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Modular forms and elliptic genera for ALE spaces

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Abstract.

When we describe string propagation on non-compact or singular Calabi–Yau manifolds by CFT, continuous as well as discrete representations appear in the theory. These representations mix in an intricate way under the modular transformations. In this article, we propose a method of combining discrete and continuous representations so that the resulting combinations have a simpler modular behavior and can be used as conformal blocks of the theory. We compute elliptic genera of ALE spaces and obtain results which agree with those suggested from the decompactification of K3 surface. Consistency of our approach is assured by some remarkable identity of theta functions.

We include in the appendix some new materials on the representation theory of $\mathcal{N} = 4$ superconformal algebra.

§1. Introduction

Description of strings propagating on non-compact curved background is a challenging problem in particular when the space-time develops a singularity. A better grasp of underlying conformal field theory (CFT) should shed light on the physics of such space-time.

When a Calabi–Yau (CY) manifold is non-compact or singular, it is necessary to introduce a CFT possessing continuous as well as discrete representations in order to describe its geometry. These CFT's have a central charge above the "threshold", i.e. $c = 3$ for $\mathcal{N} = 2$ supersymmetric case, and are of non-minimal type. We may call these theories generically as Liouville type theories. Since continuous and discrete representations mix under modular transformations, representations of Liouville theories in general do not have good modular properties. Thus

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it is a non-trivial problem to construct suitable modular invariants describing the geometry of non-compact CY.

In this paper we present an attempt at constructing (holomorphic) modular invariants for some non-compact CY manifolds. In particular we propose the elliptic genera for the ALE spaces which are the degenerate limits of K3 surface. It turns out that the consistency of our approach hinges on the validity of some theta-function identities. These non-trivial identities have been proved by D. Zagier and the proof is given in Section 2.5.

This paper is a contribution to the Proceedings of the workshop in honor of prof. A. Tsuchiya's retirement from Nagoya University on March 2007. It is an expanded version of ref [1] and contains some new materials on the representation theory of superconformal algebras, in particular on higher level $\mathcal{N} = 4$ character formulas.

1.1. Bosonic Liouville theory

We start our discussions by reviewing the simple case of bosonic Liouville theory. Its stress tensor is given by

$$T(z) = -\frac{1}{2}(\partial\phi)^2 + \frac{\mathcal{Q}}{2}\partial^2\phi \tag{1.1}$$

where $\mathcal{Q}$ is the background charge. Central charge is given by

$$c = 1 + 3\mathcal{Q}^2. \tag{1.2}$$

If we parameterize $\mathcal{Q}$ as $\mathcal{Q} = \sqrt{2}(b + 1/b)$, the vertex operator

$$\exp(\sqrt{2}b\phi) \tag{1.3}$$

has a conformal dimension $h = 1$. Liouville theory is defined as a theory perturbed by this marginal operator (Liouville potential) from free fields.

Dynamics of boundary Liouville theory became clarified in late 1990's by the method of conformal bootstrap [2]. We first reintroduce the result of conformal bootstrap using representation theory and the modular properties of character formulas.

It is known that there are two types of representations in bosonic Liouville theory: continuous and identity representation. Their character formulas and their S-transformation are given by ($q \equiv e^{2\pi i\tau}$)

continuous representations; $p > 0$

$$\chi_p(\tau) = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}(1-q^n)} = \frac{q^{\frac{p^2}{2}}}{\eta(\tau)}, \qquad h = \frac{p^2}{2} + \frac{\mathcal{Q}^2}{8}$$

$$\chi_p(-\frac{1}{\tau}) = 2\int_0^\infty dp' \cos(2\pi pp')\chi_{p'}(\tau), \tag{1.4}$$

identity representation; $h = 0$

$$\chi_{h=0}(\tau) = \frac{q^{-\frac{\mathcal{Q}^2}{8}}(1-q)}{\eta(\tau)},$$

$$\chi_{h=0}(-\frac{1}{\tau}) = 4\int_0^\infty dp \sinh(2\pi bp)\sinh(\frac{2\pi p}{b})\chi_p(\tau). \tag{1.5}$$

We identify the LHS of the above equations as describing the open string channel and RHS as the closed string channel. We then find that open and closed channels have different spectra:

open	closed
continuous rep.	continuous rep.
identity rep.	

Namely, there exist no identity representation in the closed string channel. This is consistent with the presence of mass gap and the decoupling of gravity in non-compact space-time. Indeed the conformal dimension of a vertex operator $e^{\alpha\phi}$ is given by

$$\begin{aligned} h(e^{\alpha\phi}) &= -\frac{\alpha^2}{2} + \frac{\alpha\mathcal{Q}}{2} = -\frac{(\alpha - \frac{1}{2}Q)^2}{2} + \frac{\mathcal{Q}^2}{8} \\ &= \frac{p^2}{2} + \frac{\mathcal{Q}^2}{8} \geq \frac{\mathcal{Q}^2}{8} \quad \text{for} \quad \alpha = ip + \frac{1}{2}\mathcal{Q} \end{aligned} \tag{1.6}$$

for continuous representations. Thus there is a gap of $\mathcal{Q}^2/8$ in the spectrum of continuous representations.

Let us next turn to the brane-interpretation of transformations (1.4), (1.5). We introduce ZZ and FZZT brane boundary states $|ZZ\rangle$, $|FZZT\rangle$ and identify the character functions as the inner product

$$\chi_0(-\frac{1}{\tau}) = \langle ZZ|e^{i\pi\tau H^{(c)}}|ZZ\rangle \tag{1.7}$$

$$\chi_p(-\frac{1}{\tau}) = \langle FZZT; p\,|e^{i\pi\tau H^{(c)}}|ZZ\rangle \tag{1.8}$$

where $H^{(c)} = L_0 + \bar{L}_0 - \frac{c}{12}$ is the closed string Hamiltonian. Using Ishibashi states $|p\rangle\rangle$ with momentum p which diagonalize the closed string Hamiltonian

$$\langle\langle p\,|e^{i\pi\tau H^{(c)}}|p'\rangle\rangle = \delta(p-p')\chi_p(\tau) \tag{1.9}$$

boundary states are expanded as

$$|ZZ\rangle = \int_0^\infty dp\,\Psi_0(p)\,|p\rangle\rangle \tag{1.10}$$

$$|FZZT;p\rangle = \int_0^\infty dp'\,\Psi_p(p')\,|p'\rangle\rangle. \tag{1.11}$$

We then have

$$|\Psi_0(p)|^2 = 4\sinh\sqrt{2}\pi pb\sinh\frac{\sqrt{2}\pi p}{b}, \tag{1.12}$$

$$\Psi_p(p')^*\Psi_0(p') = 2\cos 2\pi pp'. \tag{1.13}$$

Solving these relations one finds the boundary wave-functions

$$\Psi_0(p) = \frac{2\sqrt{2}\pi ip}{\Gamma(1+i\sqrt{2}pb)\Gamma(1+\frac{i\sqrt{2}p}{b})}, \tag{1.14}$$

$$\Psi_p(p') = \frac{-1}{\sqrt{2}\pi ip'}\Gamma(1-\sqrt{2}ibp')\Gamma(1-\frac{\sqrt{2}ip'}{b})\cos(2\pi pp'). \tag{1.15}$$

Up to phase factors the above results agree with those of conformal bootstrap [2].

1.2. $\mathcal{N}=2$ Liouville theory

For the sake of applications to string theory let us now consider $\mathcal{N}=2$ supersymmetric version of Liouville theory. In $\mathcal{N}=2$ system possesses two bosons, one of them coupled to background charge and the other one is a compact boson, and two free fermions. It is known that $\mathcal{N}=2$ Liouville theory is T-dual to $SL(2;\mathbb{R})/U(1)$ supercoset model which describes the space-time of the two-dimensional black hole [3]. In general $\mathcal{N}=2$ Liouville theory is geometrically interpreted as describing the radial direction of a complex cone.

In the following we concentrate on the case when $\mathcal{N} = 2$ Liouville theory has a central charge

$$\hat{c} = \frac{c}{3} = 1 + \frac{2}{N}, \quad \mathcal{Q} = \sqrt{2/N},$$

for the sake of simplicity. We denote this case as the model L_N. Here N is an arbitrary positive integer. This theory is T-dual to two-dimensional black hole with an asymptotic radius of the cigar $\sqrt{2N}$.

Unitary representations of $\mathcal{N} = 2$ superconformal algebra with $\hat{c} = 1 + \frac{2}{N}$ are given by [4]

$$\begin{cases} \text{identity rep.} & h = 0, j = 0 & \text{vacuum,} \\ \text{continuous reps.} & p > 0, j = \frac{1}{2} + i\frac{p}{\mathcal{Q}} & \text{non-BPS states,} \\ \text{discrete reps.} & 1 \leq s \leq N, j = \frac{s}{2} & \text{BPS states, chiral primaries.} \end{cases}$$

Here p and s label continuous and discrete representations of $\mathcal{N} = 2$ Liouville theory, respectively. $\mathcal{N} = 2$ representations are in one to one correspondence with those of level $k = N$ $SL(2;\mathbb{R})/U(1)$ coset theory with the value of spin j indicated as above.

In applications to string theory we consider the sum over spectral flows of each $\mathcal{N} = 2$ representation and define an extended character [5][1]

$$\chi_*^{NS}(r;\tau,z) = \sum_{n \in r+N\mathbb{Z}} q^{\frac{\hat{c}}{2}n^2} e^{2\pi i \hat{c} z n} ch_*^{NS}(\tau; z + n\tau). \tag{1.16}$$

Here $ch_*^{NS}(\tau;z)$ denotes an irreducible character of $\mathcal{N} = 2$ superconformal algebra (in NS sector). Extended characters carry some additional label

(1.17) 1. Identity representations : $\chi_{id}^{NS}(r;\tau,z)$; $\quad r \in \mathbb{Z}_N$,

(1.18) 2. Continuous representations : $\chi_{cont}^{NS}(p,\alpha;\tau,z)$;

(1.19) 3. Discrete representations : $\chi_{dis}^{NS}(s, s+2r;\tau,z)$;
$r \in \mathbb{Z}_N, 1 \leq s \leq N$.

[1]Here the spectral flow is summed over modulo N for the sake of convenience. Idea of extended character has been introduced in [6] where the irreducible characters of $\mathcal{N} = 4$ algebra are identified as extended characters of $\mathcal{N} = 2$ algebra. For related works see [7, 8, 9].

Explicit form of these characters are presented in the Appendix A. We also present the form of modular transformation. Here we recall that the S transform of these functions has the following pattern

$$\text{(1.20)} \qquad \text{(continuous rep)} \xrightarrow{S} \text{(continuous rep)},$$

$$\text{(1.21)} \qquad \text{(identity rep)} \xrightarrow{S} \text{(discrete rep)} + \text{(continuous rep)},$$

$$\text{(1.22)} \qquad \text{(discrete rep)} \xrightarrow{S} \text{(discrete rep)} + \text{(continuous rep)}.$$

Namely, a continuous representation transforms into an integral over continuous representations while an identity and discrete representation transforms into a sum of discrete representation and an integral over continuous representations. Such a pattern was first observed in $\mathcal{N} = 4$ representation theory [6].

(1) As in the bosonic Liouville theory, there appear no identity representations in the RHS of above formulas.
(2) While the identity representation disappears after a first S-transform, it comes back after a 2nd transform: this happens when one deforms the contour of momentum integration for the sake of convergence and picks up a pole in the complex plane corresponding to the identity representation. It is further possible to check that $S^2 = C$ and $(ST)^3 = C$, where C is a charge conjugation matrix which acts as $C : (\tau, z) \to (\tau, -z)$.

As compared with the case of minimal theories where only discrete representations exist which rotate into each other under the S-transform, the above transformation laws (1.20)–(1.22) are much more complex and in particular discrete representations mix with continuous representations. We can check that even under the transformation ST^2S^{-1} one can not eliminate the contribution of continuous representations in the transform of discrete representations (ST^2S^{-1} is a generator of $\Gamma(2)$ which is the subgroup of $SL(2;\mathbb{Z})$ keeping the spin-structure fixed). It seems not possible to eliminate the mixing of continuous representations under any subgroup of the modular group.

We have three types of boundary states of $\mathcal{N} = 2$ Liouville theory corresponding to each representation. The boundary wave functions are again given by the elements of the modular S matrix. We can compare our expressions with known results of $SL(2;\mathbb{R})/U(1)$ theory obtained by semi-classical method using the geometry of 2d black hole and DBI action. It is found [5, 10] that $\mathcal{N} = 2$ Liouville theory reproduces essentially the correct wave functions of D-branes of 2d black hole [11]. Thus

the representation theory seems quite consistent with the semi-classical analysis. However, the character formulas themselves do not have good modular properties and it is a non-trivial problem to construct conformal blocks with good modular behaviors.

§2. Geometry of $\mathcal{N}=2$ Liouville fields

Let us now consider models of the following type: tensor product of $\mathcal{N}=2$ Liouville theory L_N (of $\hat{c}=1+\frac{2}{N}$) and $\mathcal{N}=2$ minimal model M_k with level k [12]

$$L_N \otimes M_k. \tag{2.1}$$

If we choose

$$N = k+2 \tag{2.2}$$

the central charge becomes integral

$$c_L + c_M = 3(1+\frac{2}{N}) + 3(1-\frac{2}{k+2}) = 6 \tag{2.3}$$

and the theory (after Z_N orbifolding) describes (complex) 2 dimensional CY manifolds. They are identified as the (A-type) ALE spaces which are obtained by blowing up A_{N-1} singularities [12]. At $N=1$ (without minimal model), we have $\hat{c}=3$ and the space-time of a conifold [13]. We may as well consider the tensor products of Liouville theories and minimal models. These describe other singular geometries like A_{N-1} spaces fibered on P^1 etc. [14, 15, 16]

2.1. Elliptic genus and CY/LG correspondence

The elliptic genus is defined by taking the sum over all states in the left-moving sector of the theory while the right-moving sector is fixed at the Ramond ground states;

$$Z(\tau,z) = Tr_{R\otimes R}(-1)^{F_L+F_R} e^{2\pi i z J_o^L} q^{L_0-\frac{\hat{c}}{8}} \bar{q}^{\bar{L}_0-\frac{\hat{c}}{8}}. \tag{2.4}$$

Here J_0^L denotes the $U(1)_R$ charge in the left-moving sector. The trace is taken in the Ramond–Ramond sector. At specific values of z we have

$Z(\tau,z=0)=\chi$,	Euler number,
$Z(\tau,z=1/2)=\sigma+\mathcal{O}(q)$,	Hirzebruch signature,
$Z(\tau,z=(\tau+1)/2)=\hat{A}q^{-1/4}+\mathcal{O}(q^{1/4})$,	$\hat{A}$ genus.

The elliptic genus is an invariant under smooth variations of the parameters of the theory and is useful, for instance, in counting the number of BPS states. We compute the elliptic genus of a non-compact CY manifolds by pairing the Liouville theory with $\mathcal{N}=2$ minimal models.

Before going into the computation of elliptic genera we first recall the results of CY/LG correspondence [17]. We consider a Landau–Ginzburg (LG) theory with a superpotential

$$W = g\,(X^{k+2} + Y^2 + Z^2) \tag{2.5}$$

which in the infra-red limit acquires scale invariance and reproduces the $\mathcal{N}=2$ minimal theory with $\hat{c} = 1 - 2/k$.

In the $\mathcal{N}=2$ minimal theory M_{N-2}, the contribution to elliptic genus comes from the Ramond ground states

$$Z_{\text{minimal}}(\tau, z) = \sum_{\ell=0}^{N-2} ch^{\tilde{R}}_{\ell,\ell+1}(\tau; z). \tag{2.6}$$

Here $ch^{\tilde{R}}_{\ell,\ell+1}(\tau; z)$ denotes the character of minimal model M_k associated to the Ramond ground state labeled by $\ell = 0, 1, \ldots, N-2$. See e.g. [18, 19] for their explicit expressions. $\tilde{R}$ denotes the Ramond sector with $(-1)^F$ insertion. On the other hand as the coupling parameter is turned off $g \to 0$, LG theory becomes a free theory of chiral field X with $U(1)_R$ charge $= 1/N$. Thus the theory possesses a free boson of charge $1/N$ and free fermion of charge $1/N - 1$. Combining these contributions one obtains [20]

$$Z_{LG}(\tau, z) = \frac{\theta_1(\tau, (1 - \frac{1}{N})z)}{\theta_1(\tau, \frac{1}{N}z)}. \tag{2.7}$$

These two expressions (2.6),(2.7) in fact agree with each other

$$Z_{\text{minimal}} = Z_{LG}. \tag{2.8}$$

We would like to try a similar construction in Liouville sector as in the case of minimal models. Ramond ground states corresponds to the extended discrete characters;

$$\chi^{\tilde{R}}_{dis}(s, s-1; \tau, z), \qquad s = 1, \cdots, N \tag{2.9}$$

and the elliptic genus is expressed as their sum, which is explicitly evaluated in [21] as follows; [2]

$$(2.10)\quad Z_{\text{Liouville}} = -\sum_{s=1}^{N} \chi^{\tilde{R}}_{dis}(s, s-1; \tau, z) = -\mathcal{K}_{2N}(\tau, \frac{z}{N}) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3}.$$

Here we have introduced the notation of an Appell function $\mathcal{K}_k$ [22, 23]

$$(2.11)\qquad \mathcal{K}_k(\tau, z) \equiv \sum_{n\in\mathbb{Z}} \frac{q^{\frac{k}{2}n^2} y^{kn}}{1 - yq^n}, \qquad y = e^{2\pi i z}.$$

We also use the anti-symmetrized version of Appell function defined as

$$(2.12)\quad \widehat{\mathcal{K}}_k(\tau, z) \equiv \frac{1}{2}\left(\mathcal{K}_k(\tau, z) - \mathcal{K}_k(\tau, -z)\right) \equiv \mathcal{K}_k(\tau, z) - \frac{1}{2}\Theta_{0,\frac{k}{2}}(\tau, 2z).$$

Unlike the theta functions of the minimal models, the Appell function in Liouville theory does not have a good modular transformation law [22]. Complication comes from the non-trivial denominator of the function (2.11) which arises due to existence of fermionic singular vectors in BPS (short) representations.

The Appell function is closely related to the function used by Miki in [8]: they are transformed to each other by spectral flow. The Appell function corresponds to an expression in $\tilde{R}$ sector while Miki's function is in NS sector.

When we couple minimal and Liouville theory to compute elliptic genera of A_{N-1} spaces, we may use the orbifoldization procedure [24]

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[2]Precisely speaking, in [21] we adopt a slightly different convention for the 'boundary contribution ($s = 1, N+1$)' of discrete representations, which yields the anti-symmetrized Appell function (2.12)

$$Z_{\text{Liouville}} = -\widehat{\mathcal{K}}_{2N}(\tau, z) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3}$$

rather than (2.10). However, the difference drops off in the orbifold procedure (2.13) [21]. Namely, one may replace $\mathcal{K}_{2N}(\tau, z)$ with $\widehat{\mathcal{K}}_{2N}(\tau, z)$ in (2.13).

and we find [21]

$$
\begin{aligned}
(2.13)\quad & Z_{ALE(A_{N-1})}(\tau,z) \\
&= \frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{a^2} e^{4\pi i a z} Z_{\text{minimal}}(\tau, z+a\tau+b) Z_{\text{Liouville}}(\tau, z+a\tau+b) \\
&= -\frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{a^2}{2}} e^{2\pi i a z}(-1)^{a+b} \frac{\theta_1(\tau, \frac{N-1}{N}(z+a\tau+b))}{\theta_1(\tau, \frac{1}{N}(z+a\tau+b))} \\
&\qquad\qquad \times \mathcal{K}_{2N}(\tau, \frac{1}{N}(z+a\tau+b)) \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}.
\end{aligned}
$$

In the special case of $N=2$ we have ($y \equiv e^{2\pi i z}$)

$$
(2.14)\qquad Z_{ALE(A_1)}(\tau,z) = -\sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} y^{n+\frac{1}{2}}}{1-yq^n} \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}
$$

$$
\left(\equiv ch_0^{\tilde{R}}(I=0;\tau,z) \right).
$$

This formula coincides with a massless character of $\mathcal{N}=4$ algebra [6]. Unfortunately these formulas do not have well-behaved modular properties and we must make a suitable modification.

The elliptic genus is associated with a conformal field theory defined on the torus and hence it must be invariant under $SL(2;\mathbb{Z})$ or under one of its subgroups. Since we are dealing with superconformal field theory, it seems natural to demand invariance under the subgroup $\Gamma(2)$ which leave fixed the spin structures

$$
\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z}),\ a=d=1, b=c=0 \mod 2 \right\}.
$$

It is known that $\Gamma(2)$ is generated by T^2 and ST^2S^{-1}. In the following we construct elliptic genera which are invariant under $\Gamma(2)$.

§3. Elliptic genera and a theta-function identity

3.1. Elliptic genus of K3

A hint for our construction comes from the study of elliptic genus of K3 surface (we denote $\theta_i(\tau) \equiv \theta_i(\tau,0)$)

$$
(3.1)\qquad Z_{K3}(\tau,z) = 8\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2 + \left(\frac{\theta_2(\tau,z)}{\theta_2(\tau)}\right)^2 \right].
$$

This formula can be easily derived by orbifold calculation on $T^4/\mathbb{Z}_2$ [25] or by using LG theory and LG/CY correspondence. One can check $Z_{K3}(z=0)=24$, $Z_{K3}(z=1/2)=16+\cdots$, $Z_{K3}(z=(\tau+1)/2)=-2q^{-1/4}+\ldots$ and Z_{K3} reproduces classical topological invariants, $\chi=24$, $\sigma=16$ and $\hat{A}=-2$. For the later convenience, we also introduce the 'elliptic genus in the NS sector', which is defined by the 1/2-spectral flow $z\mapsto z'\equiv z-\left(\frac{\tau}{2}+\frac{1}{2}\right)$ in the standard way;

$$(3.2)\qquad Z_{K3}^{(NS)}(\tau,z)\equiv q^{\frac{1}{4}}y^{-1}Z_{K3}(\tau,z') = 8\left[-\left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^2+\left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^2-\left(\frac{\theta_4(\tau,z)}{\theta_2(\tau)}\right)^2\right].$$

In the case of K3 surface the manifold has a hyperKäler structure and the CFT possesses an $\mathcal{N}=4$ symmetry. Thus one can use the representation theory of $\mathcal{N}=4$ superconformal algebra [6].

At $\hat{c}=2$ $\mathcal{N}=4$ theory contains an $SU(2)$ current algebra at level 1. Unitary representations of $\mathcal{N}=4$ algebra in the NS sector are given by

$$(3.3)\qquad \text{massive rep.}: ch^{NS}(h,I=0;\tau,z)=q^{h-\frac{1}{8}}\frac{\theta_3(\tau,z)^2}{\eta(\tau)^3},$$

$$(3.4)\qquad \text{massless rep.}: ch_0^{NS}(I=0;\tau,z),\quad ch_0^{NS}(I=1/2;\tau,z).$$

Massive representations exist only for isospin $I=0$ and are analogous to continuous representations of $\mathcal{N}=2$. The $I=0$ and $I=1/2$ massless representations are analogues of identity and discrete representations. There exists a relation among them

$$(3.5)\qquad ch_0^{NS}(I=0)+2ch_0^{NS}(I=1/2)=ch^{NS}(h=0,I=0)$$

which shows that the (non-BPS) massive representation becomes reducible as $h\to 0$ and splits into a sum of massless (BPS) representations.

There are various ways of writing the massless characters , however, particularly convenient expressions for our discussion are given by [25]

$$(3.6)\qquad ch_0^{NS}(I=1/2,\tau,z)=-\left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^2+h_3(\tau)\left(\frac{\theta_3(\tau,z)}{\eta(\tau)}\right)^2,$$

$$(3.7)\qquad =\left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^2+h_4(\tau)\left(\frac{\theta_3(\tau,z)}{\eta(\tau)}\right)^2,$$

$$(3.8)\qquad =-\left(\frac{\theta_4(\tau,z)}{\theta_2(\tau)}\right)^2+h_2(\tau)\left(\frac{\theta_3(\tau,z)}{\eta(\tau)}\right)^2,$$

where the functions $h_i(\tau)$, $i = 2, 3, 4$ are defined by

$$h_3(\tau) = \frac{1}{\eta(\tau)\theta_3(\tau)} \sum_{m\in\mathbb{Z}} \frac{q^{m^2/2-1/8}}{1+q^{m-1/2}}, \tag{3.9}$$

$$h_4(\tau) = \frac{1}{\eta(\tau)\theta_4(\tau)} \sum_{m\in\mathbb{Z}} \frac{q^{m^2/2-1/8}(-1)^m}{1-q^{m-1/2}}, \tag{3.10}$$

$$h_2(\tau) = \frac{1}{\eta(\tau)\theta_2(\tau)} \sum_{m\in\mathbb{Z}} \frac{q^{m^2/2+m/2}}{1+q^m}. \tag{3.11}$$

We note that h_i's obey identities [26]

$$h_3(\tau) - h_4(\tau) = \frac{1}{4}\left(\frac{\theta_2(\tau)}{\eta(\tau)}\right)^4, \quad h_2(\tau) - h_3(\tau) = \frac{1}{4}\left(\frac{\theta_4(\tau)}{\eta(\tau)}\right)^4,$$

$$h_2(\tau) - h_4(\tau) = \frac{1}{4}\left(\frac{\theta_3(\tau)}{\eta(\tau)}\right)^4. \tag{3.12}$$

Now using (3.6–3.8) we can rewrite K3 elliptic genus $Z_{K3}^{(NS)}(\tau, z)$ (3.2) as

$$Z_{K3}^{(NS)}(\tau, z) = 24ch_0^{NS}(I = 1/2; z) - 8\sum_{i=2,3,4} h_i(\tau)\frac{\theta_3(\tau, z)^2}{\eta(\tau)^2}. \tag{3.13}$$

If one considers the product of $\eta(\tau)$ times the sum of $h_i(\tau)$ functions

$$8\,\eta(\tau)\sum_{i=2,3,4} h_i(\tau) = q^{-1/8}\left[2 - \sum_{n=1}^{\infty} a_n q^n\right] \tag{3.14}$$

one finds that the coefficients a_n of q-expansion are positive integers. Then using the relation (3.5) we can rewrite $Z_{K3}^{(NS)}$ into a sum of irreducible characters

$$Z_{K3}^{(NS)}(\tau, z) = 20ch_0^{NS}(I = 1/2; \tau, z) - 2ch_0^{NS}(I = 0; \tau, z) + \sum_{n=1}^{\infty} a_n ch^{NS}(h = n; \tau, z). \tag{3.15}$$

Under the spectral flow from NS to R sector the $I = 0$ and $1/2$ representations turn into the $I = 1/2$ and 0 representations, respectively. Thus the coefficient -2 in front of $ch_0^{NS}(I = 0)$ in the above formula comes from the multiplicity of the ground states of Ramond $I = 1/2$

representation in the right-moving sector. Therefore the net multiplicity of $I = 0$ massless representation is 1. Hence in the NS sector the theory contains

$$\begin{array}{ll} 1 & I = 0 \quad \text{rep.} \\ 20 & I = 1/2 \quad \text{reps.} \\ \infty & \text{of massive reps. } (h = 1, 2, \cdots) \end{array}$$

$I = 0$ NS representation corresponds to the gravity multiplet and $I = 1/2$ NS representation corresponds to matter multiplets (vector in IIA, tensor in IIB). This is the well-known field content in the supergravity description of string theory compactified on K3 [27]. Note that the values of the dimension h of massive representations are quantized at positive integers. This is consistent with the T-invariance of the elliptic genus.

Now let us throw away the gravity multiplet so that we can decompactify K3 into a sum of ALE spaces; it is known that K3 may be decomposed into a sum of 16 A_1 spaces [28]. Decompactification corresponds to dropping $I = 0$ massless representation. $I = 0$ representation comes from $q^{-1/8}$ piece in (3.14) which in turn originates from the $(\theta_2(\tau, z)/\theta_2(\tau))^2$ term in (3.1). This suggests the elliptic genus of the decompactified K_3

$$Z_{K3,\text{decompactified}} = 8\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]. \tag{3.16}$$

3.2. Elliptic genera of ALE spaces

We now propose the following formula for the elliptic genus of the A_1 space

$$Z_{A_1}(\tau,z) = \frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]. \tag{3.17}$$

Note that in the NS sector we have the decomposition

$$
\begin{aligned}
(3.18)\qquad & Z_{A_1}^{(NS)}(\tau,z)\ \left(\equiv q^{\frac{1}{4}}y^{-1}Z_{A_1}(\tau,z-(\frac{\tau}{2}+\frac{1}{2}))\right) \\
&= \frac{1}{2}\left[\left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^2-\left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^2\right] \\
&= ch_0^{NS}(I=1/2;\tau,z)-\frac{1}{2}\eta(\tau)\left(h_3(\tau)+h_4(\tau)\right)\frac{\theta_3(\tau,z)^2}{\eta(\tau)^3} \\
&\equiv ch_0^{NS}(I=1/2;\tau,z)+\sum_{n=1}^{\infty}b_n\,ch^{NS}(h=n;\tau,z).
\end{aligned}
$$

Here we have introduced the expansion

$$
(3.19)\qquad \frac{1}{2}\eta(\tau)\sum_{i=3,4}h_i(\tau)=-\sum_{n=1}b_nq^{n-1/8}
$$

and one can check by Maple that the expansion coefficients b_n are positive integers for lower values of n. Actually one can prove that b_n are positive integers for all values of n.[3]

We also propose that elliptic genera of A_{N-1} spaces are simply $(N-1)$ times that of A_1

$$
(3.20)\qquad Z_{A_{N-1}}(\tau,z)=(N-1)\frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right].
$$

Above construction (3.18) of $Z_{A_1}^{(NS)}$ suggests that instead of using the irreducible character $ch_0^{NS}(I=1/2)$ by itself we should use its combination with (an infinity of) massive representations defined by the R.H.S. of (3.18), which has a good modular property and is in fact invariant under $\Gamma(2)$. We call this combination as the $\Gamma(2)$-invariant completion of the massless representation and consider it as a conformal block in non-compact CFT.

3.3. Theta-function identity

It is a non-trivial problem to show that for a given BPS representation of a superconformal algebra, it is possible to define its $\Gamma(2)$-invariant completion uniquely by adding a suitable amount of non-BPS representations. According to our analysis this seems possible when we impose suitable additional conditions: all the massive contributions have their

[3] T. Eguchi and M. Jinzenji, 2008

conformal dimensions above the gap, i.e. $h = n$ with $n = 1, 2, \cdots$ and also occur with multiplicities of a definite sign.

The $\Gamma(2)$-invariant completion is the selection of a topological part of massless representations; this may be easily seen from the formula in the $\tilde{R}$ sector. For instance we consider the decompositions

$$ch_0^{\tilde{R}}(I=0,\tau,z) = \left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + h_3(\tau)\left(\frac{\theta_1(\tau,z)}{\eta(\tau)}\right)^2, \tag{3.21}$$

$$= \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2 + h_4(\tau)\left(\frac{\theta_1(\tau,z)}{\eta(\tau)}\right)^2. \tag{3.22}$$

We see at $z = 0$, the 2nd terms of (3.21), (3.22) vanish while the 1st terms give the Witten index$= 1$. Thus the 1st terms of (3.21) and (3.22) carry the topological information of the massless representation. Our prescription is to identify the $\Gamma(2)$-invariant completion as

$$\left[ch_0^{\tilde{R}}(I=0;\tau,z)\right]_{inv} = \frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]. \tag{3.23}$$

Here we take the (GSO) projection $((\theta_3(\tau,z)/\theta_3(\tau))^2+(\theta_4(\tau,z)/\theta_4(\tau))^2)/2$ since in Ramond sector q-expansion is necesarily integer-powered. We do not adopt $(\theta_2(\tau,z)/\theta_2(\tau))^2$ since in this case associated massive representations start from $h = 0$, i.e. below the threshold.

One of the most interesting examples of our analysis will be the case of the Appell function: It turns out that the desired completion is given by

$$\begin{aligned}&[\mathcal{K}_{2N}(\tau,z)]_{inv}\\ \equiv&\frac{1}{4}\frac{i\eta(\tau)^3\theta_1(\tau,2z)}{\theta_1(\tau,z)^2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2(N-1)} + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2(N-1)}\right].\end{aligned} \tag{3.24}$$

Derivation will be given in Appendix B.

One can then plug the expression (3.24) into the orbifold formula (2.13) and represent the elliptic genera for A_{N-1} spaces as

$$\begin{aligned}&Z_{A_{N-1}}(\tau,z)\\ =&\frac{1}{4N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{a^2}{2}} e^{2\pi i a z}(-1)^{a+b}\frac{\theta_1(\tau,\frac{N-1}{N}z_{a,b})\theta_1(\tau,\frac{2}{N}z_{a,b})\theta_1(\tau,z)}{\theta_1(\tau,\frac{1}{N}z_{a,b})^3}\\ &\times\left[\left(\frac{\theta_3(\tau,\frac{1}{N}z_{a,b})}{\theta_3(\tau)}\right)^{2(N-1)} + \left(\frac{\theta_4(\tau,\frac{1}{N}z_{a,b})}{\theta_4(\tau)}\right)^{2(N-1)}\right]\end{aligned} \tag{3.25}$$

where $z_{a,b} = z + a\tau + b$.

This appears to be a somewhat complicated formula. It turns out that rather strikingly this orbifold summation agrees exactly with our proposed expression for $Z_{A_{N-1}}$

$$\text{(3.26)} \qquad \text{RHS of (3.25)} = \frac{N-1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right].$$

We have proved this identity for $N = 2$ using the addition theorem of theta functions and have checked its validity by Maple for lower values of N. Mathematical proof for all values of N has been given by Zagier [29]. Thus our approach seems altogether consistent: we have arrived at the same expression (3.26) starting either from the decompactification of K3 or the pairing of $\mathcal{N} = 2$ minimal and Liouville theories. We have managed to construct holomorphic modular ($\Gamma(2)$) invariant for a class of non-compact CY manifolds.

Actually the above identity (3.26) is a special case of identities of theta functions

$$\text{(3.27)} \quad \frac{1}{2N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{a^2}{2}} e^{2\pi i a z}(-1)^{a+b} \frac{\theta_1(\tau, \frac{N-1}{N}z_{a,b})\theta_1(\tau, \frac{2}{N}z_{a,b})\theta_1(\tau,z)}{\theta_1(\tau, \frac{1}{N}z_{a,b})^3}\left(\frac{\theta_i(\tau, \frac{1}{N}z_{a,b})}{\theta_i(\tau)}\right)^{2(N-1)}$$

$$= (N-1)\left(\frac{\theta_i(\tau,z)}{\theta_i(\tau)}\right)^2, \qquad i = 2,3,4.$$

We note that the above identities (3.27) for $i = 2, 3, 4$ transform into each other under S and T transformations (more precisely under $SL(2;\mathbb{Z})/\Gamma(2) = S_3$).

A mathematical proof of these identities (3.27) has been found by D. Zagier [29]. We present his elegant proof using residue integrals in the next section.

3.4. Proof of the theta-function identity

Throughout this subsection we fix $\tau \in \mathbb{H}$ (i.e. $\mathrm{Im}\,\tau > 0$) and $N \geq 2$. Also, for convenience we abbreviate $\theta(z) \equiv \theta_1(\tau,z)/\theta_1'(\tau,0)$ and $f_i(z) \equiv \theta_i(\tau,z)/\theta_i(\tau,0)$ ($i = 2, 3, 4$). We have $\theta(z + a\tau + b) = (-1)^{a+b}q^{-a^2/2}y^{-a}\theta(z)$ and similarly for $f_i(z)$, but with $(-1)^{a+b}$ being replaced by $(-1)^b$, 1 or $(-1)^a$ for $i = 2$, 3 or 4, respectively. The identity

(3.27) can therefore be rewritten as

(3.28)
$$\frac{1}{2N}\sum_{w}\frac{\theta((N-1)w)\,\theta(2w)}{\theta(Nw)\,\theta(w)^3}\,f_i(w)^{2N-2} = (N-1)\,\frac{f_i(z)^2}{\theta(z)^2} \quad (i=2,\,3,\,4),$$

where the sum is over $w = (z + a\tau + b)/N$ with $a, b \in \mathbb{Z}_N$, or more invariantly over $w \in E_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ with $Nw = z$.

The proposed identity (3.28) is a special case of the more general ones:

$$\frac{1}{2N}\sum_{Nw=z}\frac{\theta((N-1)w)\,\theta(2w)}{\theta(Nw)}\,\theta(w)^{2a-3}f_2(w)^{2b}f_3(w)^{2c}f_4(w)^{2d}$$

(3.29)
$$= \begin{cases} \dfrac{b\,f_2(z)^2 + c\,f_3(z)^2 + d\,f_4(z)^2}{\theta(z)^2} & \text{if } a=0 \\ 1 & \text{if } a=1 \\ 0 & \text{if } a\geq 2 \end{cases}$$

for any $a, b, c, d \geq 0$ with $a + b + c + d = N - 1$.

If we write $\wp(z)$ for $\wp(z;\tau)$ and observe that $f_i(z)^2/\theta(z)^2 = \wp(z) - e_i$ where $e_2 \equiv \wp(1/2)$, $e_3 \equiv \wp((\tau+1)/2)$ and $e_4 \equiv \wp(\tau/2)$, then we find that this identity (3.29) follows from (and is in fact equivalent to) the following proposition:

Proposition 3.1. *For $N \geq 1$, let F_N be the even elliptic function*

$$F_N(w) = \frac{\theta((N-1)w)\,\theta(2w)\,\theta(w)^{2N-5}}{\theta(Nw)},$$

and $P(X) = c_0X^{N-1} + c_1X^{N-2} + \mathrm{O}(X^{N-3})$ be a polynomial of degree $\leq N-1$. Then

$$\frac{1}{2N}\sum_{Nw=z} F_N(w)\,P(\wp(w)) = (N-1)\,c_0\,\wp(z) + c_1\,. \tag{3.30}$$

Proof. Set $\zeta(z) \equiv \theta'(z)/\theta(z)$. This function satisfies $\zeta(z + a\tau + b) = \zeta(z) - 2\pi i a$ for $a, b \in \mathbb{Z}$. If we write the beginning of the Taylor expansion of $\theta(z)$ at 0 as $\theta(z) = z + Az^3 + \mathrm{O}(z^5)$ with $A = A(\tau)$ (A is a multiple of $E_2(\tau)$), then we have $\zeta(z) = z^{-1} + 2Az + \mathrm{O}(z^3)$ and $\zeta'(z) = -z^{-2} + 2A + \mathrm{O}(z^2) = -\wp(z) + 2A$. Fix $z \in \mathbb{C}$ (with $Nz \neq 0$ in E_τ) and define a function $t(w)$ by

$$t(w) \equiv \frac{1}{2}\,\{\zeta(z+Nw) - \zeta(z-Nw)\} - \zeta((N-1)w) - \zeta(w)\,.$$

From the transformation law of ζ we find that $t(w)$ is elliptic. Therefore, by the residue theorem, we have

$$\sum_{\alpha \in E_\tau} \mathrm{Res}_{w=\alpha}\big(F_N(w)\, P(\wp(w))\, t(w)\, dw\big) \;=\; 0\,,$$

where the sum is over all singularities $\alpha \in \mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$ of $F_N(w)P(\wp(w))t(w)$. These singularities occur only at $Nw = \pm z$ or $w = 0$. (The function $F_N(w)$ has further simple poles at $Nw = 0$, $w \neq 0$, but $t(w)$ vanishes at these points, and the function $t(w)$ has simple poles at $(N-1)w = 0$, $w \neq 0$, but $F_N(w)$ vanishes at these points.) Since the residue of $t(w)$ at a point w with $Nw = \pm z$ is $1/2N$ and $F_N(w)P(\wp(w))$ is even, the identity above becomes

$$\frac{1}{N} \sum_{Nw\,=\,z} F_N(w)\, P(\wp(w)) \;+\; \mathrm{Res}_{w=0}\big(F_N(w)\, P(\wp(w))\, t(w)\, dw\big) \;=\; 0\,.$$

But for $w \to 0$ we have

$$\begin{aligned}
&F_N(w)\\
&= \frac{2(N-1)}{N}\, w^{2N-4}\,\big(1 \,+\, A\,[(N-1)^2 + 2^2 + 2N - 5 - N^2]\, w^2 + \mathrm{O}(w^4)\big)\\
&= \frac{N-1}{N}\, w^{2N-4} \,+\, \mathrm{O}(w^{2N})\,,\\
&P(\wp(w)) = c_0\,\big(\frac{1}{w^2} + \mathrm{O}(w^2)\big)^{N-1} \,+\, c_1\,\big(\frac{1}{w^2} + \mathrm{O}(w^2)\big)^{N-2} \,+\, \mathrm{O}\big(\frac{1}{w^2}\big)^{N-3}\\
&\qquad = \frac{c_0}{w^{2N-2}} \,+\, \frac{c_1}{w^{2N-4}} \,+\, \mathrm{O}\big(\frac{1}{w^{2N-6}}\big)\,,\\
&t(w) = N\zeta'(z)\, w \,-\, \frac{1}{(N-1)w} \,-\, 2A(N-1)w \,-\, \frac{1}{w} \,-\, 2Aw \,+\, \mathrm{O}(w^2)\\
&\qquad = -\,\frac{N}{N-1}\, w^{-1} \,-\, N\wp(z)\, w \,+\, \mathrm{O}(w^2),
\end{aligned}$$

and hence $\mathrm{Res}_{w=0}(F_N(w)\, P(\wp(w))\, t(w)\, dw) = -2(N-1)c_0\wp(z) - 2c_1\,.$

Q.E.D.

§4. Summary

When we consider a string theory on non-compact CY manifolds it is described by a CFT possessing continuous as well as discrete representations. Characters of representations of such CFT transform in a

peculiar manner under S transformation as

$$\text{discrete} \xrightarrow{S} \sum \text{discrete} + \int \text{continuous}$$

$$\text{continuous} \xrightarrow{S} \int \text{continuous}$$

Mathematical nature of such transformation is currently not well understood. We have found an empirical method of constructing conformal blocks which have good modular behavior and obtained elliptic genera of some non-compact CY manifolds. Our method of construction of conformal blocks, however, is still provisional and needs further studies.

From the geometrical point of view it is often difficult to define topological invariants for non-compact manifolds unambiguously and results tend to depend on the choice of boundary conditions. By our proposal (3.17), topological invariants of the A_1 space is predicted to be $\chi = 1$, $\sigma = 1$ and $\hat{A} = 0$, respectively. These are more or less the standard values except that A_1 space is topologically a cotangent bundle over S^2 and the Euler number may be considered as $\chi(S^2)$=2.

In our construction discrete representations describe homology classes of H_2 with compact support while the identity representation corresponds to the classes H_0, H_4. When we decouple gravity, we are left with one compact 2-cycle in the case of A_1 space and obtain χ=1. We may take the point of view that our proposal is to impose good modular properties to fix the ambiguity of boundary conditions. In the case of complex 2-dimensions considered here the requirement of good modular behavior fixes the results uniquely. We may, for instance, consider an alternative expression for the topological part of $\mathcal{K}$ (3.24) where we replace $(\theta_3(\tau,z)/\theta_3(\tau))^{2(N-1)} + (\theta_4(\tau,z)/\theta_4(\tau))^{2(N-1)}$ by some symmetric polynomial in θ_3 and θ_4, such as, $(\theta_3(\tau,z)/\theta_3(\tau))^{2(N-2)}(\theta_4(\tau,z)/\theta_4(\tau))^2 + (\theta_4(\tau,z)/\theta_4(\tau))^{2(N-2)}(\theta_3(\tau,z)/\theta_3(\tau))^2$. Such an ambiguity disappears after orbifold summation (3.25) according to Zagier's formula (3.29) and we obtain the same elliptic genera $Z_{A_{N-1}}$ (3.26).

In the case of complex 4-dimensions, however, some ambiguity seems to remain. In complex 3-dimensions, on the other hand, elliptic genera are known to be given by the product of the Euler number times an universal function [25, 24] and this continues to be the case in non-compact manifolds [21].

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§ Appendix A: $\mathcal{N}=2$ extended characters

We first list the irreducible characters of $\mathcal{N}=2$ theory ($q \equiv e^{2\pi i\tau}$, $y \equiv e^{2\pi iz}$):

(1) continuous representations:

$$ch^{NS}(h,Q;\tau,z) = q^{h-\frac{\hat{c}-1}{8}} y^Q \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \qquad (h > \frac{|Q|}{2}) \tag{A.1}$$

(2) discrete representations:

$$ch^{NS}_{dis}(Q;\tau,z) = q^{\frac{|Q|}{2}-\frac{\hat{c}-1}{8}} y^Q \frac{1}{1+y^{sgn(Q)}q^{1/2}} \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \tag{A.2}$$

(3) identity representation:

$$ch^{NS}_{id}(\tau,z) = q^{-\frac{\hat{c}-1}{8}} \frac{1-q}{(1+yq^{1/2})(1+y^{-1}q^{1/2})} \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \tag{A.3}$$

Here $y = e^{2\pi iz}$ and Q denotes the $U(1)$ charge of $\mathcal{N}=2$ algebra.

Extended characters are given by the sum over spectral flow of irreducible characters (1.16):

(1) continuous representations:

$$\chi^{NS}_{cont}(p,\alpha;\tau,z) = q^{\frac{p^2}{2}} \Theta_{\alpha,N}(\tau,\frac{2z}{N}) \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \quad (h = \frac{p^2}{2} + \frac{4\alpha^2+1}{4N}) \tag{A.4}$$

(2) discrete representations:

$$\begin{aligned} &\chi^{NS}_{dis}(s,s+2r;\tau,z) \\ &= \sum_{m\in\mathbb{Z}} \frac{\left(yq^{N(m+\frac{2r+1}{2N})}\right)^{\frac{s-1}{N}}}{1+yq^{N(m+\frac{2r+1}{2N})}} y^{2(m+\frac{2r+1}{2N})} q^{N(m+\frac{2r+1}{2N})^2} \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \end{aligned} \tag{A.5}$$

(3) identity representations:

$$\chi_{id}^{NS}(r;\tau,z) = q^{-\frac{1}{4N}} \sum_{m\in\mathbb{Z}} q^{N(m+\frac{r}{N})^2+N(m+\frac{2r-1}{2N})} y^{2(m+\frac{r}{N})+1}$$

$$\times \frac{1-q}{\left(1+yq^{N(m+\frac{2r+1}{2N})}\right)\left(1+yq^{N(m+\frac{2r-1}{2N})}\right)} \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \tag{A.6}$$

Here $\Theta_{k,N}(\tau,z)$ is the theta function

$$\Theta_{k,N}(\tau,z) = \sum_{m\in\mathbb{Z}} q^{N(m+\frac{k}{2N})^2} y^{N(m+\frac{k}{2N})}. \tag{A.7}$$

Range of parameters r, s are

$$r \in \mathbb{Z}_N\,, \quad 1 \le s \le N, \quad (s \in \mathbb{Z}). \tag{A.8}$$

If we go to the Ramond sector with $(-1)^F$ insertion, one has

$$\chi_{dis}^{\tilde{R}}(s, s+2r;\tau,z) \tag{A.9}$$
$$= \sum_{m\in\mathbb{Z}} \frac{\left(yq^{N(m+\frac{2r+1}{2N})}\right)^{\frac{s-1}{N}}}{1-yq^{N(m+\frac{2r+1}{2N})}} y^{2(m+\frac{2r+1}{2N})} q^{N(m+\frac{2r+1}{2N})^2} \frac{i\theta_1(\tau,z)}{\eta(\tau)^3},$$

r now takes half-integer values. We find discrete representations $1 \le s \le N$ with $r = -1/2$ carry a non-zero Witten index

$$\chi_{dis}^{\tilde{R}}(s, s-1;\tau,z=0) = -1. \tag{A.10}$$

Now we discuss S-transformation of extended characters. S-transform of continuous representations remains essentially the same as the Fourier transformation

$$\chi_{cont}\left(p,m;-\frac{1}{\tau}\right) \tag{A.11}$$
$$= \sqrt{\frac{2}{N}} \sum_{m'\in Z_{2N}} e^{-2\pi i \frac{mm'}{2N}} \int_0^\infty dp' \, \cos(2\pi pp') \, \chi_{cont}(p',m';\tau).$$

S-transformation of discrete representations is given by

(A.12)

$$\chi^{NS}_{dis}\left(s,m;-\frac{1}{\tau}\right)=\frac{1}{\sqrt{2N}}\sum_{m'\in\mathbb{Z}_{2N}}e^{-2\pi i\frac{mm'}{2N}}$$
$$\times\int_0^\infty dp'\,\frac{\cosh\left(2\pi\frac{N-(s-1)}{\sqrt{2N}}p'\right)+e^{2\pi i\frac{m'}{2}}\cosh\left(2\pi\frac{s-1}{\sqrt{2N}}p'\right)}{2\left|\cosh\pi\left(\sqrt{\frac{N}{2}}p'+i\frac{m'}{2}\right)\right|^2}\chi^{NS}_{cont}(p',m';\tau)$$
$$+\frac{i}{N}\sum_{s'=1}^{N}\sum_{m'\in\mathbb{Z}_{2N}}e^{2\pi i\frac{(s-1)(s'-1)-mm'}{2N}}\chi^{NS}_{dis}(s',m';\tau)$$
$$-\frac{i}{2N}\sum_{m'\in\mathbb{Z}_{2N}}e^{-2\pi i\frac{mm'}{2N}}\chi^{NS}_{cont}(p'=0,m';\tau),$$

where $m=s+2r$. The transformation of the identity representation is

(A.13)

$$\chi_{id}\left(m;-\frac{1}{\tau}\right)=\frac{1}{\sqrt{2N}}\sum_{m'\in Z_{2N}}e^{-2\pi i\frac{mm'}{2N}}$$
$$\times\int_0^\infty dp'\,\frac{\sinh\left(\pi\mathcal{Q}p'\right)\sinh(2\pi\frac{p'}{\mathcal{Q}})}{|\cosh\pi(\frac{p'}{\mathcal{Q}}+i\frac{m'}{2})|^2}\chi_{cont}(p',m';\tau)$$
$$+\frac{2}{N}\sum_{r'\in Z_N}\sum_{s'=2}^{N}\sin\left(\frac{\pi(s'-1)}{N}\right)e^{-2\pi i\frac{m(s'+2r')}{2N}}\chi_{dis}(s',r';\tau).$$

We refer the reader to [5, 21] for more complete discussions.

§ Appendix B: $\Gamma(2)$-invariant completion of Appell function

Let us consider the representation of $\mathcal{N}=4$ theory at general values of central charge $c=6k$ where k is an arbitrary positive integer. This theory possesses an affine $SU(2)$ current of level k which is given by a diagonal sum of level $k-1$ bosonic $SU(2)$ current and level 1 current made of fermion bilinears. When we try to generalize the formula (3.21), (3.22) for a general level, we expect an expansion of the form

(B.1)

$$ch^{\tilde{R}}_0(I=0,\tau,z)=\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2k}+\sum_{j=0}^{k-1}A_{3,j}(\tau)\chi_j^{(k-1)}(\tau,z)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}.$$

where $\chi_j^{(k)}$ denotes the $SU(2)_k$ character for spin $j/2$ representation

(B.2)

$$\chi_j^{(k)}(\tau,z)=\frac{\Theta_{j+1,k+2}(\tau,2z)-\Theta_{-j-1,k+2}(\tau,2z)}{i\theta_1(\tau,2z)}\equiv 2\,\frac{\Theta^{(-)}_{j+1,k+2}(\tau,2z)}{i\theta_1(\tau,2z)}\,.$$

It turns out that expansion coefficients $A_{3,j}$ are given by

$$\text{(B.3)}\quad A_{3,j}(\tau)=2\widehat{H}_{j+1}^{(2(k+1))}(\tau)+a_{j+1}^{(k+1)}(\tau)\qquad (j=0,1,\cdots,k-1),$$

where

$$\text{(B.4)}\qquad \widehat{H}_s^{(k)}(\tau)\equiv\frac{H_s^{(k)}(\tau)}{\Theta_{s,\frac{k}{2}}(\tau,\tau)},\quad H_s^{(k)}(\tau)\equiv\sum_{n\in\mathbb{Z}}\frac{q^{\frac{k}{2}n(n+1)+(n+\frac{1}{2})s}}{1-q^{k(n+\frac{1}{2})}}$$

and $a_{j+1}^{(k+1)}$ is expressed in terms of values of theta functions and $SU(2)$ characters at special points $z=r/2(k+1)$, $(r=1,\cdots,2k+1)$. We shall prove the decomposition formula (B.1) below.

Similarly, one has the expansion

$$\text{(B.5)}\quad ch_0^{\tilde{R}}(I=0,\tau,z)=\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2k}+\sum_{j=0}^{k-1}A_{4,j}(\tau)\chi_j^{(k-1)}(\tau,z)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}.$$

$A_{4,j}(\tau)$ is determined from $A_{3,j}(\tau)$ by taking the T-transformation;

$$\text{(B.6)}\qquad A_{4,j}(\tau)=e^{2\pi i\frac{(j+1)^2}{4(k+1)}}A_{3,j}(\tau+1).$$

We keep the 1st terms of (B.1), (B.5), take the average and obtain the $\Gamma(2)$-invariant completion

$$\text{(B.7)}\qquad \left[ch_0^{\tilde{R}}(I=0,\tau,z)\right]_{inv}=\frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2k}+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2k}\right].$$

If one recall the relation

$$\text{(B.8)}\qquad (-1)^kq^{\frac{k}{4}}y^kch_0^{NS}(I=\frac{k}{2};\tau,z+\frac{(\tau+1)}{2})=ch_0^{\tilde{R}}(k,I=0;\tau,z)$$

one finds

(B.9)

$$\left[ch_0^{NS}(I=\frac{k}{2},\tau,z)\right]_{inv}=\frac{1}{2}\left[(-1)^k\left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^{2k}+\left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^{2k}\right].$$

Taking the half spectral flow $z \mapsto z - \frac{\tau}{2} - \frac{1}{2}$, we can rewrite $\frac{1}{2}$((B.1) +(B.5)) as

$$\text{(B.10)} \quad \frac{1}{2}\left[(-1)^k \left(\frac{\theta_1(\tau,z)}{\theta_3(\tau)}\right)^{2k} + \left(\frac{\theta_2(\tau,z)}{\theta_4(\tau)}\right)^{2k}\right] = ch_0^{NS}(I = k/2;\tau,z) + \sum_{j=0}^{k-1}\sum_{n=0}^{\infty} b_{j,n}\, ch^{NS}(h_j^{(0)} + n, j/2;\tau,z),$$

where the coefficients $b_{j,n}$ are defined by the q-expansion[4]

(B.11)

$$q^{h_j^{(0)} - \frac{j(j+2)}{4(k+1)} - \frac{k^2}{4(k+1)}} \sum_{n=0}^{\infty} b_{j,n} q^n = \frac{(-1)^{k+j+1}}{2}\left(A_{3,k-1-j}(\tau) + A_{4,k-1-j}(\tau)\right),$$

and the $\mathcal{N} = 4$ massive character of conformal weight h, spin $j/2$ is given by

$$\text{(B.12)} \quad ch^{NS}(h, j/2;\tau,z) \equiv q^{\frac{p^2}{2}} \chi_j^{(k-1)}(\tau,z) \frac{\theta_3(\tau,z)^2}{\eta(\tau)^3},$$
$$h \equiv \frac{p^2}{2} + \frac{j(j+2)}{4(k+1)} + \frac{k^2}{4(k+1)}.$$

We again note that $b_{j,n}$ are non-negative integers as in (3.18). We have explicitly checked this for the cases $k = 2, 3, 4$ by Maple.

Let us next study the relation of $\mathcal{N} = 4$ character and Appell function. Explicit form of $\mathcal{N} = 4$ massless character in Ramond sector is given by

(B.13)

$$ch_0^{\tilde{R}}(k, I = 0;\tau,z) = -\frac{i\theta_1(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)} \sum_{n\in\mathbb{Z}} \frac{1+yq^n}{1-yq^n} q^{(k+1)n^2} y^{2(k+1)n}.$$

---

[4] $h_j^{(0)}$ in (B.11) is found to take the following values;

$$h_j^{(0)} = \begin{cases} \frac{k}{2} & (j \equiv k \pmod 2) \\ \frac{k+1}{2} & (j \equiv k-1 \pmod 2) \end{cases}.$$

(This expression is obtained by slightly rewriting the original formula in [6]. See (C.7) in Appendix C.) Thus

$$ch_0^{\tilde{R}}(k, I=0; \tau, z) = -\frac{i\theta_1(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)}\left(\mathcal{K}_{2(k+1)}(\tau,z) - \mathcal{K}_{2(k+1)}(\tau,-z)\right)$$

$$= -2\frac{i\theta_1(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)}\widehat{\mathcal{K}}_{2(k+1)}(\tau,z). \tag{B.14}$$

By comparing (B.14) with (B.7) we obtain the invariant completion of Appell function

$$\left[\widehat{\mathcal{K}}_{2N}(\tau,z)\right]_{inv} (\equiv [\mathcal{K}_{2N}(\tau,z)]_{inv}) \tag{B.15}$$

$$\equiv \frac{1}{4}\frac{i\eta(\tau)^3\theta_1(\tau,2z)}{\theta_1(\tau,z)^2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2(N-1)} + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2(N-1)}\right].$$

Proof of the relation (B.1) Let us now prove the identity (B.1). Written in terms of the Appell function (B.1) is expressed as

$$\widehat{\mathcal{K}}_{2(k+1)}(\tau,z) \tag{B.16}$$

$$= \frac{i}{2}\frac{\eta(\tau)^3\theta_1(\tau,2z)}{\theta_1(\tau,z)^2}\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2k} + \frac{i}{2}\sum_{j=0}^{k-1} A_{3,j}(\tau)\chi_j^{(k-1)}(\tau,z)\theta_1(\tau,2z).$$

As an intermediate step we first show the following decomposition formula

$$\widehat{\mathcal{K}}_k(\tau,z) = G_k(\tau,z) + F_k(\tau,z)\ . \tag{B.17}$$

$$G_k(\tau,z) \equiv \frac{i\eta(\tau)^3}{\theta_1(\tau,z)}\frac{\theta_2(\tau,z)}{\theta_2(\tau)}\prod_{\ell=0}^{k-1}\frac{\theta_4\left(\tau,-z+\frac{\ell}{k}\right)}{\theta_4\left(\tau,\frac{\ell}{k}\right)}. \tag{B.18}$$

$$F_k(\tau,z) \equiv \sum_{s=1}^{k-1}\widehat{H}_s^{(k)}(\tau)\Theta_{s,\frac{k}{2}}(\tau,2z). \tag{B.19}$$

(B.17) is proved by checking that $\widehat{\mathcal{K}}_k(\tau,z) - F_k(\tau,z)$ has the same quasi periodicity and has the same zeros and poles in z as the function $G_k(\tau,z)$. It is easy to see that both sides of (B.17) have the same quasi-periodicity property. We also note that the function $G_k(\tau,z)$ has poles at $z = n\tau+m$ while $F_k(\tau,z)$ is regular. These poles correspond to the zero of the denominators of the Appell function. We can check that G_k and $\widehat{\mathcal{K}}_k$ have the same residues at these poles.

Let us next show that $\widehat{\mathcal{K}}_k(\tau,z)-F_k(\tau,z)$ vanishes at zeros of $G_k(\tau,z)$, i.e. $z=z_\ell\equiv\frac{\tau}{2}+\frac{\ell}{k}$, $\ell=0,\cdots,k-1$. First using the identity

$$\frac{1}{1-yq^n}=\sum_{s=0}^{k-1}\frac{(yq^n)^s}{1-y^kq^{kn}}\ ,$$

we can rewrite $\widehat{\mathcal{K}}_k$ as

$$\text{(B.20)}\quad \widehat{\mathcal{K}}_k(\tau,z) \;=\; \frac{1}{2}\sum_{s=0}^{k-1}\sum_{n\in\mathbf{Z}}\left\{\frac{\xi_{n,s}(\tau,z)}{1-y^kq^{kn}}-\frac{\xi_{n,-s}(\tau,z)}{1-y^{-k}q^{-kn}}\right\}$$

where

$$\text{(B.21)}\qquad \xi_{n,s}(\tau,z)\equiv q^{-\frac{s^2}{2k}}\cdot q^{\frac{k}{2}\left(n+\frac{s}{k}\right)^2}y^{k\left(n+\frac{s}{k}\right)}\ .$$

By setting $z=z_\ell\equiv\frac{\tau}{2}+\frac{\ell}{k}$, we obtain

$$\text{(B.22)}\qquad \xi_{n,s}(\tau,z_\ell)=e^{2\pi i\frac{\ell s}{k}}q^{\frac{k}{2}n(n+1)+s(n+\frac{1}{2})}\ ,$$

and thus,

$$\begin{aligned}\text{(B.23)}\quad &\widehat{\mathcal{K}}_k(\tau,z_\ell)\\ &=\frac{1}{2}\sum_{s=0}^{k-1}\sum_{n\in\mathbf{Z}}\left\{\frac{q^{\frac{k}{2}n(n+1)+s(n+\frac{1}{2})}}{1-q^{k(n+\frac{1}{2})}}e^{2\pi i\frac{\ell s}{k}}-\frac{q^{\frac{k}{2}n(n+1)-s(n+\frac{1}{2})}}{1-q^{-k(n+\frac{1}{2})}}e^{-2\pi i\frac{\ell s}{k}}\right\}\\ &=\frac{1}{2}\sum_{s=1}^{k-1}H_s^{(k)}(\tau)\left(e^{2\pi i\frac{\ell s}{k}}-e^{-2\pi i\frac{\ell s}{k}}\right)\ .\end{aligned}$$

We also note

(B.24)

$$\Theta_{s,\frac{k}{2}}(\tau,2z_\ell)=q^{\frac{s^2}{2k}}\sum_{n\in\mathbf{Z}}\xi_{n,s}(\tau,z_\ell)=e^{2\pi i\frac{\ell s}{k}}\Theta_{s,\frac{k}{2}}(\tau,2z_0)=e^{2\pi i\frac{\ell s}{k}}\Theta_{s,\frac{k}{2}}(\tau,\tau)\ .$$

Hence we obtain

$$\text{(B.25)}\qquad \widehat{\mathcal{K}}_k(\tau,z_\ell)=\sum_{s=1}^{k-1}H_s^{(k)}(\tau)\frac{\Theta_{s,\frac{k}{2}}(\tau,2z_\ell)}{\Theta_{s,\frac{k}{2}}(\tau,\tau)}=F_k(\tau,z_\ell),$$

where we used the identity $H_s^{(k)}(\tau)=-H_{k-s}^{(k)}(\tau)$. Thus the difference $\widehat{\mathcal{K}}_k(\tau,z)-F_k(\tau,z)$ in fact vanishes at $z=z_\ell$.

Formula (B.16) can then be derived by modifying the function G_k in (B.17) so that it has a multiple zero at $z = 1/2+\tau/2$ instead of simple zeros at $\{z_\ell\}$. Modification can be made by using a formula

$$\text{(B.26)}\quad \left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2(K-1)} - \prod_{j=1}^{K-1} \frac{\theta_4\left(\tau, z+\frac{j}{2K}\right)\theta_4\left(\tau, z-\frac{j}{2K}\right)}{\theta_4\left(\tau,\frac{j}{2K}\right)^2} = \sum_{\ell=0}^{K-2} a_{\ell+1}^{(K)}(\tau)\chi_\ell^{(K-2)}(\tau,z)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}$$

where $\chi_\ell^{(K-2)}(\tau,z)$ denotes the spin $\ell/2$ character of $SU(2)_{K-2}$. Expansion coefficients $a_{\ell+1}^{(K)}(\tau)$ are determined below.

By combining (B.17) and (B.27) we arrive at (B.16), (B.1).

Determination of coefficients $a_\ell^{(K)}$. Expansion coefficients $a_\ell^{(K)}(\tau)$ can be determined by comparing both sides of (B.27) at $K-1$ points. By choosing the reference points $z = \frac{\tau}{2} + \frac{r}{2K}$ $(r = 1, \cdots, K-1)$ which are the zeros of G_{2K} we obtain a set of linear equations;

$$\text{(B.27)}\quad \sum_{s=1}^{K-1} a_s^{(K)}(\tau)\chi_{K-1-s}^{(K-2)}\left(\tau,\frac{r}{2K}\right) = \frac{\theta_2\left(\tau,\frac{r}{2K}\right)^{2(K-1)}\eta(\tau)^3}{\theta_3(\tau)^{2(K-1)}\theta_4\left(\tau,\frac{r}{2K}\right)^2} \equiv b_r(\tau) \qquad (r = 1,\ldots,K-1).$$

Therefore, by means of the Cramer's formula we obtain

$$\text{(B.28)}\qquad a_j^{(K)}(\tau) = \frac{\det B^{(j)}(\tau)}{\det B(\tau)} \qquad (j = 1,\ldots,K-1),$$

where $B(\tau)$, $B^{(j)}(\tau)$ are $(K-1)\times(K-1)$ matrices defined by

$$\text{(B.29)}\quad B(\tau) \equiv (B_{r,s}(\tau))_{1\le r,s\le K-1}\ ,\ B_{r,s}(\tau) \equiv \chi_{K-1-s}^{(K-2)}\left(\tau,\frac{r}{2K}\right)\ ,$$
$$B^{(j)}(\tau) \equiv (B_{r,1}(\tau),\cdots,B_{r,j-1}(\tau),b_r(\tau),B_{r,j+1}(\tau),\cdots,B_{r,K-1}(\tau))\ .$$

We can simplify (B.28) as follows; First we note

$$\text{(B.30)}\qquad \chi_{s-1}^{(K-2)}(\tau,\frac{r}{K}) \equiv \frac{2\Theta_{s,K}^{(-)}\left(\tau,\frac{r}{K}\right)}{i\theta_1\left(\tau,\frac{r}{K}\right)} = a_{r,s}\frac{\Theta_{s,K}(\tau)}{g_r(\tau)}\ ,$$

where we set

$$a_{r,s} \equiv \frac{\sin\left(\pi\frac{rs}{K}\right)}{\sin\left(\pi\frac{r}{K}\right)} , \tag{B.31}$$

(B.32)

$$g_r(\tau) \equiv \frac{\theta_1\left(\tau, \frac{r}{K}\right)}{2\sin\left(\pi\frac{r}{K}\right)} \equiv q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-e^{2\pi i\frac{r}{K}}q^m)(1-e^{-2\pi i\frac{r}{K}}q^m) .$$

Therefore,

$$B_{r,s}(\tau) = a_{r,K-s}\frac{\Theta_{K-s,K}(\tau)}{g_r(\tau)} = (-1)^{r-s}a_{r,s}\frac{\Theta_{K-s,K}(\tau)}{g_r(\tau)} , \tag{B.33}$$

and the factor $\frac{\Theta_{K-s,K}}{g_r}$ is factorized from the determinant;

$$\begin{aligned}\det B(\tau) &= (-1)^{\frac{(K-1)(K-2)}{2}}\prod_{r,s=1}^{K-1}\frac{\Theta_{K-s,K}(\tau)}{g_r(\tau)}\det\hat{B} \\ &= (-1)^{\frac{(K-1)(K-2)}{2}}\left(\frac{\eta(\tau)}{\eta(K\tau)}\right)^2\prod_{s=1}^{K-1}\left(\frac{\Theta_{s,K}(\tau)}{\eta(\tau)}\right)\det\hat{B} ,\end{aligned} \tag{B.34}$$

where $\hat{B} \equiv (a_{r,s})$. In the second line we have used

$$\prod_{r=1}^{K-1} g_r(\tau) = \eta(\tau)^{K-3}\eta(K\tau)^2.$$

We similarly obtain

(B.35)

$$\det B^{(j)} = (-1)^{\frac{(K-1)(K-2)}{2}}\left(\frac{\eta(\tau)}{\eta(K\tau)}\right)^2\prod_{s=1}^{K-1}\left(\frac{\Theta_{s,K}(\tau)}{\eta(\tau)}\right)\det\hat{B}^{(j)} ,$$

where we set $\hat{B}^{(j)}(\tau) \equiv \left(a_{r,1},\cdots,a_{r,j-1},\hat{b}_{r,j}(\tau),a_{r,j+1},\cdots a_{r,K-1}\right)$,

$$\begin{aligned}\hat{b}_{r,j}(\tau) &\equiv b_r(\tau)\cdot\frac{(-1)^{r-1}g_r(\tau)}{\Theta_{K-j,K}(\tau)} \\ &= \frac{(-1)^{r-1}\theta_2\left(\tau,\frac{r}{2K}\right)^{2(K-1)}\eta(\tau)^3}{\theta_3(\tau)^{2(K-1)}\theta_4\left(\tau,\frac{r}{2K}\right)^2\Theta_{K-j,K}(\tau)}\cdot\frac{\theta_1\left(\tau,\frac{r}{K}\right)}{2\sin\left(\pi\frac{r}{K}\right)}.\end{aligned} \tag{B.36}$$

In this way we finally obtain a simplified formula;

$$a_j^{(K)}(\tau) = \frac{\det\hat{B}^{(j)}(\tau)}{\det\hat{B}} . \tag{B.37}$$

§ Appendix C : Modular property of $\mathcal{N}=4$ characters at higher level

Let us recall the formula introduced by Miki [8] which is closely related to the Appell function,

$$I(K,a,b;\tau,z)=\sum_{r\in\mathbb{Z}+1/2} e^{2\pi i a r}\,\frac{(yq^r)^b}{1+yq^r}\,y^{Kr}\,q^{Kr^2/2}. \tag{C.1}$$

Its S-transform is given by

$$\begin{aligned}
&\frac{i}{\tau}e^{-iK\frac{\pi z^2}{\tau}}I(K,a,b;\frac{-1}{\tau},\frac{z}{\tau}) \qquad (C.2)\\
&=\sum_{r\in\mathbb{Z}+a} e^{i\pi(r-a)}y^r q^{\frac{r^2}{2K}}\frac{1}{\sqrt{K}}\int_{-\infty}^{+\infty}\frac{e^{-2\pi b\left(\frac{p}{\sqrt{K}}+i\frac{r}{K}\right)}}{1+e^{-2\pi\left(\frac{p}{\sqrt{K}}+i\frac{r}{K}\right)}}q^{\frac{p^2}{2}}\\
&\quad+i\sum_{\substack{s\in\mathbb{Z}+1/2\\ \delta(a,s)\neq 0}} e^{i\pi(\delta(a,s)-a)}e^{2\pi i(\frac{K}{2}-b)s}\frac{(yq^s)^{\delta(a,s)}}{1+yq^s}y^{Ks}q^{\frac{K}{2}s^2}\\
&\quad+\frac{i}{2}\sum_{\substack{s\in\mathbb{Z}+1/2\\ \delta(a,s)=0}} e^{-i\pi a}e^{2\pi i(\frac{K}{2}-b)s}\frac{1-yq^s}{1+yq^s}y^{Ks}q^{\frac{K}{2}s^2}.
\end{aligned}$$

Here $\delta(a,s)$ is a real number determined by the conditions

$$\delta(a,s)\equiv a-Ks \pmod{\mathbb{Z}},\quad 0\le\delta(a,s)<1. \tag{C.3}$$

We also recall the higher level $\mathcal{N}=4$ massless character of spin $\ell/2$ ($\ell=0,1,\dots,k$) [6] given by

$$\begin{aligned}
&ch_0^{NS}(k,\ell/2;\tau,z) \qquad (C.4)\\
&=q^{-6k/24}q^{\ell/2}y^\ell\prod_{n=1}\frac{(1+yq^{n-\frac{1}{2}})^2(1+y^{-1}q^{n-\frac{1}{2}})^2}{(1-q^n)^2(1-y^2q^n)(1-y^{-2}q^{n-1})}\\
&\quad\times\sum_m q^{(k+1)m^2+(\ell+1)m}y^{2(k+1)m}\frac{1-(yq^{m+\frac{1}{2}})^{2(k-\ell+1)}}{(1+yq^{m+\frac{1}{2}})^2}
\end{aligned}$$

which may be rewritten as

$$
\begin{aligned}
&= -i\frac{\theta_3(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)} \\
&\quad \sum_m q^{(k+1)m^2+(\ell+1)m+\ell/2+1/4-k/4} y^{2(k+1)m+\ell+1} \frac{1-(yq^{m+\frac{1}{2}})^{2(k-\ell+1)}}{(1+yq^{m+\frac{1}{2}})^2} \\
&= -i\frac{\theta_3(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)} \\
&\quad \sum_m q^{(k+1)(m+1/2)^2} y^{2(k+1)(m+1/2)} (yq^{m+1/2})^{\ell-k} \frac{1-(yq^{m+\frac{1}{2}})^{2(k-\ell+1)}}{(1+yq^{m+\frac{1}{2}})^2}.
\end{aligned}
\tag{C.5}
$$

Here the factor $(yq^{m+1/2})^{\ell-k}$ is extracted so that we can apply Miki's formula. The last term in (C.5) can be expanded as

$$
\frac{1-(yq^{m+\frac{1}{2}})^{2(k-\ell+1)}}{(1+yq^{m+\frac{1}{2}})^2} = \sum_{i=0}^{2(k-\ell)+1} \frac{(-1)^i (yq^{m+1/2})^i}{1+yq^{m+1/2}}. \tag{C.6}
$$

Thus altogether the massless characters are written as

$$
\begin{aligned}
ch_0^{NS}(k,\ell;\tau,z) &= -i\frac{\theta_3(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)} \\
&\quad \times \sum_m q^{(k+1)(m+1/2)^2} y^{2(k+1)(m+1/2)} \sum_{i=0}^{2(k-\ell)+1} \frac{(-1)^i (yq^{m+1/2})^{i+\ell-k}}{1+yq^{m+1/2}}.
\end{aligned}
\tag{C.7}
$$

Terms in the 2nd line above have exactly the same form as the functions $I(2(k+1),a,b;\tau,z)$ and we have

$$
\begin{aligned}
&ch_0^{NS}(k,\ell/2;\tau,z) \\
&\quad = -i\frac{\theta_3(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)} \sum_{i=0}^{2(k-\ell)+1} I(K=2(k+1),i,i+\ell-k;\tau,z).
\end{aligned}
\tag{C.8}
$$

Now we can apply the modular transformation law (C.2). We first note that the factor $\delta(a,s)$ of (C.3) vanishes

$$
\delta(a,s) = a - 2(k+1)s \equiv 0 \mod \mathbb{Z} \tag{C.9}
$$

where s =1/2+integer. Therefore only the 1st (continuous rep) and the 3rd term remain in (C.2).

We also note that the 3rd term has exactly the from as the $\ell = k$ representation

(C.10)

$$\text{3rd term} = (-1)^{a+b+k+1} \sum_m \frac{1 - yq^{m+1/2}}{1 + yq^{m+1/2}} y^{2(k+1)(m+1/2)} q^{(k+1)(m+1/2)^2}.$$

Therefore we arrive at the result: under the S-tranformation $\mathcal{N} = 4$ massless characters produce only $\ell = k$ massless representation besides the continuous ones.

(C.11)
$$ch_0^{NS}(k, \ell/2; -1/\tau, z/\tau)$$
$$= \text{continuous reps} + (-1)^\ell (k - \ell + 1) ch_0^{NS}(k, k/2, \tau, z).$$

Let us next examine the part of the continuous representations. We introduce the notation

(C.12)
$$X_r \equiv e^{-2\pi\left(\frac{p}{\sqrt{2(k+1)}} + i\frac{r}{2(k+1)}\right)}.$$

Continuous representations contained in (C.8) are given by

(C.13)
$$\sum_{i=0}^{2(k-\ell)+1} \sum_r (-1)^i (-1)^r y^r q^{\frac{r^2}{4(k+1)}} \frac{1}{\sqrt{2(k+1)}} \int_{-\infty}^{\infty} \frac{X_r^{i+\ell-k}}{1 + X_r} q^{\frac{p^2}{2}}.$$

After the sum over i

(C.14)
$$X_r^{\ell-k} \cdot \sum_{i=0}^{2(k-\ell)+1} (-1)^i \frac{X_r^i}{1 + X_r} = \frac{X_r^{\ell-k} - X_r^{k-\ell+2}}{(1 + X_r)^2},$$

(C.13) is written as

(C.15)
$$\sum_r (-1)^r (y^r - y^{-r}) q^{\frac{r^2}{4(k+1)}}$$
$$\times \frac{1}{\sqrt{2(k+1)}} \times \int \frac{e^{-2\pi(\ell-k)\left(\frac{p}{\sqrt{2(k+1)}} + i\frac{r}{2(k+1)}\right)}}{\left[1 + e^{-2\pi\left(\frac{p}{\sqrt{2(k+1)}} + i\frac{r}{2(k+1)}\right)}\right]^2} q^{\frac{p^2}{2}} dp.$$

Then by summing over r modulo $2(k+1)$ the above formula is transformed to

$$\text{(C.16)} \qquad \sum_{j=0}^{2k+1} \frac{(-1)^j}{\sqrt{2(k+1)}} [\Theta_{j,k+1}(\tau,2z) - \Theta_{-j,k+1}(\tau,2z)]$$
$$\times \int \frac{e^{-2\pi(\ell-k)\left(\frac{p}{\sqrt{2(k+1)}}+i\frac{j}{2(k+1)}\right)}}{\left[1+e^{-2\pi\left(\frac{p}{\sqrt{2(k+1)}}+i\frac{j}{2(k+1)}\right)}\right]^2} q^{\frac{p^2}{2}} dp$$
$$= \sum_{j=1}^{k} (-1)^j \sqrt{\frac{2}{k+1}} [\Theta_{j,k+1}(\tau,2z) - \Theta_{-j,k+1}(\tau,2z)]$$
$$\times i\,Im \int \frac{e^{-2\pi(\ell-k)\left(\frac{p}{\sqrt{2(k+1)}}+i\frac{j}{2(k+1)}\right)}}{\left[1+e^{-2\pi\left(\frac{p}{\sqrt{2(k+1)}}+i\frac{j}{2(k+1)}\right)}\right]^2} q^{\frac{p^2}{2}} dp.$$

If one restores the overall factor $\theta_3(\tau,z)^2/\eta(\tau)^3\theta_1(\tau,z)$, we recognize the combination $(\Theta_{j,k+1}(\tau,2z) - \Theta_{-j,k+1}(\tau,2z))/\theta_1(\tau,2z)$ as the level $k-1$ $SU(2)$ character. Thus (C.17) has the form of an integral over $\mathcal{N}=4$ continuous representations.

Then the modular transformation of massless representation is given by

$$\text{(C.17)} \qquad ch_0^{NS}(k,\ell/2;-1/\tau,z/\tau)$$
$$= e^{\frac{2\pi i k z^2}{\tau}} \left[\sum_{0\le\ell'\le k-1} \int_{-\infty}^{\infty} dp' \mathcal{S}(\ell|p',\ell') ch^{NS}(k,p',\ell'/2;\tau,z) \right.$$
$$\left. + (-1)^\ell (k-\ell+1) ch_0^{NS}(k,k/2;\tau,z) \right]$$

where the massive character is defind as in (B.12) and the coefficient $\mathcal{S}(\ell|p',\ell')$ is given by

$$\mathcal{S}(\ell|p',\ell') = (-1)^\ell \sqrt{\frac{2}{(k+1)}}\, \mathrm{Im} \frac{e^{-2\pi(\ell'-k)\left(\frac{p'}{\sqrt{2(k+1)}}+i\frac{\ell'+1}{2(k+1)}\right)}}{\left[1+e^{-2\pi\left(\frac{p'}{\sqrt{2(k+1)}}+i\frac{\ell+1}{2(k+1)}\right)}\right]^2}$$

(C.18)

$$\equiv \frac{e^{2\pi\frac{k-\ell+2}{\sqrt{2(k+1)}}p'}S^{(k-1)}_{\ell,\ell'} + 2e^{2\pi\frac{k-\ell+1}{\sqrt{2(k+1)}}p'}S^{(k-1)}_{\ell-1,\ell'} + e^{2\pi\frac{k-\ell}{\sqrt{2(k+1)}}p'}S^{(k-1)}_{\ell-2,\ell'}}{\left|2\cosh\pi\left(\frac{p'}{\sqrt{2(k+1)}} + i\frac{\ell'+1}{2(k+1)}\right)\right|^4}.$$

Here $S^{(k-1)}_{\ell,\ell'} \equiv \sqrt{\frac{2}{k+1}}\sin\left(\pi\frac{(\ell+1)(\ell'+1)}{k+1}\right)$ is the modular coefficients of $SU(2)_{k-1}$.

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