Advanced Studies in Pure Mathematics 61, 2011
New Structures and Natural Constructions in Mathematical Physics
pp. 327–347

# On Frenkel–Mukhin algorithm for $q$-character of quantum affine algebras

**Wakako Nakai and Tomoki Nakanishi**

***Dedicated to Professor Akihiro Tsuchiya***

**Abstract.**

The $q$-character is a strong tool to study finite-dimensional representations of quantum affine algebras. However, the explicit formula of the $q$-character of a given representation has not been known so far. Frenkel and Mukhin proposed the iterative algorithm which generates the $q$-character of a given irreducible representation starting from its highest weight monomial. The algorithm is known to work for various classes of representations. In this note, however, we give an example in which the algorithm fails to generate the $q$-character.

## §1. Background

### 1.1. Finite-dimensional representations of quantum affine algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $U_q(\hat{\mathfrak{g}})$ be the untwisted quantum affine algebra of $\mathfrak{g}$ by Drinfeld and Jimbo [D1, D2, J].

The following are the most basic facts on the finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$, due to Chari–Pressley [CP1, CP2]:

(i) The isomorphism classes of the irreducible finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ are parametrized by an $n$-tuple of polynomials of constant term 1, $\mathbf{P} = (P_i(u))_{i\in I}$, where $I = \{1,\dots,n\}$ and $n = \operatorname{rank}\mathfrak{g}$. The polynomials $\mathbf{P}$ are often called the *Drinfeld polynomials* because an analogous result for Yangian was obtained earlier by Drinfeld [D2].

(ii) For given Drinfeld polynomials $\mathbf{P}$, let $V(\mathbf{P})$ denote the corresponding irreducible representation. For a pair of Drinfeld polynomials $\mathbf{P} = (P_i(u))_{i\in I}$ and $\mathbf{Q} = (Q_i(u))_{i\in I}$, let $\mathbf{PQ} := (P_i(u)Q_i(u))_{i\in I}$. Then, $V(\mathbf{PQ})$ is a subquotient of $V(\mathbf{P}) \otimes V(\mathbf{Q})$.

Received January 15, 2008.
Revised April 24, 2008.

(iii) A representation $V(\mathbf{P})$ is called the $i$th *fundamental representation* and denoted by $V_{\omega_i}(a)$ if $P_i(u) = 1 - au$ and $P_j(u) = 1$ for any $j \neq i$. Suppose that Drinfeld polynomials $\mathbf{P}$ are in the form

$$P_i(u) = \prod_{k=1}^{n_i}(1 - a_k^{(i)}u). \tag{1.1}$$

Namely, $a_k^{(i)}$ are the inverses of the zeros of $P_i(u)$. Then, as a consequence of (ii), $V(\mathbf{P})$ is a subquotient of the tensor product of fundamental representations $\bigotimes_{i \in I} \bigotimes_{k=1}^{n_i} V_{\omega_i}(a_k^{(i)})$.

For $\mathfrak{g}$ of type $A_1$, the structure of $V(\mathbf{P})$ for an arbitrary $\mathbf{P}$ is known [CP1]. Also, when $\mathfrak{g}$ is simply-laced, the relation between $V(\mathbf{P})$ and the so-called standard representations is described by an analogue of the Kazhdan–Lusztig polynomials [N1]. So far, no more general results are known for the structure of $V(\mathbf{P})$.

### 1.2. $q$-Character

To study the structure of $V(\mathbf{P})$, the *$q$-character* $\chi_q$ was introduced by Frenkel and Reshetikhin [FR]. It is an injective ring homomorphism from the Grothendieck ring of the finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ to the Laurent polynomial ring of infinitely-many variables $Y_{i,a}$, $i \in I, a \in \mathbb{C}^\times$,

$$\chi_q : \operatorname{Rep} U_q(\hat{\mathfrak{g}}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times}. \tag{1.2}$$

The variables $Y_{i,a}$ are regarded as affinizations of the formal exponentials $\exp(\omega_i)$ of the fundamental weights $\omega_i$ of $U_q(\mathfrak{g})$. By replacing $Y_{i,a}$ with $\exp(\omega_i)$, $\chi_q(V(\mathbf{P}))$ reduces to the underlying $U_q(\mathfrak{g})$-character of $V(\mathbf{P})$ with respect to the standard embedding $U_q(\mathfrak{g}) \subset U_q(\hat{\mathfrak{g}})$.

There are several equivalent ways to define the $q$-character.

(i) *By universal R-matrix.* This is the original definition of [FR]. The idea originates from the *transfer matrix*, which plays the central role in the *quantum inverse scattering method*, or the *Bethe ansatz method* for integrable spin chains such as the Heisenberg $XXX$ model [TF]. The $q$-character $\chi_q(V)$ of a representation $V$ is defined as a partial trace of the universal $R$-matrix of $U_q(\hat{\mathfrak{g}})$ on $V$.

(ii) *By weight decomposition.* It is shown also in [FR] that $\chi_q(V)$ is regarded as the formal character of the weight decomposition of $V$ with respect to certain elements in the Cartan subalgebra in the 'second realization' of $U_q(\hat{\mathfrak{g}})$ [D2]. Hernandez extended this definition of $\chi_q$ to the affinizations of the full family of the quantum Kac–Moody algebras [H2, H4].

(iii) *By quiver varieties.* When $\mathfrak{g}$ is simply-laced, Nakajima [N1, N2] geometrically defined a $t$-analogue of $q$-character $\chi_{q,t}$ (the $q,t$-*character*) as the generating function of the Poincaré polynomials of graded quiver varieties. Then, $\chi_q$ is obtained by $\chi_q = \chi_{q,1}$. The algorithm of calculating $\chi_{q,t}$ is given based on the analogue of the Kazhdan–Lusztig polynomials.

(iv) *By axiom.* In [N2] the axiom which characterizes $\chi_{q,t}$ in (iii) is given. The axiom is further extended for non simply-laced cases in [H1]. Then, $\chi_q$ is obtained by $\chi_q = \chi_{q,1}$.

Before the introduction of the $q$-character, the spectrum of the transfer matrix defined by the trace on a so-called *Kirillov–Reshetikhin (KR) representation* [KR] of $U_q(\hat{\mathfrak{g}})$ was extensively studied by the Bethe ansatz method ([R1, R2, R3, BR, KR, KNS, KNH, KOS, KS, TK], etc.). The fundamental representations $V_{\omega_i}(a)$, for example, are special cases of the KR representations. Because of Definition (i) above, these results, including many conjectures, are naturally translated and restudied in the context of the $q$-character [FR, FM1, CM, KOSY, N3, H3, H6]. As a result, the $q$-characters of the KR representations are, not fully, but rather well understood now.

However, beyond the KR representations, not much is known for the explicit formula of the $q$-character except for some partial results and conjectures (e.g., [H5, NN1, NN2, NN3]).

### 1.3. Frenkel–Mukhin algorithm

We say that a monomial in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times}$ is *dominant* if it is a monomial of variables $Y_{i,a}$, $i \in I, a \in \mathbb{C}^\times$, i.e., without $Y_{i,a}^{-1}$. Suppose that Drinfeld polynomials $\mathbf{P}$ are in the form (1.1). Then, $\chi_q(V(\mathbf{P}))$ contains a dominant monomial

$$m_+ = \prod_{i\in I}\prod_{k=1}^{n_i} Y_{i,a_k^{(i)}} \tag{1.3}$$

called the *highest weight monomial* of $V(\mathbf{P})$ [FR]. Since $\mathbf{P}$ and $m_+$ are in one-to-one correspondence, we parametrize the irreducible representations of $U_q(\hat{\mathfrak{g}})$ by their highest weight monomials as $V(m_+)$, instead of $V(\mathbf{P})$, from now on.

Frenkel and Mukhin [FM1] introduced the iterative algorithm which generates a polynomial, say, $\chi(m_+) \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times}$ from a given dominant monomial $m_+$. We call it the *FM algorithm* here. *A priori*, it is not clear whether the algorithm does not fail (i.e., it is not halted halfway); also it is not clear whether it stops at finitely many steps. It was conjectured that

**Conjecture 1.1** ([FM1], Conjecture 5.8). *For any dominant monomial $m_+$, the algorithm never fails and stops after finitely many steps. Moreover, the result $\chi(m_+)$ equals to $\chi_q(V(m_+))$.*

The algorithm is fairly practical so that, assuming the conjecture, one can explicitly calculate the $q$-characters of representations, by hand, or by computer, when the dimensions are small.

Conjecture 1.1 is partially proved by [FM1] as we shall explain now. We say a representation $V(m_+)$ is *special* if its highest weight monomial $m_+$ is the unique dominant monomial occurring in $\chi_q(V(m_+))$. For example, the fundamental representations are special [FM1]. More generally, the KR representations are special [N3, H3, H6]. (See [H5] for further examples of special representations.)

**Theorem 1.2** ([FM1], Theorem 5.9). *If $V(m_+)$ is special, then Conjecture 1.1 is true.*

In particular, the FM algorithm is applicable to the fundamental representations and the KR representations, and provides the aforementioned results for their $q$-characters. We note that there are also many *nonspecial* representations for which Conjecture 1.1 is true; *e.g.*, $\mathfrak{g}$ of type $A_2$ with $m_+ = Y_{1,1}^2 Y_{1,q^2}$, where $V(m_+) \simeq V(Y_{1,1}) \otimes V(Y_{1,1}Y_{1,q^2})$.

The purpose of this note is to give a counterexample of Conjecture 1.1. More precisely, it is an example where the algorithm *fails* in the sense of [FM1] (see Definition 2.7).

In Section 2 the FM algorithm is recalled. In Section 3, as a warmup, we give two examples in which the algorithm works well. Then, a counterexample is given in Section 4. Taking this opportunity, we also demonstrate the synthesis of the FM algorithm and Young tableaux in [BR, KOS, KS, NT, NN1, NN2, NN3] by these examples.

We thank D. Hernandez for his helpful comments.

## §2. FM algorithm

Here we recall the FM algorithm. The presentation here is minimal to describe the counterexample in Section 4. We faithfully follow [FM1, Section 5.5], so that the reader is asked to consult it for more details.

### 2.1. Preliminary: $q$-character of $U_q(\hat{\mathfrak{sl}}_2)$

The FM algorithm is based on the explicit formula of the $q$-characters of the irreducible representations of $U_q(\hat{\mathfrak{sl}}_2)$ [CP1, FR].

**Example 2.1.** Let $W_r(a)$ be the irreducible representation $U_q(\hat{\mathfrak{sl}}_2)$ with highest weight monomial

$$m_+ = \prod_{k=1}^{r} Y_{aq^{r-2k+1}}, \tag{2.1}$$

where we set $Y_a := Y_{1,a}$. Then, its $q$-character is given by

$$\chi_q(W_r(a)) = m_+ \sum_{i=0}^{r} \prod_{j=1}^{i} A^{-1}_{aq^{r-2j+2}}, \quad A_a := Y_{aq^{-1}} Y_{aq}. \tag{2.2}$$

Generally, the $q$-character of any irreducible representation of $U_q(\hat{\mathfrak{sl}}_2)$ is given by a product of (2.2) as follows [CP1]: Let $\Sigma_{a,r}$ be the set of the indices of the variables $Y_b$ in (2.1), i.e., $\Sigma_{a,r} = \{aq^{r-2k+1}\}_{k=1,\dots,r}$. We call it a *q-string.* We say that two $q$-strings $\Sigma_{a,r}$ and $\Sigma_{a',r'}$ are in *general position* if either (i) the union $\Sigma_{a,r} \cup \Sigma_{a',r'}$ is not a $q$-string, or (ii) $\Sigma_{a,r} \subset \Sigma_{a',r'}$ or $\Sigma_{a',r'} \subset \Sigma_{a,r}$. Then,

**Example 2.2.** Let $m_+ \in \mathbb{Z}[Y_a^{\pm 1}]_{a\in\mathbb{C}^\times}$ be a given dominant monomial. Then, one can uniquely (up to permutations) factorize $m_+$ as

$$m_+ = \prod_{i=1}^{k} \left( \prod_{b\in\Sigma_{a_i,r_i}} Y_b \right), \tag{2.3}$$

where $\Sigma_{a_1,r_1}, \dots, \Sigma_{a_k,r_k}$ are $q$-strings which are pairwise in general position. The $q$-character of $V(m_+)$ is given by

$$\chi_q(V(m_+)) = \prod_{i=1}^{k} \chi_q(W_{r_i}(a_i)). \tag{2.4}$$

## 2.2. Algorithm

Let us start from some key definitions.

**Definition 2.3.** (i) We say that a monomial $m \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times}$ is *i-dominant* if it does not contain variables $Y_{i,a}^{-1}$, $a \in \mathbb{C}^\times$.

(ii) For a polynomial $\chi \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times}$ and a monomial $m$ occurring in $\chi$ with coefficient $s$, a *coloring* of $m$ is a set of integers $\{s_i\}_{i\in I}$ such that $0 \le s_i \le s$. We say that a polynomial $\chi$ is *colored* if all monomials occurring in $\chi$ have colorings.

(iii) Let $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times}$ be a colored polynomial, and let $m$ be a monomial occurring in $\chi$ with coefficient $s \in \mathbb{Z}_{\ge 0}$ and coloring $\{s_i\}_{i\in I}$. We say that $m$ is *admissible* if, for any $i \in I$ such that $s_i < s$, $m$ is $i$-dominant.

Let

$$A_{i,a} = Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{j:C_{ji}=-1} Y_{j,a}^{-1} \times \prod_{j:C_{ji}=-2} Y_{j,aq^{-1}}^{-1} Y_{j,aq}^{-1} \prod_{j:C_{ji}=-3} Y_{j,aq^{-2}}^{-1} Y_{j,a}^{-1} Y_{j,aq^2}^{-1}, \tag{2.5}$$

where $C_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ is the Cartan matrix of $\mathfrak{g}$. The monomials $A_{i,a}$ are regarded as affinizations of the formal exponentials $\exp(\alpha_i)$ of the simple roots $\alpha_i$ of $U_q(\mathfrak{g})$.

The FM algorithm is an iterative algorithm, and its main routine utilizes the following procedure called the *i-expansion*:

**Definition 2.4.** Let $i \in I$, $\chi$ be a colored polynomial, and $m$ be an admissible monomial occurring in $\chi$ with coefficient $s$ and coloring $\{s_j\}_{j\in I}$. Then, a new colored polynomial $i_m(\chi)$, called the *i-expansion of $\chi$ with respect to $m$*, is defined as follows:

(i) If $s_i = s$, then $i_m(\chi) = \chi$.

(ii) If $s_i < s$, we define $i_m(\chi)$ in the following two steps.

First, we obtain a colored polynomial $\mu$ which depends on $m$ and $i$ (but not on $\chi$) as follows: Let $\overline{m}$ be the $i$th projection of $m$, i.e., $\overline{Y}_{i,a}^{\pm 1} = Y_a^{\pm 1}$ and $\overline{Y}_{j,a}^{\pm 1} = 1$ for any $j \neq i$. Let $\chi_{q_i}(V(\overline{m})) = \overline{m}(1+\sum_p \overline{M}_p)$ be the $q$-character of the irreducible representation $V(\overline{m})$ of $U_{q_i}(\hat{\mathfrak{sl}}_2)$ with highest weight monomial $\overline{m}$, where $\overline{M}_p$ is a product of $\overline{A}_{i,a}^{-1}$ (see (2.2), (2.4), and (2.5)). Then,

$$\mu = m(1 + \sum_p M_p), \tag{2.6}$$

where $M_p$ is obtained from $\overline{M}_p$ by replacing all $\overline{A}_{i,a}^{-1}$ by $A_{i,a}^{-1}$ in (2.5).

Next, we obtain $i_m(\chi)$ by adding the monomials occurring in $\mu$ to $\chi$ as follows: Suppose that a monomial $n$ occurs in $\mu$ with coefficient $t$. If $n$ does not occur in $\chi$, we add $n$ to $\chi$ with coefficient $t(s - s_i)$ and set its coloring $\{s'_j\}_{j\in I}$ as $s'_j = 0$ for any $j \neq i$ and $s'_i = t(s - s_i)$. If $n$ occurs in $\chi$ with coefficient $r$ and coloring $\{r_j\}_{j\in I}$, we set the coefficient $s'$ and the coloring $\{s'_j\}_{j\in I}$ of $n$ in $i_m(\chi)$ as $s' = \max\{r, r_i + t(s - s_i)\}$, $s'_j = r_j$ for any $j \neq i$, and $s'_i = r_i + t(s - s_i)$. The coefficients and the colorings of other monomials occurring in $\chi$ are unchanged in $i_m(\chi)$.

Note that the $i$-expansion is defined only if $m$ is admissible.

**Remark 2.5.** In (2.6), the coefficient of $m$ in $\mu$ is always 1. Therefore, both the coefficient and the $i$th coloring of $m$ in $i_m(\chi)$ are $s$ in

Definition 2.4. In other words, the $i$-expansion of $\chi$ with respect to $m$ is designed to saturate the $i$th coloring of $m$ to its coefficient.

**Definition 2.6.** (i) The $U_q(\mathfrak{g})$-*weight* of a monomial

$$\prod_{i\in I}\left(\prod_{r=1}^{k_i} Y_{i,a_{ir}} \prod_{s=1}^{l_i} Y_{i,b_{is}}^{-1}\right)$$

is defined by $\sum_{i\in I}(k_i - l_i)\omega_i$.

(ii) We equip the $U_q(\mathfrak{g})$-weight lattice $P := \bigoplus_{i\in I}\mathbb{Z}\omega_i$ with a partial order such that $\lambda \geq \lambda'$ if $\lambda - \lambda' = \sum_i a_i\alpha_i$, $a_i \in \mathbb{Z}_{\geq 0}$, and call it the *natural partial order* in $P$.

Now let us define the FM algorithm. It is an algorithm generating a colored polynomial $\chi(m_+) \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times}$ from a given dominant monomial $m_+$.

**Definition 2.7** (The FM algorithm). Let $m_+$ be a given dominant monomial in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times}$, and $\lambda_+$ be the $U_q(\mathfrak{g})$-weight of $m_+$. Choose any total order in the set $P_{\leq\lambda_+} := \{\mu \in P \mid \mu \leq \lambda_+\}$ such that it is compatible with the natural partial order in $P$; then enumerate the elements in $P_{\leq\lambda_+}$ as $\lambda_1 = \lambda_+ > \lambda_2 > \lambda_3 > \dots$.

*Step 1.* We set the colored polynomial $\chi$ by $\chi = m_+$ with the $i$th coloring of $m_+$ being 0 for any $i \in I$.

*Step 2.* Repeat the following steps (i)–(iii) for $\lambda = \lambda_1, \lambda_2, \lambda_3, \dots$.

- (i) Let $\chi$ be the colored polynomial obtained in the previous step. Let $m_1, \dots, m_t$ be all the monomials occurring in $\chi$ whose $U_q(\mathfrak{g})$-weights are $\lambda$. If there is at least one *non-admissible* monomial among them, then the algorithm halted halfway. We say that *the algorithm fails at* $m_i$ if $m_i$ is one of such non-admissible monomials.
- (ii) Repeat the following for all $i \in I$ and all $k \in 1, \dots, t$: Replace $\chi$ with the $i$-expansion $i_{m_k}(\chi)$ of $\chi$ with respect to $m_k$.
- (iii) If there is no monomial occurring in $\chi$ whose $U_q(\mathfrak{g})$-weight is less than $\lambda$ in the total order of $P_{\leq\lambda_+}$, then set $\chi(m_+) = \chi$ and the algorithm *stops* (i.e., completes).

It follows from Remark 2.5 that, if the algorithm successfully stops, the $i$th coloring of any monomial $m$ occurring in $\chi(m_+)$ equals to the coefficient of $m$ for any $i \in I$. Thus, once $\chi(m_+)$ is obtained, one can safely forget the coloring.

## §3. Examples

Let us see how the FM algorithm works in good situations. This is a warmup to understand the 'bad situation' in the next section.

### 3.1. Example 1

Let us consider the case where $\mathfrak{g}$ is of type $A_2$ and the representation $V(m_+)$ has the highest weight monomial

$$m_+ = Y_{1,q^2} Y_{2,q^{-1}}. \tag{3.1}$$

The $U_q(\mathfrak{g})$-weight of $m_+$ is $\lambda_+ = \omega_1 + \omega_2$. It is well known that $V(m_+)$ is an *evaluation representation* of the adjoint representation $V_{\omega_1+\omega_2}$ of $U_q(\mathfrak{g})$. As a $U_q(\mathfrak{g})$-representation, it is isomorphic to $V_{\omega_1+\omega_2}$. It is also known that $V(m_+)$ is special [H5] so that the FM algorithm is applicable.

We use the following data: $q_1 = q_2 = q$, $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = -\omega_1 + 2\omega_2$, and

$$A_{1,a}^{-1} = Y_{1,aq^{-1}}^{-1} Y_{1,aq}^{-1} Y_{2,a}, \quad A_{2,a}^{-1} = Y_{2,aq^{-1}}^{-1} Y_{2,aq}^{-1} Y_{1,a}. \tag{3.2}$$

Now let us execute the FM algorithm step by step. We choose a total order in $P_{\leq \lambda_+}$ as

$$\begin{aligned} &\lambda_1 = \lambda_+, \quad \lambda_2 = \lambda_+ - \alpha_1, \quad \lambda_3 = \lambda_+ - \alpha_2, \lambda_4 = \lambda_+ - 2\alpha_1, \\ &\lambda_5 = \lambda_+ - \alpha_1 - \alpha_2, \quad \lambda_6 = \lambda_+ - 2\alpha_2, \quad \lambda_7 = \lambda_+ - 2\alpha_1 - \alpha_2, \\ &\lambda_8 = \lambda_+ - \alpha_1 - 2\alpha_2, \quad \lambda_9 = \lambda_+ - 2\alpha_1 - 2\alpha_2, \quad \dots, \end{aligned} \tag{3.3}$$

where the rest of the order is irrelevant.

*Step 1.* Set $\chi = m_+ = Y_{1,q^2} Y_{2,q^{-1}}$ with the coloring of $m_+$ being $(0,0)$.

*Step 2.* (1) $\lambda = \lambda_1 = \omega_1 + \omega_2$.

The 1-expansion of $\chi$ with respect to $m_+ = Y_{1,q^2} Y_{2,q^{-1}}$ is done as follows: Since $\overline{Y_{1,q^2} Y_{2,q^{-1}}} = Y_{q^2}$, we have

$$\begin{aligned} \chi_q(V) &= Y_{q^2}(1 + \overline{A}_{1,q^3}^{-1}), \\ \mu &= Y_{1,q^2} Y_{2,q^{-1}}(1 + A_{1,q^3}^{-1}) = Y_{1,q^2} Y_{2,q^{-1}} + Y_{1,q^4}^{-1} Y_{2,q^{-1}} Y_{2,q^3}, \\ 1_{m_+}(\chi) &= \underset{(1,0)}{Y_{1,q^2} Y_{2,q^{-1}}} + \underset{(1,0)}{Y_{1,q^4}^{-1} Y_{2,q^{-1}} Y_{2,q^3}}, \end{aligned}$$

where $(1,0)$ represents the coloring. Then, $\chi$ is replaced with $1_{m_+}(\chi)$.

Similarly, the 2-expansion of $\chi$ with respect to $m_+$ is calculated as

$$\begin{aligned}\mu &= Y_{1,q^2}Y_{2,q^{-1}}(1+A_{2,1}^{-1}) = Y_{1,q^2}Y_{2,q^{-1}} + Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1},\\ \chi &= \underset{(1,1)}{Y_{1,q^2}Y_{2,q^{-1}}} + \underset{(1,0)}{Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}} + \underset{(0,1)}{Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1}}.\end{aligned}$$

(2) $\lambda = \lambda_2 = -\omega_1 + 2\omega_2$. From now on, we only write down the nontrivial $i$-expansions, i.e., the cases where $s_i < s$.

The 2-expansion w.r.t. $Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}$:

$$\begin{aligned}\mu &= Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}(1+A_{2,1}^{-1})(1+A_{2,q^4}^{-1})\\ &= Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3} + Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q}^{-1}Y_{2,q^3} + Y_{2,q^{-1}}Y_{2,q^5}^{-1} + Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1},\\ \chi &= \underset{(1,1)}{Y_{1,q^2}Y_{2,q^{-1}}} + \underset{(1,1)}{Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}} + \underset{(0,1)}{Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1}} + \underset{(0,1)}{Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q}^{-1}Y_{2,q^3}}\\ &\quad + \underset{(0,1)}{Y_{2,q^{-1}}Y_{2,q^5}^{-1}} + \underset{(0,1)}{Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1}}.\end{aligned}$$

(3) $\lambda = \lambda_3 = 2\omega_1 - \omega_2$.

The 1-expansion w.r.t. $Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1}$:

$$\begin{aligned}\mu &= Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1}(1+A_{1,q^3}^{-1}+A_{1,q}^{-1}A_{1,q^3}^{-1})\\ &= Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1} + Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q}^{-1}Y_{2,q^3} + Y_{1,q^2}^{-1}Y_{1,q^4}^{-1}Y_{2,q^3},\\ \chi &= \underset{(1,1)}{Y_{1,q^2}Y_{2,q^{-1}}} + \underset{(1,1)}{Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}} + \underset{(1,1)}{Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1}} + \underset{(1,1)}{Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q}^{-1}Y_{2,q^3}}\\ &\quad + \underset{(0,1)}{Y_{2,q^{-1}}Y_{2,q^5}^{-1}} + \underset{(0,1)}{Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1}} + \underset{(1,0)}{Y_{1,q^2}^{-1}Y_{1,q^4}^{-1}Y_{2,q^3}}.\end{aligned}$$

(4) $\lambda = \lambda_4 = -3\omega_1 + 3\omega_2$. No nontrivial $i$-expansions.

(5) $\lambda = \lambda_5 = 0$.

The 1-expansion w.r.t. $Y_{2,q^{-1}}Y_{2,q^5}^{-1}$:

$$\begin{aligned}\mu &= Y_{2,q^{-1}}Y_{2,q^5}^{-1},\\ \chi &= \underset{(1,1)}{Y_{1,q^2}Y_{2,q^{-1}}} + \underset{(1,1)}{Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}} + \underset{(1,1)}{Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1}} + \underset{(1,1)}{Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q}^{-1}Y_{2,q^3}}\\ &\quad + \underset{(1,1)}{Y_{2,q^{-1}}Y_{2,q^5}^{-1}} + \underset{(0,1)}{Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1}} + \underset{(1,0)}{Y_{1,q^2}^{-1}Y_{1,q^4}^{-1}Y_{2,q^3}}.\end{aligned}$$

(6) $\lambda = \lambda_6 = 3\omega_1 - 3\omega_2$. No nontrivial $i$-expansions.
(7) $\lambda = \lambda_7 = -2\omega_1 + \omega_2$.
The 2-expansion w.r.t. $Y^{-1}_{1,q^2}Y^{-1}_{1,q^4}Y_{2,q^3}$:

$$\begin{aligned}
\mu &= Y^{-1}_{1,q^2}Y^{-1}_{1,q^4}Y_{2,q^3}(1 + A^{-1}_{2,q^4}) = Y^{-1}_{1,q^2}Y^{-1}_{1,q^4}Y_{2,q^3} + Y^{-1}_{1,q^2}Y^{-1}_{2,q^5},\\
\chi &= \underset{(1,1)}{Y_{1,q^2}Y_{2,q^{-1}}} + \underset{(1,1)}{Y^{-1}_{1,q^4}Y_{2,q^{-1}}Y_{2,q^3}} + \underset{(1,1)}{Y_{1,1}Y_{1,q^2}Y^{-1}_{2,q}} + \underset{(1,1)}{Y_{1,1}Y^{-1}_{1,q^4}Y^{-1}_{2,q}Y_{2,q^3}}\\
&\quad + \underset{(1,1)}{Y_{2,q^{-1}}Y^{-1}_{2,q^5}} + \underset{(0,1)}{Y_{1,1}Y^{-1}_{2,q}Y^{-1}_{2,q^5}} + \underset{(1,1)}{Y^{-1}_{1,q^2}Y^{-1}_{1,q^4}Y_{2,q^3}} + \underset{(0,1)}{Y^{-1}_{1,q^2}Y^{-1}_{2,q^5}}.
\end{aligned}$$

(8) $\lambda = \lambda_8 = \omega_1 - 2\omega_2$.
The 1-expansion w.r.t. $Y_{1,1}Y^{-1}_{2,q}Y^{-1}_{2,q^5}$:

(3.4)
$$\begin{aligned}
\mu &= Y_{1,1}Y^{-1}_{2,q}Y^{-1}_{2,q^5}(1 + A^{-1}_{1,q}) = Y_{1,1}Y^{-1}_{2,q}Y^{-1}_{2,q^5} + Y^{-1}_{1,q^2}Y^{-1}_{2,q^5},\\
\chi &= \underset{(1,1)}{Y_{1,q^2}Y_{2,q^{-1}}} + \underset{(1,1)}{Y^{-1}_{1,q^4}Y_{2,q^{-1}}Y_{2,q^3}} + \underset{(1,1)}{Y_{1,1}Y_{1,q^2}Y^{-1}_{2,q}} + \underset{(1,1)}{Y_{1,1}Y^{-1}_{1,q^4}Y^{-1}_{2,q}Y_{2,q^3}}\\
&\quad + \underset{(1,1)}{Y_{2,q^{-1}}Y^{-1}_{2,q^5}} + \underset{(1,1)}{Y_{1,1}Y^{-1}_{2,q}Y^{-1}_{2,q^5}} + \underset{(1,1)}{Y^{-1}_{1,q^2}Y^{-1}_{1,q^4}Y_{2,q^3}} + \underset{(1,1)}{Y^{-1}_{1,q^2}Y^{-1}_{2,q^5}}.
\end{aligned}$$

(9) $\lambda = \lambda_9 = -\omega_1 - \omega_2$. There is no nontrivial $i$-expansions; furthermore, there is no monomial occurring in $\chi$ in (3.4) whose $U_q(\mathfrak{g})$-weight is less than $\lambda_9$ in the total order. Therefore, we set $\chi(m_+)$ to be $\chi$ in (3.4), and the algorithm stops.

Thus, we obtain the $q$-character of $V(m_+)$ as $\chi$ in (3.4) by forgetting coloring.

Next, let us introduce a diagrammatic notation of monomials by Young tableaux, following [BR, KOS, KS, NT, FR, FM2, NN1].

To each letter $a = 1, 2, 3$ within a box of a Young diagram $Y$, we assign a monomial as (cf. [FR, Section 5.4.1])

(3.5)
$$\begin{aligned}
&\boxed{1}_{\,ij} = Y_{1,q^{-2i+2j}}, \quad \boxed{2}_{\,ij} = Y^{-1}_{1,q^{-2i+2j+2}}Y_{2,q^{-2i+2j+1}},\\
&\boxed{3}_{\,ij} = Y^{-1}_{2,q^{-2i+2j+3}},
\end{aligned}$$

where the subscription '$ij$' indicates that the box is located in the $i$th row and $j$th column of $Y$. To each tableau $T$ on $Y$, we assign a monomial $m(T)$ by multiplying the monomials assigned to all the boxes in $T$. For example, the first and the second monomials in $\chi$ in (3.4) are represented

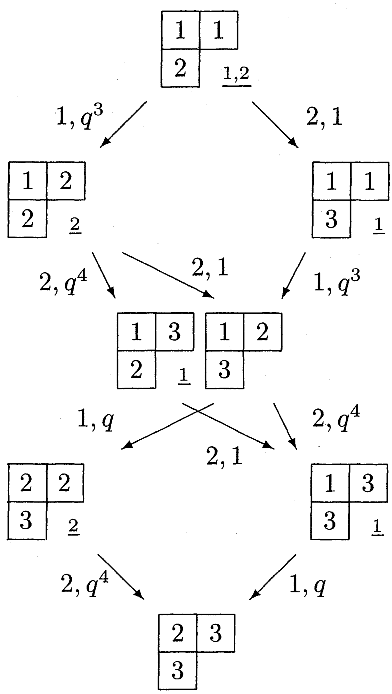


Fig. 1. The flow of the FM algorithm by Young tableaux for Example 1. The symbol $i, q^k$ at an arrow represents the action of $A^{-1}_{i,q^k}$. The suffix $\underline{i}$ at a tableau indicates that $s_i < s$ when $\chi$ is to be $i$-expanded with respect to the corresponding monomial.

by tableaux as

$$m\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right) = Y_{1,1}Y_{1,q^2}(Y^{-1}_{1,1}Y_{2,q^{-1}}) = Y_{1,q^2}Y_{2,q^{-1}}, \tag{3.6}$$

$$\begin{aligned} m\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\right) &= Y_{1,1}(Y^{-1}_{1,q^4}Y_{2,q^3})(Y^{-1}_{1,1}Y_{2,q^{-1}}) \\ &= Y^{-1}_{1,q^4}Y_{2,q^{-1}}Y_{2,q^3}. \end{aligned} \tag{3.7}$$

The definition (3.5) is designed so that the following equalities hold [FR, Section 5.4.1]:

$$A^{-1}_{1,q^{-2i+2j+1}}\ \boxed{1}_{\,ij} = \boxed{2}_{\,ij}, \quad A^{-1}_{2,q^{-2i+2j+2}}\ \boxed{2}_{\,ij} = \boxed{3}_{\,ij}. \tag{3.8}$$

Namely, the multiplication of $A_{i,a}^{-1}$ is regarded as the 'action' of changing the letter $i$ to $i+1$ in a tableau, if $a$ is appropriately chosen.

With this notation, one can concisely keep track and express the whole process of the algorithm presented above by the *semistandard tableaux* of shape $(2,1)$ as in Figure 1. Moreover, as a corollary of Figure 1, we obtain the *tableaux expression* of the $q$-character

$$\chi_q(V(m_+)) = \sum_{T\in \mathrm{SST}(2,1)} m(T), \tag{3.9}$$

where $\mathrm{SST}(2,1)$ is the set of all the semistandard tableaux of shape $(2,1)$.

**Remark 3.1.** For $\mathfrak{g}$ of classical type, similar tableaux expressions to (3.9) have been conjectured and partially proved for a large class of irreducible representations $V(\lambda/\mu)$ parametrized by skew Young diagrams $\lambda/\mu$ [BR, KOS, KS, FR, FM2, NN1, NN2, NN3, H5]. More precisely, there is a tableaux expression for the 'Jacobi–Trudi-type determinant' $\chi(\lambda/\mu)$, which lies in the image of the homomorphism $\chi_q$. For types $A_n$ and $B_n$, it is known that $\chi(\lambda/\mu) = \chi_q(V(\lambda/\mu))$ for any skew Young diagram [H5, H7]. For type $A_n$, this was also shown in the context of the character of Yangian $Y(\mathfrak{gl}_n)$ [NT]. For types $C_n$ and $D_n$ [NN1, NN2, NN3], for a (non-skew) Young diagram $\lambda$, it was conjectured that $\chi(\lambda) = \chi_q(V(\lambda))$. In general, $\chi(\lambda/\mu)$ is conjectured to be the $q$-character of, not $V(\lambda/\mu)$ itself, but some representation which has $V(\lambda/\mu)$ as a subquotient. Using this opportunity, let us withdraw our false claim for types $C_n$ and $D_n$ in [NN1, NN2, NN3] that *we expect that* $\chi(\lambda/\mu) = \chi_q(V(\lambda/\mu))$, *if* $\lambda/\mu$ *is connected.* A counterexample is given by $\mathfrak{g}$ of type $C_2$ with $\lambda = (2,2,1), \mu = (1)$.

**Remark 3.2.** The underlying $U_q(\mathfrak{g})$-character of $V(m_+)$ is symmetric under the Dynkin diagram automorphism $1 \leftrightarrow 2$. However, we see in Figure 1 that the $U_q(\hat{\mathfrak{g}})$-structure of $V(m_+)$ is not so. Of course, this is not a contradiction, because the highest weight monomial $m_+$ in (3.1) are not symmetric under the automorphism.

### 3.2. Example 2

To convince the reader further that the FM algorithm is well designed, let us give another, a little more complicated example, where the coefficients of some monomials in the $q$-character are greater than one. We consider the case where $\mathfrak{g}$ is of type $C_2$ and the representation $V(m_+)$ has the highest weight monomial

$$m_+ = Y_{2,q^{-1}} Y_{2,q}. \tag{3.10}$$

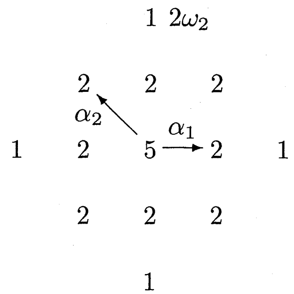


Fig. 2. The $U_q(\mathfrak{g})$-weight diagram of $V(m_+)$ in Example 2. The numbers represent the weight multiplicities.

The $U_q(\mathfrak{g})$-weight of $m_+$ is $\lambda_+ = 2\omega_2$. We faithfully follow the convention in [FR, FM1]; in particular, $\alpha_2$ is the long root.

Since any monomial occurring in $\chi_q(V(m_+))$ for (3.10) should occur in the product $\chi_q(V(Y_{2,q^{-1}}))\chi_q(V(Y_{2,q}))$, and

$$\begin{aligned}(3.11)\qquad \chi_q(V(Y_{2,q^{-1}})) &= Y_{2,q^{-1}} + Y_{1,1}Y_{1,q^2}Y_{2,q^3}^{-1} + Y_{1,1}Y_{1,q^4}^{-1} \\ &\quad + Y_{1,q^2}^{-1}Y_{1,q^4}^{-1}Y_{2,q} + Y_{2,q^5}^{-1},\\ (3.12)\qquad \chi_q(V(Y_{2,q})) &= Y_{2,q} + Y_{1,q^2}Y_{1,q^4}Y_{2,q^5}^{-1} + Y_{1,q^2}Y_{1,q^6}^{-1} \\ &\quad + Y_{1,q^4}^{-1}Y_{1,q^6}^{-1}Y_{2,q^3} + Y_{2,q^7}^{-1},\end{aligned}$$

one can immediately see that $m_+$ is the only possible dominant monomial in $\chi_q(V(m_+))$. Thus, $V(m_+)$ is special, and the FM algorithm is applicable. It also implies that $V(Y_{2,q^{-1}}) \otimes V(Y_{2,q})$ is irreducible and isomorphic to $V(m_+)$. In particular, as a $U_q(\mathfrak{g})$-representation, $V(m_+)$ is decomposed as $V_{\omega_2} \otimes V_{\omega_2} \simeq V_{2\omega_2} \oplus V_{2\omega_1} \oplus V_0$ with dimension $5 \times 5 = 14 + 10 + 1$, and its $U_q(\mathfrak{g})$-weight diagram is given in Figure 2.

Keep in mind that (Definition 2.4 (ii)) the $i$-expansion should be done, *not with* $U_q(\hat{\mathfrak{sl}}_2)$, *but with* $U_{q_i}(\hat{\mathfrak{sl}}_2)$. Then, the algorithm can be straightforwardly executed with the data: $q_1 = q$, $q_2 = q^2$, and

$$(3.13)\qquad A_{1,a}^{-1} = Y_{1,aq^{-1}}^{-1}Y_{1,aq}^{-1}Y_{2,a}, \quad A_{2,a}^{-1} = Y_{2,aq^{-2}}^{-1}Y_{2,aq^2}^{-1}Y_{1,aq^{-1}}Y_{1,aq}.$$

Again, the flow of the algorithm can be expressed with tableaux of shape $(2,2)$. We assign a monomial to each letter $a = 1, 2, \overline{2}, \overline{1}$ within the box at position $(i,j)$ as (cf. [FR, Section 5.4.3]).

$$(3.14)\qquad \begin{aligned} \boxed{1}_{\,ij} &= Y_{1,q^{-2i+2j}}, & \boxed{\overline{2}}_{\,ij} &= Y_{1,q^{-2i+2j+4}}Y_{2,q^{-2i+2j+5}}^{-1},\\ \boxed{2}_{\,ij} &= Y_{1,q^{-2i+2j+2}}^{-1}Y_{2,q^{-2i+2j+1}}, & \boxed{\overline{1}}_{\,ij} &= Y_{1,q^{-2i+2j+6}}^{-1}. \end{aligned}$$

For example, the highest weight monomial $m_+$ is represented as

$$(3.15)\quad m\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}\right) = Y_{1,1}Y_{1,q^2}(Y_{1,1}^{-1}Y_{2,q^{-1}})(Y_{1,q^2}^{-1}Y_{2,q}) = Y_{2,q^{-1}}Y_{2,q}.$$

The 'action' of $A_{i,a}^{-1}$ on a box is given by

$$(3.16)\quad \boxed{1}_{ij} \xrightarrow{A^{-1}_{1,q^{-2i+2j+1}}} \boxed{2}_{ij} \xrightarrow{A^{-1}_{2,q^{-2i+2j+3}}} \boxed{\overline{2}}_{ij} \xrightarrow{A^{-1}_{1,q^{-2i+2j+5}}} \boxed{\overline{1}}_{ij}.$$

Then, the flow and the result of the algorithm is expressed by tableaux in Figure 3.

We note that two monomials occur in $\chi_q(V(m_+))$ with coefficient two, and, in Figure 3, each monomial is represented by two different tableaux such as

(3.17)
$$m\left(\begin{array}{|c|c|}\hline \overline{2} & 2 \\ \hline \overline{2} & 2 \\ \hline\end{array}\right) = m\left(\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \overline{2} & \overline{2} \\ \hline\end{array}\right),\quad m\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \overline{1} & \overline{1} \\ \hline\end{array}\right) = m\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \overline{1} \\ \hline\end{array}\right).$$

Purely from the point of view of the FM algorithm, this is redundant, because the FM algorithm does not distinguish tableaux if they represents the same monomial. However, by doing this, we have the following *tableaux expression* of the $q$-character

$$(3.18)\quad \chi_q(V(m_+)) = \sum_{T\in \mathrm{Tab}} m(T),$$

where Tab is the set of the tableaux occurring in Figure 3. Remarkably, the formula (3.18) exactly coincides with the tableaux expression in [NN1, NN3] based on the Jacobi–Trudi-type determinant, thereby showing nice compatibility between two approaches.

**Remark 3.3.** The implementation of the FM algorithm with tableaux (or, equivalently, with paths) of [NN1, NN3] demonstrated here, can be generalized to the skew diagram representations of type $C_n$ [NN4]. See also Remark 4.4.

## §4. Counterexample

Now we are ready to present an example where the FM algorithm *fails* in the sense of Step 2 (i) of Definition 2.7.

We consider the case where $\mathfrak{g}$ is of type $C_3$ and the representation $V(m_+)$ has the highest weight monomial

$$(4.1)\quad m_+ = Y_{1,q^4}Y_{2,q}Y_{3,q^{-2}}.$$

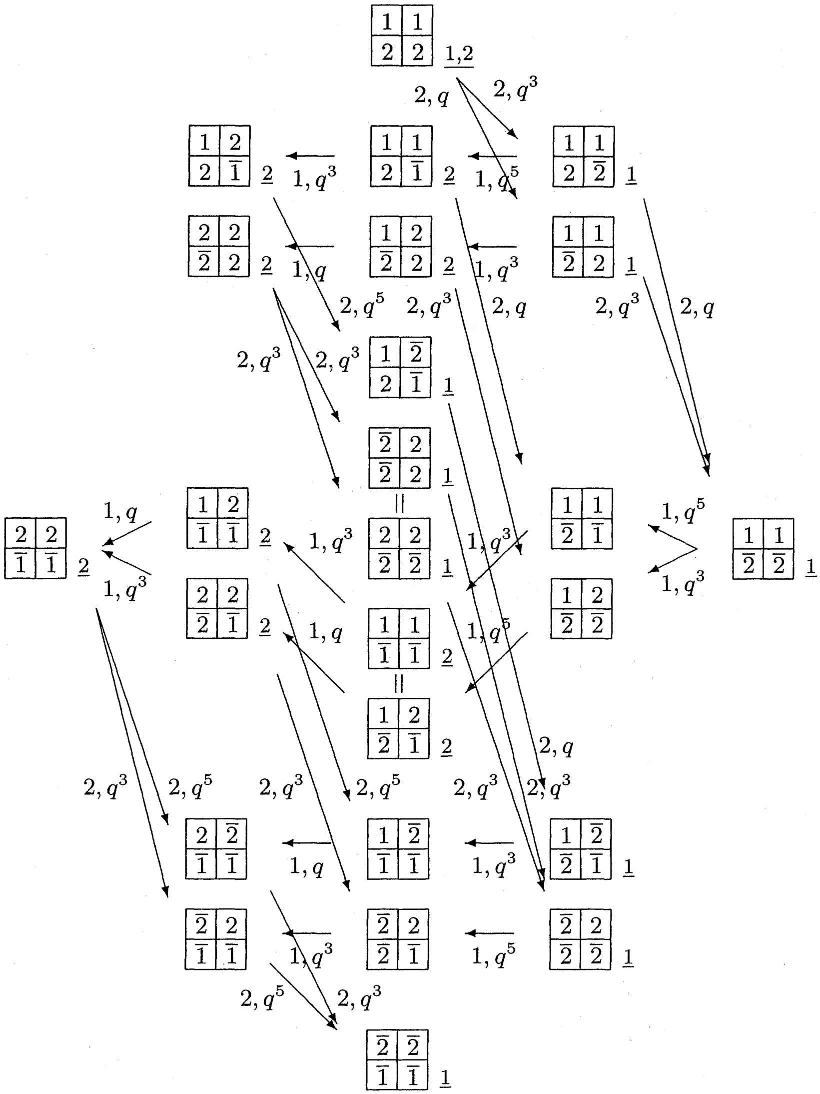

Fig. 3. The flow of the FM algorithm by Young tableaux for Example 2. The equality between two tableaux means that they represent the same monomial.

According to [NN1, Conjecture 2.2, Theorem A.1], it is expected to be decomposed into $V_{\omega_1+\omega_2+\omega_3} \oplus V_{2\omega_1+\omega_2} \oplus V_{2\omega_2} \oplus V_{\omega_1+\omega_3} \oplus V_{2\omega_1} \oplus V_{\omega_2}$ as a $U_q(\mathfrak{g})$-representation, with dimension $512+189+90+70+21+14 = 896$. The algorithm is executed with the data: $q_1 = q_2 = q$, $q_3 = q^2$, and

$$A_{1,a}^{-1} = Y_{1,aq^{-1}}^{-1} Y_{1,aq}^{-1} Y_{2,a}, \quad A_{2,a}^{-1} = Y_{2,aq^{-1}}^{-1} Y_{2,aq}^{-1} Y_{1,a} Y_{3,a},$$
$$A_{3,a}^{-1} = Y_{3,aq^{-2}}^{-1} Y_{3,aq^2}^{-1} Y_{2,aq^{-1}} Y_{2,aq}. \tag{4.2}$$

Again, the process of the algorithm can be expressed by Young tableaux of shape $(3,2,1)$. We assign a monomial to each letter $a = 1, 2, 3, \overline{3}, \overline{2}, \overline{1}$ within the box at position $(i,j)$ as

$$\boxed{1}_{ij} = Y_{1,q^{-2i+2j}}, \qquad \boxed{\overline{3}}_{ij} = Y_{2,q^{-2i+2j+5}} Y_{3,q^{-2i+2j+6}}^{-1},$$
$$\boxed{2}_{ij} = Y_{1,q^{-2i+2j+2}}^{-1} Y_{2,q^{-2i+2j+1}}, \quad \boxed{\overline{2}}_{ij} = Y_{1,q^{-2i+2j+6}} Y_{2,q^{-2i+2j+7}}^{-1}, \tag{4.3}$$
$$\boxed{3}_{ij} = Y_{2,q^{-2i+2j+3}}^{-1} Y_{3,q^{-2i+2j+2}}, \quad \boxed{\overline{1}}_{ij} = Y_{1,q^{-2i+2j+8}}^{-1}.$$

For example, the highest weight monomial $m_+$ is represented as

$$m\left(\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2&2& \\ \cline{1-2} 3& & \\ \cline{1-1}\end{array}\right)$$
$$= Y_{1,1}Y_{1,q^2}Y_{1,q^4}(Y_{1,1}^{-1}Y_{2,q^{-1}})(Y_{1,q^2}^{-1}Y_{2,q})(Y_{2,q^{-1}}^{-1}Y_{3,q^{-2}})$$
$$= Y_{1,q^4}Y_{2,q}Y_{3,q^{-2}}. \tag{4.4}$$

The 'action' of $A_{i,a}^{-1}$ on a box is given by

$$\boxed{1}_{ij} \xrightarrow{A_{1,q^{-2i+2j+1}}^{-1}} \boxed{2}_{ij} \xrightarrow{A_{2,q^{-2i+2j+2}}^{-1}} \boxed{3}_{ij} \xrightarrow{A_{3,q^{-2i+2j+4}}^{-1}}$$
$$\boxed{\overline{3}}_{ij} \xrightarrow{A_{2,q^{-2i+2j+6}}^{-1}} \boxed{\overline{2}}_{ij} \xrightarrow{A_{1,q^{-2i+2j+7}}^{-1}} \boxed{\overline{1}}_{ij}. \tag{4.5}$$

**Theorem 4.1.** *The FM algorithm fails for $m_+$ in (4.1).*

Let us prove the theorem. We set

$$m_1 := A_{3,1}^{-1}m_+ = Y_{1,q^4}(Y_{2,q^{-1}}Y_{2,q}^2)Y_{3,q^2}^{-1}, \tag{4.6}$$
$$m_2 := A_{2,q^2}^{-1}A_{3,1}^{-1}m_+ = (Y_{1,q^2}Y_{1,q^4})(Y_{2,q^{-1}}Y_{2,q}Y_{2,q^3}^{-1}), \tag{4.7}$$
$$m_3 := A_{2,q^2}^{-2}A_{3,1}^{-1}m_+ = (Y_{1,q^2}^2Y_{1,q^4})(Y_{2,q^{-1}}Y_{2,q^3}^{-2})Y_{3,q^2}, \tag{4.8}$$
$$m_4 := A_{1,q^3}^{-1}A_{2,q^2}^{-2}A_{3,1}^{-1}m_+ = Y_{1,q^2}(Y_{2,q^{-1}}Y_{2,q^3}^{-1})Y_{3,q^2},, \tag{4.9}$$
$$m_5 := A_{1,q^3}^{-1}A_{2,q^2}^{-1}A_{3,1}^{-1}m_+ = Y_{2,q^{-1}}Y_{2,q}, \tag{4.10}$$
$$m_6 := A_{2,q^2}^{-1}m_+ = (Y_{1,q^2}Y_{1,q^4})(Y_{3,q^{-2}}Y_{3,q^2}). \tag{4.11}$$

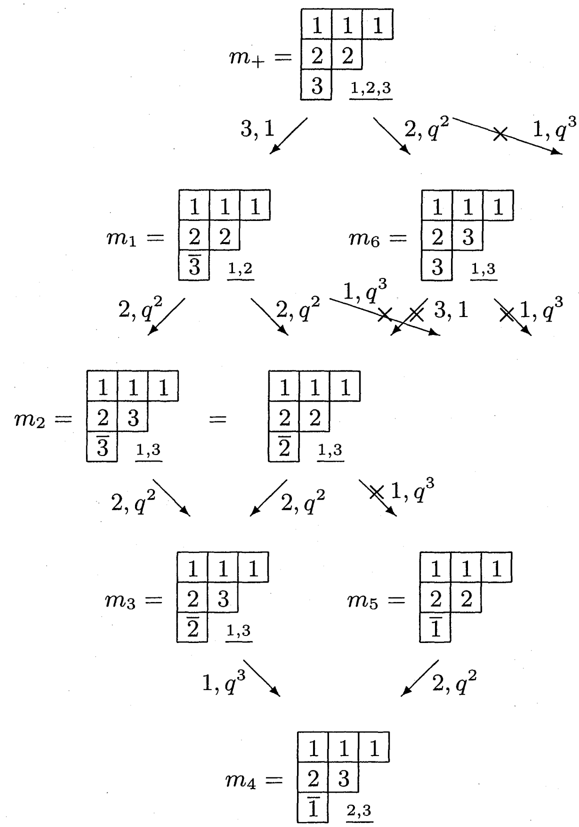


Fig. 4. The diagram explaining how the FM algorithm fails for $m_+$ in (4.1).

We show below that the algorithm fails at $m_4$. See Figure 4 for the outline of the proof in terms of tableaux.

**Lemma 4.2.** *The monomial $m_4$ occurs in $\chi$ at some step in the algorithm.*

*Proof.* By (4.1), the 3-expansion of $\chi$ with respect to $m_+$ gives $\mu = m_+(1 + A_{3,1}^{-1})$, where $\mu$ is the polynomial in (2.6). Therefore, $m_1$ occurs in $\chi$ after the expansion. Next, by (4.6), $m_1$ is admissible, and the 2-expansion of $\chi$ with respect to $m_1$ gives $\mu = m_1(1 + A_{2,q^2}^{-1} + A_{2,1}^{-1}A_{2,q^2}^{-1})(1+A_{2,q^2}^{-1})$. Thus, $m_3$ occurs in $\chi$ after the expansion. Finally, by (4.8), $m_3$ is admissible, and the 1-expansion of $\chi$ with respect to $m_3$

gives $\mu = m_1(1 + A^{-1}_{1,q^5} + A^{-1}_{1,q^3}A^{-1}_{1,q^5})(1 + A^{-1}_{1,q^3})$. In particular, $m_4$ occurs in $\chi$ after the expansion. Q.E.D.

Let $\lambda\,(= \omega_1 + \omega_3)$ denote the $U_q(\mathfrak{g})$-weight of $m_4$. Let us show that the monomial $m_4$ is *not* admissible when $\chi$ is going to be expanded at $\lambda$; hence, the algorithm fails at $m_4$. To see it, suppose that $m_4$ is admissible when $\chi$ is going to be expanded $\chi$ at $\lambda$. Since $m_4$ is not 2-dominant, it should occur in the 2-expansion with respect to some 2-dominant monomial, say, $n$ whose $U_q(\mathfrak{g})$-weight is greater than $\lambda$. Since $\{A_{i,a}\}_{i\in I; a\in\mathbb{C}^\times}$ are algebraically independent, $n$ should be either $m_5 = A_{2,q^2}m_4$ or $m'_5 = A^2_{2,q^2}m_4$. Then, one can easily check that the 2-expansion with respect to $m_5$ generates $m_4$, while the 2-expansion with respect to $m'_5$ does not so. Therefore, $n = m_5$. However,

**Lemma 4.3.** *The monomial $m_5$ does not occur in $\chi$ at any step in the algorithm.*

*Proof.* By (4.10), there are six possible routes to obtain $m_5$ from $m_+$ by $i$-expansions: (The symbol $\xrightarrow{i,q^k}$ represents the action of $A^{-1}_{i,q^k}\cdot$.)

(i) $m_+ \xrightarrow{1,q^3} * \xrightarrow{2,q^2} * \xrightarrow{3,1} m_5$. The 1-expansion of $\chi$ with respect to $m_+$ gives $\mu = m_+(1 + A^{-1}_{1,q^5})$. So, it does not happen.

(ii) $m_+ \xrightarrow{1,q^3} * \xrightarrow{3,1} * \xrightarrow{2,q^2} m_5$. By the same reason as above, it does not happen.

(iii) $m_+ \xrightarrow{2,q^2} m_6 \xrightarrow{1,q^3} * \xrightarrow{3,1} m_5$. The 2-expansion of $\chi$ with respect to $m_+$ gives $\mu = m_+(1 + A^{-1}_{2,q^2})$. So, $m_6$ occurs in $\chi$. Then, the 1-expansion of $\chi$ with respect to $m_6$ gives $\mu = m_6(1 + A^{-1}_{1,q^5} + A^{-1}_{1,q^3}A^{-1}_{1,q^5})$. So, it does not happen.

(iv) $m_+ \xrightarrow{2,q^2} m_6 \xrightarrow{3,1} m_2 \xrightarrow{1,q^3} m_5$. The 3-expansion of $\chi$ with respect to $m_6$ gives $\mu = m_6(1 + A^{-1}_{3,q^4} + A^{-1}_{3,1}A^{-1}_{3,q^4})$. So, it does not happen.

(v) $m_+ \xrightarrow{3,1} m_1 \xrightarrow{1,q^3} * \xrightarrow{2,q^2} m_5$. The 1-expansion of $\chi$ with respect to $m_1$ gives $\mu = m_1(1 + A^{-1}_{1,q^5})$. So, it does not happen.

(vi) $m_+ \xrightarrow{3,1} m_1 \xrightarrow{2,q^2} m_2 \xrightarrow{1,q^3} m_5$. The 1-expansion of $\chi$ with respect to $m_2$ gives $\mu = m_2(1 + A^{-1}_{1,q^5} + A^{-1}_{1,q^3}A^{-1}_{1,q^5})$. So, it does not happen.

Therefore, $m_5$ does not occur in $\chi$ at any step. Q.E.D.

This completes the proof of Theorem 4.1.

Shortly speaking, the algorithm fails because it fails to generate $m_5$ which is an extra *dominant* monomial in $\chi_q(V(m_+))$.

It is not difficult to find some other examples where similar phenomena happen. For example, it is a good exercise to check that, if $\mathfrak{g}$ is of type $D_4$ and the representation has the highest weight monomial

$$m\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}\right) = Y_{1,q^2}Y_{3,q^{-2}}Y_{4,q^{-2}}, \tag{4.12}$$

the FM algorithm fails at the monomial

$$m\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{\overline{1}} \\ \cline{1-1}\end{array}\right) = Y_{1,1}Y_{2,q}^{-1}Y_{3,1}Y_{4,1}, \tag{4.13}$$

where we use the diagrammatic notation in [FR, NN1, NN2].

We conclude with a remark on a modification of the FM algorithm.

**Remark 4.4.** Actually, in the counterexample above, the FM algorithm *almost* works except for missing one monomial $m_5$. It suggests the following *modification* of the algorithm: when we encounter the non-admissible monomial $m_4$ in the algorithm, one simply adds $m_5$ (the '2-ancestor' of $m_4$) to $\chi$ with coloring $(0,0,0)$, then restart the expansions from $\lambda = 2\omega_2$. Then, we have checked by computer that the modified algorithm stops and certainly generates monomials represented by 896 tableaux as expected in [NN1, NN3]. For general representations, this *trace-back* procedure is, *a priori*, not well-defined, because one cannot uniquely determine the '$i$-ancestor' of a given monomial. However, for the family of the skew diagram representations of type $C_n$ in [NN1, NN3], one can do so *with help of tableaux representation (or, more conveniently, paths representation)* of monomials. Observe Figure 4 as a simple example. By modifying the FM algorithm with the trace-back procedure, we expect that Conjecture 1.1 is true for these representations, and it is supported by our computer experiment. The detail will be published elsewhere [NN4].

## References

[BR] V. V. Bazhanov and N. Reshetikhin, Restricted solid-on-solid models connected with simply laced algebras and conformal field theory, J. Phys. A: Math. Gen., **23** (1990), 1477–1492.

[CM] V. Chari and A. Moura, Characters of fundamental representations of quantum affine algebras, Acta Appl. Math., **90** (2006), 43–63.

[CP1] V. Chari and A. Pressley, Quantum affine algebras, Comm. Math. Phys., **142** (1991), 261–283.

[CP2] V. Chari and A. Pressley, Quantum affine algebras and their representations, In: Proc. of Representations of groups, Banff, 1994, CMS Conf. Proc., **16**, 1995, pp. 59–78.

[D1] V. Drinfel'd, Hopf algebras and the quantum Yang–Baxter equation, Soviet. Math. Dokl., **32** (1985), 254–258.

[D2] V. Drinfel'd, A new realization of Yangians and quantized affine algebras, Soviet. Math. Dokl., **36** (1988), 212–216.

[FM1] E. Frenkel and E. Mukhin, Combinatorics of $q$-characters of finite-dimensional representations of quantum affine algebras, Comm. Math. Phys., **216** (2001), 23–57.

[FM2] E. Frenkel and E. Mukhin, The Hopf algebra Rep $U_q(\hat{\mathfrak{gl}}_\infty)$, Selecta Math. (N.S.), **8** (2002), 537–635.

[FR] E. Frenkel and N. Reshetikhin, The $q$-characters of representations of quantum affine algebras and deformations of $W$-algebras, Contemp. Math., **248** (1999), 163–205.

[H1] D. Hernandez, Algebraic approach to $q,t$-characters, Adv. Math., **187** (2004), 1–52.

[H2] D. Hernandez, Representations of quantum affinizations and fusion product, Transformation Groups, **10** (2005), 163–200.

[H3] D. Hernandez, The Kirillov–Reshetikhin conjecture and solutions of T-systems, J. Reine Angew. Math., **596** (2006), 63–87.

[H4] D. Hernandez, Drinfeld coproduct, quantum fusion tensor category and applications, Proc. London Math. Soc., **95** (2007), 567–608.

[H5] D. Hernandez, On minimal affinizations of representations of quantum groups, Comm. Math. Phys., **277** (2007), 221–259.

[H6] D. Hernandez, Kirillov–Reshetikhin conjecture: the general case, arXiv:0704.2838.

[H7] D. Hernandez, private communication.

[J] M. Jimbo, A $q$-difference analogue of $U(\hat{\mathfrak{g}})$ and the Yang–Baxter equation, Lett. Math. Phys., **10** (1985), 63–69.

[KR] A. N. Kirillov and N. Reshetikhin, Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor' product of representations of simple Lie algebras, J. Sov. Math., **52** (1990), 3156–3164.

[KNS] A. Kuniba, T. Nakanishi and J. Suzuki, Functional relations in solvable lattice models: I. Functional relations and representation theory, Internat. J. Modern Phys. A, **9** (1994), 5215–5266.

[KNH] A. Kuniba, S. Nakamura and R. Hirota, Pfaffian and determinant solutions to a discretized Toda equation for $B_r$, $C_r$, and $D_r$, J. Phys. A: Math. Gen., **29** (1996), 1759–1766.

[KOS] A. Kuniba, Y. Ohta and J. Suzuki, Quantum Jacobi–Trudi and Giambelli formulae for $U_q(B_r^{(1)})$ from the analytic Bethe ansatz, J. Phys. A: Math. Gen., **28** (1995), 6211–6226.

[KOSY] A. Kuniba, M. Okado, J. Suzuki and Y. Yamada, Difference $L$ operators related to $q$-characters, J. Phys. A: Math. Gen., **35** (2002), 1415–1435.

[KS] A. Kuniba and J. Suzuki, Analytic Bethe ansatz for fundamental representations of Yangians, Comm. Math. Phys., **173** (1995), 225–264.

[N1] H. Nakajima, $t$-analogue of the $q$-characters of finite dimensional representations of quantum affine algebras, In: Physics and Combinatorics, Proc. of Nagoya 2000 International Workshop, Nagoya, 2000, World Scientific, 2001, pp. 181–212.

[N2] H. Nakajima, Quiver varieties and $t$-analogues of $q$-characters of quantum affine algebras, Ann. of Math. (2), **160** (2004), 1057–1097.

[N3] H. Nakajima, $t$-analogs of $q$-characters of Kirillov–Reshetikhin modules of quantum affine algebras, Represent. Theory, **7** (2003), 259–274.

[NN1] W. Nakai and T. Nakanishi, Paths, tableaux and $q$-characters of quantum affine algebras: the $C_n$ case, J. Phys. A: Math. Phys., **39** (2006), 2083–2115.

[NN2] W. Nakai and T. Nakanishi, Paths and tableaux descriptions of Jacobi–Trudi determinant associated with quantum affine algebra of type $C_n$, J. Alg. Combin., **26** (2007), 253–290.

[NN3] W. Nakai and T. Nakanishi, Paths and tableaux descriptions of Jacobi–Trudi determinant associated with quantum affine algebra of type $D_n$, SIGMA, **3** (2007) 078, 20 pages.

[NN4] W. Nakai and T. Nakanishi, in preparation.

[NT] M. Nazarov and V. Tarasov, Representations of Yangians with Gelfand–Zetlin basis, J. Reine Angew. Math., **496** (1998), 181–212.

[R1] N. Reshetikhin, A method of functional equations in the theory of exactly solvable quantum systems, Lett. Math. Phys., **7** (1983), 205–213.

[R2] N. Reshetikhin, Integrable models of quantum one-dimensional magnets with $O(n)$ and $Sp(2k)$ symmetry, Theor. Math. Phys., **63** (1985), 559–569.

[R3] N. Reshetikhin, The spectrum of the transfer matrices connected with Kac–Moody algebras, Lett. Math. Phys., **14** (1987), 235–246.

[TF] L. Takhatajan and L. Faddeev, The quantum inverse problem method and the Heisenberg $XYZ$ model, Russ. Math. Survey, **34** (1979), 11–68.

[TK] Z. Tsuboi and A. Kuniba, Solutions of a discretized Toda field equation for $D_r$ from analytic Bethe ansatz, J. Phys. A: Math. Phys., **29** (1996), 7785–7796.

Wakako Nakai
*Kaiyo Academy*
*Kaiyo-cho, Gamagori, 443-8588*
*Japan*

Tomoki Nakanishi
*Graduate School of Mathematics, Nagoya University*
*Chikusa-ku, Nagoya, 464-8604*
*Japan*
*E-mail address*: nakanisi@math.nagoya-u.ac.jp