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The triplet vertex operator algebra $W(p)$ and the restricted quantum group $\bar{U}_q(sl_2)$ at $q = e^{\frac{\pi i}{p}}$

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Abstract.

We study the abelian category $W(p)$-mod of modules over the triplet W algebra $W(p)$. We construct the projective covers $\mathcal{P}_s^{\pm}$ of all the simple objects $\mathcal{X}_s^{\pm}$, $1 \leq s \leq p$, in the category $W(p)$-mod. By using the structure of these projective modules, we show that $W(p)$-mod is a category which is equivalent to the abelian category of the finite-dimensional modules for the restricted quantum group $\bar{U}_q(sl_2)$ at $q = e^{\frac{\pi i}{p}}$. This Kazdan–Lusztig type correspondence was conjectured by Feigin et al. [FGST1], [FGST2].

§1. Introduction

The theory of vertex operator algebra (VOA) is an algebraic counterpart of comformal field theory. About general facts around VOA, see [FrB]. Up to now, examples of conformal field theory over general Riemann surfaces are constructed by using lattice VOAs, VOAs associated with integrable representations of affine Lie algebras with the positive integer level, or VOAs associated with the minimal series of the Virasoro algebra. The abelian category of modules over these VOA's are all semisimple and the number of simple objects is finite. In order to define a conformal field theory on Riemann surfaces associated with a VOA, it is necessary that this VOA has some finiteness condition. Zhu found such a finiteness condition on a VOA called the C_2-finiteness condition, and showed that the abelian category of modules over a VOA satisfying C_2-finiteness condition is Artinian and Noetherian, moreover, the number of simple objects is finite [FrZ], [Zhu].

Associated to a VOA which has C_2-finiteness condition, Zhu developed the theory of conformal blocks on Riemann surfaces, and showed that the dimension of conformal blocks are finite for genus one case, and

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found the Knizhnik–Zamolodchikov type differential equations satisfied by conformal blocks in the genus one case [Zhu].

It is not obvious to construct examples of VOA which satisfy C_2-finiteness condition. Only few examples with C_2-finiteness condition are known. One of them is a series of VOA called $W(p)$, $p = 2, 3, \dots$, which was constructed by H. G. Kausch about twenty years ago [Kau]. It is very recently proved that VOA $W(p)$ satisfies C_2-finiteness conditions by D. Adamovic [AM2] and [CaN]. It is known that the abelian categories $W(p)$-mod are not semi-simple.

Conformal field theory associated to a VOA $W(p)$ gives a logarithmic conformal field theory, because zero mode operator $T(0)$ of energy-momentum tensor is not diagonalizable, and therefore N-points functions may have logarithmic parts [Gab].

Quite recently it is observed that $W(p)$ type conformal field theory appears as the scaling limit of some boundary conditions of the integrable lattice models, (c.f. Pearce et al. [PRZ], Bushlanov et al. [BFGT]).

The purpose of this paper is to analyze the structure of the abelian category of $W(p)$-modules. In order to solve this problem, we construct $W(p)$-modules $\mathcal{P}_s^{\pm}$, $s = 1, \dots, p-1$, and prove that these are in fact projective $W(p)$-modules.

In the papers [FGST1], [FGST2], Feigin et al. conjectured that two abelian categories $W(p)$-mod and $\bar{U}_q(sl_2)$-mod are categorically equivalent as abelian categories. By using the structure theorems of these projective modules $\mathcal{P}_s^{\pm}$ obtained in this paper we prove the conjecture of Feigin et al.

The VOA $W(p)$ are constructed by using the free field realization of the Virasoro algebra with central charge $c_p = 13 - 6(p + \frac{1}{p})$, $p = 2, 3, \dots$, and screening operators. There are two screening operators $Q_+(z)$ and $Q_-(z)$. For each integer $1 \le s \le p-1$ and $\varepsilon = \pm$, we define the screening operator $Q_-^{[d_s^{\varepsilon}]}(z)$ from $Q_-(z)$ by using the iterated integral on a twisted local system. The screening operators $Q_-^{[d_s^{\varepsilon}]}(z)$, $1 \le s \le p-1$ and $\varepsilon = \pm$, play a very important role in this paper.

In §2 we collect some structures of Fock space representations of the Virasoro algebra by using intertwing operators arising from $Q_+(z)$ and $Q_-^{[d_s^{\varepsilon}]}(z)$. The results are well known in [FF1], [FF2], [Fel] and [TsK].

Our VOA $W(p)$ are defined from the lattice vertex operator algebra V_L using the screening operator $Q_-^{[d_s^{\varepsilon}]}(z)$. The C_2-finiteness condition of $W(p)$ is already known in [Ada], [AM1], [AM2] and [CaN]. These facts will be stated in §3.

In §4 we construct $W(p)$-modules $\mathcal{P}_s^{\pm}$, $1 \le s \le p-1$, by using the method of J. Fjelsted et al. [FFHST]. The $W(p)$-modules $\mathcal{P}_s^{\pm}$ $(1 \le s \le p-1)$, which we will construct in this paper is obtained by deforming a $W(p)$-module $\mathcal{V}_s^{+} \oplus \mathcal{V}_s^{-}$ by using screening operators $Q_{-}^{[d_s^{\varepsilon}]}(z)$. The construction of $\mathcal{P}_s^{\pm}$, and an analysis of these $W(p)$-module are the most important parts of this paper. The structure of $\mathcal{P}_s^{\pm}$, $1 \le s \le p-1$, is described in Theorem 4.3 and Theorem 4.4. These two theorems are a part of the main results of this paper. By using these structure theorems, we determine completely Zhu's algebra $A_0(W(p))$ of the VOA $W(p)$, which is stated in Theorem 4.6.

In §5, we determine the Ext^1 group between simple objects $\mathcal{X}_s^{\pm}$, $s = 1, \ldots, p$, the results is given in Theorem 5.1 and 5.2. By using the both theorems, we show that the abelian category $W(p)$-mod of $W(p)$-modules has the block decomposition

$$W(p)\text{-mod} = \bigoplus_{s=0}^{p} C_s.$$

The subcategories C_0 and C_p are semi-simple consisting of simple objects $\mathcal{X}_0$ and $\mathcal{X}_p$, respectively. But for $1 \le s \le p-1$, C_s is not semi-simple. The set of simple objects of C_s consists of two elements $\{\mathcal{X}_s^{+}, \mathcal{X}_s^{-}\}$. These results are stated at the first part of §5.

Finaly we show that the $W(p)$ module $\mathcal{P}_s^{\pm}$, $1 \le s \le p-1$, is a projective cover of $\mathcal{X}_s^{\pm}$, self-dual, and therefore injective. On the module $\mathcal{P}_s^{\pm}$, the zero mode operator $T(0)$ of the energy-momentum tensor is not diagonizable. To prove the projectivity of $\mathcal{P}_s^{\pm}$, we must know the detailed structure of $\mathcal{P}_s^{\pm}$, and show that Ext^1 groups between $\mathcal{P}_s^{\pm}$ and simple modules $\mathcal{X}_s^{\pm}$ are zero.

On the very final step for $1 \le s \le p-1$ we compute the endmorphism algebra,

$$B_s = \mathrm{End}_{C_s}(\mathcal{P}_s), \qquad \mathcal{P}_s = \mathcal{P}_s^{+} \oplus \mathcal{P}_s^{-}.$$

The structure of B_s is given in Theorem 6.4. They are eight dimensional basic Artinian algebras, mutually isomorphic to the basic algebra arising from $\bar{U}_q(sl_2)$-mod computed by Feigin [FGST1], [FGST2].

The structures of these basic Artinian algebras are explicitly described. This is stated in Theorem 6.2.

Using the fact that two basic algebras coming from $W(p)$ and $\bar{U}_q(sl_2)$ are isomorphic, it is easy to prove by the conjectures of Feigin [FGST1], [FGST2]:

$$W(p)\text{-mod} \simeq \bar{U}_q(sl_2)\text{-mod}.$$

Since the abelian category $W(p)$-mod is not semi-simple, it is very interesting and important to analyze the structures of: (1) the fusion tensor products, (2) the monodromy representations mapping class group, braid group, and (3) genus one and higher genus conformal blocks occuring in the conformal field theory associated with the VOA $W(p)$. Having the results obtained in this paper we are now ready to study these problems.

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§2. Free fields and screening charge operators

In this section we give a free field realization of the Virasoro algebra and intertwining operators.

2.1. Notations

We fix an integer $p \geq 2$, and set $\alpha_+ = \sqrt{2p}$, $\alpha_- = -\sqrt{2/p}$ and $\alpha_0 = \alpha_+ + \alpha_-$. Then we have $\alpha_+ \cdot \alpha_- = -2$, $\alpha_+ = -p\alpha_-$, $\frac{1}{\alpha_+} = -\frac{\alpha_-}{2}$, $\frac{\alpha_0^2}{2} = \frac{(p-1)^2}{p}$, $\alpha_+ \cdot \alpha_+ = 2p$ and $\alpha_- \cdot \alpha_- = \frac{2}{p}$.

Let us introduce an even integral lattice and its dual;

$$L = \mathbb{Z}\alpha_+, \tag{2.1}$$

$$L^\vee = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = \mathbb{Z} \cdot \frac{\alpha_-}{2}. \tag{2.2}$$

For any integers $r, s \in \mathbb{Z}$, we set

$$\alpha_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_-$$

and for any integers $s, n \in \mathbb{Z}$, we set

$$\lambda_s(n) = \frac{1-s}{2}\alpha_- + n\alpha_+ = \alpha_{1-2n,s}, \tag{2.3}$$

and

$$\Lambda_s = \{\lambda_s(n); n \in \mathbb{Z}\}. \tag{2.4}$$

Then we see that, if $s_1 - s_2 \neq 0 \pmod{2p}$, then $\Lambda_{s_1} \cap \Lambda_{s_2} = \emptyset$, $\lambda_{s+2p}(n) = \lambda_s(n+1)$, $\Lambda_{s+2p} = \Lambda_s$. We have the L-orbit decomposition of $L^\vee$ as follows,

$$L^\vee = \bigsqcup_{-(p-1)\le s\le p} \Lambda_s. \tag{2.5}$$

We set for $1 \le s \le p-1$

$$\begin{aligned} \Lambda_s^+ &= \Lambda_s,\ \Lambda_s^- = \Lambda_{-s}, \\ \Lambda_p^- &= \Lambda_0,\ \Lambda_s^+ = \Lambda_p. \end{aligned} \tag{2.6}$$

For each $\mu \in \mathbb{C}$ we set

$$h_\mu = \frac{1}{2}\left(\mu - \frac{1}{2}\alpha_0\right)^2 - \frac{1}{8}\alpha_0^2. \tag{2.7}$$

Then we have

$$h_\lambda = h_{\alpha_0-\lambda} = h_{\lambda^+}, \tag{2.8}$$

where for $\lambda \in \mathbb{C}$ we denote

$$\lambda^+ = \alpha_0 - \lambda. \tag{2.9}$$

Then we have

$$h_{\lambda_s(n)} = \frac{1}{4p}\left\{(2np+s-p)^2 - (p-1)^2\right\}, \tag{2.10}$$

and the following formulas hold, for all s $(0 \le s \le p-1)$;

$$h_{\lambda_{-s}}(n) = h_{\lambda_s(1-n)},\quad \lambda_{-s}(n) + \lambda_s(1-n) = \alpha_0,$$

$$h_{\lambda_p}(n) = h_{\lambda_p(-n)},\quad \lambda_p(n) + \lambda_p(-n) = \alpha_0.$$

We introduce the following sequence of numbers, for $1 \le s \le p-1$, $n \ge 0$;

(2.11)

$$h_s(2n) = h_{\lambda_s(-n)} = h_{\lambda_{-s}(n+1)} = \frac{1}{4p}\{((2n+1)p-s)^2 - (p-1)^2\},$$

$$h_s(2n+1) = h_{\lambda_s(n+1)} = h_{\lambda_{-s}(-n)} = \frac{1}{4p}\{((2n+1)p+s)^2 - (p-1)^2\},$$

$$h_0(n) = h_{\lambda_0(n+1)} = h_{\lambda_0(-n)} = \frac{1}{4p}\{((2n+1)p)^2 - (p-1)^2\},$$

$$h_p(n) = h_{\lambda_p(n)} = h_{\lambda_p(-n)} = \frac{1}{4p}\{(2np)^2 - (p-1)^2\}.$$

Then we have an increasing and a decreasing series of rational numbers, for $0 \leq s \leq p$;

(2.12) $h_s(0) < h_s(1) < h_s(2) < \dots,\ h_s(n+1) - h_s(n) \in \mathbb{Z}_{\geq 1},\ n \geq 0,$

(2.13)
$$h_{p-1}(1) > h_{p-2}(1) > \cdots > h_1(1) > h_0(0) > h_1(0) > \cdots > h_p(0).$$

We see $h_p(0) = -\dfrac{(p-1)^2}{4p}$, $h_1(0) = 0$ and $h_0(0) = \dfrac{p^2-(p-1)^2}{4p}$.

We also define the sets of rational numbers, for $1 \leq s \leq p-1$:

$$H_s^+ = \{h_s(2n); n \geq 0\}, H_s^- = \{h_s(2n+1); n \geq 0\}, H_s = H_s^+ \cup H_s^-,$$

$$H_p = H_p^+ = \{h_p(n); n \geq 0\},$$

$$H_0 = H_p^- = \{h_0(n); n \geq 0\}.$$

Then we see $H_s \cap H_{s'} = \emptyset$ if $s \neq s'$. We set

(2.14)
$$H = \bigsqcup_{s=0}^{p} H_s.$$

2.2. Free field realization of the Virasoro algebra

First we introduce the free Bosonic field as follows:

(2.15)
$$\varphi(z) = \hat{a} + a(0)\log z - \sum_{n\neq 0} \frac{a(n)}{n} z^{-n},$$

(2.16)
$$a(z) = \partial\varphi(z) = \sum_{n\in z} a(n) z^{-n-1}.$$

This field is characterized by the operator product expansions (OPE)

$$\varphi(z)\varphi(w) \sim \log(z-w),$$

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}.$$

The operators $\hat{a}$ and $a(n)$ satisfy the following commutator relations

(2.17)
$$[a(n), a(n)] = m\delta_{m+n,0}\ \mathrm{id},$$

$$[a(n), \hat{a}] = \delta_{n,0}\hat{a}.$$

Set

$$\varphi_\pm(z) = \mp \sum_{n\geq 1} \frac{a(\pm n)}{n} z^{\mp n},$$

then we have

$$\varphi(z) = \varphi_-(z) + \hat{a} + a(0)\log z + \varphi_+(z).$$

For each $\lambda \in \mathbb{C}$ we define the left and the right Fock module by following relations

$$F_\lambda \ni |\lambda\rangle \neq 0, \quad a(n)|\lambda\rangle = \delta_{n,0}\lambda|\lambda\rangle, \quad n \geq 0, \tag{2.18}$$
$$F_\lambda^\dagger \ni \langle\lambda| \neq 0, \quad \langle\lambda|a(-n) = \delta_{n,0}\lambda\langle\lambda|, \quad n \geq 0.$$

Then we have a unique non-degenerate pairing

$$\langle\ |\ \rangle : F_\lambda^\dagger \times F_\lambda \longrightarrow \mathbb{C} \tag{2.19}$$

such that

$$\langle\lambda|\lambda\rangle = 1, \quad \langle va(n)|u\rangle = \langle v|a(n)u\rangle,$$

for $n \in \mathbb{Z},\ u \in F_\lambda,\ v \in F_\lambda^+$.

Define the energy-momentum tensor

$$T(z) = \frac{1}{2} : \partial\varphi(z)^2 : + \frac{\alpha_0}{2}\partial^2\varphi(z) = \sum_{n\in\mathbb{Z}} T(n) z^{-n-2}, \tag{2.20}$$

then we have OPE of the Virasoro field with central charge c_p;

$$T(z)T(w) \sim \frac{\frac{1}{2}c_p}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) \tag{2.21}$$

where $c_p = 13 - 6\left(p + \dfrac{1}{p}\right)$ as usual.

For each $\mu \in \mathbb{C}$ we set

$$V_\mu(z) = : e^{\mu\varphi(z)} : = e^{\mu\varphi_+(z)} e^{\mu\varphi_-(z)} e^{\mu\hat{a}} e^{\mu a(a)\log z}. \tag{2.22}$$

Then we have

$$V_\mu(z) : F_\lambda \longrightarrow F_{\lambda+\mu},$$
$$V_{\mu_1}(z_1)V_{\mu_2}(z_2) = (z_1 - z_2)^{\mu_1\cdot\mu_2} : V_{\mu_1}(z_1)V_{\mu_2}(z_2),$$

and the following operator product expansion

$$T(z)V_\mu(w) \sim \frac{h_\mu}{(z-w)^2}V_\mu(w) + \frac{1}{(z-w)}\partial_w V_\mu(w). \tag{2.23}$$

Here h_μ is defined in (2.7) and called conformal dimension of the field operator $V_\mu(z)$.

For $1 \le s \le p-1$ we set

$$\mathcal{V}_s^{\pm} = \sum_{\lambda \in \Lambda_s^{\mp}} F_\lambda,$$

$$\mathcal{V}_p = \mathcal{V}_p^{+} = \sum_{\lambda \in \Lambda_p} F_\lambda, \quad \mathcal{V}_0 = \mathcal{V}_p^{-} = \sum_{\lambda \in \Lambda_p^{-}} F_\lambda. \tag{2.24}$$

Then for each $\lambda \in L$ we get

$$V_\lambda(z) \subset \mathrm{End}_{\mathbb{C}}(\mathcal{V}_s^{\pm})[[z, z^{-1}]] \tag{2.25}$$

for all $1 \le s \le p$.

2.3. Screening operators

Since $h_{\alpha_\pm} = 1$, the field $Q_\pm(z) = V_{\alpha_\pm}(z)$ has the conformal dimension 1 with respect to the Virasoro field $T(z)$. For each $\lambda \in L^\vee$, since $\alpha_+ \cdot \lambda \in \mathbb{Z}$ we see that

$$Q_+(z) \in \mathrm{Hom}(F_\lambda, F_{\lambda+\alpha_+})[[z, z^{-1}]]. \tag{2.26}$$

Remark that

$$Q_+(0) = Q_+ = \int dz\, Q_+(z) : F_\lambda \longrightarrow F_{\lambda+\alpha_+} \tag{2.27}$$

commutes with $T(z)$. Here we consider $\int dz$ as taking residue at $z = 0$.

While for $\lambda \in L^\vee$, $\alpha_- \cdot \lambda \notin \mathbb{Z}$ in general, therefore $\int Q_-(z)dz$: $F_\lambda \to F_{\lambda+\alpha_-}$ cannot be defined for $\alpha_- \cdot \lambda \notin \mathbb{Z}$.

To construct an intertwining operator from $Q_-(z)$, we have to use an iterated integration on the twisted cycles. To this end, we prepare some notations.

For $d \ge 1$, consider the product of screening operators

(2.28)

$$\begin{aligned} &Q_-(w_1) \dots Q_-(w_d) \\ =&e^{d\hat{a}} \prod_{1 \le i < j \le d} (w_i - w_j)^{\frac{2}{p}} \prod_{j=1}^{d} w_j^{\alpha_- a(0)} e^{\alpha_- \sum_{j=1}^{d} \varphi_-(w_j)} e^{\alpha_- \sum_{j=1}^{d} \varphi_+(w_j)}, \end{aligned}$$

acting on

$$\mathcal{V}_{[\lambda]} \longrightarrow \mathcal{V}_{[\lambda+d\alpha_-]}.$$

For $d \geq 2$, define the complex manifold

$$X_d = \{(w_1, \ldots, w_d) \in (\mathbb{C}^\times)^d;\ w_i \neq w_j\}, \tag{2.29}$$

$$Z_{d-1} = \{(\xi_1, \ldots, \xi_{d-1}) \in (\mathbb{C} \setminus \{0,1\})^{d-1};\ \xi_i \neq \xi_j\}, \tag{2.30}$$

and define a map

$$\begin{aligned} \mathbb{C}^\times \times Z_{d-1} &\to X_d \\ (w; \xi_1, \ldots, \xi_{d-1}) &\mapsto (w\xi_0, w\xi_1, \ldots, w\xi_{d-1}) \end{aligned} \tag{2.31}$$

where we put $\xi_0 = 1$

Then this map is $\mathbb{C}^\times$-equivariant isomorphism where

$$\begin{aligned} \lambda(w; \xi_1, \ldots, \xi_{d-1}) &= (\lambda w; \xi_1, \ldots, \xi_{d-1}), \\ \lambda(w_1, \ldots, w_d) &= (\lambda w_1, \ldots, \lambda w_d). \end{aligned} \tag{2.32}$$

For each $\lambda \in L^\vee$ and $d \geq 2$, we define multivalent functions respectively on X_d, X_{d-1}, by

$$\Phi_d^\lambda(w_1, \ldots, w_d) = \prod_{1 \leq i < j \leq d} (w_i - w_j)^{\frac{2}{p}} \prod_{j=1}^{d} w_j^{\alpha_- \lambda} \tag{2.33}$$

and

$$\bar{\Phi}_{d-1}^\lambda(\xi_1, \ldots, \xi_{d-1}) = \prod_{0 \leq i < j \leq d-1} (\xi_i - \xi_j)^{\frac{2}{p}} \prod_{j=1}^{d-1} \xi_j^{\alpha_- \lambda} \tag{2.34}$$

where we set $\xi_0 = 1$. Then we have the formula

$$\Phi_d^\lambda(w_1, \ldots, w_d) = \bar{\Phi}_{d-1}^\lambda(\xi_1, \ldots, \xi_{d-1}) \cdot w^{\Delta_d(\lambda)} \tag{2.35}$$

where

$$\Delta_d(\lambda) = \frac{1}{p} d(d-1) + d\alpha_- \lambda \in \frac{1}{p}\mathbb{Z}. \tag{2.36}$$

For $\lambda \in L^\vee$ and $d \geq 2$, we denote $\bar{S}_{d-1}^{\lambda,*}$, the local system on Z_{d-1}, determined by the monodromy of $\bar{\Phi}_{d-1}^\lambda$, and also denote $\bar{S}_{d-1}^\lambda$, the dual local system of $\bar{S}_{d-1}^{\lambda,*}$. Then these local systems depend only on the class $[\lambda] \in L^\vee/L$ of $\lambda \in L^\vee$. Therefore we can write $\bar{S}_{d-1}^{[\lambda]}$ etc.

If we take an element $[\bar{\Gamma}] \in H_{d-1}(Z_{d-1}, \bar{S}^{[\lambda]}_{d-1})$, the integral

$$\int_{[\bar{\Gamma}]} Q_-(w\xi_0), \dots, Q_-(w\xi_{d-1}) w^{d-1} d\xi_1 \dots d\xi_{d-1} \tag{2.37}$$

define an element of

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_{[\lambda]}, \mathcal{V}_{[\lambda+d\alpha_{-1}]})[w, w^{-1}], \tag{2.38}$$

if $\Delta_d(\lambda) \in Z$.

For $1 \le s \le p-1$ and $\varepsilon = \pm$, we define an integer $d_s^{\pm}$ by the following way;

$$d_s^+ = p - s \text{ and } d_s^- = s.$$

And we denote

$$\lambda_s^+ = \lambda_{-s}(1) \in \Lambda_s^+, \quad \lambda_s^- = \lambda_s(0) \in \Lambda_s^-.$$

Then we have the following;

$$\Delta_{d_s^+}(\lambda_s^+) \in \mathbb{Z}, \quad \Delta_{d_s^-}(\lambda_s^-) \in \mathbb{Z}.$$

For $1 \le s \le p-1$ we define operators

$$Q_-^{[d_s^{\pm}]}(w) \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_s^{\pm}, \mathcal{V}_s^{\mp})[[w, w^{-1}]]$$

by the following way;

(1) For $d_s^{\pm} = 1$, we set

$$Q_-^{[d_s^{\pm}]}(z) = Q_-(z).$$

(2) For $2 \le d_s^{\pm} \le p-1$, we set

$$Q_-^{[d_s^{\pm}]}(z) = \int_{\bar{\Gamma}} Q_-(w\xi_1) \cdots Q_-(w\xi_{d_s^{\pm}-1}) w^{d_s^{\pm}-1} d\xi_1 \dots d\xi_{d_s^{\pm}-1}.$$

We fix a cycle

$$[\bar{\Gamma}] \in H_{d_s^{\pm}-1}(Z_{d_s^{\pm}-1}, S^{[\lambda_s^{\pm}]}_{d_s^{\pm}-1})$$

which satisfies the following normalized conditions [TsK];

$$\int_{\bar{\Gamma}} \bar{\Phi}^{\lambda_s^{\pm}}_{d_s^{\pm}-1}(\xi_1 \dots \xi_{d_s^{\pm}-1}) d\xi_1 \dots d\xi_{d_s^{\pm}-1} = 1.$$

Proposition 2.1. *For $1 \le s \le p-1$ and $\varepsilon = \pm$, we have*

(1) $\displaystyle T(z) Q_-^{[d_s^{\varepsilon}]}(w) \sim \frac{1}{(z-w)^2} Q_-^{[d_s^{\varepsilon}]}(w) + \frac{1}{(z-w)^2} \partial_w Q_-^{[d_s^{\varepsilon}]}(w),$

(2) $[Q_-^{[d_s^\varepsilon]}(0),\ T(z)] = 0$,

(3) $[Q_+,\ Q_-^{[d_s^\varepsilon]}(0)] = 0$.

Proof. It can be proved in the standard way. We give a proof of (3) only.

Since $Q_+(z)Q_-(w) = (z-w)^{-2} : e^{\alpha_+\phi(z)+\alpha_-\phi(w)} :$, we have

$$[Q_+, Q_-(w)] = \frac{\alpha_+}{\alpha_+ + \alpha_-}\frac{\partial}{\partial w}V_{\alpha_+ + \alpha_-}(w).$$

Therefore we get

$$\left[Q_+, Q_-^{[d_\lambda]}\right] = \left[Q_+, \int_\Gamma dw_1 \cdots dw_{d_\lambda} Q_-(w_1)\cdots Q_-(wd_\lambda)\right]$$

$$= \frac{1}{\alpha_+ + \alpha_-}\int_\Gamma d\,[\sum_{j=1}^{d_\lambda}(-1)^{j+1}V_{\alpha_-}(w_1)\cdots V_{\alpha_+ + \alpha_-}(w_j)\cdots V_{\alpha_-}(w_{d_\lambda})$$

$$dw_1 \cdots \overset{\vee}{dw_j} \cdots dw_{d_\lambda}]$$

$= 0$.

Q.E.D.

2.4. Abelian category $\mathcal{L}_{c_p}$-mod

Let us consider the Virasoro algebra

$$\mathcal{L} = \sum_{n \in Z} \mathbb{C}T(n) \oplus \mathbb{C}c \tag{2.38}$$

with $c = c_p$ id. Define Lie subalgebra as $\mathcal{L}_{>0}$ and $\mathcal{L}_{<0}$ of $\mathcal{L}$

$$\mathcal{L}_{>0} = \sum_{n \geq 1} \mathbb{C}T(n), \quad \mathcal{L}_{<0} = \sum_{n \geq 1} \mathbb{C}T(-n), \tag{2.39}$$

and we define involtive anti-astromorphism of Lie algebra $\mathcal{L}$ by $\sigma(T(n)) = T(-n)$ and $\sigma(c) = c$.

Consider $\mathcal{L}$-module M with the following properties.

(1) $c = c_p$ id on M.

(2) M has the following decomposition $M = \sum_{h \in H(M)} M[h]$, where $H(M) = H_0(M) + \mathbb{Z}_{\geq 0}$, for some finite subset $H_0(M)$ of $\mathbb{C}$, and for $h \in H(M)$, set $M[h] = \{m \in M : (T(0) - h)^n m = 0$ for some $n \geq 0\}$. We further assume $\dim_{\mathbb{C}} M[h] < \infty$.

Let us denote

$$\mathcal{L}_{c_p}\text{-mod}, \tag{2.40}$$

the abelian category of left $\mathcal{L}$-modules which satisfy the above conditions (1) and (2).

For each $h \in \mathbb{C}$, let $M_h \ni |h\rangle$ be the Verma module of the highest weight h, the highest vector $|h\rangle$, and the central charge $c = c_p$ id, let L_h be the irreducible quotient of M_h.

These $\mathcal{L}_{c_p}$-module M_h, L_h, and Fock module F_λ are objects of $\mathcal{L}_{c_p}$-mod.

For each $\lambda \in \mathbb{C}$, there exists a unique $\mathcal{L}_{c_p}$-module map

$$M_{h_\lambda} \longrightarrow F_\lambda \tag{2.41}$$

such that $|h_\lambda\rangle$ is mapped to $|\lambda\rangle$.

The facts which we are going to use can be found in Feigin and Fuchs [FF1], [FF2], Felder [Fel], and Tsuchiya and Kanie [TsK]. By using Kac determinant formula for the Virasoro algebra, it is easy to show the following.

Proposition 2.2. *For $h \in \mathbb{C} \setminus H$, the Virasoro module M_h is a simple object in $\mathcal{L}_{c_p}$-mod, where H is defined in §2-1, (2.10).*

Proposition 2.3. *Fix $0 \leq s \leq p$, for $m, n \in \mathbb{Z}$. Then*

(1)

$$\mathrm{Hom}_{C_s}(M_{h_s(m)}, M_{h_s(n)}) \simeq \begin{cases} \mathbb{C} & m \geq n \\ 0 & m < n \end{cases}$$

(2) *For $m \geq n$, the Virasoro sequence*

$$0 \longrightarrow M_{h_s(m)} \longrightarrow M_{h_s(n)}$$

is exact.

(3) *For $n \geq 0$, the Virasoro sequence*

$$0 \longrightarrow M_{h_s(n+1)} \longrightarrow M_{h_s(n)} \longrightarrow L_{h_s(n)} \longrightarrow 0$$

is exact.

We define the following notations for the later use. For $0 \leq s \leq p$ there exists a singular vector element

$$\eta_s |h_s(0)\rangle \in M_{h_s(0)}[h_s(1)] \tag{2.42}$$

which is uniquely determined up to constant. Where η_s is an element

$$\eta_s \in U(\mathcal{L}_{<0})[s], \tag{2.43}$$

we define

$$\eta_s^\vee = \sigma(\eta_s) \in U(\mathcal{L}_{>0})[-s]. \tag{2.44}$$

Proposition 2.4. (1) *For each $1 \le s \le p-1$ the followings hold:*

(a) $Q_+|\lambda_s(n)\rangle = 0$, $Q_+^{2n}|\lambda_s(-n)\rangle \neq 0$ *and* $Q_+^{2n+1}|\lambda_s(-n)\rangle = 0$ *for* $n \ge 0$.

(b) $Q_+|\lambda_{-s}(n+1)\rangle = 0$, $Q_+^{2n+1}|\lambda_{-s}(-n)\rangle \neq 0$ *and* $Q_+^{2n+2}|\lambda_{-s}(-n)\rangle = 0$ *for* $n \ge 0$.

(2) $Q_+|\lambda_0(n+1)\rangle = 0$, $Q_+^{2n+1}|\lambda_0(-n)\rangle \neq 0$ *and* $Q_+^{2n+2}|\lambda_0(-n)\rangle = 0$ *for* $n \ge 0$.

(3) $Q_+|\lambda_p(n)\rangle = 0$, $Q_+^{2n}|\lambda_p(-n)\rangle \neq 0$ *and* $Q_+^{2n+1}|\lambda_p(-n)\rangle = 0$ *for* $n \ge 0$.

Proposition 2.5. *For $1 \le s \le p-1$, we have:*

(1) $Q_-^{[s]}|\lambda_s(-n)) = 0$, $Q_-^{[s]}|\lambda_s(n+1)\rangle \neq 0$ *for* $n \ge 0$.

(2) $Q_-^{[p-s]}|\lambda_{-s}(-n)\rangle = 0$, $Q_-^{[p-s]}|\lambda_{-s}(n+1)\rangle \neq 0$ *for* $n \ge 0$.

(3) $Q_-^{[p-s]}|\lambda_{-s}(1)\rangle = c|\lambda_s(0)\rangle$ $(c \neq 0)$.

Proposition 2.6. *We have the following exact sequences of Virasoro modules with $c = c_p$.*

(1) *For $1 \le s \le p-1$, $n \ge 0$:*

(a) $0 \longrightarrow M_{h_s(2n+1)} \longrightarrow M_{h_s(2n)} \longrightarrow F_{\lambda_s(-n)}$,

(b) $0 \longrightarrow M_{h_s(2n+3)} \longrightarrow M_{h_s(2n+1)} \longrightarrow F_{\lambda_s(n+1)}$,

(c) $0 \longrightarrow M_{h_s(2n+2)} \longrightarrow M_{h_s(2n)} \longrightarrow F_{\lambda_{-s}(n+1)}$,

(d) $0 \longrightarrow M_{h_s(2n+2)} \longrightarrow M_{h_s(2n+1)} \longrightarrow F_{\lambda_{-s}(-n)}$.

(2) *For $s = 0$, $n \ge 0$:*

(a) $0 \longrightarrow M_{h_0(n+1)} \longrightarrow M_{h_0(n)} \longrightarrow F_{\lambda_0(-n)}$,

(b) $0 \longrightarrow M_{h_0(n+1)} \longrightarrow M_{h_0(n)} \longrightarrow F_{\lambda_0(n+1)}$.

(3) *For $s = p$, $n \ge 0$:*

(a) $0 \longrightarrow M_{h_p(n+1)} \longrightarrow M_{h_p(n)} \longrightarrow F_{\lambda_p(-n)}$,

(b) $0 \longrightarrow M_{h_p(n+1)} \longrightarrow M_{h_p(n)} \longrightarrow F_{\lambda_p(n)}$.

As a consequence we obtain the so-called Felder complex [Fel].

Theorem 2.7. *For $1 \le s \le p-1$, the following is an exact sequence of Virasoro modules:*

$$\cdots \xrightarrow{Q_-^{[p-s]}} \mathcal{V}_s^- \xrightarrow{Q_-^{[s]}} \mathcal{V}_s^+ \xrightarrow{Q_-^{[p-s]}} \mathcal{V}_s^+ \longrightarrow \cdots.$$

We define

$$\mathcal{X}_s^{\pm} = \ker Q^{[d_s^{\mp}]}, \quad Q^{[d_s^{\mp}]} : \mathcal{V}_s^{\mp} \longrightarrow \mathcal{V}_s^{\pm}.$$

Then we also have exact sequences of Virasoro modules

$$0 \longrightarrow \mathcal{X}_s^{\mp} \longrightarrow \mathcal{V}_s^{\pm} \longrightarrow \mathcal{X}_s^{\pm} \longrightarrow 0.$$

Virasoro modules $\mathcal{X}_s^{\pm}$ and $\mathcal{X}_p^{\pm}$ are decomposed into the sum of Virasoro submodules.

We define $\mathcal{X}_p^{\pm} = \mathcal{V}_p^{\pm} \in \mathcal{L}_{c_p}$-mod.

Theorem 2.8. (1) *For* $1 \leq s \leq p-1$,

$$\mathcal{X}_s^{+} = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} U(\mathcal{L}) Q_{+}^{m} |\lambda_s(-n)\rangle,\ U(\mathcal{L}) Q_{+}^{m} |\lambda_s(-n)\rangle \simeq L_{h_s(2n)}.$$

(2) *For* $1 \leq s \leq p-1$,

$$\mathcal{X}_s^{-} = \sum_{n=0}^{\infty} \sum_{m=0}^{2n+1} U(\mathcal{L}) Q_{+}^{m} |\lambda_{-s}(-n)\rangle,\ U(\mathcal{L}) Q_{+}^{m} |\lambda_{-s}(-n)\rangle \simeq L_{h_s(2n+1)}.$$

(3) $$\mathcal{X}_p^{+} = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} U(\mathcal{L}) Q_{+}^{m} |\lambda_p(-n)\rangle,\ U(\mathcal{L}) Q_{+}^{m} |\lambda_p(-n)\rangle \simeq L_{h_p(n)}.$$

(4) $$\mathcal{X}_p^{-} = \sum_{n=0}^{\infty} \sum_{m=0}^{2n+1} U(\mathcal{L}) Q_{+}^{m} |\lambda_0(-n)\rangle,\ U(\mathcal{L}) Q_{+}^{m} |\lambda_0(-n)\rangle \simeq L_{h_0(n)}.$$

2.5. Block structure of $\mathcal{L}_{c_p}$-mod

Consider the decomposition of $\mathbb{C} = \bigsqcup_{b \in B} b$, where $b = \{h\}$ for $h \in \mathbb{C} \setminus H$ or $b = H_s$, $0 \leq s \leq p$. Let us consider the abelian subcategory $C_b(\mathcal{L}_{c_p})$ of $\mathcal{L}_{c_p}$-mod, which is parametrised by $b \in B$ as follows.

(1) $b = \{h\}$, $h \in \mathbb{C} \setminus H$, then $M \in \mathcal{L}_{c_p}$-mod belongs to $C_b(\mathcal{L}_{c_p})$ if and only if M is a direct sum of the Verma modules M_h.

(2) $b = H_s$, $0 \leq s \leq p$, then $M \in \mathcal{L}_{c_p}$-mod belongs to $C_b(\mathcal{L}_{c_p})$ if and only if the irreducible sub-quotient of M is isomorphic to $L_{h_s(n)}$ $(n \geq 0)$.

Theorem 2.9. *The abelian category* $\mathcal{L}_{c_p}$*-mod has the following decomposition of abelian category*

$$\mathcal{L}_{c_p}\text{-}mod = \bigoplus_{b \in B} C_b(\mathcal{L}_{c_p}).$$

For $b \neq b'$, $b, b' \in B$ *and* $M \in C_b(\mathcal{L}_{c_p})$, $N \in C_{b'}(\mathcal{L}_{c_p})$, *we have the following facts,*

$$\mathrm{Ext}^{i}_{\mathcal{L}_{c_p}}(M, N) = 0. \qquad i = 0, 1, \ldots.$$

Homological properties of the abelian categories $C_s(\mathcal{L}_{c_p}) = C_{H_s}(\mathcal{L}_{c_p})$, $0 \le s \le p-1$, are very important in this paper. Here we state the required results as Proposition 2.10, 2.11, 2.12.We can not find the results in the litrature. But these results can be proved by using Janzen filtrations in Verma modules and Fock modules of Virasoro algebra and Kac determinant formula which are given in [FF1] and [FF2].

At first we fix s, $1 \le s \le p-1$, and consider the abelian category $C_s(\mathcal{L}_{c_p})$. We use the following notations

$$\begin{aligned} M_n &= M_{h_s(n)}, \quad n \ge 0, \\ L_n &= M_{h_s(n)}/M_{h_s(n+1)}, \quad n \ge 0, \\ L_n^{(1)} &= M_{h_s(n)}/M_{h_s(n+2)}, \quad n \ge 0, \\ L_n^{(1)\vee} &= D(L_n^{(1)}), \quad n \ge 0. \end{aligned} \tag{2.45}$$

Proposition 2.10. *For each s, $1 \le s \le p-1$, we have the following.*

(1) *The set of equivalense classes of simple objects in $C_s(\mathcal{L}_{c_p})$ are $\{L_n : n \ge 0\}$.*
(2) *The module L_n are self dual $D(L_n) \simeq L_n$.*
(3) *For $m, n \in \mathbb{Z}_{\ge 0}$, we have*

$$\mathrm{Ext}^1(L_m, L_n) \simeq \begin{cases} C & m = n \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

(4) $\mathrm{Ext}^1(L_n, L_{n+1}) \ni [L_n^{(1)}] \ne 0$,
$\mathrm{Ext}^1(L_{n+1}, L_n) \ni [L_n^{(1)\vee}] \ne 0$.

Now we restrict our attention to $1 \le s \le p-1$, and fix the following highest weight vectors;

$$\begin{aligned} &u \in L_0[h_s(0)], \; u_0^{(1)} \in L_0^{(1)}[h_s(0)], \\ &v \in L_1[h_s(1)], \\ &\eta_s(u_0^{(1)}) = v \quad \text{in } L_0^{(1)}. \end{aligned} \tag{2.46}$$

Then we have the following exact sequences;

$$\begin{aligned} 0 \longrightarrow L_1 \longrightarrow\ & L_0^{(1)} \longrightarrow L_0 \longrightarrow 0 \\ & u_0^{(1)} \mapsto u. \end{aligned}$$

Proposition 2.11. *Fix s, $1 \le s \le p-1$, then the followings hold.*

(1) $\mathrm{Ext}^1(L_0, L_0^{(1)}) = 0$, $\mathrm{Ext}^1(L_0^{(1)\vee}, L_0) = 0$.

(2) $\mathrm{Ext}^1(L_0^{(1)}, L_0) \simeq \mathbb{C}$, $\mathrm{Ext}^1(L_0, L_0^{(1)\vee}) \simeq \mathbb{C}$.

(3) *Fix a generator* $K^{(1)}$ *of* $\mathrm{Ext}^1(L_0^{(1)}, L_0) \simeq \mathbb{C}$. *Then the followings hold.*

(a) $D(K^{(1)}) \simeq K^{(1)}$.

(b) $K^{(1)}$ *is a generator of* $\mathrm{Ext}^1(L_0, L_0^{(1)\vee}) \simeq \mathbb{C}$,

$$[K^{(1)}] \in \mathrm{Ext}^1(L_0, L_0^{(1)\vee}).$$

(4) *We can take elements* $u_0, u_1 \in K^{(1)}[h_s(0)]$ *and* $v_0 \in K^{(1)}[h_s(1)]$ *with the following properties*

(2.47)
$$v_0 = \eta_s(u_0),$$
$$u_1 = \eta_s^\vee(v_0) \in L_0 \subseteq K_0^{(1)},$$
$$0 \longrightarrow L_0 \longrightarrow K^{(1)} \longrightarrow L_0^{(1)} \longrightarrow 0,$$
$$u_0 \mapsto u_0^{(1)}$$
$$v_0 \mapsto v.$$

Then the following relation holds;

(2.48)
$$(T(0) - h_s(0))u_0 = cu_1, \quad c \neq 0.$$

Proposition 2.12. *Fix* s, $1 \leq s \leq p-1$, *then the followings hold.*

(1) $\mathrm{Ext}^1(L_0^{(1)}, L_1) = 0$, $\mathrm{Ext}^1(L_1, L_0^{(1)\vee}) = 0$.

(2) $\mathrm{Ext}^1(L_1, L_0^{(1)}) \simeq \mathbb{C}$, $\mathrm{Ext}^1(L_0^{(1)\vee}, L_1) \simeq \mathbb{C}$.

(3) *Fix the generator* $K_{(1)}$ *of* $\mathrm{Ext}^1(L_1, L_0^{(1)}) \simeq \mathbb{C}$, *the following facts hold.*

(a) $D(K_{(1)}) \simeq K_{(1)}$.

(b) $K_{(1)}$ *is a generator of* $Ext^1(L_0^{(1)\vee}, L_1) \simeq \mathbb{C}$,

$$[K_{(1)}] \in \mathrm{Ext}^1(L_0^{(1)\vee}, L_1).$$

(4) *We can take elements* u_0, v_0, *and* $v_1 \in K_{(1)}$ *such that*

$$u_0 \in L_0^{(1)}[h_s(0)] = K_{(1)}[h_s(0)],$$
$$v_1 \in L_0^{(1)}[h_s(1)] \subseteq K_{(1)}[h_s(1)], \ v_0 \in K_{(1)}[h_s(1)].$$
$$\eta_s^\vee(v_0) = u_0, \ \eta_s(u_0) = v_1$$
$$0 \longrightarrow L_0^{(1)} \longrightarrow K_{(1)} \longrightarrow L_1 \longrightarrow 0,$$

$$v_0 \mapsto v.$$

Then the following relation holds;

$$(T(0) - h_s(1))v_0 = cv_1, \quad c \neq 0.$$

§3. The triplet VOA $W(p)$

In this section, we define the so-called triplet VOA $W(p)$ and show that it satisfies Zhu's C_2-finiteness condition [AM2].

3.1. Vertex operator algebras

In this paper the notion of vertex operator algebra (VOA) plays an important role. For definitions and properties of VOA, we follow [FrB] and [Kac]. We use the notations of [NaT].

Roughly speeking, a vertex operator algebra is a quadruple $(V, |0\rangle, T, J)$ such that

$$V = \bigoplus_{\Delta \geq 0} V[\Delta] \tag{3.1}$$

which is a $\mathbb{Z}_{\geq 0}$ graded $\mathbb{C}$-vector space with the properties $V[0] = \mathbb{C}|0\rangle \neq 0$, $\dim_{\mathbb{C}} V[\Delta] < \infty$, and with an distinguished element $T \in V[2]$, $T \neq 0$. The element $|0\rangle$ is called the vacum element and the element T is called the Virasoro element.

There exists a degree preserving linear map

$$J : V \longrightarrow \mathrm{End}(V)[[z, z^{-1}]], \tag{3.2}$$

$$A \mapsto J(A, z),$$

where we set degree of $z = -1$. These must satisfy some compatibility conditions. The most important properties are the locality of any two operators $J(A, z)$ and $J(B, w)$, and their operator product expansions (OPE). For details of OPE, we refer [MaN].

For each $A \in V[\Delta]$, we denote

$$J(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1} = \sum_{n \in \mathbb{Z}} A[n] z^{-n-\Delta} \tag{3.3}$$

where $A_{(n)} = A[n - \Delta + 1]$, $A[n] = A_{(n+\Delta-1)}$, $\deg A_{(n)} = -n + \Delta - 1$ and $\deg A[n] = -n$. Sometimes we write $J_n(A) = A[n]$.

We denote

$$J(T, z) = T(z) = \sum_{n \in \mathbb{Z}} T(n) z^{-n-2}. \tag{3.4}$$

Then $\deg T(n) = -n$, and we have the following operator product expansion (OPE);

$$T(z)T(w) \sim \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2}{(z-w)^2}T(a) + \frac{1}{(z-w)}\partial_w T(w) \tag{3.5}$$

where c is a some complex number. The operator $T(z)$ is called the energy-momentum tensor.

For $A, B \in V$, we denote the OPE for $J(A,z)$ and $J(B,z)$ by the following way.

$$\begin{aligned} J(A,z)J(B,z) &= \sum_{n\in\mathbb{Z}} J(J_n(A)B,w)(z-w)^{-n-\Delta_A} \\ &= \sum_{n\in\mathbb{Z}} J(A_{(n)}B,w)(z-w)^{-n-1}. \end{aligned} \tag{3.6}$$

And sometimes we use the following notations

$$J(A_{(n)}B,w) = (A_{(n)}B)(w), \tag{3.7}$$

$$(A_{(n)}B)(w) = \mathrm{Res}_{z=w}(z-w)^n A(z)B(w)\,dz.$$

The representation (M, J^M) of a VOA is a degree preserving linear map

$$J^M : V \longrightarrow \mathrm{End}(M)[[z,z^{-1}]] \tag{3.8}$$

such that $M = \sum_{h\in H(M)} M[h]$ with $M[h] = \{m \in M; (T(0)-h)^n m = 0$ for some $n \geq 0\}$ and some compatibility conditions. In this paper we assume that $H(M) = H_0(M) + \mathbb{Z}_{\geq 0}$ for a finite set $H_0(M)$, and also assume that for any $h \in H(M)$, $\dim_{\mathbb{C}} M[h] < \infty$. In general $\dim_{\mathbb{C}} M[h] < \infty$ is a too strong condition. However, since in this paper we mainly deal with VOA's which satisfy C_2-finiteness conditions, this condition is not restrictive. We denote

$$V\text{-mod} \tag{3.9}$$

the abelian category of left V-modules which satisfy the above conditions.

For a VOA V, its universal enveloping algebra

$$U(V) = \sum_d U(V)[d] \tag{3.10}$$

is introduced in [FrZ], [NaT] and [MNT].

The algebra $U(V)$ is a degreewise completed linear topological algebra generated by $A[n]$, $A \in V$ and $n \in \mathbb{Z}$, $\deg A[n] = -n$. A representation of VOA V is a representation of $U(V)$, and vice versa.

We define a subalgebra $F_0(U(V)) = \sum_{d \leq 0} U(V)[d]$ of $U(V)$ and a closed left ideal $I_0(V)$ of $U(V)$, which is generated by $\sum_{d \leq -1} U(V)[d]$. Then $F_0(U(V)) \cap I_0(V)$ is a closed two-sided ideal of $F_0(U(V))$. The Zhu's algebra $A_0(V)$ of V is defined as the quotient algebra of $F_0(U(V))$,

$$A_0(V) = F_0(U(V))/F_0(U(V)) \cap I_0(V). \tag{3.11}$$

For any $A \in V$, let $[A[0]]$ be the element of $A_0(V)$ represented by $A[0]$ mod $I_0(V)$. The algebra $A_0(V)$ also can be defined as a quotient space of V itself [Zhu].

The algebra $A_0(V)$ is called zero mode algebra or Zhu's algebra of VOA V.

For each $M \in V$-mod, define

$$HW(M) = \{m \in M : J_n(A)m = 0, \quad n \geq 1, A \in V\}. \tag{3.12}$$

Then the Zhu algebra $A_0(V)$ act on $HW(M)$.

Here we introduce one important notion called Zhu's C_2-finiteness condition.

For each vertex operator algebra V, we define a graded subspace $C_2(V)$ of V by

$$C_2(V) = \text{ span of } \{A_{(n)}B : A, B \in V,\ n \leq -2\}. \tag{3.13}$$

Then the quotient space

$$\mathfrak{p}(V) = V/C_2(V) \tag{3.14}$$

is graded, and has a Poisson algebra structure defined by for any $A, B \in V$;

$$\begin{aligned} [A] \cdot [B] &= [A_{(-1)}B], \\ \{[A], [B]\} &= [A_{(0)}B] \end{aligned} \tag{3.15}$$

where $[A]$ denote the equivalent class of A in $\mathfrak{p}(V)$.

Definition 3.1. *The following condition of V is called Zhu's C_2-finiteness condition;*

$$\dim_{\mathbb{C}} \mathfrak{p}(V) < \infty. \tag{3.16}$$

When $\dim_{\mathbb{C}} \mathfrak{p}(V) < \infty$, we are able to prove $\dim_{\mathbb{C}} A_0(V) \leq \dim_{\mathbb{C}} \mathfrak{p}(V)$. Thus in this case we would like to study $A_0(V)$-mod that is the abelian category consisting of all finite dimensional left $A_0(V)$-modules. Then the covariant functor HW maps any V-modules to a $A_0(V)$-module,

$$HW : V\text{-mod} \longrightarrow A_0(V)\text{-mod}, \tag{3.17}$$

and it has the adjoint functor

$$X \in A_0(V)\text{-mod} \mapsto U(V) \underset{F_0(U(V))}{\otimes} X \in V\text{-mod}, \tag{3.18}$$

where X is considered as a $F_0(U(V))$-module through the map $F_0(U(V)) \to A_0(V)$.

The following important theorem is proved in [MNT].

Theorem 3.2. *Suppose that V satisfies Zhu's C_2-finiteness condition. Then we have:*

(1) *The abelian category V-mod is Artinian and Noetherian.*
(2) *The number of equivalence classes of simple V-modules is finite.*
(3) *The number of simple $A_0(V)$-modules is equal to the number of simple V-modules.*

3.2. The lattice vertex operator algebra V_L

Define

$$V_L = \sum_{\lambda \in L} F_\lambda \tag{3.19}$$

and set $T = \frac{1}{2}a(-1)^2|0\rangle - \frac{\alpha_0}{2}a(-2)|0\rangle \in V_L$, then the following is well known [FrB].

Theorem 3.3. (1) *There exists a unique vertex operator algebra structure on V_L such that*

$$J(a(-1)|0\rangle : z) = a(z),$$
$$J(|\lambda\rangle : z) = V_\lambda(z) \quad \text{for } \lambda \in V_L,$$
$$J(T : z) = T(z).$$

(2) *For each $1 \leq s \leq p$, $\mathcal{V}_s^{\pm}$ is an irreducible V_L-module.*
(3) *The abelian category of V_L-modules is semi-simple and its inequivalent simple objects are $\mathcal{V}_s^{\pm}$, $1 \leq s \leq p$.*

Then $(F_0, |0\rangle, T, J)$ is a vertex operator subalgebra of V_L. We remark that Fock space F_0 is not the one which appears in the filtration of $U(V)$ and think this may not make any confusions. The VOA F_0 is generated by fields $a(z)$ and the associated Virasoro field is $T(z) = \frac{1}{2} : a(z)^2 : +\alpha_0/2\ \partial a(z)$.

Note that $V(\mathcal{L}_{c_p}) := U(\mathcal{L})|0\rangle \subseteq F_0$ contains $|0\rangle$ and T, therefore $V(\mathcal{L}_{c_p})$ is a sub VOA of F_0. The abelian category $V(\mathcal{L}_{c_p})$-mod is nothing but the abelian category $\mathcal{L}_{c_p}$-mod. The $\mathcal{L}$-module $V(\mathcal{L}_{c_p})$ is isomorphic to $L_{h_1(0)}$ as $\mathcal{L}$-modules. Note that $h_1(0) = 0$.

3.3. Duality in V-mod

The duality functor in VOA was introduced in [FHL]. The universal enveloping algebra $U(V)$ has an involutive anti-automorphism of the topological algebra $U(V)$:

(3.20) $$\sigma : U(V) \longrightarrow U(V), \quad (\text{write } \sigma(A) = A^\sigma \text{ for short}),$$

such that $\sigma(U(V)[d]) = U(V)[-d]$. For $A \in V[\Delta_A]$, we define

(3.21) $$J^\sigma(A; z) = \sum_n A[n]^\sigma z^{-n-\Delta_A},$$

which is given by

(3.22) $$J^\sigma(A; z) = J(e^{zT(1)}(-z^{-2})^{T(0)}A; z^{-1}).$$

For $M \in V$-mod, its dual $D(M) \in V$-mod is defined by

(3.23) $$D(M) = \sum_{h \in H(M)} \mathrm{Hom}_{\mathbb{C}}(M[h], \mathbb{C})$$

as $\mathbb{C}$-vector space and the action is defined by

(3.24) $$\langle A\phi, u\rangle = \langle \phi, A^\sigma u\rangle$$

for all $A \in U(V)$, $\phi \in D(M)$ and $u \in M$.

The following gives the duality of the VOA V_L.

Proposition 3.4. (1) *$a(n)^\sigma = -a(-n) + \delta_{n,0}\alpha_0\ id$, $V_\lambda(n)^\sigma = -V_\lambda(-n)$ for $\lambda \in L$, $\sigma(T(n)) = T(-n)$ for $n \in \mathbb{Z}$.*
(2) *$D(\mathcal{V}_s^\pm) = \mathcal{V}_s^\mp$ for $1 \le s \le p-1$.*
(3) *The sub-VOA F_0 is closed under the duality and $D(F_\lambda) = F_{\alpha_0-\lambda}$ for any $\lambda \in \mathbb{C}$.*

3.4. Construction of $W(p)$

Recall that $\mathcal{V}_1^- = V_L$ carries a VOA structure. Then the intertwining operator $Q_-^{[1]} = Q_-$ defines a subspace

$$W(p) = \mathcal{X}_1^+ = \ker\ (Q_- : V_L \longrightarrow \mathcal{V}_1^+).$$

The space $W(p)$ contains $|0\rangle$ and T. Thus $W(p) = (W(p), |0\rangle, T, J)$ defines a sub VOA of V_L. This VOA W(p) is called the triplet VOA.

We denote

$$W^- = |-\alpha_+\rangle, \quad W^0 = Q_+|-\alpha_+\rangle, \quad W^+ = Q_+^2|-\alpha_+\rangle.$$

Then we see that $W^a \in W(p)$, $a \in \{\pm, 0\}$ by Theorem 2.7 and that its conformal dimension is $2p-1$.

Proposition 3.5. *The VOA $W(p)$ is generated by $T(z)$, $W^a(z) = J(W^a : z)$, $a \in \{\pm, 0\}$ as a VOA.*

For the proof we refer the readers to [AM2].

3.5. C_2-finiteness of $W(p)$

We define $W_0(p) = W(p) \cap F_0$. Then $W_0(p)$ is a sub-VOA both of F_0 and $W(p)$. It is easy to see that $T, W^0 \in W_0(p)$.

Proposition 3.6. *The VOA $W_0(p)$ is generated by $T(z)$ and $W^0(z)$ as a VOA.*

For the proof we see [Ada].

Now we denote Zhu's algebra of $W_0(p)$ by $A_0(W_0(p))$. Then by Proposition 3.6 $A_0(W_0(p))$ is a quotient algebra of polynomial ring $C[[T(0)], [W^0(0)]]$.

Then the following important proposition is proved in [Ada].

Proposition 3.7. *Zhu's algebra $A_0(W_0(p))$ is isomorphic to*

$$\mathbb{C}[[T(0)], [W^0(0)]]/\langle G\rangle$$

where $\langle G\rangle$ is the ideal generated by an element

$$G = ([W^0(0)]^2 - c([T(0)] - h_p(0))) \prod_{s=1}^{p-1} ([T(0)] - h_s(0))^2$$

where $c = \dfrac{(4p)^{2p-1}}{((2p-1)!)^2}$.

Now recall that

$$\mathfrak{p} = \mathfrak{p}(W(p)) = W(p)/C_2W(p)$$

where

$$C_2(W(p)) \equiv \{A_{(n)}B : A, B \in W(p) \quad n \leq -2\}.$$

It is known that the associative algebra $A_0(W(p))$ has a filtration $G_\bullet A_0(W(p))$, so that we have a surjection

$$\mathfrak{p}(W(p)) \longrightarrow Gr^G_\bullet A_0(W(p)) \longrightarrow 0$$

as Poisson algebras, by $[T(0)] \mapsto [T]$, $[W^a(0)] \mapsto [W^a]$, $a \in \{\pm, b\}$ [MNT].

The following proposition and corollary are proved in [AM2].

Proposition 3.8. *There exist the following relations on the Poisson algebra* $\mathfrak{p}(W(p))$.

(1) $[W^\pm]^2 = 0, \quad [W^0]^2 + [W^-][W^+] = 0, \quad [W^0][W^\pm] = 0.$
(2) $[W^0]^2 = c[T]^{2p-1} \quad (c \neq 0).$
(3) $[T]^p[W^a] = 0 \quad (a \in \{\pm, 0\}).$
(4) $[T]^{3p-1} = 0.$
(5) $\{[T], [W^a]\} = c_a[T]^p \ (c_0 \neq 0, c_\pm = 0).$
(6) *Other Poisson brackets are zero.*

Corollary 3.9. *We have the following:*

(1) $\dim \mathfrak{p}(W(p)) \leq 6p - 1.$
(2) $\dim A_0(W(p)) \leq 6p - 1.$
(3) $W(p)$ *satisfies Zhu's* C_2*-finiteness condition.*

3.6. The abelian category of $W(p)$-modules

Now we denote by $W(p)$-mod the abelian category of left $W(p)$-modules.

By Theorem 3.2, we have the following.

Proposition 3.10. *The abelian category* $W(p)$*-mod has following properties.*

(1) *The category* $W(p)$*-mod is Noetherian and Artinian, i.e., if* $M_0 \subset M_1 \subset \ldots$ *is an increasing sequence of objects of* $W(p)$*-mod then* $M_n = M_{n-1} = \ldots$ *for some* $n \geq 0$*, and if* $M_0 \supset M_1 \supset \ldots$ *is a decreasing sequence of objects of* $W(p)$*-mod then* $M_n = M_{n+1} = \ldots$ *for some* $n \geq 0$.
(2) *The number of isomorphism classes of simple objects in* $W(p)$*-mod is finite.*

Proposition 3.11. *For $1 \le s \le p-1$ the linear maps*

$$Q_-^{[d_s^\pm]} : \mathcal{V}_s^\pm \longrightarrow \mathcal{V}_s^\mp$$
$$d_s^+ = p - s, \quad d_s^- = s$$

are $W(p)$-module maps. We define $W(p)$-module $\mathcal{X}_s^\pm$ by the formulas;

$$\mathcal{X}_s^\pm = \ker Q_-^{[d_s^\mp]}(\mathcal{V}_s^\mp \longrightarrow \mathcal{V}_s^\pm).$$

Then we have the following exact sequences of $W(p)$-modules;

$$0 \longrightarrow \mathcal{X}_s^\mp \longrightarrow \mathcal{V}_s^\pm \longrightarrow \mathcal{X}_s^\pm \longrightarrow 0.$$

We denote $\mathcal{X}_p^\pm = \mathcal{V}_p^\pm$, $\mathcal{X}_0 = \mathcal{X}_p^-$, $\mathcal{X}_p = \mathcal{X}_p^+$ where those are viewed $W(p)$-modules. The duality on $W(p)$ is given as follows:

Proposition 3.12. *On $W(p)$ the following formulas hold.*

(1) $T(n)^\sigma = T(-n), \quad n \in \mathbb{Z},$
$W^a(n)^\sigma = -W^a(-n), \quad a \in \{\pm, 0\},\ n \in \mathbb{Z}.$
(2) $D(\mathcal{V}_s^\pm) \simeq \mathcal{V}_s^\mp, \quad 1 \le s \le p-1,$
$D(\mathcal{X}_s^\pm) \simeq \mathcal{X}_s^\pm, \quad 1 \le s \le p.$

Define $\bar{X}_s^\pm \in A_0(W(p))$-mod by the following;

$$\begin{aligned}
&\bar{X}_s^+ \equiv \mathcal{X}_s^+[h_s(0)] = C|\lambda_s(0)\rangle, \quad 1 \le s \le p-1,\\
&\bar{X}_s^- \equiv \mathcal{X}_s^-[h_s(1)] = C|\lambda_{-s}(0)\rangle \oplus CQ_+|\lambda_{-s}(0)\rangle, \quad 1 \le s \le p-1,\\
&\bar{X}_p = \bar{X}_p^+ = \mathcal{X}_p^+[h_p(0)] = C|\lambda_p(0)\rangle,\\
&\bar{X}_0 = \bar{X}_p^- = \mathcal{X}_0^-[h_0(0)] = C|\lambda_0(0)\rangle \oplus CQ_+|\lambda_0(0)\rangle.
\end{aligned}$$

Proposition 3.13. *$A_0(W(p))$-modules*

$$\bar{X}_s^\varepsilon : 1 \le s \le p, \varepsilon = \pm$$

are irreducible $A_0(W(p))$-modules, and all are mutually inequivalent among themselves.

Proof. By the definition of h_s, it holds that

$$h_{p-1}(1) > \cdots > h_1(1) > h_0(0) > h_1(0) > \cdots > h_p(0).$$

Therefore all $\bar{X}_s^\pm$ $(1 \le s \le p)$ are inequivalent.

For $0 \le s \le p-1$, by direct calculations we have

$$
\begin{aligned}
W^0[0]|\lambda_{-s}(0)\rangle &= \binom{-s-1}{2p-1}|\lambda_{-s}(0)\rangle,\\
W^0[0]Q_+|\lambda_{-s}(0)\rangle &= -\binom{-s-1}{2p-1}Q_-|\lambda_{-s}(0)\rangle,\\
W^+[0]|\lambda_{-s}(0)\rangle &= 2\binom{-s-1}{2p-1}Q_+|\lambda_{-s}(0)\rangle,\\
W^+[0]Q_+|\lambda_{-s}(0)\rangle &= 0,\\
W^-[0]|\lambda_{-s}(0)\rangle &= 0,\\
W^-[0]Q_+|\lambda_{-s}(0)\rangle &= -\binom{-s-1}{2p-1}|\lambda_{-s}(0)\rangle.
\end{aligned}
$$

Therefore these are all irreducible $A_0(W(p))$-modules. Q.E.D.

By the Proposition 3.12, we have a family of irreducible $W(p)$-modules $\mathcal{X}_s^\varepsilon$, $1 \le s \le p$ and $\varepsilon = \pm$.

The structure of $\mathcal{X}_s^\pm$ as $\mathcal{L}$-modules are described in Theorem 2.7.

3.7. The structure of $W(p)$-modules $\mathcal{V}_s^\pm$, $1 \le s \le p-1$

The structure of $W(p)$-modules $\mathcal{V}_s^\pm$ is described as follows.

Proposition 3.14. *For $1 \le s \le p-1$, we have:*

(1) *The following equations are satisfied on $\mathcal{V}_s^+$.*
  (a) $\eta_s|\lambda_{-s}(1)\rangle = Q_+|\lambda_{-s}(0) = cW^+(0)|\lambda_{-s}(0)\rangle \quad (c \ne 0)$,
  (b) $W^-(-s)|\lambda_{-s}(1)\rangle = |\lambda_{-s}(0)\rangle$,
  (c) $W^0(-s)|\lambda_{-s}(1)\rangle = c'Q_+|\lambda_{-s}(0)\rangle \quad (c' \ne 0)$.

(2) *The following equations are satisfied on $\mathcal{V}_s^-$.*
  (a) $\eta_s^\vee W^-(0)|\lambda_s(1)\rangle = c|\lambda_s(0)\rangle \quad (c \ne 0)$,
  (b) $W^-(s)|\lambda_s(1)\rangle = c''|\lambda_s(0)\rangle \quad (c'' \ne 0)$,
  (c) $W^0(s)W^-(0)|\lambda_s(1)\rangle = c'''|\lambda_s(0)\rangle \quad (c''' \ne 0)$.

§4. Construction of Log $W(p)$-modules and structure of $W(p)$-mod

In this section, we construct $W(p)$-modules, $\mathcal{P}_s^\varepsilon$, $1 \le s \le p-1$, $\varepsilon = \pm$, which we call log $W(p)$-modules, by using the logarithmic deformation of VOA $W(p)$ which is given in J. Fjeistad et al. [FFHST]. We show that the dimension of $A_0(W(p))$ is equal to $6p-1$, and give the block decomposition of $A_0(W(p))$-mod.

4.1. Construction of $\mathcal{P}_s^{\pm}$, $1 \leq s \leq p-1$

Let us fix s such that $1 \leq s \leq p-1$, and set

$$\mathcal{P}_s = \mathcal{V}_s^+ \oplus \mathcal{V}_s^-. \tag{4.1}$$

For each $A \in V_L$ we denote

$$A(z) = J^{\mathcal{V}_s^+}(A:z) \oplus J^{\mathcal{V}_s^-}(A:z) \in \mathrm{End}_{\mathbb{C}}(\mathcal{P}_s)[[z,z^{-1}]]. \tag{4.2}$$

Then $\mathcal{P}_s$ becomes V_L-module by $A(z)$ for any s.

We define operators

$$E_s^{\pm}(z) \in \mathrm{End}_{\mathbb{C}}(\mathcal{P}_s)[[z,z^{-1}]]$$

by the following way;

$$E_s^{\pm}(z)|_{\mathcal{V}_s^{\pm}} = Q_-^{[d_s^{\pm}]}(z), \ E_s^{\pm}(z)|_{\mathcal{V}_s^{\mp}} = 0.$$

Then we have

$$E_s^{\pm}(z) \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_s^{\pm}, \mathcal{V}_s^{\mp})[[z,z^{-1}]] \subseteq \mathrm{End}_{\mathbb{C}}(\mathcal{P}_s)[[z,z^{-1}]].$$

For each $P \in U(V)$, we denote

$$P = \rho^{\mathcal{V}_s^+}(P) + \rho^{\mathcal{V}_s^-}(P) \in \mathrm{End}(\mathcal{P}_s). \tag{4.3}$$

Then on $\mathrm{End}(\mathcal{P}_s)[[z,z^{-1}]]$, the following properties are satisfied. The two family of operators

$$\{E_s^+(z), A(z) : A \in V_L\}, \quad \{E_s^-(z), A(z) : A \in V_L\}, \tag{4.4}$$

are mutually local among themselves. Also we have

$$E_s^+(z)E_s^+(w) = 0, \quad E_s^-(z)E_s^-(w) = 0. \tag{4.5}$$

Moreover, we have

$$T(z)E_s^{\pm}(w) \sim \frac{1}{(z-w)^2}E_s^{\pm}(w) + \frac{1}{(z-w)}\partial_w E_s^+(w). \tag{4.6}$$

For each $A \in V_L$, we define

(4.7)

$$\Delta_s^{\pm}(A:z) = (E_s(0)^{\pm}A)(z)\log z + \sum_{n\geq 1} \frac{(-1)^n}{n}(E_s(n)^{\pm}A)(z)z^{-n} \in \mathrm{End}(\mathcal{P}_s)[[z,z^{-1}]][\log z].$$

Remark that for any $A \in W(p)$, we have $(E_s(0)^{\pm}A)(z) = 0$.

The following two theorems can be proved easily by the methods given in [FFHST]. The construction of $W(p)$-module $\mathcal{P}_s^{\pm}$ is our first main result. The analysis of the module structure of $W(p)$-modules $\mathcal{P}_s^{\pm}$ will be a main subjects of this paper.

Theorem 4.1. *There exists a unique degree preserving linear maps*

$$\Delta_s^{\pm} : U(V_L) \longrightarrow \mathrm{End}(\mathcal{P}_s)[\log z], \qquad P \mapsto \Delta_s^{\pm}(P), \tag{4.8}$$

which satisfies the following conditions.

(a) *For any $A \in V_L$, $m \in \mathbb{Z}$,*

$$\Delta_s^{\pm}(A_{(m)}) = \left[\int dz\ z^m (E_s(0)^{\pm}A)(z)\right] \log z + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \int dz\ z^{m-n}(E_s(n)^{\pm}A)(z). \tag{4.9}$$

(b) *For all $P, Q \in U(V_L)$,*

$$\Delta_s^{\pm}(PQ) = \Delta_s^{\pm}(P)Q + P\Delta_s^{\pm}(Q). \tag{4.10}$$

(c) *For all $P, Q \in U$,*

$$\Delta_s^{\pm}(P)\Delta_s^{\pm}(Q) = 0. \tag{4.11}$$

Theorem 4.2. (a) *For $A \in W(p)$, define operators by*

$$J^{\mathcal{P}_s^{\pm}}(A : z) = J^{\mathcal{P}_s}(A : z) + \Delta_s^{\pm}(A : z) \in End_{\mathbb{C}}(\mathcal{P}_s)[[z, z^{-1}]]. \tag{4.12}$$

Then these introduce a $W(p)$-module structure on $\mathcal{P}_s$ for any s. We denote these $W(p)$-modules by

$$(\mathcal{P}_s^{\pm}, J^{\mathcal{P}_s^{\pm}}). \tag{4.13}$$

(b) *We have the following exact sequence of $W(p)$-modules*

$$0 \longrightarrow \mathcal{V}_s^{\mp} \longrightarrow \mathcal{P}_s^{\pm} \longrightarrow \mathcal{V}_s^{\pm} \longrightarrow 0.$$

(c)

$$J^{\mathcal{P}_s^{\pm}}(T, z) = T(z) + E_s^{\pm}(z)z, \tag{4.14}$$

consequently we have

$$\rho^{\mathcal{P}_s^+}(T(0)) = \begin{cases} T(0) + Q_-^{[p-s]}(0) & \text{on } \mathcal{V}_s^+, \\ T(0) & \text{on } \mathcal{V}_s^-, \end{cases} \tag{4.15}$$

$$\rho^{\mathcal{P}_s^-}(T(0)) = \begin{cases} T(0) + Q_-^{[s]}(0) & \text{on } \mathcal{V}_s^-, \\ T(0) & \text{on } \mathcal{V}_s^+. \end{cases} \tag{4.16}$$

4.2. Structure of $\mathcal{P}_s^\pm$, $1 \le s \le p-1$

In this subsection, we fix s such that $1 \le s \le p-1$.

The following two theorems concerning the structures of $W(p)$-modules $\mathcal{P}_s^\pm$ are the most important results of this paper.

Theorem 4.3. *On the $W(p)$-module $\mathcal{P}_s^+ = \mathcal{V}_s^+ \oplus \mathcal{V}_s^-$, the following relations hold.*

$$\begin{aligned} (T(0) - h_s(0))|\lambda_{-s}(1)\rangle &= |\lambda_s(0)\rangle, \\ (T(0) - h_s(0))|\lambda_s(0)\rangle &= 0. \end{aligned} \tag{4.17}$$

$$\begin{aligned} \eta_s|\lambda_{-s}(1)\rangle &= \rho^{\mathcal{V}_s^+}(\eta_s)|\lambda_{-s}(1)\rangle + \Delta_s^+(\eta_s)|\lambda_{-s}(1)\rangle \\ &= Q_+|\lambda_{-s}(0)\rangle + \Delta_s^+(\eta_s)|\lambda_{-s}(1)\rangle. \end{aligned} \tag{4.18}$$

$$Q_+\Delta_s^+(\eta_s)|\lambda_{-s}(1)\rangle = c|\lambda_s(1)\rangle \quad (c \neq 0), \tag{4.19}$$

$$\eta_s|\lambda_s(0)\rangle = \rho^{\mathcal{V}_s^-}(\eta_s)|\lambda_s(1)\rangle = 0. \tag{4.20}$$

$$\eta_s^\vee|\lambda_{-s}(0)\rangle = 0, \tag{4.21}$$

$$\begin{aligned} \eta_s^\vee W^+(0)|\lambda_{-s}(0)\rangle &= \Delta_s^+(C_s^\vee)Q_+|\lambda_{-s}(0)\rangle \\ &= c|\lambda_s(0)\rangle \quad (c \neq 0), \end{aligned} \tag{4.22}$$

$$\eta_s^\vee|\lambda_s(1)\rangle = 0. \tag{4.23}$$

$$\begin{aligned} \eta_s^\vee W^-(0)|\lambda_s(1)\rangle &= \rho^{\mathcal{V}_s^-}(\eta_s^\vee)\rho^{\mathcal{V}_s^-}(W^-(0))|\lambda_s(1)\rangle \\ &= c|\lambda_s(0)\rangle \quad (c \neq 0). \end{aligned}$$

Theorem 4.4. *On the $W(p)$-module $\mathcal{P}_s^- = \mathcal{V}_s^+ \oplus \mathcal{V}_s^-$, the following relations hold.*

(4.24)
$$(T(0) - h_s(0))|\lambda_s(0)\rangle = 0,$$
$$(T(0) - h_s(0))|\lambda_{-s}(1)\rangle = 0,$$

(4.25)
$$\begin{aligned}(T(0) - h_s(1))|\lambda_s(1)\rangle &= Q_+|\lambda_{-s}(0)\rangle \\ &= cW^+(0)|\lambda_{-s}(0)\rangle \quad (c \neq 0),\end{aligned}$$

$$(T(0) - h_s(1))W^-(0)|\lambda_s(1)\rangle = c|\lambda_{-s}(0)\rangle \quad (c \neq 0).$$

(4.26)
$$\begin{aligned}\eta_s|\lambda_s(0)\rangle &= \rho^{\mathcal{V}_s^-}(\eta_s)|\lambda_s(0)\rangle + \Delta_s^-(\eta_s)|\lambda_s(0)\rangle \\ &= \Delta_s^-(\eta_s)|\lambda_s(0)\rangle \\ &= c|\lambda_{-s}(0)\rangle \quad (c \neq 0),\end{aligned}$$

(4.27)
$$\begin{aligned}\eta_s|\lambda_{-s}(1)\rangle &= \rho^{\mathcal{V}_s^+}(\eta_s)|\lambda_{-s}(1)\rangle \\ &= Q_+|\lambda_{-s}(0)\rangle.\end{aligned}$$

$$\begin{aligned}\eta_s^\vee|\lambda_s(1)\rangle &= \rho^{\mathcal{V}_s^-}(\eta_s^\vee)|\lambda_s(1)\rangle + \Delta_s^-(\eta_s^\vee)|\lambda_s(1)\rangle \\ &= \Delta_s^-(\eta_s^\vee)|\lambda_s(1)\rangle \\ &= c|\lambda_{-s}(1)\rangle \quad (c \neq 0).\end{aligned}$$

$$\begin{aligned}\eta_s^\vee|W^-(0)|\lambda_s(1)\rangle &= \rho^{\mathcal{V}_s^-}(\eta_s^\vee)W^-(0)|\lambda_s(1)\rangle + \Delta_s^-(\eta_s^\vee)W^-(0)|\lambda_s(1)\rangle \\ &= \rho^{\mathcal{V}_s^-(\eta_s^\vee)}\rho^{\mathcal{V}_s^-}W^-(0)|\lambda_s(1)\rangle \\ &= c|\lambda_s(1)\rangle \quad (c \neq 0).\end{aligned}$$

$$\eta_s^\vee|\lambda_{-s}(0)\rangle = 0,$$
$$\eta_s^\vee W^+(0)|\lambda_{-s}(0)\rangle = 0.$$

Proof. We will prove Theorem 4.3. Theorem 4.4 can be proved in the same way.

In oder to prove Theorem 4.3, we express the element $\eta_s \in U(\mathcal{L}_{<0})[s]$ by using Bosonic operators $a(-1), a(-2), \ldots$. Consider the vector space

(4.28)
$$\mathfrak{a}_\pm = \sum_{n \geq 1} \mathbb{C}a(\pm n),$$

so that

$$\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-.$$

These elements satisfy the following commutator relations

$$[a(m), a(n)] = m\delta_{m+n,0}.$$

Define degree of $a(n)$ as $-n$, and consider the degreewise completed universal enveloping algebra

$$U(\mathfrak{a}) = \sum_{d \in Z} U(\mathfrak{a})[d]. \tag{4.29}$$

Then $U(\mathfrak{a})$ is a degreewise completed tensor product of two commutative algebras such that

$$\begin{aligned} U(\mathfrak{a}) &= U(\mathfrak{a}_-)\hat{\otimes}U(\mathfrak{a}_+), \\ U(\mathfrak{a}_\pm) &= \mathbb{C}[a(\pm 1), a(\pm 2), \ldots]. \end{aligned} \tag{4.30}$$

Bosonic realization of the energy-momentum tensor

$$T(z) = \frac{1}{2} : a(z)^2 : + \frac{\alpha_0}{2} \partial a(z)$$

defines an algebra homomorphism

$$U(\mathcal{L}_{c_p}) \longrightarrow U(\mathfrak{a}) \otimes \mathbb{C}[a(0)]. \tag{4.31}$$

Consider a Virasoro module map

$$\begin{gathered} M_{h_s(0)} \longrightarrow F_{\lambda_{-s}(1)}, \\ |h_s(0)\rangle \mapsto |\lambda_{-s}(1)\rangle, \end{gathered} \tag{4.32}$$

which is $\mathbb{C}$-linear isomorphism up to $T(0)$ degree $h - h_s(0) \leq s$. Consider $\mathbb{C}$-linear isomorphism (4.33) and (4.34);

$$\begin{gathered} U(\mathcal{L}_{<0}) \longrightarrow M_{h_s(0)} \\ A \mapsto A|h_s(0)\rangle \end{gathered} \tag{4.33}$$

and

$$\begin{gathered} U(\mathfrak{a}_-) \longrightarrow F_{\lambda_{-s}(1)} \\ A \mapsto A|\lambda_{-s}(1)\rangle. \end{gathered} \tag{4.34}$$

By using (4.32), (4.33) and (4.34), we get an algebra homomorphism, for $\phi_{\lambda_{-s}(1)} : U(\mathcal{L}_{<0}) \longrightarrow U(\mathfrak{a}_-)$

$$\begin{array}{ccc} U(\mathcal{L}_{<0}) & \longrightarrow & U(\mathfrak{a}_-) \\ \simeq\downarrow & & \downarrow\simeq \\ M_{h_s(0)} & \longrightarrow & F_{\lambda_{-s}(1)}. \end{array} \tag{4.35}$$

The algebra homomorphism $\phi_{\lambda_{-s}(1)}$ is a $\mathbb{C}$-linear isomorphism in degree $\leq s$.

In $F_{\lambda_{-s}(1)}$, the singular vector for $\mathcal{L}$ in degree $h_s(1)$ is represented by using the screening operator as

$$\begin{aligned} Q_+|\lambda_{-s}(0)\rangle &= \int dz\ e^{\alpha_+\varphi(z)}|\lambda_{-s}(0)\rangle \\ &= e^{\alpha_+\hat{a}} \int dz\ z^{s-1} e^{\alpha_+\varphi_-(z)}|\lambda_{-s}(0)\rangle \\ &= \int dz\ z^{s-1} e^{\alpha_+\varphi_-(z)}|\lambda_{-s}(1)\rangle. \end{aligned} \tag{4.36}$$

Therefore by the map $\phi_{\lambda_{-s}(1)}$, the element η_s is mapped to

$$\phi_{\lambda_{-s}(1)}(\eta_s) = \int dz\ z^{s-1} e^{\alpha_+\varphi_-(z)} \in U(\mathfrak{a}_-). \tag{4.37}$$

About the element

$$\bar{V}_{\alpha_+}(z) = e^{\alpha_+\varphi_-(z)} e^{\alpha_+\varphi_+(z)} \in U(\mathfrak{a})[[z, z^{-1}]]$$

we have the following formula in $U(\mathfrak{a})$.

$$\int dz\ \bar{V}_{\alpha_+}(z) z^{s-1} = \int dz\ z^{s-1} e^{\alpha_+\varphi_-(z)} + \sum_{n\geq 1} B_n a_n$$

where $B_n \in U(\mathfrak{a})[s+n]$.

The map

$$\Delta_s^+ : U(\mathfrak{a}) \longrightarrow \mathrm{End}(\mathcal{P}_s^+)[\log z]$$

factors through

$$\Delta_s^+ : U(\mathfrak{a}) \longrightarrow U(V_L) \longrightarrow \mathrm{End}(\mathcal{P}_s^+)[\log z]$$

and Δ_s^+ is a degree preserving map which satisfies

$$\Delta_s^+(P \cdot Q) = \Delta_s^+(P) J^{P_s}(Q) + J^{P_s}(P) \Delta_s^+(Q).$$

Then we have

$$\Delta_s^+(B_n\ a_n)|\lambda_{-s}(1)\rangle = 0,$$

and therefore

$$\begin{aligned}\Delta_s^+(\eta_s)|\lambda_{-s}(1)\rangle &= \Delta_s^+\left(\int dz\ z^{s-1}e^{\alpha_+\varphi_-(z)}\right)|\lambda_{-s}(1)\rangle \\ &= \Delta_s^+\left(\int dz\ z^{s-1}\bar{V}_{\alpha_+}(z)\right)|\lambda_{-s}(1)\rangle.\end{aligned}$$

Now we see

$$\begin{aligned}&\langle\lambda_s(1)|Q_+\Delta_s^+(\eta_s)|\lambda_{-s}(1)\rangle \\ &= \int dz\ z^{s-1}\langle\lambda_s(1)|Q_+\Delta_s^+(\bar{V}_{\alpha_+}(z))|\lambda_{-s}(1)\rangle \\ &= \int dy\int dw\int dz\ z^{s-1}\langle\lambda_s(1)|V_{\alpha_+}(y)Q_+^{[p-s]}(w)\bar{V}_{\alpha_+}(z)|\lambda_{-s}(1)\rangle \\ &= \int dy\int dw_1\cdots dw_{p-s}\int dz\ z^{s-1}\langle\lambda_s(1)|V_{\alpha_+}(y)V_{\alpha_-}(w_1) \\ &\qquad\qquad\cdots V_{\alpha_-}(w_{p-s})\bar{V}_{\alpha_+}(z)|\lambda_{-s}(1)\rangle \\ &\neq 0.\end{aligned}$$

Consequently we get

$$Q_+\Delta_s^+(\eta_s)|\lambda_{-s}(1)\rangle = \text{const.}|\lambda_s(1)\rangle \quad (\text{const.} \neq 0).$$

Proof of $\Delta_s^+(C_s^V)Q_+|\lambda_{-s}(0)\rangle = \text{const.}|\lambda_s(0)\rangle$ (const. $\neq 0$) can be done exactly in the same way. Q.E.D.

By using the results of §4-2, it is easy to verify the following structure of projective module $\mathcal{P}_s^\pm$.

Proposition 4.5. (1) *The socles sequence of* $\mathcal{P}_s^+$ *is*

$$S_1(\mathcal{P}_s^+) \subseteq S_2(\mathcal{P}_s^+) \subseteq S_3(\mathcal{P}_s^+) = \mathcal{P}_s^+,$$

$$S_1(\mathcal{P}_s^+) \simeq \mathcal{X}_s^+,\ S_2(\mathcal{P}_s^+)/S_1(\mathcal{P}_s^+) \simeq \mathcal{X}_s^- \oplus \mathcal{X}_s^-,\ S_3(\mathcal{P}_s^+)/S_2(\mathcal{P}_s^+) \simeq \mathcal{X}_s^+.$$

(2) *The socles sequence of* $\mathcal{P}_s^-$ *has following structures*

$$S_1(\mathcal{P}_s^-) \subseteq S_2(\mathcal{P}_s^-) \subseteq S_3(\mathcal{P}_s^-) = \mathcal{P}_s^-,$$

$$S_1(\mathcal{P}_s^-) \simeq \mathcal{X}_s^-,\ S_2(\mathcal{P}_s^-)/S_1(\mathcal{P}_s^-) \simeq \mathcal{X}_s^+ \oplus \mathcal{X}_s^+,\ S_3(\mathcal{P}_s^-)/S_2(\mathcal{P}_s^-) \simeq \mathcal{X}_s^-.$$

4.3. Structure of $A_0(W(p))$

Let us consider the $W(p)$-module $\mathcal{P}_s^+$, $1 \le s \le p-1$. Define

$$\bar{P}_s^+ = \mathcal{P}_s^+[h_s(0)] = \mathbb{C}|\lambda_s(0)\rangle + \mathbb{C}|\lambda_{-s}(0)\rangle.$$

Then $\bar{P}_s^+$ is a $A_0(W(p))$-module. By Theorem 4.3 we see

$$\rho_s^{\mathcal{P}_s^+}(T(0) - h_s(0))|\lambda_{-s}(1)\rangle = |\lambda_s(0)\rangle,$$

$$\rho_s^{\mathcal{P}_s^+}(T(0) - h_s(0))|\lambda_s(0)\rangle = 0.$$

For each $1 \le s \le p$ and $\varepsilon = \pm$, we define finite dimensional algebra I_s^ε by the following way.

(1) Case 1. $1 \le s \le p-1$, $\varepsilon = 1$.
Consider the algebra homomorphism

$$\rho_s^+ : A_0(W(p)) \longrightarrow \mathrm{End}(\bar{P}_s^+) = M_2(\mathbb{C}).$$

Then Image $(\rho_s^+) \subseteq M_2(\mathbb{C})$ contain two dimensional algebra

$$I_s^+ = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

(2) Case 2. $s = p$, $\varepsilon = \pm$ or $1 \le s \le p-1$, $\varepsilon = -1$.
Consider the algebra homomorphism for any s

$$\rho_s^\varepsilon : A(W(p)) \longrightarrow \mathrm{End}(\bar{X}_s^\varepsilon).$$

Since $\bar{X}_s^\varepsilon$ is an irreducible $A_0(W(p))$-module, the map ρ_s^ε is surjective. Set

$$I_s^\varepsilon = \mathrm{Image}(\rho_s^\varepsilon) = \mathrm{End}(\bar{X}_s^\varepsilon).$$

Theorem 4.6. (1) *The algebra $A_0(W(p))$ is isomorphic to* $I = \sum_{s=1}^{p} \sum_{\varepsilon=\pm} I_s^\varepsilon$.

(2) *The center of $A_0(W(p))$ is generated by $[T(0)]$.*

(3) *The set of inequivalent irreducible $A_0(W(p))$-modules is $\{ \bar{X}_s^\varepsilon, 1 \le s \le p, \varepsilon = \pm\}$.*

Proof. We see $\dim_{\mathbb{C}} I = 6p - 1$ and $\dim_{\mathbb{C}} A_0(p) \le 6p - 1$. On the other hand, by definition we see $\dim_{\mathbb{C}} I \le \dim_{\mathbb{C}} A_0(p)$. So we get the proposition. Q.E.D.

4.4. $A_0(W(p))$-mod

For each $s = 1, \dots, p$ and $\varepsilon = \pm$, we define the full abelian subcategories $\bar{C}_s^\varepsilon$ of $A_0(W(p))$-mod such that an element $M \in A_0(W(p))$-mod belongs to $\bar{C}_s^\varepsilon$, if and only if the irreducible components of M are $\bar{X}_s^\varepsilon$. Then we have the following theorem from Theorem 4.6.

Theorem 4.7. (1) *The abelian category $A_0(W(p))$-mod has the following block decomposition*

$$A_0(W(p))\text{-}mod = \sum_{s=1}^{p} \sum_{\varepsilon=\pm} \bar{C}_s^\varepsilon.$$

(2) *For $s = p$, $\varepsilon = \pm$ or $1 \leq s \leq p-1$, $\varepsilon = -$, the abelian category $\bar{C}_s^\varepsilon$ is semi-simple with a simple object $\bar{X}_s^\varepsilon$.*

(3) *For $1 \leq s \leq p-1$, the set of indecomposable objects in the abelian category $\bar{C}_s^+$ is $\{\bar{X}_s^+, \bar{P}_s^+\}$. Moreover, we have the non-trivial exact sequence of $A_0(W(p))$-mod*

$$0 \longrightarrow \bar{X}_s^+ \longrightarrow \bar{P}_s^+ \longrightarrow \bar{X}_s^+ \longrightarrow 0.$$

4.5. Block decomposition of the abelian category $W(p)$-mod

The following Theorem 4.8 is proved in [AM1], [AM2]

Theorem 4.8. $\mathrm{Ext}^1_{W(p)}(\mathcal{X}_{s_1}^{\varepsilon_1}, \mathcal{X}_{s_2}^{\varepsilon_2}) = 0$ *for $1 \leq s_1 \neq s_2 \leq p$ and $\varepsilon_1, \varepsilon_2 = \pm$.*

For each $0 \leq s \leq p$ we denote by C_s the full abelian category of $W(p)$-mod such that $M \in W(p)$-mod belong to C_s if and only if M has Jordan–Hölder sequence whose factors are $\mathcal{X}_s^\pm$ if $1 \leq s \leq p-1$, and $\mathcal{X}_s$ if $s = 0$ or p, respectively.

Then by virtue of Theorem 4.8, we have the following.

Theorem 4.9. *The abelian category $W(p)$-mod has the following block decomposition*

$$W(p)\text{-}mod = \sum_{s=0}^{p} C_s,$$

with the properties:

(1) *Each element of $W(p)$-mod has the unique decomposition*

$$M = \sum_{s=0}^{p} C_s \quad \text{with} \quad M_s \in C_s.$$

(2) *For any* $M \in C_s$, $N \in C_{s'}$,

$$\mathrm{Ext}^\bullet(M,N) = 0 \quad \textit{if} \quad s \neq s'.$$

Proposition 4.10. *For each* $0 \leq s \leq p$, *any element* $M \in C_s$ *has following eigenspace decompositions*

$$M = \sum_{h \in H_s} M[h],$$

where $M[h] = \{m \in M : (T(0) - h)^n m = 0$ *for some* $n \geq 1\}$, *and* $\dim_{\mathbb{C}} M[h] < \infty$ *for all* $h \in \mathbb{C}$.

The following Proposition is very important in this paper.

Proposition 4.11. *Let* s *be an integer such that* $1 \leq s \leq p-1$, *and let* $M, N \in C_s$, *and* $f : M \to N$ *be a* $W(p)$*-module map. If* f *is a vector space isomorphism of degree* h, *for* $h - h_s(0) \leq s$, *then* f *is a* $W(p)$*-module isomorphism.*

Proof. Category C_s has simple objects $\mathcal{X}_s^{\pm}$, and the highest weight of $\mathcal{X}_s^+$ and $\mathcal{X}_s^-$ are $h_s(0)$ and $h_s(1)$, respectively. Note that $h_s(1) - h_s(0) = s$.

Consider the kernel and the cokernel of f, then by the condition of $\mathcal{P}$ the weight h satisfies $h - h_s(0) > s$. This shows that the kernel and the cokernel of f must be zero. Q.E.D.

Proposition 4.12. (1) *For* $s = p$ *and* $\varepsilon = \pm$ *or* $1 \leq s \leq p-1$, $\varepsilon = -1$, *we have isomorphism of* $W(p)$*-modules*

$$U(W(p)) \underset{F_0(W(p))}{\otimes} \bar{X}_s^{\varepsilon} \simeq \mathcal{X}_s^{\varepsilon}.$$

(2) *For* $1 \leq s \leq p-1$, $\varepsilon = \pm$, *an element* M *of* C_s *is a direct sum of* $\mathcal{X}_s^{\varepsilon}$ *if and only if* $M = \sum_{h \in H_s^{\varepsilon}} M[h]$.

§5. Projectivity of $\mathcal{P}_s^{\pm}$

In this section we show that $\mathcal{P}_s^{\pm}$, $1 \leq s \leq p$, are projective covers of simple modules $\mathcal{X}_s^{\pm}$, $1 \leq s \leq p$.

5.1. The structure of $\mathrm{Ext}^1(\mathcal{X}_s^{\varepsilon}, \mathcal{X}_{s'}^{\varepsilon'})$

The following two theorems are part of our main results.

Theorem 5.1. *For* $1 \leq s \leq p$, $\varepsilon = \pm$

$$\mathrm{Ext}^1(\mathcal{X}_s^{\varepsilon}, \mathcal{X}_s^{\varepsilon}) = 0. \tag{5.1}$$

Proof. We divide the proof into two cases.

Case 1. $s = p$ and $\varepsilon = \pm$, or $1 \leq s \leq p-1$, $\varepsilon = -1$.
We denote $X = \mathcal{X}_s^\varepsilon$ for simplicity, and consider an exact sequence of $W(p)$-modules

$$0 \longrightarrow X \longrightarrow E \longrightarrow X \longrightarrow 0.$$

The lowest $T(0)$ degree part of this exact sequence gives an exact sequence of $A_0(W(p))$-modules

$$0 \longrightarrow \bar{X} \longrightarrow \bar{E} \longrightarrow \bar{X} \longrightarrow 0.$$

Then this exact sequence belongs to the block $\bar{C}_s^\varepsilon$. This case $\bar{C}_s^\varepsilon$ is a semi-simple category whose simple object is X. So we get $E = X \oplus X$ as $W(p)$-module by Proposition 4.11(1).

Case 2. $1 \leq s \leq p-1$, $\varepsilon = +$.
We denote $X = \mathcal{X}_s^+$ and consider an exact sequence of $W(p)$-modules

$$0 \longrightarrow X \longrightarrow E \longrightarrow X \longrightarrow 0.$$

Then the lowest $T(0)$ degree part of this sequence gives an exact sequence of $A_0(W(p))$-modules

$$0 \longrightarrow \bar{X} \longrightarrow \bar{E} \longrightarrow \bar{X} \longrightarrow 0.$$

This sequence belongs to the block $\bar{C}_s^+$, $1 \leq s \leq p-1$, and $E = X \oplus X$ as $W(p)$-modules if and only if $\bar{E} = \bar{X} \oplus \bar{X}$ in $\bar{C}_s^+$, that is, $[T(0)]$ acts on $\bar{E}$ semi-simple. Consider the $\mathcal{L}$-module $M = U(\mathcal{L})(\bar{E}) \subseteq E$. Then as $\mathcal{L}$-modules it has the following exact sequence

$$0 \longrightarrow L_{h_s(0)} \longrightarrow M \longrightarrow L_{h_s(0)} \longrightarrow 0.$$

Then by Proposition 2.11 for Virasoro modules gives $\bar{E} = \bar{X} \oplus \bar{X}$ as $A_0(W(p))$-modules. Thus by Proposition 4.11(1) we have

$$E = X \oplus X$$

as $W(p)$-modules. Q.E.D.

We define the $W(p)$-module $\mathcal{Y}_s^+$, $1 \leq s \leq p-1$, by the following exact sequence;

$$0 \longrightarrow \mathcal{X}_s^+ \longrightarrow \mathcal{P}_s^+ \longrightarrow \mathcal{Y}_s^+ \longrightarrow 0. \tag{5.2}$$

Theorem 5.2. *For* $1 \leq s \leq p-1$ *we have*

$$\mathrm{Ext}^1(\mathcal{X}_s^\pm, \mathcal{X}_s^\mp) = \mathbb{C}^2.$$

Proof. We first prepare some notations. By the duality in $W(p)$-mod, we have $D(\mathcal{X}_s^\varepsilon) \cong \mathcal{X}_s^\varepsilon$, thus it is suficient to prove $\mathrm{Ext}^1(\mathcal{X}_s^+, \mathcal{X}_s^-) \simeq \mathbb{C}^2$.

First we fix the following elements of $\mathcal{X}_s^\pm$;

$$u = |\lambda_s(0)\rangle \in \mathcal{X}_s^+ \subseteq \mathcal{V}_s^-,$$

$$v_- = |\lambda_{-s}(0)\rangle, \quad v_+ = Q_+|\lambda_{-s}(0)\rangle \in \mathcal{X}_s^- \subseteq \mathcal{V}_s^+.$$

The element

$$v_+ = Q_+|\lambda_{-s}(0)\rangle \in F_{\lambda_{-s}(1)}[h_s(1)]$$

is a singlar vector of Virasoro module $F_{\lambda_{-s}(1)}[h_s(1)]$, and the sequence of $\mathcal{L}_{c_p}$-module maps;

$$M_{h_s(2)} \longrightarrow M_{h_s(0)} \longrightarrow F_{\lambda_{-s}(1)}$$

is exact, so we have

$$v_+ = \eta_s|\lambda_{-s}(1)\rangle.$$

Now we give a proof of Theorem 5.2.

Consider an exact sequence and the elements in E, and fix element $\tilde{u} \in E$,

$$0 \longrightarrow \mathcal{X}_s^- \longrightarrow E \longrightarrow \mathcal{X}_s^+ \longrightarrow 0$$

$$\mathcal{X}_s^- \ni v_+, v_-, \qquad E \ni \tilde{u} \mapsto u$$

such that elements $\tilde{u} \in E[h_s(0)]$ is mapped to u in $\mathcal{X}_s^+$, which is uniquely determined. Set

$$\eta_s \tilde{u} = a_+ v_+ + a_- v_-.$$

Note that if $a_+ = 0$ and $a_- = 0$, then $[E] = 0$ in $\mathrm{Ext}^1(\mathcal{X}_s^+, \mathcal{X}_s^-)$.

By the definition of $W(p)$-module $\mathcal{P}_s^+$ we see the exact sequence

$$0 \longrightarrow \mathcal{V}_s^- \longrightarrow \mathcal{P}_s^+ \longrightarrow \mathcal{V}_s^+ \longrightarrow 0.$$

We define two $W(p)$-submodules E_1 and E_2 of $\mathcal{Y}_s$ by

$$E_1 = \mathcal{V}_s^-/\mathcal{X}_s^+ \hookrightarrow \mathcal{Y}_s^+ = \mathcal{P}_s^+/\mathcal{X}_s^+,$$
$$E_2 = U(W(p))(\mathbb{C}|\lambda_{-s}(0)\rangle \oplus \mathbb{C}Q_+|\lambda_{-s}(0)\rangle) \subseteq \mathcal{Y}_s^+.$$

Then by using Theorem 4.4, it is easy to show that the $W(p)$-modules E_1 and E_2 are both isomorphic to the $W(p)$-module $\mathcal{X}_s^+$.

Consequently, the $W(p)$-module $\mathcal{Y}_s/E_1$ is canonically isomorphic to $\mathcal{V}_s^+$. Let us introduce a $W(p)$-module $\mathcal{Y}_s^+/E_2 = \mathcal{V}_s^{+\vee}$. Then we have exact sequences

$$0 \longrightarrow \mathcal{X}_s^- \longrightarrow \mathcal{V}_s^+ \longrightarrow \mathcal{X}_s^+ \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{X}_s^- \longrightarrow \mathcal{V}_s^{+\vee} \longrightarrow \mathcal{X}_s^+ \longrightarrow 0.$$

For $\mathcal{V}_s^+$, we have $a_+ = 1$ and $a_- = 0$. Also for $\mathcal{V}_s^{+\vee}$, we have $a_+ = 0$ and $a_- = 1$. Consequently $[\mathcal{V}_s^+]$ and $[\mathcal{V}_s^-]$ are linearly independent in $\mathrm{Ext}^1(\mathcal{X}_s^+, \mathcal{X}_s^-)$. Q.E.D.

Proposition 5.3. *The subcategories C_0 and C_p of $W(p)$-mod are semi-simple with only one simple object $\mathcal{X}_0$, and $\mathcal{X}_p$, respectively.*

Proof. By Theorem 5.2 we have $\mathrm{Ext}^1(\mathcal{X}_0, \mathcal{X}_0) = 0$, $\mathrm{Ext}^1(\mathcal{X}_p, \mathcal{X}_p) = 0$. Therefore we have proved the statement. Q.E.D.

We denote $\mathcal{P}_p = \mathcal{P}_p^+ = \mathcal{V}_p = \mathcal{X}_p$ and $\mathcal{P}_0 = \mathcal{P}_p^- = \mathcal{V}_0 = \mathcal{X}_0$. Then these two modules are projective modules in $W(p)$-mod.

5.2. Projectivity of $\mathcal{P}_s^+$, $1 \le s \le p-1$

We fix s such that $1 \le s \le p-1$.

Proposition 5.4. *One has*

$$\mathrm{Hom}(\mathcal{V}_s^+, \mathcal{X}_s^+) = \mathbb{C}, \quad \mathrm{Hom}(\mathcal{V}_s^{+\vee}, \mathcal{X}_s^+) \simeq \mathbb{C},$$
$$\mathrm{Hom}(\mathcal{V}_s^+, \mathcal{X}_s^-) = 0, \quad \mathrm{Hom}(\mathcal{V}_s^{+\vee}, \mathcal{X}_s^-) = 0,$$
$$\mathrm{Hom}(\mathcal{Y}_s^+, \mathcal{X}_s^+) = \mathbb{C}, \quad \mathrm{Hom}(\mathcal{Y}_s^+, \mathcal{X}_s^-) = 0, \tag{5.3}$$

$$\mathrm{Ext}^1(\mathcal{V}_s^+, \mathcal{X}_s^-) \simeq \mathbb{C},$$
$$\mathrm{Ext}^1(\mathcal{V}_s^{+\vee}, \mathcal{X}_s^-) \simeq \mathbb{C}, \tag{5.4}$$

$$\mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^-) = 0. \tag{5.5}$$

Proof. These follow from the definitions and results obtained in §5-1. Here we only prove (5.5). Consider the exact sequence

$$0 \longrightarrow (\mathcal{X}_s^-)^2 \longrightarrow \mathcal{Y}_s^+ \longrightarrow \mathcal{X}_s^+ \longrightarrow 0,$$

which gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}((\mathcal{X}_s^-)^2, \mathcal{X}_s^-) \longrightarrow \mathrm{Ext}^1(\mathcal{X}_s^+, \mathcal{X}_s^-) \longrightarrow \mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^-) \longrightarrow 0.$$

Therefore as discussed in §5-1, we have

$$\mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^-) = 0.$$

Q.E.D.

Proposition 5.5. *The linear map*

$$U(W(p)) \underset{F_0(U(W(p)))}{\otimes} \bar{X}_s^+ \longrightarrow \mathcal{Y}_s^+ \tag{5.6}$$

is an isomorphism of $W(p)$*-modules.*

Proof. Consider canonical map

$$U(W(p)) \underset{F_0(U(W(p)))}{\otimes} \bar{X}_s^+ \longrightarrow \mathcal{Y}_s^+.$$

This map is surjective, and the kernel is isomorphic to $(\mathcal{X}_s^-)^l$ for some $l \geq 0$. But $\mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^-) = 0$. Therefore we have $l = 0$. Q.E.D.

Proposition 5.6.

$$\mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^+) \simeq \mathbb{C}. \tag{5.7}$$

Proof. Since $[\mathcal{P}_s^+] \neq 0$ in $\mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^+)$, we have $\dim_{\mathbb{C}} \mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^+) \geq 1$. Consider an exact sequence of $W(p)$-modules

$$0 \longrightarrow \mathcal{X}_s^+ \longrightarrow E \longrightarrow \mathcal{Y}_s^+ \longrightarrow 0,$$

and fix elements $u_0 \in \mathcal{X}_s^+[h_s(0)] \simeq \mathbb{C}$ and $u_1 \in \mathcal{X}_s^+[h_s(0)] \simeq \mathbb{C}$. Take an element $\tilde{u}_0 \in E[h_s(0)] \simeq \mathbb{C}^2$ which is mapped to u_0 in $\mathcal{Y}_s^+$. Then we have $(T(0) - h_s(0))\tilde{u} = cu$ for some $c \in \mathbb{C}$. We assume $c = 0$, then by Proposition 5.4 we have a following $W(p)$-module map

$$\mathcal{Y}_s^+ = U(W(p)) \underset{F_0(U(W(p)))}{\otimes} \bar{X}_s^+ \longrightarrow E,$$

which is a lifting of $E \to \mathcal{Y}_s^+ \to 0$. Thus $[E] = 0$ in $\mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^+)$. This shows that $\dim_{\mathbb{C}} \mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^+) \leq 1$. Therefore we get the result. Q.E.D.

Proposition 5.7.

$$\mathrm{Ext}^1(\mathcal{P}_s^+, \mathcal{X}_s^+) = 0. \tag{5.8}$$

Proof. This follows easily by the following exact sequence and Proposition 5.6,

$$0 \longrightarrow \mathcal{X}_s^+ \longrightarrow \mathcal{P}_s^+ \longrightarrow \mathcal{Y}_s^+ \longrightarrow 0.$$

Q.E.D.

Proposition 5.8. *For any* s *we have*

$$\mathrm{Ext}^1(\mathcal{P}_s^+, \mathcal{X}_s^-) = 0. \tag{5.9}$$

Proof. Consider an exact sequence

$$0 \longrightarrow \mathcal{X}_s^+ \longrightarrow \mathcal{P}_s^+ \longrightarrow \mathcal{Y}_s^+ \longrightarrow 0.$$

Since $\mathrm{Ext}^1(\mathcal{Y}_s^+, \mathcal{X}_s^-) = 0$ this gives

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{P}_s^+, \mathcal{X}_s^-) \longrightarrow \mathrm{Ext}^1(\mathcal{X}_s^+, \mathcal{X}_s^-). \tag{5.10}$$

Let us consider an exact sequence

$$0 \longrightarrow \mathcal{X}_s^- \longrightarrow E \longrightarrow \mathcal{P}_s^+ \longrightarrow 0, \tag{5.11}$$

and define $\bar{u}_1 = |\lambda_s(0)\rangle$, $\bar{u}_0 = |\lambda_{-s}(0)\rangle$ in $\mathcal{P}_s^+$. Take the elements $u_i \in E$ which are mapped to $\bar{u}_i \in \mathcal{P}_s^+$ for $i = 0, 1$.

Then we have $(T(0) - h_1(0))u_0 = u_1$, $(T(0) - h_1(0))u_1 = 0$ and $T(n)u_i = 0$, for $n \geq 1$, $i = 0, 1$. By Proposition 2.11, we have $\eta_s(u_1) = 0$. This shows that $[E] = 0$ in $\mathrm{Ext}^1(\mathcal{X}_s^+, \mathcal{X}_s^-)$. Consequently $[E] = 0$ in $\mathrm{Ext}^1(\mathcal{P}_s^+, \mathcal{X}_s^-)$. Q.E.D.

Theorem 5.9. (1) *$\mathcal{P}_a^+$ are projective $W(p)$-modules.*
(2) *For all s, $\mathcal{P}_s^+ \to \mathcal{X}_s^+ \to 0$ are projective covers.*

Proof. These are direct consequences of Proposition 5.7 and Theorem 5.8. Q.E.D.

Proposition 5.10. (1) $\mathrm{Ext}^1(\mathcal{V}_s^+, \mathcal{V}_s^-) \simeq \mathbb{C}$, $\mathrm{Ext}^1(\mathcal{V}_s^{+\vee}, \mathcal{V}_s^{-\vee}) \simeq \mathbb{C}$.
(2) *These two vector spaces in (1) have generators $[\mathcal{P}_s^+]$.*

Proof. Consider exact sequences

$$0 \longrightarrow \mathcal{V}_s^- \longrightarrow \mathcal{P}_s^+ \longrightarrow \mathcal{V}_s^+ \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{V}_s^{-\vee} \longrightarrow \mathcal{P}_s^+ \longrightarrow \mathcal{V}_s^{+\vee} \longrightarrow 0.$$

Then statements follow immediately. Q.E.D.

Proposition 5.11. *One has*

$$\mathrm{Ext}^1(\mathcal{V}_s^+, \mathcal{V}_s^{-\vee}) = 0, \tag{5.12}$$

$$\mathrm{Ext}^1(\mathcal{V}_s^{+\vee}, \mathcal{V}_s^-) = 0.$$

Proof. The same as the one for Proposition 5.10. Q.E.D.

Proposition 5.12. (1) $D(\mathcal{P}_s^+) \simeq \mathcal{P}_s^+$.
(2) *$\mathcal{P}_s^+$ is an injective module.*

Proof. (2) follows from (1), since $\mathcal{P}_s^+$ is a generator of $\mathrm{Ext}^1(\mathcal{V}_s^+,\mathcal{V}_s^-)\simeq\mathbb{C}$. But $D(\mathcal{V}_s^{\pm})\simeq\mathcal{V}_s^{\mp}$, and then we have $D(\mathcal{P}_s^+)\simeq\mathcal{P}_s^+$. Q.E.D.

Proposition 5.13.

$$\mathrm{Ext}^1(\mathcal{V}_s^+,\mathcal{X}_s^+)\simeq 0, \tag{5.13}$$

$$\mathrm{Ext}^1(\mathcal{X}_s^{+\vee},\mathcal{X}_s^+)\simeq 0.$$

Proof. Consider the exact sequence

$$0\longrightarrow\mathcal{V}_s^-\longrightarrow\mathcal{P}_s^+\longrightarrow\mathcal{V}_s^+\longrightarrow 0.$$

Then we have the exact sequence

$$0=\mathrm{Hom}(\mathcal{V}_s^-,\mathcal{X}_s^+)\xrightarrow{\sim}\mathrm{Ext}^1(\mathcal{V}_s^+,\mathcal{X}_s^+).$$

Then we can prove $\mathrm{Ext}^1(\mathcal{V}_s^+,\mathcal{X}_s^+)\simeq 0$ similary. Q.E.D.

5.3. Projectivity of $\mathcal{P}_s^-$, $1\leq s\leq p-1$

We fix $1\leq s\leq p-1$, and define the $W(p)$-module $\mathcal{Y}_s^-$ by the exact sequence

$$0\longrightarrow\mathcal{X}_s^-\longrightarrow\mathcal{P}_s^-\longrightarrow\mathcal{Y}_s^-\longrightarrow 0. \tag{5.14}$$

Proposition 5.14. *We have*

(1) $\mathrm{Ext}^1(\mathcal{V}_s^-,\mathcal{X}_s^+)\simeq\mathbb{C}$, $\mathrm{Ext}^1(\mathcal{V}_s^{-\vee},\mathcal{X}_s^+)\simeq\mathbb{C}$,
(2) $\mathrm{Ext}^1(\mathcal{Y}_s^-,\mathcal{X}_s^+)\simeq 0$.

Proof. Consider an exact sequence

$$0\longrightarrow\mathcal{X}_s^+\longrightarrow\mathcal{V}_s^-\longrightarrow\mathcal{X}_s^-\longrightarrow 0.$$

This gives an exact sequence

$$0\longrightarrow\mathrm{Hom}(\mathcal{X}_s^+,\mathcal{X}_s^+)\longrightarrow\mathrm{Ext}^1(\mathcal{X}_s^-,\mathcal{X}_s^+)\longrightarrow\mathrm{Ext}^1(\mathcal{V}_s^-,\mathcal{X}_s^+)\longrightarrow 0,$$

and then we get $\mathrm{Ext}^1(\mathcal{V}_s^-,\mathcal{X}_s^+)\simeq\mathbb{C}$.

In the same way, we can conclude $\mathrm{Ext}^1(\mathcal{V}_s^{-\vee},\mathcal{X}_s^+)\simeq\mathbb{C}$.

Consider the exact sequence

$$0\longrightarrow\mathcal{X}_s^+\longrightarrow\mathcal{Y}_s^-\longrightarrow\mathcal{V}_s^-\longrightarrow 0.$$

Then we have an exact sequence

$$0\longrightarrow\mathrm{Hom}(\mathcal{X}_s^+,\mathcal{X}_s^+)\longrightarrow\mathrm{Ext}^1(\mathcal{V}_s^-,\mathcal{X}_s^+)\longrightarrow\mathrm{Ext}^1(\mathcal{Y}_s^-,\mathcal{X}_s^+)\longrightarrow 0.$$

The statement (2) follows from this sequence. Q.E.D.

Proposition 5.15.

$$\mathrm{Ext}^1(\mathcal{P}_s^-, \mathcal{X}_s^+) = 0.$$

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{V}_s^+ \longrightarrow \mathcal{P}_s^- \longrightarrow \mathcal{V}_s^- \longrightarrow 0.$$

Then we have an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{V}_s^+, \mathcal{X}_s^+) \to \mathrm{Ext}^1(\mathcal{V}_s^-, \mathcal{X}_s^+) \to \mathrm{Ext}^1(\mathcal{P}_s^-, \mathcal{X}_s^+) \to \mathrm{Ext}^1(\mathcal{V}_s^+, \mathcal{X}_s^+).$$

By Proposition 5.14, we have $\mathrm{Ext}^1(\mathcal{V}_s^-, \mathcal{X}_s^+) \simeq \mathbb{C}$. Therefore by Proposition 5.13 we have $\mathrm{Ext}^1(\mathcal{V}_s^+, \mathcal{X}_s^+) = 0$. Q.E.D.

Proposition 5.16. (1) $\mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-) \simeq \mathbb{C}$.
(2) *The element* $[\mathcal{P}_s^-]$ *is a generator of* $\mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-)$.

Proof. Since the element $[\mathcal{P}_s^-]$ is non-zero element of $\mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-)$, it is sufficient to prove $\dim_{\mathbb{C}} \mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-) \leq 1$.

We fix elements of $\mathcal{Y}_s^- = \mathcal{P}_s^-/\mathcal{X}_s^-$ by the following way.

$$v_+ = |\lambda_s(1)\rangle, \ v_- = W^-(0)v_+ \in \mathcal{Y}_s^-[h_s(1)],$$
$$u_+ = \eta_s^\vee v_+, \ u_- = \eta_s^\vee u_- \in \mathcal{Y}_s^-[h_s(0)].$$

Then we have

$$W^+(0)v_+ = 0.$$

Let $[E] \in \mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-)$, then we have an exact sequence of $W(p)$-module,

$$0 \longrightarrow E_0 \longrightarrow E \xrightarrow{\pi} \mathcal{Y}_s^- \longrightarrow 0,$$

where E_0 is isomorphic to $\mathcal{X}_s^-$. Fix elements of E_0 by the following way,

$$v_+^{(1)}, \ v_-^{(1)} = W^-(0)v_+^{(1)} \in E_0[h_s(1)] \simeq \mathbb{C}^2.$$

Then we have $W^+(0)v_+^{(1)} = 0$. Moreover, take elements of E by the following way,

$$\tilde{v}_+ \in E[h_s(1)] \to v_+ \in \mathcal{Y}_s^-,$$
$$\tilde{v}_- = W^-(0)\tilde{v}_+,$$
$$\tilde{u}_\pm = \eta_s^\vee \tilde{v}_\pm.$$

Then we have $\pi(\tilde{v}_\pm) = u_\pm$ and $W^+(0)\tilde{v}_+ = 0$.

For $W(p)$-module $M \in W(p)$-mod, we define $Q : M \to M$ by

$$Q|_{M[h]} = (T(0) - h)\ \mathrm{id}.$$

Then the $\mathbb{C}$-linear map $Q : M \to M$ is an $W(p)$-module map, and satisfies $Q^n = 0$ for some $n \geq 1$. Then we have a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & E & \longrightarrow & \mathcal{Y}_s^- & \longrightarrow & 0 \\ & & \downarrow Q & & \downarrow Q & & \downarrow Q & & \\ 0 & \longrightarrow & E_0 & \longrightarrow & E & \longrightarrow & \mathcal{Y}_s^- & \longrightarrow & 0. \end{array} \tag{5.15}$$

Since the map $Q = 0$ on $\mathcal{Y}_s^-$ and on E_0, we have

$$Q(E) \subseteq E_0 \subseteq E$$

and $Q^2 = 0$ on E. Therefore Q factors through

$$Q : E \xrightarrow{\bar{\pi}} \mathcal{X}_s^- \xrightarrow{\bar{Q}} E_0$$

where $\bar{\pi} : E \xrightarrow{\pi} \mathcal{Y}_s^- \to \mathcal{X}_s^-$. Since Q is a $W(p)$-module map there exists a constant γ such that

$$Q(\tilde{v}_\pm) = \gamma\ \tilde{v}_\pm^{(1)}. \tag{5.16}$$

We show that if $\gamma = 0$, then $[E] = 0$ in $\mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-)$. Consider the exact sequence

$$0 \longrightarrow (\mathcal{X}_s^+)^2 \longrightarrow \mathcal{Y}_s^- \longrightarrow \mathcal{X}_s^- \longrightarrow 0,$$

then we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-) \longrightarrow \mathrm{Ext}^1((\mathcal{X}_s^+)^2, \mathcal{X}_s^-).$$

Therefore to prove $[E] = 0$, it is sufficient to prove that

$$\eta_s(\tilde{u}_\pm) = 0.$$

By Proposition 2.12, (2.40) we see that

$$\begin{aligned} \eta_s(\tilde{u}_\pm) &= \eta_s \eta_s^\vee(\tilde{v}_\pm) \\ &= c(T(0) - h_s(1))\tilde{v}_\pm, \end{aligned}$$

for some $c \neq 0$. By the assumption $\gamma = 0$, we have $\eta_s(\tilde{u}_\pm) = 0$.

This shows that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-) \leq 1.$$

Q.E.D.

Proposition 5.17.

$$\mathrm{Ext}^1(\mathcal{P}_s^-, \mathcal{X}_s^-) = 0.$$

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{X}_s^- \longrightarrow \mathcal{P}_s^- \xrightarrow{\pi} \mathcal{Y}_s^- \longrightarrow 0.$$

Then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathcal{X}_s^-, \mathcal{X}_s^-) \longrightarrow \mathrm{Ext}^1(\mathcal{Y}_s^-, \mathcal{X}_s^-) \longrightarrow \mathrm{Ext}^1(\mathcal{P}_s^-, \mathcal{X}_s^-) \longrightarrow 0.$$

By Proposition 5.16, we have $\mathrm{Ext}^1(\mathcal{P}_s^-, \mathcal{X}_s^-) = 0$. Q.E.D.

Proposition 5.18. *$\mathcal{P}_s^-$ is projective cover of simple $W(p)$-module $\mathcal{X}_s^-$.*

Proof. By Proposition 5.15 and Proposition 5.17, the $W(p)$-module $\mathcal{P}_s^-$ is projective. Q.E.D.

The following propositions can be easily proved by using the above propositions.

Proposition 5.19. *We have*

$$\mathrm{Ext}^1(\mathcal{V}_s^-, \mathcal{X}_s^-) = 0,$$

$$\mathrm{Ext}^1(\mathcal{V}_s^{-\vee}, \mathcal{X}_s^-) = 0.$$

Proposition 5.20. (1) $\mathrm{Ext}^1(\mathcal{V}_s^-, \mathcal{V}_s^+) \simeq \mathbb{C}$, $\mathrm{Ext}^1(\mathcal{V}_s^{-\vee}, \mathcal{V}_s^{+\vee}) \simeq \mathbb{C}$ *and these two vector spaces are generated by* $[\mathcal{P}_s^-]$.
(2) $\mathrm{Ext}^1(\mathcal{V}_s^-, \mathcal{V}_s^{+\vee}) = 0$, $\mathrm{Ext}^1(\mathcal{V}_s^{-\vee}, \mathcal{V}_s^+) = 0$.

Proposition 5.21. *We have*

(1) $D(\mathcal{P}_s^-) \simeq \mathcal{P}_s^-$.
(2) $\mathcal{P}_s^-$ *is an injective module.*

§6. Category equivqlent of $W(p)$-mod and $\bar{U}_q(sl_2)$-mod

In this section we prove that two abelian categories $W(p)$-mod and $\bar{U}_q(sl_2)$-mod are equivalent as abelian categories. This is conjected in [FGST1], [FGST2].

6.1. The quantum group $\bar{U}_q(sl_2)$

We fix positive integer $p \geq 2$, and set $q = e^{\pi i/p}$. We introduce the restricted quantum group $\bar{U}_q(sl_2) = \bar{U}$. We will follow the articles of Feigin et al. [FGST1], [FGST2] and Kondo and Saito [KoS].

For each integer n, we set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{6.1}$$

The restricted quantum group $\bar{U}_q(sl_2)$ is an associative $\mathbb{C}$-algebra with the unit, which is generated by E, F, K, K^{-1} satisfying the following fundamental relations

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F, \tag{6.2}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

$$K^{2p} = 1,\ E^p = F^p = 0.$$

This is a finite dimensional $\mathbb{C}$-algebra, and has a structure of Hopf-algebra.

Let $\bar{U}$-mod be the abelian category of finite dimensional $\bar{U}$-modules. Then it is known in [FGST1], [FGST2], [KoS]:

Proposition 6.1. *The abelian category $\bar{U}$-mod has the block decomposition*

$$\bar{U}\text{-}mod = \sum_{s=0}^{p} C_s(\bar{U}),$$

where $C_0(\bar{U})$ and $C_p(\bar{U})$ are semi-simple categories whose have only one simple object, respectively. For $1 \leq s \leq p-1$, $C_s(\bar{U})$ are all isomorphic each other as abelian categories.

The category $C_s(\bar{U})$ is Artinian and Neotherian, and the number of simple object is two. We denote this abelian category $C(\bar{U})$, and denote simple object $\mathcal{X}^{\pm}(\bar{U}) = \mathcal{X}^{\pm}$ and their projective cover $\mathcal{P}^{\pm}(\bar{U}) = \mathcal{P}^{\pm}$. Set $P(\bar{U}) = P^{+}(\bar{U}) \oplus P^{-}(\bar{U}) = P^{+} \oplus P^{-}$, and consider the finite dimensional $\mathbb{C}$-algebra

$$B(\bar{U}) = \mathrm{End}_{C(\bar{U})}(\bar{P}(\bar{U})). \tag{6.3}$$

The following proposition is known in [FGST1], [FGST2], [KoS].

Theorem 6.2. *$B(\bar{U})$ is an eight dimensional algebra of the form;*

$$B(\bar{U}) = \mathrm{End}_C(\mathcal{P}_s^+, \mathcal{P}_s^+) \oplus \mathrm{End}_C(\mathcal{P}_s^-, \mathcal{P}_s^-) \oplus \mathrm{Hom}_C(\mathcal{P}_s^+, \mathcal{P}_s^-) \oplus \mathrm{Hom}_C(\mathcal{P}_s^-, \mathcal{P}_s^+), \tag{6.4}$$

and generated by

$$\mathrm{Hom}_C(\mathcal{P}_s^\pm, \mathcal{P}_s^\mp) = C\tau_1^\pm \oplus C\tau_2^\pm, \tag{6.5}$$

with the relations;

$$\begin{aligned} \tau_i^\pm \tau_i^\mp &= 0, \quad i = 1, 2, \\ \tau_1^\pm \tau_2^\mp &= \tau_2^\pm \tau_1^\mp. \end{aligned} \tag{6.6}$$

Now we consider a $\mathbb{C}$-linear abelian category C with the following properties;

(1) C is Neotherian and Artinian.
(2) The set of equivalence classes of simple objects is finite, say $\{S_1, \ldots, S_N\}$.

Denote the projective cover of S_i by P_i. And set $P = \sum_{i=1}^N P_i$. Consider the Endmorphism algebra of $\mathcal{P}$,

$$B(C) = \mathrm{End}_C(P).$$

Then $B(C)$ is a finite dimensional algebra over $\mathbb{C}$.

Denote by mod $B(C)$, the abelian category of finite dimensional right $B(C)$-modules. Then the following proposition is well known.

Proposition 6.3.

$$\begin{aligned} \Phi : C &\longrightarrow \text{mod } B(C) \\ M &\mapsto \mathrm{Hom}_C(P, M) \end{aligned}$$

is equivalence of abelian categories.

6.2. Categorical equivalence of two abelian category $W(p)$-mod and $\bar{U}$-mod

We showed that the abelian category $W(p)$-mod has the block decomposition

$$W(p)\text{-mod} = \sum_{s=0}^{p} C_s,$$

and that C_0 and C_p are semi-simple categories whose simple objects are $\mathcal{X}_0$ and $\mathcal{X}_p$, respectively. On the other hand for $1 \le s \le p-1$, each abelian category C_s has two simple objects $\mathcal{X}_s^+$ and $\mathcal{X}_s^-$.

Now for each $1 \leq s \leq p-1$, consider $\mathcal{P}_s = \mathcal{P}_s^+ \oplus \mathcal{P}_s^- \in C_s$, and define finite dimensional $\mathbb{C}$-algebra B_s as follows

$$B_s = \mathrm{End}_{\mathbb{C}}(C_s).$$

Theorem 6.4. *For each $1 \leq s \leq p-1$, the finite dimensional algebra B_s is isomorphic to $B(\bar{U})$.*

Proof. By Proposition 4.5, it is easy to show that B_s is isomorphic to $B(\bar{U})$ as an algebra over $\mathbb{C}$. Q.E.D.

Hence by Proposition 6.3, we have the following main theorem of this section.

Theorem 6.5. *Two abelian categories $W(p)$-mod and $\bar{U}_q(sl_2)$-mod are equivalent.*

6.3. Length of the Jordan blocks

For each $M \in W(p)$-mod, we define $l(M) \in \mathbb{Z}_{\geq 0}$ by

$$l(M) = \max\{n \in \mathbb{Z}_{\geq 0}\ ;\ (T(0)-h)^n v \neq 0 \text{ for some } h \in \mathbb{C},\ v \in M[h]\}.$$

Then we obtain the following proposition.

Proposition 6.6. (1) *For each $M \in W(p)$-mod we have $l(M) \leq 1$.*

(2) *Any indecomposable module M in $W(p)$-mod such that $l(M) = 1$ is equivalent to $M \simeq \mathcal{P}_s^+$ for some s such that $1 \leq s \leq p-1$.*

References

[Ada] D. Adamovic, Classification of irreducible modules of certain subalgebras of free boson vertex algebra, arXiv:math/0207155.

[AM1] D. Adamovic and A. Milas, Logarithmic interwining operators and $W(2, 2p-1)$algebras, J. Math. Phys., **48** (2007), 073503; arXiv:math/0702081.

[AM2] D. Adamovic and A. Milas, On the triplet vertex algebra $W(p)$, arXiv:math/0707.1857.

[BFGT] P. V. Bushlanov, B. L. Feigin, A. M. Gainutdinov and I. Yu. Tipunin, Lusztig limit of Quantum sl(2) at root of unity and fusion of (1,p) virasoro logarithmic minimal models, arXiv:hep-th/0901.1602.

[CaN] N. Carqueville and M. Flohr, Nonmeromorphic operator product expansion and C_2-cofiniteness for a family of W-algebras, J. Phys. A, **39** (2006), 951–966.

[FFHST] J. Fjelstad, J. Fuchs, S. Hwang, A. M. Semikhakov and I. Yu. Tipunin, Logarithmic conformal field theories via logarithmic deformation, Nuclear Phys. B, **633** (2002), 379–413.

[FF1] B. I. Feigin and D. B. Fuchs, Verma modules over the Virasoro algebra, Funkts. Anzl. Prilozhen, **17** (1983), 91–92.

[FF2] B. I. Feigin and D. B. Fuchs, Representations of the Virasoro algebra, In: Representation of Lie Groups and Related Topics, Adv. Stud. Contemp. Math., **7**, Gordan and Breach Science Publ., New York, 1990, pp. 465–554.

[FGST1] B. I. Feigin, A. M. Gainutdinov, A. M. Semikhatov and I. Yu. Tipunin, Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center, Commun. Math. Phys., **265** (2006), 47–93; arXiv:hep-th/0504093.

[FGST2] B. I. Feigin, A. M. Gainutdinov, A. M. Semikhatov and I. Yu. Tipunin, Logarithmic extensions of minimal models: characters and modular transformations, Nuclear Phys. B, **757** (2006), 303–343; arXiv:hep-th/0606196.

[Fel] G. Felder, BRST approach to minimal models, Nuclear Phys. B, **317** (1989), 215–236.

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On Axiomatic Approaches to Vertex Operator Algebras and Modules, Mem. Amer. Math. Soc., **104**, Amer. Math. Soc., Providence, RI, 1993.

[FrZ] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representation of affine and Virasoro algebras, Duke Math. J., **66** (1992), 123–168.

[FrB] E. Frenkel and D. Ben-Zvi, Vertex Algebras and Algebraic Curves, Math. Surveys Monogr., **88**, Amer. Math. Soc., Providence, RI, 2001.

[FHST] J. Fuchs, S. Hwang, A. M. Semikhakov and I. Yu. Tipunin, Nonsemisimple fusion algebras and the verlinde formula, Comm. Math. Phys., **247** (2004), 713–742.

[Gab] M. R. Gaberdiel, An algebraic approach to logarithmic conformal field theory, Internat. J. Modern Phys. A, **18** (2003), 4593–4638; arXiv:hep-th/0111260.

[Kac] V. Kac, Vertex Algebras for Beginners, Univ. Lecture Ser., **10**, Amer. Math. Soc., Providence, RI, 1998.

[Kau] H. G. Kausch, Extended conformal algebras generated by multiplet of primary fields, Phys. Lett. B, **259** (1991), 448–455.

[KoS] H. Kondo and Y. Saito, Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to sl_2, arXiv:math.QA/0901.4221.

[MaN] A. Matsuo and K. Nagatomo, Axioms for a Vertex Algebra and the Locality of Quantum Fields, MSJ Memoirs, **4**, Math. Soc. Japan, 1999.

[MNT] A. Matsuo, K. Nagatomo and A. Tsuchiya, Quasi-finite algebras graded by Hamiltonian and vertex operator algebras, arXiv:math.QA/0505071.

[NaT] K. Nagatomo and A. Tsuchiya, Conformal field theories associated to regular chiral vertex operator algebras. I. Theories over the projective line, Duke Math. J., **128** (2005), 393–471.

[PRZ] P. A. Pearce, J. Rasmussen and J.-B. Zuber, Logarithmic minimal models, J. Stat. Math., **0611** (2006), P017; arXiv:hep-th/0607232.

[TsK] A. Tsuchiya and Y. Kanie, Fock space representation of the Virasoro algebra, Publ. Res. Inst. Math. Sci., **22** (1986), 259–327.

[Zhu] Y. C. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc., **9** (1996), 237–302.

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