y}$. Then $\overline{Y}_{\mathrm{reg}} = \overline{Y} \cap \mathrm{Fitt}_p(J_\Lambda) - \mathrm{Fitt}_{p+1}(J_\Lambda)$, where $p = \dim \overline{Y}$.

If $Y$ is smooth, then $V(\tau_a)$ is a Fitting variety. More generally:

**Corollary 3.11.** $V(\tau_a) \cap \overline{Y}_{\mathrm{reg}} = \mathrm{Fitt}_p(a, I) \cap \overline{Y}_{\mathrm{reg}}$.

*Proof.* At any point of $\overline{Y}_{\mathrm{reg}}$ the rank of $J_\Lambda$ is equal to $q-p = \operatorname{codim}\overline{Y}$, so the rank of matrix (3.1) is at least $q-p$. Then $\begin{bmatrix} \frac{a_0}{y_0} & \cdots & \frac{a_q}{y_q} \end{bmatrix}$ is in the row space of $J(y)$ if and only if (3.1) has rank $q-p$, if and only if all $(q-p+1)\times(q-p+1)$ minors vanish. Q.E.D.

**Remark 3.12.** In fact, the intersections $\overline{Y}\cap \mathrm{Fitt}_i(J_\Lambda)$, $p\le i\le q$ determine a stratification of $\overline{Y}$ by locally-closed subvarieties, and $V(\tau_a)$ coincides with $\mathrm{Fitt}_i(a,I)$ on the stratum $\mathrm{Fitt}_i(J_\Lambda)-\mathrm{Fitt}_{i+1}(J_\Lambda)$, by the same argument.

In view of Proposition 2.1 we study the zeros of cocycles for $\tau_a$. Since $\dim Y = p$, every $\psi\in\Omega^p(Y)$ is a cocycle for $\tau_a$. For $0\le i\le q$, set $\tau_i = \frac{\mathrm{d}y_i}{y_i}$, so that $\tau_a = \sum_{i=0}^q a_i\tau_i = \sum_{i=1}^q a_i(\tau_i-\tau_0) = \sum_{i=1}^q \mathrm{d}\log(y_i/y_0)$.

**Proposition 3.13.** *The intersection*

$$\bigcap\{V(\xi)\mid \xi\in\Omega^p_{\mathbb{C}}(Y), \tau_a\wedge\xi = 0\}$$

*is contained in* $\mathrm{Sing}(\overline{Y})$.

*Proof.* For $I=\{i_1<\cdots<i_p\}_<\subseteq\{1,\ldots,q\}$, consider the $p$-form

$$\xi_I = (\tau_{i_1}-\tau_0)\wedge\cdots\wedge(\tau_{i_p}-\tau_0).$$

Then $\tau_a\wedge\xi_I = 0$. But $\dim\overline{Y} = p$, so at each point of $\overline{Y}_{\mathrm{reg}}$, $\xi_I$ must be nonzero for some $I$. Q.E.D.

### 3.3. The case $q = p+1$

Suppose $D$ is a singular subspace of rank $\dim(D)-1$, with integral basis $\Lambda = \{\omega_{\xi_1},\ldots,\omega_{\xi_q}\}$. Then $\overline{Y}$ is defined by a single homogeneous polynomial $P(z_0,\ldots,z_q)$. This hypothesis holds for all the examples we know, with one exception: the Hessian arrangement, which supports a rational singular subspace of dimension three and rank one—see Examples 2.8 and 4.11.

Consider the rational mapping

$$\rho = [z_0\partial_0 P : \cdots : z_q\partial_q P] \colon \mathbb{P}^q \rightarrowtail \mathbb{P}^q. \tag{3.2}$$

This map has poles along $\mathrm{Sing}(\overline{Y})$ and $\mathrm{Sing}(\overline{Y}\cap\mathbb{C}^I)$ where $\mathbb{C}^I$ is the coordinate subspace $z_i = 0$, $i\in I$. It is regular on $\overline{Y}_{\mathrm{reg}}\cap(\mathbb{C}^*)^q$. By Euler's formula, $\sum_{j=0}^q z_j\partial_j P = \deg(P)P$, so the image of $\overline{Y}$ under $\rho$ is contained in the hyperplane $\sum_{j=0}^q z_j = 0$.

**Proposition 3.14.** *Suppose $\overline{Y}$ is the hypersurface given by $P=0$, and $\rho$ is as given in* (3.2). *Then*

(i) *$V(\tau_a) \neq \emptyset$ if and only if $a \in \rho(Y)$.*
(ii) *If $a \in \rho(Y)$ then $V(\tau_a) = \rho^{-1}(a)$.*
(iii) *For generic $a \in \rho(Y)$, $V(\tau_a)$ is nonempty and every component has codimension equal to* $\dim \rho(\overline{Y})$.

*Proof.* The first two assertions follow from Lemma 3.10, and the third follows from the second. Q.E.D.

**Corollary 3.15.** *Suppose*

(i) *$\Phi_\Lambda$ is nonsingular on $U$,*
(ii) *$\overline{Y}$ is nonsingular and intersects the coordinate hyperplanes transversely,*
(iii) $\dim \rho(\overline{Y}) = \dim \overline{Y}$, *and*
(iv) $\Phi_\Lambda(U) = \overline{Y} \cap (\mathbb{C}^*)^q$.

*If* $\mathrm{d}\log(\Phi) = \sum_{i=1}^q a_i \omega_{\xi_i}$ *and $a_i \neq 0$ for all $i$, then* $\mathrm{crit}(\Phi)$ *is nonempty and every component has codimension $q-1$.*

*Proof.* We have observed that $\rho(\overline{Y})$ is contained in the hyperplane $\Delta$ defined by $\sum_{i=0}^q z_i = 0$. The second hypothesis ensures that $\mathrm{Sing}(Y \cap \mathbb{C}^I)$ is empty for every $I \subseteq \{0, \ldots, q\}$. Then $\rho$ is regular on $\overline{Y}$. By the third condition, $\dim \rho(\overline{Y}) = q - 1 = \dim \Delta$. Since $\Delta$ is irreducible, we conclude $\rho(\overline{Y}) = \Delta$. Since $Y = \overline{Y} \cap (\mathbb{C}^*)^q$ by (iii), $\Delta \cap (\mathbb{C}^*)^q = \rho(Y)$. The result then follows from Proposition 3.14 and Theorem 3.5, since (i) and (ii) imply $S_\Lambda = \emptyset$ and the fibers of $\Phi_\Lambda$ all have codimension $q-1$. Q.E.D.

The hypotheses in Corollary 3.15 are satisfied in many examples. The third condition is automatic in case $q = 2$. The last two conditions hold in every example we know where (i) and (ii) hold.

### 3.4. Examples

**Example 3.16** (Example 2.6, continued)**.** Let $D$ be the rational singular subspace of $A^1 \cong \mathbb{C}^5$ with basis $\Lambda = \{\omega_{010} - \omega_{100},\ \omega_{001} - \omega_{100}\}$.

We have

$$\Phi_\Lambda = [\Phi_{100} : \Phi_{010} : \Phi_{001}] = [x(y-z) : y(x-z) : z(x-y)].$$

$\Phi_\Lambda$ is nonsingular on $U$, and $\overline{Y} = Y \cap (\mathbb{C}^*)^5$. The components of $\Phi_\Lambda$ satisfy the homogeneous relation $\Phi_{100} - \Phi_{010} + \Phi_{001} = 0$, so $\overline{Y}$ is the line $z_0 - z_1 + z_2 = 0$ in $\mathbb{P}^2$. Corollary 3.15 implies $\mathrm{crit}(\Phi_{abc})$ is nonempty if and only if $a + b + c = 0$ and $a, b$ and $c$ are nonzero. In this case,

$$\mathrm{crit}(\Phi_{abc}) = (\rho \circ \Phi_\Lambda)^{-1}([a : b : c]) = \Phi_\Lambda^{-1}([a : -b : c]),$$

that is, $\mathrm{crit}(\Phi_{abc})$ is given by

$$[x(y-z):y(x-z):z(x-y)]=[a:-b:c], \text{ or, equivalently}$$
$$[x(y-z):y(z-x)]=[a:b]$$

as we found earlier. It has codimension one in $\mathbb{P}^2$. In this example the map $\Phi_\Lambda$ has connected generic fiber, hence $\mathrm{crit}(\Phi_{abc})$ is connected. If $a, b$, or $c$ is zero, and $a+b+c=0$, then $\mathrm{crit}(\Phi_{abc})$ is empty.

The basis $\Lambda$ above has special properties that resulted in the linear syzygy of master functions: we will revisit this in the next section. By way of comparison, consider the basis $\Lambda'=\{\omega_{120}-\omega_{012},\omega_{300}-\omega_{012}\}$. Then

$$\begin{aligned}\Phi_{\Lambda'}&=[\Phi_{010}\Phi_{001}^2:\Phi_{100}\Phi_{010}^2:\Phi_{100}^3]\\&=[yz^2(x-y)^2(x-z):xy^2(x-z)^2(y-z):x^3(y-z)^3].\end{aligned}$$

A *Macaulay 2* calculation [12] shows that $\Phi_{\Lambda'}$ is nonsingular on $U$. Using the identity $\Phi_{100}-\Phi_{010}+\Phi_{001}=0$, one finds that the Zariski closure $\overline{Y'}$ of $Y'=\Phi_{\Lambda'}(U)$ is defined by

$$z_1^3-z_0^2z_2-4z_0z_1z_2-2z_1^2z_2+z_1z_2^2=0.$$

Then $\overline{Y'}$ is an irreducible cubic with a node at $[z_0:z_1:z_2]=[-2:1:-1]$. This is a point of $Y'$. It is also in $\widetilde{\Phi}_{\Lambda'}(E)$, where $\widetilde{\Phi}_{\Lambda'}$ is the lift of $\Phi_{\Lambda'}$ to the blow-up of $\mathbb{P}^2$ at the four base points, and $E$ is the exceptional divisor over $[0:1:0]$.

The image of $Y'$ under $\rho$ misses the three points $[0:-1:1]$, $[-2:1:1]$, and $[-2:3:-1]$, corresponding to the three one-forms $\omega_{100}-\omega_{010}$, $\omega_{100}-\omega_{001}$, and $\omega_{010}-\omega_{001}$ in $D$ that have empty zero locus. In particular $Y'\neq\overline{Y'}\cap(\mathbb{C}^*)^5$.

**Example 3.17.** Let $\mathcal{A}$ be the arrangement with defining equation $Q=Q_0Q_1Q_2$ where

$$\begin{aligned}Q_0&=(x+z)(2x-y-z)(2x+y-z)\\Q_1&=(x-z)(2x+y+z)(2x-y+z), \quad\text{and}\\Q_2&=(y+z)(y-z)z\end{aligned}$$

The image of $\Phi_\Lambda=[Q_0:Q_1:Q_2]\colon\mathbb{P}^2\rightarrowtail\mathbb{P}^2$ is the line $z_0-z_1+2z_2=0$. The fibers are the cubics passing through nine points, three on each of three concurrent lines—$\mathcal{A}$ is a specialization of the Pappus arrangement. One of these cubics is $x(4x^2-y^2-3z^2)=0$. Although $\overline{Y}$ is smooth, $\Phi_\Lambda$ is singular at two points of $U$, given by $x=y^2+3z^2=0$. These two points lie in $\mathrm{crit}(Q_0^aQ_1^bQ_2^c)$ for every $a,b,c$ with $a+b+c=0$.

Similarly, if $\mathcal{A}$ the subarrangement of the Hessian arrangement (2.8) defined by $P_1P_2P_3 = 0$, then every critical set $\mathrm{crit}(P_1^aP_2^bP_3^c)$, $a+b+c=0$, contains the three points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ of $U$ where the fourth special fiber $xyz = 0$ is singular. The map $\Phi_\Lambda = [P_1 : P_2 : P_3]$ is singular at these points, although $\overline{Y}$ is smooth, given by $\zeta P_1 + \zeta^2 P_2 + P_3 = 0$. (See also Example 4.11.)

**Remark 3.18.** In fact, Corollary 3.9 can be used to detect Čeva pencils [10] (see 4.4). For instance the master function

$$\Phi = \frac{x(y-z)}{y(x-z)}$$

has critical points $[0:1:0]$ and $[1:1:0]$, the singular points of the third completely decomposable fiber in Example 2.6.

**Example 3.19** (Example 2.7, continued)**.** In this example, the one-form $\omega$ has no decomposable 2-cocycle. Indeed there are no singular subspaces of $A^1$ of rank $p=1$ or $p=2$. For $p=1$ this holds because $\mathcal{A}$ is 2-generic. For $p=2$ one can verify the statement computationally using the approach of [15]; in [2] we give a combinatorial argument based on Theorem 5.4. Setting

$$\Psi = [\frac{x}{y+z+w} : \frac{y}{x+z+w} : \frac{z}{x+y+w} : \frac{w}{x+y+z}] : \mathbb{P}^3 \rightarrowtail \mathbb{P}^3,$$

we have $\omega = \mathrm{d}\log(\Phi) = \Psi^*(\tau)$ where

$$\tau = a\,\mathrm{d}\log(y_0) + b\,\mathrm{d}\log(y_1) + c\,\mathrm{d}\log(y_2) + d\,\mathrm{d}\log(y_3).$$

The map $\Psi$ is dominant. The one-dimensional critical locus of $\Phi$ is a fiber of a different map

$$[x(y+z+w) : y(x+z+w) : z(x+y+w) : w(x+y+z)].$$

## §4. Linear dependence among master functions

In this section we consider a singular subspace $D$ with an integral basis $\Lambda$ such that $Y_\Lambda$ is linear, i.e., the syzygy ideal $I_\Lambda$ is generated by homogeneous linear forms. We saw this phenomenon in Example 3.16. We start with a trivial example that will be useful for what follows.

### 4.1. Example: equations for the critical locus

Suppose $\mathcal{A}$ is an essential arrangement of $n+1$ hyperplanes in $\mathbb{P}^\ell$, with $n > \ell$. Then $D = A^1$ is a singular subspace, with integral basis

$\{\omega_i - \omega_0 \mid 1 \leq i \leq n\}$. The corresponding rational mapping is

$$\Phi = \Phi_\Lambda = [\alpha_0 : \cdots : \alpha_n] \colon \mathbb{P}^\ell \rightarrowtail \mathbb{P}^n,$$

and $\overline{Y} = \overline{Y}_\Lambda$ is a linear subvariety of $\mathbb{P}^n$. The reader will recognize that this is the usual identification of a labelled vector configuration, $(\alpha_0, \ldots, \alpha_n)$, with a point in the Grassmannian of $\ell$-planes in $\mathbb{P}^n$. Let us denote $\overline{Y}_\Lambda$ by $L_\mathcal{A}$. (This is an abuse of notation; $\overline{Y}_\Lambda$ depends on the choice of defining forms $\alpha_i$.)

Most of the results of the previous section are vacuous in this situation, but Lemma 3.10 tells us something:

**Theorem 4.1.** *Let $B = [b_{ij}]$ be an $(\ell+1) \times (n+1)$ matrix such that $L_\mathcal{A}$ is the kernel of $B$. Then, for any $\lambda \in \mathbb{C}^n$, the critical locus of $\Phi_\lambda$ is defined by the $\binom{n+1}{\ell+1}$ equations*

$$\sum_{i \in I} \sigma(i, I) b_{I - \{i\}} \frac{\lambda_i}{\alpha_i(x)} = 0$$

*where $I$ ranges over the subsets of $\{0, \ldots, n\}$ of size $\ell+1$, the coefficient $b_J$ is given by $b_J = \det [b_{ij} \mid j \in J]$, and $\sigma(i, I) = \pm 1$, depending on the position of $i$ in $I$.*

*Proof.* The linear forms defined by the rows of $B$ generate the syzygy ideal $I_\Lambda$. Then the Jacobian $J_\Lambda$ is equal to $B$. With the observation that $S_\Lambda = \emptyset$, setting $a = \lambda$ in Lemma 3.10 and applying Theorem 3.5 yields the claim. Q.E.D.

The columns of the matrix $B$ above define a realization of the matroid dual to the matroid of $\mathcal{A}$. In Example 2.6,

$$B = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(The matroid of $\mathcal{A}$ is self-dual.) Theorem 4.1 says that $\operatorname{crit}(\Phi_{abc})$ is defined by the $4 \times 4$ minors of

$$\begin{bmatrix} \frac{a}{x} & \frac{b}{y} & \frac{c}{z} & \frac{c}{x-y} & \frac{b}{x-z} & \frac{a}{y-z} \\ 0 & -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

These 15 equations reduce to the single equation found in Example 2.6.

### 4.2. Linear hypersurfaces

Suppose $\overline{Y}$ is a linear and $p = \operatorname{rank}(D) = q - 1$, i.e., $\overline{Y}$ is a linear hyperplane in $\mathbb{P}^q$. Let $P(z) = \sum_{j=0}^{q} b_j z_j$ be a generator for the syzygy ideal. It is no loss to assume $b_j \neq 0$ for all $j$, or equivalently, $\overline{Y}$ is not contained in any coordinate hyperplane. Otherwise some proper subset of $\{\Phi_{\nu_0}, \ldots, \Phi_{\nu_q}\}$ is linearly dependent.

**Proposition 4.2.** *Suppose $\overline{Y}$ is a hyperplane not contained in any coordinate hyperplane in $\mathbb{P}^q$. Let $\lambda = \sum_{i=0}^{q} a_i \nu_i$. Then for generic $a$ satisfying $\sum_{i=0}^{q} a_i = 0$,* $\operatorname{crit}(\Phi_\lambda) - S_\Lambda$ *is nonempty and every component of* $\operatorname{crit}(\Phi_\lambda) - S_\Lambda$ *has codimension equal to the rank of $D$.*

*Proof.* The syzygy ideal $I_\Lambda$ is generated by a linear polynomial $P(z) = \sum_{j=0}^{q} b_j z_j$, and $b_j \neq 0$ for $0 \leq j \leq q$ by hypothesis. The map $\rho = [b_0 z_0 : \cdots : b_q z_q]$ of (3.2) is an automorphism of $\mathbb{P}^q$ since all of the $b_j$ are nonzero. Consequently, $\rho$ maps the hyperplane $\overline{Y}$ isomorphically to the hyperplane $\Delta$ defined by $\sum_{i=0}^{q} z_i = 0$. The result then follows from Proposition 3.14. Q.E.D.

### 4.3. The general case

Suppose the singular subspace $D \subseteq A^1$ has an integral basis $\Lambda$ for which $\overline{Y} = \overline{Y}_\Lambda$ is a linear variety in $\mathbb{P}^q$. Choose a linear isomorphism $\varphi : \mathbb{P}^p \to \overline{Y}$, given by a $(q+1) \times (p+1)$ matrix $B = \left[b_{ij}\right]$. Assume $D$ is not contained in any coordinate hyperplane. Then the intersections of $\overline{Y}$ with the coordinate hyperplanes in $\mathbb{P}^q$ determine an essential arrangement $\mathcal{B}$ of $q+1$ not necessarily distinct hyperplanes in $\mathbb{P}^p$, with defining forms $\beta_i(x) = \sum_{i=0}^{p} b_{ij} x_j$, for $0 \leq i \leq q$. By construction, the subspace $L_{\mathcal{B}}$ of $\mathbb{P}^q$, as described in Section 4.1, is equal to $\overline{Y}$.

**Theorem 4.3.** *Suppose $\overline{Y}_\Lambda = L_{\mathcal{B}}$ is a linear subspace not contained in any coordinate hyperplane in $\mathbb{P}^q$, and $Y = \overline{Y} \cap (\mathbb{C}^*)^q$. Let $\lambda = \sum_{i=0}^{q} a_i \nu_i$ with $\sum_{i=0}^{q} a_i = 0$. Let $\Psi_a = \prod_{i=0}^{q} \beta_i^{a_i}$ be the master function on the complement of the arrangement $\mathcal{B}$ in $\mathbb{P}^p$ corresponding to $a$. Then*

(i) $\operatorname{crit}(\Phi_\lambda) - S_\Lambda \neq \emptyset$ *if and only if* $\operatorname{crit}(\Psi_a) \neq \emptyset$,
(ii) $\operatorname{crit}(\Phi_\lambda) - S_\Lambda = \Phi_\Lambda^{-1}(\rho(\operatorname{crit}(\Psi_a)))$, *and*
(iii) $\operatorname{codim}(\operatorname{crit}(\Phi_\lambda) - S_\Lambda) \leq \operatorname{codim} \operatorname{crit}(\Psi_a)$.

*Proof.* Just as in Section 4.1, the one-form $\tau_a$ on $\overline{Y}$ pulls back to $\mathrm{d}\log(\Psi_a)$ under the isomorphism $\varphi$, and the assertions follow from Proposition 3.14. Q.E.D.

**Example 4.4** (Example 2.6, continued). We saw in Example 3.16 that the variety $\overline{Y}$ is given by $z_0 - z_1 + z_2 = 0$ in $\mathbb{P}^2$, and $Y = \overline{Y} \cap (\mathbb{C}^*)^2$.

We can take $\varphi\colon \mathbb{P}^1 \xrightarrow{\cong} \overline{Y}$ to be given by the matrix

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then associated arrangement $\mathcal{B}$ consists of three points $[1:0]$, $[1:1]$, $[0:1]$ in $\mathbb{P}^1$. The complement of $\mathcal{B}$ has Euler characteristic $-1$, so a generic $\mathcal{B}$-master function $\Psi_{abc}$ has a single nondegenerate critical point. A computation shows that this holds if $a, b$, and $c$ are nonzero. We reach the same conclusion as before, that $\mathrm{crit}(\Phi_{abc})$ has codimension one.

### 4.4. Multinets and codimension-one critical sets

Next we use the main result of [10] to give a complete description of $\mathrm{crit}(\Phi)$ for any $\omega = \mathrm{d}\log(\Phi) \in \mathcal{R}^1(\mathcal{A})$. As we observed earlier, if $\omega \in \mathcal{R}^1(\mathcal{A})$, then $\omega$ has a nontrivial decomposable 1-cocycle $\psi$. The statement that $\omega \wedge \psi = 0$ means, for each $x \in U$, $\{\omega(x), \psi(x)\}$ is linearly dependent. Then there are functions $a$ and $b$ on $U$ such that $a(x)\omega(x) + b(x)\psi(x) = 0$ for all $x \in U$. This implies $V(\omega)$ contains $V(b) - V(a)$, which is a hypersurface unless it is empty. It was this observation that led to the current research.

According to [14], the maximal isotropic subspaces $D$ of $A^1$ of dimension at least two are the components of $\mathcal{R}^1(\mathcal{A})$, and they intersect trivially. By [10, Theorem 3.11], such a component has an integral basis $\Lambda = (\omega_{\xi_1}, \ldots, \omega_{\xi_q})$, with the property that the corresponding polynomial master functions $\Phi_{\nu_0}, \ldots, \Phi_{\nu_q}$ are all collinear in the space of degree $d$ polynomials. Then $\overline{Y} = \overline{Y}_\Lambda$ is a line in $\mathbb{P}^q$. The homogenized basis $\{\omega_{\nu_0}, \ldots, \omega_{\nu_q}\}$ corresponds to the characteristic vectors $\nu_i$ of the blocks in a multinet structure on a subarrangement of $\mathcal{A}$, as defined below.

For $X \subseteq \mathbb{P}^\ell$, write $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$. A *rank-two flat* of $\mathcal{A}$ is a subspace $X$ of the form $H \cap K$ for some $H, K \in \mathcal{A}, H \neq K$. If $\mathcal{P}$ is a partition of $\mathcal{A}$, the *base locus* of $\mathcal{P}$ is the set of rank-two flats of $\mathcal{A}$ obtained by intersecting hyperplanes from different blocks of $\mathcal{P}$.

**Definition 4.5.** A $(q+1, d)$-multinet on $\mathcal{A}$ is a pair $(\mathcal{P}, m)$ where $\mathcal{P}$ is a partition $\{\mathcal{A}_0, \ldots, \mathcal{A}_q\}$ of $\mathcal{A}$ into $q+1$ blocks, with base locus $\mathcal{X}$, and $m\colon \mathcal{A} \to \mathbb{Z}_{>0}$ is a multiplicity function, satisfying

(i) $\sum_{H\in\mathcal{A}_i} m(H) = d$ for every $i$.
(ii) For each $X \in \mathcal{X}$, $\sum_{H\in\mathcal{A}_i\cap\mathcal{A}_X} m(H) = n_X$ for some integer $n_X$, independent of $i$.
(iii) For each $i$, $\bigcup \mathcal{A}_i - \bigcup \mathcal{X}$ is connected.

The third condition says that $\mathcal{P}$ cannot be refined to a $(q', d)$-multinet with the same multiplicity function, with $q' > q+1$. Given a multinet on $\mathcal{A}$, let $\nu_i = \sum_{H_i \in \mathcal{A}_i} m(H_i)e_i$, for $0 \le i \le q$, and $\xi_i = \nu_i - \nu_0$ for $1 \le i \le q$. We call $\nu_0, \dots, \nu_q$ the *characteristic vectors* of $(\mathcal{P}, m)$. We have the following results from [10].

**Theorem 4.6** ([10, Corollary 3.12]). *Suppose $D$ is a maximal isotropic subspace of $A^1$ of dimension $q \ge 2$. Then there is a subarrangement $\mathcal{A}'$ of $\mathcal{A}$ and a $(q+1, d)$-multinet on $\mathcal{A}'$ whose characteristic vectors $\nu_0, \dots, \nu_q$ yield an integral basis $\omega_{\xi_1}, \dots, \omega_{\xi_q}$ of $D$.*

**Theorem 4.7** ([10, Theorem 3.11]). *Suppose $\nu_0, \dots, \nu_q$ are the characteristic vectors of a $(q+1, d)$-multinet structure on $\mathcal{A}$. Then each of the master functions $\Phi_{\nu_i}$, $i \ge 2$, is a linear combination of $\Phi_{\nu_0}$ and $\Phi_{\nu_1}$. Moreover every fiber of the mapping $[\Phi_{\nu_0} : \Phi_{\nu_1}] \colon \mathbb{P}^\ell \rightarrowtail \mathbb{P}^1$ is connected.*

Given this result, the analysis of critical sets proceeds exactly as in the Example 4.4. First, we need a lemma about isolated critical points.

**Lemma 4.8.** *Suppose $\mathcal{A}$ is an affine arrangement of $n$ hyperplanes in $\mathbb{C}^\ell$, and $W$ is a nonempty Zariski-open subset of $\mathbb{C}^\ell$. Then there is a nonempty Zariski-open subset $L$ of $\mathbb{C}^n$ such that $W \cap \operatorname{crit}(\Phi_\lambda)$ consists of $|\chi(U)|$ points, for each $\lambda \in L$.*

*Proof.* By [20, Theorem 1.1], there is a nonempty Zariski-open subset $L'$ of $\mathbb{C}^n$ for which $\operatorname{crit}(\Phi_\lambda)$ is isolated and consists of $|\chi(U)|$ points, for $\lambda \in L'$. Let

$$\Sigma = \{(\lambda, v) \in \mathbb{C}^n \times U \colon \omega_\lambda(v) = 0\},$$

an $n$-dimensional smooth complex variety by [20, Prop. 4.1]. Let $\pi_i$ for $i = 1, 2$ denote its projections onto $\mathbb{C}^n$ and $U$, respectively. Then $\pi_2^{-1}(U \cap W)$ and $\pi_1^{-1}(L')$ are each nonempty Zariski-open subsets of $\Sigma$, as is their intersection $Z$. Then $\pi_1(Z)$ is a finite union of locally-closed subsets of $\mathbb{C}^n$—see [13, Exercise 3.19]. Since $Z$ is dense in $\Sigma$, $\pi_1(Z)$ is dense in $\mathbb{C}^n$. Hence $\pi_1(Z)$ contains a Zariski-open subset $L$, which has the required property. Q.E.D.

**Theorem 4.9.** *Suppose $\omega = \mathrm{d}\log(\Phi) \in \mathcal{R}^1(\mathcal{A})$, and $D$ is the maximal isotropic subspace of $A^1$ containing $\omega$. Let $\Lambda$ be the integral basis of $D$ arising from the associated multinet. Then*

(i) *For every $\omega \in D$, $\operatorname{Sing}(\Phi_\Lambda) \subseteq \operatorname{crit}(\Phi)$, and,*
(ii) *For generic $\omega \in D$, $\operatorname{crit}(\Phi) - \operatorname{Sing}(\Phi_\Lambda)$ is a union of $\dim(D) - 1$ connected smooth hypersurfaces of the same degree.*

*Proof.* Write $q = \dim(D)$ and let $\nu_0, \dots, \nu_q$ be the characteristic vectors of the $(q+1, d)$-multinet corresponding to $D$. Write $\omega =$

$\sum_{i=0}^{q} a_i \omega_{\nu_i}$. By the preceding theorem, for each $2 \leq k \leq q$, there is a linear relation $\Phi_{\nu_k} = b_k \Phi_{\nu_0} + c_k \Phi_{\nu_1}$. Then $\overline{Y}$ is a line in $\mathbb{P}^q$. The first assertion follows from Corollary 3.9.

We can choose the isomorphism $\varphi \colon \mathbb{P}^1 \xrightarrow{\cong} \overline{Y} \subset \mathbb{P}^q$ to be given by the matrix

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ b_2 & c_2 \\ \vdots & \vdots \\ b_q & c_q \end{bmatrix}.$$

The corresponding arrangement $\mathcal{B}$ consists of $q+1$ distinct points in $\mathbb{P}^1$. The Euler characteristic of the complement of $\mathcal{B}$ is $1-q$. Then for generic $a$ the $\mathcal{B}$-master function $\Psi_a$ has $q-1$ isolated, nondegenerate critical points in $\overline{Y} \cap (\mathbb{C}^*)^q$. In fact, since $Y$ is dense in $\overline{Y} \cap (\mathbb{C}^*)^q$, we apply Lemma 4.8 to see that, for generic $a$, $\Psi_a$ has $q-1$ critical points in $Y$. Then Corollary 3.7 implies $\operatorname{crit}(\Phi)$ is the union of $q-1$ fibers of $\Phi_\Lambda$.

The projection $\mathbb{P}^q \rightarrowtail \mathbb{P}^1$ along $z_0 = z_1 = 0$ restricts to an isomorphism on $\overline{Y}$. Then the last statement of Theorem 4.7 implies the fibers of $\Phi_\Lambda$ are connected. These fibers are given by $[\Phi_{\nu_0} : \Phi_{\nu_1}] = [a_0 : a_1]$. Since the $\Phi_{\nu_i}$ have degree $d$, $\operatorname{crit}(\Phi_\Lambda)$ is a union of $q-1$ connected hypersurfaces of degree $d$. The generic fiber of $\Phi_\Lambda$ is smooth by Bertini's Theorem [13, Corollary III.10.9]. Q.E.D.

**Corollary 4.10.** *For generic $\omega = \mathrm{d}\log(\Phi) \in \mathcal{R}^1(\mathcal{A})$, the number of connected components of* $\operatorname{crit}(\Phi_\lambda)$ *is equal to the dimension of $H^1(A^{\cdot}, \omega)$.*

Example 3.17 shows that $\operatorname{crit}(\Phi)$ need not be smooth or irreducible for all $\omega = \mathrm{d}\log(\Phi) \in \mathcal{R}^1(\mathcal{A})$.

**Example 4.11.** By [22, 31], the maximum number of blocks in a multinet is equal to four. The only known example with four blocks is the multinet on the Hessian arrangement corresponding to the Hesse pencil, Example 2.8. The factors of the polynomial master functions $P_0 = xyz, P_1, P_2, P_3$ define the blocks of a multinet on $\mathcal{A}$ with all multiplicities equal to one. These master functions satisfy two linear syzygies:

$$P_2 = 3(1-\zeta)P_0 + P_1$$
$$P_3 = 3(1-\zeta^2)P_0 + P_1.$$

Then the variety $\overline{Y}_\Lambda$ corresponding to the basis

$$\Lambda = \{\omega_{100}, \omega_{010}, \omega_{001}\}$$

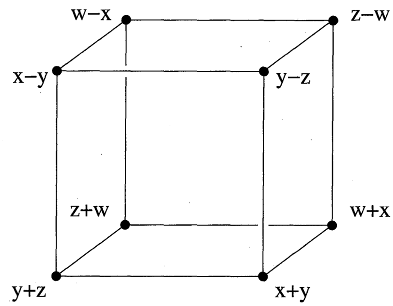


Fig. 2. A rank-four matroid with a linear syzygy of master functions

is the line given by $z_2 = 3(1-\zeta)z_0 + z_1,\ z_3 = 3(1-\zeta^2)z_0 + z_1$ in $\mathbb{P}^3$, which meets the coordinate hyperplanes in the four points corresponding to the singular fibers. The corresponding arrangement $\mathcal{B}$ consists of four points in general position in $\mathbb{P}^1$, given by the rows of the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3(1-\zeta) & 1 \\ 3(1-\zeta^2) & 1 \end{bmatrix}.$$

The complement of $\mathcal{B}$ has Euler characteristic $-2$, hence a generic $\mathcal{B}$-master function has two isolated critical points. Then, for generic $a = (a_1, a_2, a_3)$, the critical locus of the $\mathcal{A}$-master function $\Phi = \Phi_{a_1 a_2 a_3}$ has two components and codimension one, as found by direct calculation in Example 2.8. This example shows that Theorem 4.8(ii) and Corollary 4.9 may not hold under the weaker hypothesis that $a_i \neq 0$ for all $i$.

Here is a rank-four example, that has appeared in different form in the lecture of A. Libgober in this volume.

**Example 4.12.** Let $\mathcal{A}$ be the arrangement with defining polynomial

$$Q = (x+y)(x-y)(y+z)(y-z)(z+w)(z-w)(w+x)(w-x),$$

with the hyperplanes numbered according to the order of factors in $Q$. Then $\mathcal{A}$ is a 2-generic subarrangement of the Coxeter arrangement of type $D_4$. Up to lattice-isotopy, the dual projective point configuration consists of the eight vertices of a cube—see Figure 2. Let $D$ be the subspace of $A^1$ with basis

$$\Lambda = \{(\omega_0+\omega_1)-(\omega_6+\omega_7),\ (\omega_2+\omega_3)-(\omega_6+\omega_7), (\omega_4+\omega_5)-(\omega_6+\omega_7)\}.$$

Then $D$ is a rational singular subspace of rank two; if $\omega \in D$, then $H^1(A^\cdot, \omega) = 0$ and $H^2(A^\cdot, \omega) \cong D/\mathbb{C}\omega$ has dimension 2. $\overline{Y}_\Lambda$ is the linear hyperplane in $\mathbb{P}^3$ defined by $z_0 + z_1 + z_2 + z_3 = 0$, reflecting the linear syzygy of polynomial master functions

$$(x+y)(x-y) + (y+z)(y-z) + (z+w)(z-w) + (w+x)(w-x) = 0.$$

From Proposition 4.2 we see that

$$\Phi = \left(\frac{x^2-y^2}{w^2-x^2}\right)^{a_1} \left(\frac{y^2-z^2}{w^2-x^2}\right)^{a_2} \left(\frac{z^2-w^2}{w^2-x^2}\right)^{a_3}$$

has nonempty critical set of codimension two in $\mathbb{P}^3$, for generic $(a_1, a_2, a_3)$.

## §5. The rank condition

We are left with the problem of finding rational singular subspaces of $A^1$. The theory of multinets gives a method to find such subspaces of rank one. In this section we give a combinatorial condition for a set $\Lambda$ of linearly independent integral one-forms to span a singular subspace of $A^1$ of arbitrary rank, using tropical implicitization and nested sets.

### 5.1. Tropicalization

The *tropicalization* of a projective variety $V$ in $\mathbb{P}^q$ is a polyhedral fan $\operatorname{trop}(V)$ in tropical projective space $\mathbb{TP}^q = \mathbb{R}^{q+1}/\mathbb{R}(1, \ldots, 1)$, associated to a homogeneous defining ideal $I$ of $V$. If $V$ is a hypersurface with defining equation $f = 0$, then $\operatorname{trop}(V)$ is the image in $\mathbb{TP}^q$ of the union of the cones of codimension at least one in the normal fan of the Newton polytope of $f$. In general, $\operatorname{trop}(V)$ is the image of the union of the cones of codimension at least one in the Gröbner fan of $I$. The set $\operatorname{trop}(V)$ arises geometrically from the lowest-degree terms in Puiseux expansions of curves lying in $V$. See [7] and the references therein for background on tropical varieties. See [21] for matroid terminology.

We will need several results from tropical geometry. The first is a theorem of Bieri and Groves [3].

**Theorem 5.1.** *The maximal cones in* $\operatorname{trop}(V)$ *have dimension equal to* $\dim(V)$.

If $V$ is an $\ell$-dimensional linear subvariety of $\mathbb{P}^n$, given as the column space of a matrix $R$, then the tropicalization $\operatorname{trop}(V)$ depends only on the dependence matroid $\mathfrak{G}$ on the rows of $R$. In our setting the rows of $R$ give the defining forms of a hyperplane arrangement $\mathcal{A}$. The *matroid* *polytope* $\Delta(\mathfrak{G})$ of $\mathfrak{G}$ is the convex hull of the set

$$\Big\{\sum_{i\in B} e_i \mid B \text{ is a basis of } \mathfrak{G}\Big\}.$$

The tropicalization $\operatorname{trop}(V)$, called the *Bergman fan* of $\mathfrak{G}$, is the image in $\mathbb{TP}^n$ of the union of the cones of codimension at least one in the normal fan of $\Delta(\mathfrak{G})$. We denote it by $B(\mathfrak{G})$.

In [11] the Bergman fan is described in terms of *nested set cones.* Let $\mathcal{G}$ be the set of proper connected (i.e., irreducible) flats of $\mathfrak{G}$. These are the flats corresponding to the dense edges of the projective arrangement $\mathcal{A}$. A collection $\mathcal{S} = \{X_1, \dots, X_p\}$ of subsets of $\mathcal{G}$ is a *nested set* if, for every set $\mathcal{T}$ of pairwise incomparable elements of $\mathcal{S}$, the join $\bigvee \mathcal{T}$ is not an element of $\mathcal{G}$. The nested sets form a simplicial complex $\Delta = \Delta(\mathfrak{G})$, the *nested set complex*, which is pure of dimension $r = \ell - 1$. It is the coarsest of a family of nested set complexes, obtained by replacing $\mathcal{G}$ with larger "building sets." All of these complexes are subdivided by the order complex of the poset of nonempty flats of $\mathfrak{G}$.

If $S \subseteq \mathcal{A}$, set $e_S = \sum_{H_i \in S} e_i$. The *nested set fan* $N(\mathfrak{G})$ is the image in $\mathbb{TP}^n$ of the union of the cones generated by

$$\{e_S \mid S \in \mathcal{S}\}$$

for $\mathcal{S} \in \Delta(\mathfrak{G})$. From [11] we have the following result.

**Theorem 5.2.** *The nested set fan $N(\mathfrak{G})$ subdivides the Bergman fan $\mathcal{B}(\mathfrak{G})$.*

### 5.2. Singular subspaces

Let $\mathcal{A}$ be an arrangement in $\mathbb{P}^\ell$ with homogeneous defining linear forms $\alpha_0, \dots, \alpha_n$. Let $\mathfrak{G}$ be the underlying matroid of $\mathcal{A}$, the dependence matroid on $\{\alpha_0, \dots, \alpha_n\}$. Suppose $D$ is a rational subspace of $A^1(\mathcal{A})$, with integral basis $\Lambda = \{\omega_{\xi_1}, \dots, \omega_{\xi_q}\}$. We identify $\Lambda$ with the $q \times (n+1)$ matrix of integers $\big[\xi_{ij}\big]$, and recall that $\sum_{j=0}^n \xi_{ij} = 0$ for $1 \le i \le q$. Let $\overline{Y}_\Lambda$ be the Zariski closure of the image of the associated rational map $\Phi_\Lambda = [1 : \Phi_{\xi_1} : \cdots : \Phi_{\xi_q}] \colon \mathbb{P}^\ell \rightarrowtail \mathbb{P}^q$.

The main observation is that $\Phi_\Lambda$ can be factored as a linear map followed by a monomial map. Assume $\mathcal{A}$ is essential, and let

$$\alpha = \big[\alpha_0 : \cdots : \alpha_n\big] : \mathbb{P}^\ell \to \mathbb{P}^n.$$

Let $\mu = \mu_\Lambda \colon \mathbb{P}^n \rightarrowtail \mathbb{P}^q$ be given by

$$\mu([t_0 : \cdots : t_n]) = [1 : t^{\xi_1} : \cdots : t^{\xi_q}],$$

where we use the usual vector notation for monomials: $t^{(i_0,\dots,i_n)} = t_0^{i_0}\cdots t_n^{i_n}$. Then the following diagram commutes.

$$\begin{tikzcd} & \mathbb{P}^n \arrow[dr, "\mu_\Lambda"] & \\ \mathbb{P}^\ell \arrow[ur, "\alpha"] \arrow[rr, tail, "\Phi_\Lambda"] & & \mathbb{P}^q \end{tikzcd}$$

In this situation the diagram tropicalizes faithfully, in the following sense.

**Theorem 5.3** ([7, Theorem 3.1]). *The tropicalization* $\operatorname{trop}(\overline{Y}_\Lambda)$ *is equal to the image of the Bergman fan* $B(\mathfrak{G})$ *under the linear map*

$$\mathbb{TP}^n \to \mathbb{TP}^q$$

*with matrix* $\Lambda$.

We obtain the following characterization. Write $\Lambda = \big[\Lambda_0 \,|\, \cdots \,|\Lambda_n\big]$ with $\Lambda_j \in \mathbb{Z}^q$ for each $j$. For $S = \{H_{j_1},\dots,H_{j_k}\} \subseteq \mathcal{A}$, let $\Lambda_S = \sum_{r=1}^k \Lambda_{j_r}$.

**Theorem 5.4.** *The subspace* $D$ *is singular if and only if the rank of the matrix*

$$\Lambda_{\mathcal{S}} = \big[\Lambda_{S_1} \,|\, \cdots \,|\Lambda_{S_{\ell-1}}\big]$$

*is less than* $q$*, for each maximal nested set* $\mathcal{S} \in N(\mathfrak{G})$*. In this case the rank of* $D$ *is the maximal rank of* $\Lambda_{\mathcal{S}}$ *for* $\mathcal{S} \in N(\mathfrak{G})$*.*

*Proof.* The subspace $D$ is singular if and only if $\dim \overline{Y}_\Lambda < q$. By Theorem 5.1, this occurs if and only if $\dim \operatorname{trop}(\overline{Y}_\Lambda) < q$. The cones of $\operatorname{trop}(\overline{Y}_\Lambda)$ are images of the cones of $B(\mathfrak{G})$ under $\Lambda$, by Theorem 5.3. The linear hulls of the cones in $B(\mathfrak{G})$ are the images in $\mathbb{TP}^n$ of the linear spans of the sets $\{e_S \mid S \in \mathcal{S}\}$, for $\mathcal{S} \in N(\mathfrak{G})$, by Theorem 5.2. Since $\Lambda(e_S) = \Lambda_S$, the result follows. The last statement holds because the rank of $D$ is equal to $\dim \overline{Y}_\Lambda$. Q.E.D.

**Example 5.5.** Consider the arrangement of rank four with defining polynomial

$$Q = xyz(x+y+z)w(x+y+w),$$

with hyperplanes ordered according to the given factorization of $Q$. The dual point configuration consists of the six vertices of a triangular prism in $\mathbb{P}^3$.

For generic $(a,b,c)$, the master function $\Phi = x^a y^{-a} z^b (x+y+z)^{-b} w^c (x+y+w)^{-c}$ has critical set of codimension two, and $H^2(A,\omega) \cong$

$\mathbb{C}$ for $\omega = d\log(\Phi)$. Based on our other examples, one might suspect that the subspace $D$ with basis $\Lambda = \{\omega_0 - \omega_1,\ \omega_2 - \omega_3,\ \omega_4 - \omega_5\}$ is singular. Among the nested sets of $\mathcal{A}$ is the set $\mathcal{S} = \{0, 02, 024\}$, and

$$\Lambda_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

does not have rank two. Then by Theorem 5.4, $D$ is not singular. In fact, $\psi = a \cdot \partial(\omega_0\omega_1\omega_5) + b \cdot \partial(\omega_2\omega_3\omega_5)$ is the unique 2-cocycle for $\omega$. $\psi$ is trivial if $a$ or $b$ is zero, and our argument shows that $\psi$ is not decomposable if $a$ and $b$ are both nonzero.

In the forthcoming paper [2], Theorem 5.4 is used to derive combinatorial conditions for $p$-generic arrangements to support singular subspaces of rank $p$. Using that approach one can show by combinatorial means that there are no singular subspaces of $A^1$ of rank two in Example 5.5.

Theorem 5.4 also has the following corollary.

**Corollary 5.6.** *If $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are loop-free matroids on the ground set $\{1, \ldots, n\}$ and $B(\mathfrak{G}_1) = B(\mathfrak{G}_2)$, then $\mathfrak{G}_1 = \mathfrak{G}_2$.*

*Proof.* Let $\mathfrak{G}$ be a loop-free matroid on $\{1, \ldots, n\}$, with Orlik–Solomon algebra $A^{\cdot} = A^{\cdot}(\mathfrak{G})$. Let $e_1, \ldots, e_n \in A^1$ denote the canonical generators. Then $S = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ is dependent in $\mathfrak{G}$ if and only if $e_{i_1} \wedge \cdots \wedge e_{i_k} = 0$ in $A^k$. (This statement holds even if $\mathfrak{G}$ has multiple points.) Equivalently, $S$ is dependent if and only if the coordinate subspace $D \subseteq A^1$ spanned by $\{e_{i_1}, \ldots, e_{i_k}\}$ is singular. By Theorem 5.3, $D$ is singular if and only if the image of the Bergman fan $B(\mathfrak{G}) \subseteq \mathbb{TP}^n$ under the projection $\mathbb{TP}^n \to \mathbb{TP}^S \cong \mathbb{TP}^{k-1}$ has dimension less than $k - 1$. Thus $B(\mathfrak{G})$ determines $\mathfrak{G}$. Q.E.D.

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