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Quantum invariance under $\mathbb{P}^1$ flops of type $(k+2, k)$

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Abstract.

In the joint paper [8] with Y.-P. Lee and C.-L. Wang, we have shown that the big quantum ring is invariant under $\mathbb{P}^r$ flops of splitting type, after an analytic continuation over the extended Kähler moduli space. It is a generalization of our previous work for the case of simple $\mathbb{P}^r$ flops [7]. In this note, I would like to outline the results and concentrate mainly on the detailed study of a simple type, called $\mathbb{P}^1$ flops of type $(k+2, k)$.

§1. Introduction

To study topological problems arising from the non-uniqueness of minimal models in higher dimensional birational geometry, C.-L. Wang in 1998 raised the notion of K-equivalent varieties to generalize birational minimal models [14]. Two ($\mathbb{Q}$-Gorenstein) varieties X and X' are *K-equivalent* if there exist birational morphisms $\phi : Y \to X$ and $\phi' : Y \to X'$ with Y smooth

$$\begin{array}{ccccc} & & Y & & \\ & \phi \swarrow & & \searrow \phi' & \\ X & & & & X' \end{array}$$

such that

$$\phi^* K_X = \phi'^* K_{X'}.$$

Two birational minimal models are automatically K-equivalent, so we turn our attention to study K-equivalent varieties.

V. Batyrev [1] and C.-L. Wang [14] showed that K-equivalent smooth varieties have the same Betti numbers. However, the cohomology ring

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structures are in general different. Y. Ruan [13] and C.-L. Wang [15] posed the following conjecture to modify this drawback.

Conjecture 1.1. *K-equivalent smooth varieties have canonically isomorphic quantum cohomology rings over the extended Kähler moduli spaces.*

For threefolds, this Conjecture was proved by A. Li and Y. Ruan [11]. For higher dimensional case, we achieved the following result in [7].

Theorem 1.1. The big quantum cohomology ring is invariant under simple ordinary flops, after an analytic continuation over the extended Kähler moduli space.

Ordinary flops are the simplest type of flops and also crucial to the general theory of minimal models and K-equivalence, so it is natural to work on them first. We recall the definition and construction as follows.

Let X be a smooth complex projective manifold and $\psi : X \to \bar{X}$ a flopping contraction in the sense of minimal model theory, with $\bar{\psi} : Z \to S$ the restriction map on the exceptional loci. Assume that

(i) $\bar{\psi}$ equips Z with a $\mathbb{P}^r$-bundle structure $\bar{\psi} : Z = \mathbb{P}_S(F) \to S$ for some rank $r+1$ vector bundle F over a smooth base S,

(ii) $N_{Z/X}|_{Z_s} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$ for each $\bar{\psi}$-fiber Z_s, $s \in S$.

Then there is another rank $r+1$ vector bundle F' over S such that

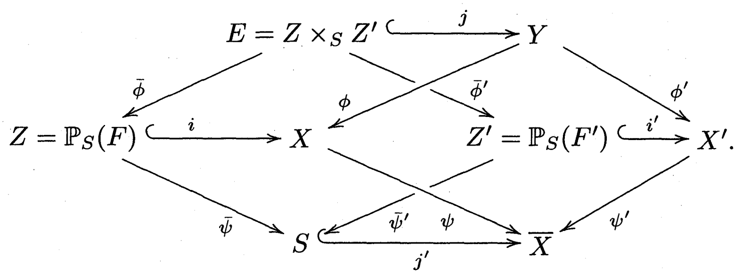
$$N_{Z/X} \cong \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* F'.$$

We may blow up X along Z to get $\phi : Y \to X$. The exceptional divisor

$$E = \mathbb{P}_Z(N_{Z/X}) \cong \mathbb{P}_Z(\bar{\psi}^* F') = \bar{\psi}^* \mathbb{P}_S(F') = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F')$$

is a $\mathbb{P}^r \times \mathbb{P}^r$-bundle over S. We may then blow down E along another fiber direction $\phi' : Y \to X'$ to get another contraction $\psi' : X' \to \bar{X}$, with exceptional loci $\bar{\psi}' : Z' = \mathbb{P}_S(F') \to S$ and $N_{Z'/X'}|_{\psi'\text{-fiber}} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$.

We call $f : X \dashrightarrow X'$ constructed as above an *ordinary* $\mathbb{P}^r$ *flop.* The various sets and maps are summarized in the following commutative diagram.



When S consists of a point, we call f a *simple* $\mathbb{P}^r$ *flop*.

So far, we have completely solved the problem on the invariance of Gromov–Witten theory under simple $\mathbb{P}^r$ flops, including the higher genus case [9]. To be precise, the canonical correspondence is given by the graph closure $\mathcal{F} = [\bar{\Gamma}_f]$. The crucial idea behind is to interpret $\mathcal{F}$-*invariance* in terms of *analytic continuations in Gromov–Witten theory.*

Analogous to the procedures used in our previous work [7], for a $\mathbb{P}^r$ flop over general base, we have determined the *defect formula for 3-point functions*, shown that the 3-point *extremal quantum corrections* exactly remedy the defect, and gotten functional equations for $n \geq 3$ point extremal functions. For the invariance of big quantum product on non-extremal curve classes, we still used degeneration to the normal cone [4] together with the degeneration formula [10] to perform *cohomology reduction to local models*, which reduces the problem to the double projective bundles. After that, we applied further reduction to *quasi-linearity* via reconstruction and WDVV equation and finally provided a proof of quasi-linearity for *split flops* i.e. $F = \bigoplus_{i=0}^r L_i$ and $F' = \bigoplus_{i=0}^r L'_i$ where L_i, L'_i are line bundles over S.

To complete the last step, we need to apply *generalized mirror transformations* to relate I functions with J functions [8], [2], [3]. The procedure is somehow quite complicated, so it is worthwhile to provide simple examples which only involve classical mirror transformations so that the proof in [8] becomes more transparent.

The purpose of this note is to give an introduction to [8] and to give a complete proof of the last step for a series of examples, namely ordinary $\mathbb{P}^1$ flops over base $S = \mathbb{P}^1$ with $F = \mathcal{O} \oplus \mathcal{O}(-(k+2))$ and $F' = \mathcal{O} \oplus \mathcal{O}(k)$ for any $k \in \mathbb{Z}$:

Theorem 1.2. Let $f : X \dashrightarrow X'$ be an ordinary $\mathbb{P}^1$ flop of type $(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}(-(k+2)), \mathcal{O} \oplus \mathcal{O}(k))$. The canonical correspondence $\mathcal{F}$ induces an isomorphism of big quantum rings $QH(X)$ and $QH(X')$ in the sense of analytic continuations over the extended Kähler moduli space.

We refer to [7] for the basic definition of quantum cohomology and the notion of analytic continuations over the Kähler moduli. In Section 2 and Section 3 we give a survey of part of the results in [8] without proofs.

The detailed proof of Theorem 1.2 is given in Section 4.

§2. Functional equations for n-point extremal functions

2.1. Defect of the classical product

In [7], for a general $\mathbb{P}^r$ flop, we have established the canonical correspondence between their cohomology groups.

Theorem 2.1. For an ordinary $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$, the graph closure $\mathcal{F} := [\bar{\Gamma}_f]$ induces $\hat{X} \cong \hat{X}'$ via $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$ and $\mathcal{F} \circ \mathcal{F}^* = \Delta_{X'}$. Moreover, $\mathcal{F}$ preserves the Poincaré pairing, that is, if $\dim \alpha_1 + \dim \alpha_2 = \dim X$, then

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2) = (\alpha_1.\alpha_2).$$

In practice, the correspondence $\mathcal{F}$ associates a map on Chow groups:

$$\mathcal{F} : A^*(X) \to A^*(X'); \quad W \mapsto p'_*(\bar{\Gamma}_f.p^*W) = \phi'_*\phi^*W.$$

Let $h = c_1(\mathcal{O}_{\mathbb{P}_S(F)}(1))$ and $h' = c_1(\mathcal{O}_{\mathbb{P}_S(F')}(1))$. For a simple $\mathbb{P}^r$ flop we have the basic transformation formula

$$\mathcal{F}(i_*h^k) = (-1)^{r-k} i'_* h'^k.$$

However, for a general $\mathbb{P}^r$ flop this does not hold anymore. We may show, by induction on k, that for all $k \in \mathbb{N}$,

$$\mathcal{F}(i_*h^k) = (-1)^{r-k} i'_*(a_0 h'^k + a_1 h'^{k-1} + \cdots + a_k) \in A^*(Z')$$

where $a_0 = 1$ and $a_k \in A^*(S)$ are determined by the recursive relations:

$$c'_k = a_k - c_1 a_{k-1} + c_2 a_{k-2} + \cdots + (-1)^k c_k.$$

Here, we abuse notations to denote $c_i(F)$, $\bar{\psi}^* c_i(F)$ and $\bar{\psi}'^* c_i(F)$ by the same symbol c_i. Similarly we denote $c_i(F')$, $\bar{\psi}^* c_i(F')$ and $\bar{\psi}'^* c_i(F')$ by c'_i. We use this abbreviation for any class in $A^*(S)$.

The similar formula turns out to be achieved by replacing h^k, h'^k with H_k, H'_k.

Proposition 2.1. For all positive integers $k \le r$,

$$\mathcal{F}(i_*H_k) = (-1)^{r-k} i'_* H'_k$$

where $H_k = h^k + c_1 h^{k-1} + \cdots + c_k$ and $H'_k = h'^k + c'_1 h'^{k-1} + \cdots + c'_k$.

Here, the discovery of H_k, H'_k is crucial to the whole calculation with a general base S. The most important fact is that we can express the dual basis of the canonical basis $\{t_i^k h^j\}$ in $A^*(Z)$ in terms of H_k.

Lemma 2.1. The basis $\{t_i^{k-j}h^j\}_{j\le\min\{k,r\}}$ of $A^k(Z)$ has its dual basis $\{\hat{t}_i^{k-j}H_{r-j}\}_{j\le\min\{k,r\}}$ in $A^{r+s-k}(Z)$ where $\{t_i^k\}$ is a basis of $A^k(S)$ and $\{\hat{t}_i^k\}$ in $A^{s-k}(S)$ is its dual basis where $s=\dim S$.

Through direct computation, we succeed in writing down the defect of the triple products of classes in X and X':

Theorem 2.2. Let $\alpha_i\in A^{k_i}(X)$ for $i=1,2,3$ with $k_1+k_2+k_3=\dim X=s+2r+1$. Then

$$\begin{aligned}(\mathfrak{F}\alpha_1.\mathfrak{F}\alpha_2.\mathfrak{F}\alpha_3)^{X'} &= (\alpha_1.\alpha_2.\alpha_3)^X+(-1)^r\times\\ &\sum(\alpha_1.\hat{t}_{i_1}^{k_1-j_1}H_{r-j_1})^X(\alpha_2.\hat{t}_{i_2}^{k_2-j_2}H_{r-j_2})^X(\alpha_3.\hat{t}_{i_3}^{k_3-j_3}H_{r-j_3})^X\\ &\qquad\times(\tilde{s}_{j_1+j_2+j_3-2r-1}t_{i_1}^{k_1-j_1}t_{i_2}^{k_2-j_2}t_{i_3}^{k_3-j_3})^S,\end{aligned}$$

where the sum is over all possible i_1,i_2,i_3 and j_1,j_2,j_3 subject to constraint: $1\le j_p\le\min\{r,k_p\}$ for $p=1,2,3$ and $j_1+j_2+j_3\ge 2r+1$. Here

$$\tilde{s}_i := s_i(F+F'^*)$$

is the i-th Segre class of $F+F'^*$.

2.2. Quantum corrections attached to the extremal ray

Let $\alpha_i\in A^{k_i}(X)$, $i=1,\dots,n$, with

$$\sum_{i=1}^n k_i = 2r+1+s+(n-3).$$

Since

$$\alpha_i|_Z=\sum_{s_i}\sum_{j_i\le\min\{k_i,r\}}(\alpha_i.\hat{t}_{s_i}^{k_i-j_i}H_{r-j_i})t_{s_i}^{k_i-j_i}h^{j_i},$$

we compute

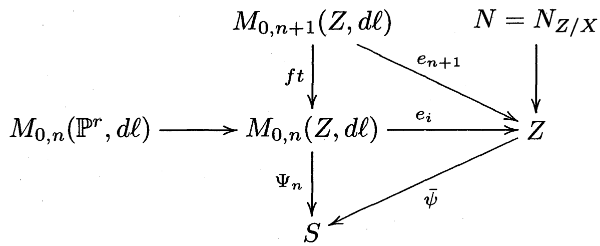
$$\begin{aligned}&\langle\alpha_1,\dots,\alpha_n\rangle_{0,n,d\ell}^X\\ &=\sum_{\vec{s},\vec{j}}\int_{M_{0,n}(Z,d\ell)}\prod_{i=1}^n\Big((\alpha_i.\hat{t}_{s_i}^{k_i-j_i}H_{r-j_i})\,e_i^*(\bar{\psi}^*t_{s_i}^{k_i-j_i}.h^{j_i})\Big).e(R^1ft_*e_{n+1}^*N)\\ &=\sum_{\vec{s},\vec{j}}\prod_{i=1}^n(\alpha_i.\hat{t}_{s_i}^{k_i-j_i}H_{r-j_i})\left[\prod_{i=1}^n t_{s_i}^{k_i-j_i}.\Psi_{n*}\Big(\prod_{i=1}^n e_i^*h^{j_i}.e(R^1ft_*e_{n+1}^*N)\Big)\right]_S,\end{aligned}$$

with the sum over all $\vec{s} = (s_1 \dots, s_n)$ and admissible $\vec{j} = (j_1, \dots, j_n)$. By the fundamental class axiom, we must have $j_i \geq 1$ for all i.

Here we make use of

$$[M_{0,n}(X, d\ell)]^{virt} = [M_{0,n}(Z, d\ell)] \cap e(R^1 ft_* e_{n+1}^* N)$$

and the fiber bundle diagram over S



as well as the fact that classes in S are constants among bundle morphisms (by the projection formula applies to $\Psi_n = \bar{\psi} \circ e_i$ for each i).

We must have $\sum(k_i - j_i) \leq s$ to get nontrivial invariants. That is,

$$\sum_{i=1}^{n} j_i \geq 2r + 1 + n - 3.$$

If the equality holds, then $\prod_{i=1}^{n} t_{s_i}^{k_i - j_i}$ is a zero dimensional cycle in S and the invariant readily reduces to the corresponding one on any fiber, namely the simple case, which is completely determined in [7]:

$$(t_{s_1}^{k_1-j_1} \cdots t_{s_n}^{k_n-j_n})^S \langle h^{j_1}, \dots, h^{j_n} \rangle_{0,n,d\ell}^{\text{simple}} = (\prod t_{s_i})^S N_{\vec{j}}\, d^{n-3}.$$

Generally, we can show that for a $\mathbb{P}^r$ flop with $n = 3$, when $\sum_{i=1}^{3} j_i = 2r + 1 + \mu$ ($\mu \leq r - 1$), there is a degree μ cohomology valued polynomial $W_\mu^{F,F'}(d) = \sum_{i=0}^{\mu} w_{\mu,i}(F, F')\, d^i$ with coefficients $w_{\mu,i} \in A^\mu(S) \otimes \mathbb{Q}$ such that for any class $t \in A^{r-\mu}(S)$,

$$\langle h^{j_1}, h^{j_2}, th^{j_3} \rangle_d = (-1)^{(d-1)(r+1)} (W_\mu^{F,F'}(d).t)^S.$$

For example, when $\mu = 1$,

$$W_1^{F,F'}(d) = (-c_1 + c_1') - d(c_1 + c_1') = \tilde{s}_1 - d(c_1(F + F')).$$

This implies that the 3-point *extremal quantum corrections* for X and X' remedy the defect of classical cup product for the cases $\mu = 1$.

To see this, it is convenient to consider the basic rational function

$$f(q) := \frac{q}{1-(-1)^{r+1}q} = \sum_{d\geq 1}(-1)^{(d-1)(r+1)}q^d,$$

which is the 3-point extremal correction for the case $\mu = 0$. It is clear that

$$f(q) + f(q^{-1}) = (-1)^r.$$

Here, the class th^j is regarded as $\alpha|_Z$ for some $\alpha \in H^*(X)$, so if $j \leq r$, we have the familiar formula $\mathcal{F}\alpha|_{Z'} = (-1)^j th'^j$ and thus $\mathcal{F}(th^j) = (-1)^j th'^j$ for $j \leq r$. Hence the geometric series on X

$$\sum_{d\geq 1}(-1)^{(d-1)(r+1)}(\tilde{s}_1.t)^S\, q^{d\ell} = (\tilde{s}_1.t)^S f(q^\ell)$$

together with its counterpart on X' *exactly* correct the classical term via

$$\begin{aligned}&(\tilde{s}_1.t)^S f(q^\ell) - (-1)^{j_1+j_2+j_3}(\tilde{s}'_1.t)^S f(q^{\ell'})\\ &= (\tilde{s}_1.t)^S(f(q^\ell) + f(q^{-\ell})) = (-1)^r(\tilde{s}_1.t)^S.\end{aligned}$$

The new feature for $\mu = 1$ is that we also have contributions involving the differential operator $\delta_h = q^\ell\, \partial/\partial q^\ell$, namely

$$-(c_1(F+F').t)^S \sum_{d\geq 1}(-1)^{(d-1)(r+1)} d q^{d\ell} = -(c_1(F+F').t)^S\, \delta_h f(q^\ell).$$

Since $\delta_h^i f(q^\ell) = (-\delta_{h'})^i((-1)^r - f(q^{\ell'})) = (-1)^{i+1}\delta_{h'}^i f(q^{\ell'})$, this higher order term agrees with one on the X' side, as required.

For general μ, the 3-point extremal correction is given by

$$\langle h^{j_1}, h^{j_2}, th^{j_3}\rangle_+ := \sum_{d\geq 1}\langle h^{j_1}, h^{j_2}, th^{j_3}\rangle_d\, q^{d\ell} = (W_\mu^{F,F'}.t)^S(\delta_h) f(q^\ell).$$

By using the divisor relation [6]

$$e_i^* h = e_j^* h + \sum_{d_1+d_2=d} (d_2[D_{ik,d_1|j,d_2}]^{virt} - d_1[D_{i,d_1|jk,d_2}]^{virt}),$$

any invariants $\langle t_1 h^{j_1}, t_2 h^{j_2}, t_3 h^{j_3}\rangle_d$ with $1 \leq j_i \leq r$ and $\sum j_i = (2r+1)+\mu$ may be inductively switched into $\langle h^{\mu+1}, h^r, th^r\rangle_d$ with $t = t_1t_2t_3 \in A^{s-\mu}(S)$. Define the extremal corrections $W_\mu := \langle h^{\mu+1}, h^r, h^r\rangle_+^{/S} =$

$W_\mu^{F,F'}(\delta_h)f(q^\ell)$. They turn out to be determined by the recursive formula:

$$W_\mu = s_\mu f + \sum_{j=1}^{\mu} W_{\mu-j}\big((-1)^r c_j f - (-1)^{r+j} c'_j f - c_j\big).$$

We can show that if W'_μ is defined on X' side similarly, then they satisfy the functional equation

$$W_\mu - (-1)^{\mu+1} W'_\mu = (-1)^r \tilde{s}_\mu$$

for $0 \le \mu \le r-1$. That is, the classical defect is corrected, so the following theorem follows.

Theorem 2.3. For any ordinary flop over a smooth base, we have

$$\mathcal{F}\langle \alpha_1, \alpha_2, \alpha_3 \rangle^X \cong \langle \mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3 \rangle^{X'}$$

modulo non-extremal curve classes.

By more complicated analysis, the above theorem extends to all $n \ge 4$. Namely

$$\mathcal{F}\langle \alpha_1, \cdots, \alpha_n \rangle^X \cong \langle \mathcal{F}\alpha_1, \cdots, \mathcal{F}\alpha_n \rangle^{X'}$$

modulo non-extremal curve classes.

§3. Degeneration analysis

Our next task is to *compare* the Gromov–Witten invariants of X and X' for all genera and for curve classes other than the flopped curve. As in [7], we use the degeneration formula [11], [10], [5] to reduce the problem to local models. This has been achieved for *simple* ordinary flops in [7] for *genus zero* invariants. In this section we extend the argument to the general case.

3.1. Absolute invariants to relative local invariants

Given a $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$, the *deformations to the normal cone* on X is the blowing-up $\Phi : W \to X \times \mathbb{A}^1$ along $Z \times \{0\}$. $W_t \cong X$ for all $t \ne 0$ and $W_0 = Y_1 \cup Y_2$ with $j_i : Y_i \hookrightarrow W_0$ the inclusion maps for $i = 1, 2$. Here $Y_1 = Y$ with $\phi = \Phi|_Y : Y \to X$ is the blowing-up along Z and $Y_2 = \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O})$ where $p = \Phi|_{\tilde{E}} : \tilde{E} \to Z \subset X$ is the compactified normal bundle. $Y \cap \tilde{E} = E = \mathbb{P}_Z(N_{Z/X})$ is the ϕ-exceptional divisor which consists of the infinity part of $\tilde{E}$. Similarly we have $\Phi' : W' \to X' \times \mathbb{A}^1$ and $W'_0 = Y' \cup \tilde{E}'$. By definition of ordinary flops, $Y = Y'$ and $E = E'$. In fact $\tilde{E} \cong \tilde{E}'$ too, but they are glued into Y in a different manner (up to a twist), thus $W_0 \ncong W'_0$.

Since the family $W \to \mathbb{A}^1$ comes from a trivial family, all cohomology classes $\alpha \in H^*(X, \mathbb{Z})^{\oplus n}$ have global liftings and the restriction $\alpha(t)$ on W_t is defined for all t. The class $\alpha(0)$ can be represented by $(j_1^*\alpha(0), j_2^*\alpha(0)) = (\alpha_1, \alpha_2)$ with $\alpha_i \in A^*(Y_i)$ such that

$$\iota_1^*\alpha_1 = \iota_2^*\alpha_2 \quad \text{and} \quad \phi_*\alpha_1 + p_*\alpha_2 = \alpha.$$

Such representatives are not unique. The flexibility on different choices is of key importance. Actually for e being a class in E, if $\alpha(0) = (\alpha_1, \alpha_2)$ then it can also be represented by

$$\alpha(0) = (\alpha_1 - \iota_{1*}e, \alpha_2 + \iota_{2*}e).$$

We start with the representative $(\phi^*\alpha, p^*(\alpha|_Z))$ for $\alpha(0)$ and the representative $(\phi'^*\mathfrak{F}\alpha, p'^*(\mathfrak{F}\alpha|_{Z'}))$ for $\mathfrak{F}\alpha(0)$. Then we can modify the choices $\phi^*\alpha$ and $\phi'^*\mathfrak{F}\alpha$ by adding suitable classes in E to make them equal. This is possible since

$$\phi^*\alpha - \phi'^*\mathfrak{F}\alpha \in \iota_{1*}H^*(E).$$

Finally, we can show that for representatives $\alpha(0) = (\alpha_1, \alpha_2)$ and $\mathfrak{F}\alpha(0) = (\alpha'_1, \alpha'_2)$,

$$\text{if} \quad \alpha_1 = \alpha'_1 \quad \text{then} \quad \mathfrak{F}\alpha_2 = \alpha'_2.$$

Here we must mention that the ordinary flop f induces an ordinary flop

$$\tilde{f} : \tilde{E} \dashrightarrow \tilde{E}'$$

on the local model, so the graph closure $\mathfrak{F}$ of $\tilde{f}$ also gives a correspondence of $H^*(\tilde{E})$ and $H^*(\tilde{E}')$.

The *degeneration formula* expresses the absolute invariants of X in terms of the relative invariants of the two smooth pairs (Y_1, E) and (Y_2, E):

$$\langle \alpha \rangle^X_{g,n,\beta} = \sum_I \sum_{\eta \in \Omega_\beta} C_\eta \left\langle j_1^*\alpha(0) \,\middle|\, e_I, \mu \right\rangle^{\bullet(Y_1,E)}_{\Gamma_1} \left\langle j_2^*\alpha(0) \,\middle|\, e^I, \mu \right\rangle^{\bullet(Y_2,E)}_{\Gamma_2}$$

where $\{e_i\}$ is a basis of $H^*(E)$ with $\{e^i\}$ its dual basis and $\{e_I\}$ forms a basis of $H^*(E^\rho)$ with dual basis $\{e^I\}$, $|I| = \rho$, $e_I = e_{i_1} \otimes \cdots \otimes e_{i_\rho}$.

Here $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ is an *admissible triple* which consists of (possibly disconnected) topological types

$$\Gamma_i = \coprod_{\pi=1}^{|\Gamma_i|} \Gamma_i^\pi$$

with the same partition μ of contact order under the identification I_ρ of contact points. The gluing $\Gamma_1 +_{I_\rho} \Gamma_2$ has type (g, n, β) and is connected. In particular, $\rho = 0$ if and only if that one of the Γ_i is empty. The total genus g_i, total number of marked points n_i and the total degree $\beta_i \in NE(Y_i)$ satisfy the splitting relations

$$\begin{aligned} g - 1 &= \sum_{\pi=1}^{|\Gamma_1|} (g_1(\pi) - 1) + \sum_{\pi=1}^{|\Gamma_2|} (g_2(\pi) - 1) + \rho \\ &= g_1 + g_2 - |\Gamma_1| - |\Gamma_2| + \rho, \\ n &= n_1 + n_2, \\ \beta &= \phi_* \beta_1 + p_* \beta_2. \end{aligned}$$

(The first one is the arithmetic genus relation for nodal curves.)

The constants $C_\eta = m(\mu)/|\mathrm{Aut}\,\eta|$, where $m(\mu) = \prod \mu_i$ and $\mathrm{Aut}\,\eta = \{\sigma \in S_\rho \mid \eta^\sigma = \eta\}$. We denote by Ω the set of equivalence classes of all admissible triples; by Ω_β and Ω_μ the subset with fixed degree β and fixed contact order μ respectively.

Define the generating series for genus g (connected) invariants

$$\langle A \mid \varepsilon, \mu \rangle_g^{(\tilde{E}, E)} := \sum_{\beta_2 \in NE(\tilde{E})} \frac{1}{|\mathrm{Aut}\,\mu|} \langle A \mid \varepsilon, \mu \rangle_{g,\beta_2}^{(\tilde{E}, E)} q^{\beta_2}$$

and the similar one with possibly disconnected domain curves

$$\langle A \mid \varepsilon, \mu \rangle^{\bullet(\tilde{E}, E)} := \sum_{\Gamma;\, \mu_\Gamma = \mu} \frac{1}{|\mathrm{Aut}\,\Gamma|} \langle A \mid \varepsilon, \mu \rangle_\Gamma^{\bullet(\tilde{E}, E)} q^{\beta^\Gamma} \kappa^{g^\Gamma - |\Gamma|}.$$

For connected invariants of genus g we assign the κ-weight κ^{g-1}, while for disconnected ones we simply assign the product weights.

To compare $\mathfrak{F}\langle \alpha \rangle^{\bullet X}$ and $\langle \mathfrak{F}\alpha \rangle^{\bullet X'}$, by the cohomology reduction we may assume that $\alpha_1 = \alpha_1'$ and $\alpha_2' = \mathfrak{F}\alpha_2$, so the relative terms for (Y, E) are identical. It remains to compare

$$\langle \alpha_2 \mid \varepsilon, \mu \rangle^{\bullet(\tilde{E}, E)} \quad \text{and} \quad \langle \mathfrak{F}\alpha_2 \mid \varepsilon, \mu \rangle^{\bullet(\tilde{E}', E)}.$$

Proposition 3.1. To prove $\mathfrak{F}\langle \alpha \rangle_g^X \cong \langle \mathfrak{F}\alpha \rangle_g^{X'}$ for all α up to genus $g \le g_0$, it is enough to show that

$$\mathfrak{F}\langle A \mid \varepsilon, \mu \rangle_g^{(\tilde{E}, E)} \cong \langle \mathfrak{F}A \mid \varepsilon, \mu \rangle_g^{(\tilde{E}', E)}$$

for all A, ε, μ and $g \leq g_0$.

3.2. Relative local back to absolute local

Now let $X = \tilde{E}$. We shall further reduce the relative cases to the absolute cases with at most descendent insertions along E. This has been done in [7] for genus zero invariants under simple flops. Here we extend the argument to ordinary flops over any smooth base S and to all genera.

The local model

$$\bar{p} := \bar{\psi} \circ p : \tilde{E} \xrightarrow{p} Z \xrightarrow{\bar{\psi}} S$$

as well as the flop $f : \tilde{E} \dashrightarrow \tilde{E}'$ are all over S, with each fiber isomorphic to the simple case. Thus the map on numerical one cycles

$$\bar{p}_* : N_1(\tilde{E}) \to N_1(S)$$

has kernel spanned by the p-fiber line class γ and $\bar{\psi}$-fiber line class ℓ, which is the flopping log-extremal ray. Notice that for general S the structure of $NE(Z)$ could be complicated and $NE(\tilde{E})$ is in general larger than $i_*NE(Z) \oplus \mathbb{Z}^+\gamma$. For $\beta = \beta_Z + d_2(\beta)\gamma \in NE(\tilde{E})$, it could happen that β_Z is not effective or $d_2(\beta)$ is non-positive. At least the following is true:

Lemma 3.1. The correspondence $\mathcal{F}$ is compatible with $N_1(S)$. Namely

$$\begin{array}{ccc} N_1(\tilde{E}) & \xrightarrow{\quad\mathcal{F}\quad} & N_1(\tilde{E}') \\ & {\scriptstyle d_2 \oplus \bar{p}_*} \searrow \quad \swarrow {\scriptstyle d_2' \oplus \bar{p}'_*} & \\ & \mathbb{Z} \oplus N_1(S) & \end{array}$$

is commutative.

This leads to the following key observation, which applies to both absolute and relative invariants:

Proposition 3.2. Functional equation of a generating series $\langle A \rangle$ over Mori cone on local models $f : \tilde{E} \dashrightarrow \tilde{E}'$ is equivalent to functional equations of its various subseries (fiber series) $\langle A \rangle_{d_2, \beta_S}$ labelled by $\mathbb{Z} \oplus NE(S)$. The fiber series is a sum over the affine ray $\beta \in (d_2\gamma + \bar{\psi}^*\beta_S + \mathbb{Z}\ell) \cap NE(\tilde{E})$.

Given insertions $A = (\alpha_1, \ldots, \alpha_n) \in H^*(\tilde{E})^{\oplus n}$ and weighted partition $(\varepsilon, \mu) = \{(\varepsilon_1, \mu_1), \ldots, (\varepsilon_\rho, \mu_\rho)\}$, the genus g relative invariant

$\langle A \mid \varepsilon, \mu\rangle_g$ is summing over classes $\beta = \beta_Z + d_2\gamma \in NE(\tilde{E})$ with

$$\sum_{j=1}^{n} \deg \alpha_j + \sum_{j=1}^{\rho} \deg \varepsilon_j = (c_1(\tilde{E}).\beta) + (\dim \tilde{E} - 3)(1-g) + n + \rho - |\mu|.$$

In this case $d_2 = (E.\beta) = |\mu|$ is already fixed and non-negative.

Proposition 3.3. For an ordinary flop $\tilde{E} \dashrightarrow \tilde{E}'$, to prove

$$\mathcal{F}\langle A \mid \varepsilon, \mu\rangle_{g,\beta_S} \cong \langle \mathcal{F}A \mid \varepsilon, \mu\rangle_{g,\beta_S}$$

for any A, $\beta_S \in NE(S)$ and (ε, μ) up to genus $g \leq g_0$, it is enough to show that

$$\mathcal{F}\langle A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho\rangle^{\tilde{E}}_{g,d_2,\beta_S} \cong \langle \mathcal{F}A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho\rangle^{\tilde{E}'}_{g,d_2,\beta_S}$$

for any $A \in H^*(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$, $\varepsilon_j \in H^*(E)$ and $d_2 \geq 0$, $\beta_S \in NE(S)$ up to genus $g \leq g_0$.

3.3. Analysis on local models

In this section, X is the local model

$$X = \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O}).$$

To get the invariance of quantum rings under ordinary $\mathbb{P}^r$ flop, the remaining job we need to do is the following theorem.

Theorem 3.1. For any $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in H^*(X) \cup \tau_\bullet H^*(E)$, if $\beta_S \neq 0$ in $NE(S)$, then

$$\mathcal{F}\langle\alpha\rangle^X_{0,d_2,\beta_S} \cong \langle\mathcal{F}\alpha\rangle^{X'}_{0,d_2,\beta_S}.$$

We will prove the theorem by induction on (d_2, β_S), n and then m which is the number of insertions not coming from base classes. This is based on the following observations: (1) By the analysis in Proposition 3.2, we just need to take care of the subseries with some fixed d_2 and fixed β_S each time (2) Under divisor relations the degree β is either preserved or split into effective classes $\beta = \beta_1 + \beta_2$ (3) When summing over $\beta \in (d_2\gamma + \bar{\psi}^*\beta_S + \mathbb{Z}\ell) \cap NE(X)$, the splitting terms can usually be written as the product of two generating series with no more marked points in a manner which will be clear in each context during the proof.

For the excluded case $(d_2, \beta_S) = (0, 0)$, since $\xi|_Z = 0$ and the extremal curves will always stay in Z, we get trivial invariant if one of the insertions involves ξ. Hence by the general form of Theorem 2.3, the statement in the theorem holds for this initial case except for the unique case $\langle t_1 h^r, t_2 h^r \rangle$. In this case, by the divisor axiom

$$\delta_h \langle t_1 h^r, t_2 h^r \rangle = \langle h, t_1 h^r, t_2 h^r \rangle_+,$$

which will satisfy the functional equation up to analytic continuation only after incorporated with classical defect. Thus we may base our induction on $(d_2, \beta_S) = (0, 0)$ with special care to treat this case.

Let $\beta_S \neq 0$. The case $n = 1$ is covered by the following Conjecture which has been proved by us for the case of the vector bundles F, F' being split into line bundles and the genus $g = 0$. Here we refer to [3] for the definition of J function.

Conjecture 3.1. (Quasi-linearity)

(1) If $d_2 < 0$, then

$$\mathfrak{F} J^X_\beta = J^{X'}_{\mathfrak{F}\beta}$$

term-wise. And for any $\alpha \in H^*(X)$, $t_i \in H^*(S)$,

$$\langle t_1, \dots, t_{n-1}, \tau_k \alpha \rangle^X_\beta = \langle t_1, \dots, t_{n-1}, \tau_k \mathfrak{F}\alpha \rangle^{X'}_{\mathfrak{F}\beta}.$$

(2) If there is no restriction on d_2, then

$$\mathfrak{F}(J^X_\beta . \xi) \cong J^{X'}_{\mathfrak{F}\beta} . \xi'.$$

And thus for any $\alpha \in H^*(X)$, $t_i \in H^*(S)$,

$$\langle t_1, \dots, t_{n-1}, \tau_k \alpha . \xi \rangle^X_\beta \cong \langle t_1, \dots, t_{n-1}, \tau_k \mathfrak{F}\alpha . \xi' \rangle^{X'}_{\mathfrak{F}\beta}.$$

Let $n \geq 2$ here. First we handle the case of ξ appearing in some α_i. For $d_2 \neq 0$, if not, then, there will be no descendent insertions and we may write

$$\langle \alpha_1, \dots, \alpha_n \rangle = \langle \alpha_1, \dots, \alpha_n, \xi \rangle / d_2$$

by the divisor axiom. By reordering we may assume that $\alpha_n = \tau_s \xi a$, $s \geq 0$. Write $\alpha_1 = t_1 \tau_k h^l \xi^j$. The induction procedure is to move divisors in α_1 into α_n in the order of ψ, h and ξ. That is we use induction on the following six ingredients in the alphabetical order:

$$((d_2, \beta_S), n, m, k, l, j).$$

By applying the induction process, we can reduce the invariants to be of the form $\langle t_1, \dots, t_{n-1}, \tau_k \alpha . \xi \rangle^X_{d_2, \beta_S}$ which are also taken care of in Conjecture 3.1.

Up to now we show that Theorem 3.1 holds for either ξ appearing in the insertion classes or $d_2 \neq 0$ by using induction hypothesis on all cases.

The final remaining cases are *twisted extremal functions* of the form

$$\langle t_1, \ldots, t_n h^j \rangle^X_{\beta_S, d_2=0} \tag{3.1}$$

with $\beta_S \neq 0$. The analytic continuations of them from X to X' are solved by induction on Mori cone $\beta_S \in NE(S)$, using *Birkhoff factorizations* and *generalized mirror maps*. As this step is rather technical, it would be helpful to give explicit proofs in some simple examples.

§4. $\mathbb{P}^1$ flops of type $(k+2, k)$

4.1. The toric geometric setup

We study the local model of a non-simple ordinary flop with base $S = \mathbb{P}^1$. The data consists of two vector bundles $F \to S$ and $F' \to S$. By the Grothendieck Lemma, we may assume that F and F' are of splitting type.

We consider the case

$$F = \mathcal{O} \oplus \mathcal{O}(-(k+2)), \quad F' = \mathcal{O} \oplus \mathcal{O}(k) \tag{4.1}$$

for some $k \in \mathbb{Z}$. For symmetric reason we may assume that $k \geq -1$. In this case $\mathcal{O}(-(k+2))$ is a negative line bundle. In order to make our discussions consistent we further assume that $k \geq 0$. The remaining case $k = -1$ corresponding to $F = \mathcal{O} \oplus \mathcal{O}(-1)$, $F' = \mathcal{O} \oplus \mathcal{O}(-1)$ can be treated in essentially the same (in fact much simpler) way and is postponed till the end of this section.

Here, $Z = \mathbb{P}_S(F) \xrightarrow{\bar\psi} S$ and $Z' = \mathbb{P}_S(F') \xrightarrow{\bar\psi'} S$. By Leray–Hirsch, the cohomology algebras are given by

$$\begin{aligned} H(Z) &= H(S)[h] = \mathbb{Z}[p][h], \quad h^2 - (k+2)hp = 0, \\ H(Z') &= H(S)[h'] = \mathbb{Z}[p][h'], \quad h'^2 + kh'p = 0, \end{aligned} \tag{4.2}$$

where $h = c_1(\mathcal{O}_Z(1))$ and $h' = c_1(\mathcal{O}_{Z'}(1))$.

Also $X = \mathbb{P}_Z(N \oplus \mathcal{O})$ where $N = N_{Z/X} = \bar\psi^* F' \otimes \mathcal{O}_Z(-1)$. The Chern roots of N are given by $-h$ and $kp - h$, thus

$$\begin{aligned} &H(X) = H(Z)[\xi] = \mathbb{Z}[p, h][\xi], \\ &\quad \xi^3 + (kp - 2h)\xi^2 + 2hp\xi = 0. \end{aligned} \tag{4.3}$$

Similarly, $X' = \mathbb{P}_{Z'}(N' \oplus \mathcal{O})$ where $N' = N_{Z'/X} = \bar{\psi}'^* F \otimes \mathcal{O}_{Z'}(-1)$ with Chern roots $-h'$ and $-(k+2)p - h'$ and

$$\begin{aligned} H(X') = H(Z')[\xi'] = \mathbb{Z}[p', h'][\xi'], \\ \xi'^3 - ((k+2)p + 2h')\xi'^2 + 2h'p\xi' = 0. \end{aligned} \tag{4.4}$$

The induced ordinary $\mathbb{P}^1$ flop $f : X \dashrightarrow X'$ is given by blowing up $\phi : Y = \mathrm{Bl}_Z X \to X$ followed by the blowing down map $\phi' : Y \to X'$ which contracts the divisor $E \cong \mathbb{P}_Z(N)$ (a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over S) to Z' along the $\bar{\psi}$-fiber direction. We call this flop of type $(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}(-(k+2)), \mathcal{O} \oplus \mathcal{O}(k))$, abbreviated as type $(k+2,k)$.

The purpose of this section is to achieve the invariance of quantum product for flops of type $(k+2,k)$.

Theorem 4.1. (= Theorem 1.2) Let $f : X \dashrightarrow X'$ be an ordinary $\mathbb{P}^1$ flop of type $(k+2,k)$. The canonical correspondence $\mathcal{F}$ induces an isomorphism of big quantum rings $QH(X)$ and $QH(X')$ in the sense of analytic continuations over the extended Kähler moduli space.

Notice that we will only need to consider genus zero n-point fiber functions of the form

$$\langle t_1, \ldots, t_n h^j \rangle^X_{\beta_S, d_2 = 0} \tag{4.5}$$

where $t_i \in A^*(S)$, $j \le r$ and $\beta = \beta_S + d\ell + d_2\gamma$. By the virtual dimension count

$$d^v = c_1(X).\beta + \dim X + n - 3 = \sum |t_i| + j. \tag{4.6}$$

Since $\dim X = 4$ and $c_1(X).\beta = (c_1 + c_1' + c_1(S)).\beta = 0$, we get $n + 1 = \sum |t_i| + j$. The only possibility is that $t_i = p$ and $j = 1$, that is, the only such invariant is $\langle p, \ldots, p, hp \rangle_{\beta_S, 0}$ where $p \in A^{\dim S}(S)$ is the point class.

The analytic continuation is induced by $\mathcal{F}$ on the 1-cycles of X and X':

$$q^\beta \mapsto q^{\mathcal{F}\beta}. \tag{4.7}$$

The log-extremal flopping contractions $\psi : X \to \bar{X}$ and $\psi' : X' \to \bar{X}$ are given by contracting the $\mathbb{P}^1$ rulings $Z \to S$ and $Z' \to S$ respectively. Denote the ruling class (extremal ray) of ψ (resp. ψ') by ℓ (resp. by ℓ').

The contraction ray of ϕ is the ruling of $\bar{\phi} : E \to Z$ whose class is denoted by γ. Similarly we have the ruling class γ' of $\bar{\phi}' : E \to Z'$ for ϕ'.

Denote by $b = [S]$ the fundamental (curve) class of the base S. Then

$$N_1(X) = \mathbb{Z}b \oplus \mathbb{Z}\ell \oplus \mathbb{Z}\gamma, \tag{4.8}$$
$$N_1(X') = \mathbb{Z}b \oplus \mathbb{Z}\ell' \oplus \mathbb{Z}\gamma',$$

where the 1-cycle $b \in N_1(X)$ is understood as the canonical lifting $\bar{\psi}^*b.H_1 = [Z].(h^1 + c_1(F)) = h - (k+2)p$, followed by the inclusion $i : Z \hookrightarrow X$. Since $(h-(k+2)p).h = h^2 - (k+2)ph = 0$, we see that $b = h-(k+2)p$ is the zero section $S \hookrightarrow Z$ of the bundle $\mathcal{O}(-(k+2)) \to S$.

As a toric variety, the Mori cones (effective cycles) admits a slightly involved description. Yet by a general reduction result, we only need to consider the class

$$\beta = sb + d\ell + d_2\gamma \in NE(X). \tag{4.9}$$

In the case of $d_2 = 0$, we have

Lemma 4.1. $\beta = sb + d\ell \in NE(X)$ if and only if $s, d \geq 0$.

Proof. Both b and ℓ are effective classes, so $\beta = sb + d\ell \in NE(X)$ if $s, d \geq 0$.

Conversely since the bundle $\mathcal{O}(-(k+2)) \to S$ is negative, we know that $NE(Z) = \mathbb{Z}_{\geq 0}b \oplus \mathbb{Z}_{\geq 0}\ell$. If $\beta = sb + d\ell \in NE(X)$, then under the bundle projection map $p : X \to Z$, $\beta = p_*\beta \in NE(Z)$ and we must have $s, d \geq 0$. Q.E.D.

Similarly,

Lemma 4.2. $\beta' = sb + d\ell' \in NE(X')$ if and only if $s \geq 0$ and $d \geq -ks$.

Proof. Instead of using the zero section b, now we should use the infinity section $S_\infty \sim h'$ of $Z' \to S$ to form the generator of the Mori cone $NE(Z')$ since $N_{S_\infty/Z'} = \mathcal{O}(-k)$ while $N_{S/Z'} = \mathcal{O}(k)$. (If $k = 0$, then these two choices are equivalent.)

Since $b = h' + kp$ and $p \sim \ell'$ in Z', by the same proof as above, we get

$$\beta' = sb + d\ell' = sh' + (d+ks)\ell' \in NE(X') \tag{4.10}$$

if and only if $s, (d+ks) \geq 0$. Q.E.D.

Lemma 4.3. $\mathcal{F}(sb + d\ell) = sb - d\ell'$ for all $s, d \in \mathbb{Z}$.

Proof. The fact that $T\ell = -\ell'$ is well known. For $\mathcal{F}b$, recall that $\mathcal{F}H_j = (-1)^{r-j}H'_j$, so for $r = 1$, $\mathcal{F}H_1 = H'_1$. Also $\mathcal{F}$ is $H(S)$-linear, hence

$$\mathcal{F}\bar{\psi}^*b.H_1 = \bar{\psi}'^*b.\mathcal{F}H_1 = \bar{\psi}'^*b.H'_1, \tag{4.11}$$

as expected. Q.E.D.

We always identify divisors with their pull backs.

Lemma 4.4. $\operatorname{Pic} X = \mathbb{Z}p \oplus \mathbb{Z}h \oplus \mathbb{Z}\xi$. In fact, $\{p, h, \xi\}$ is the dual basis of $\{b, \ell, \gamma\}$.

Proof. First of all, we have that $b.p = 1$ by the projection formula. $b.h = (\bar{\psi}^* b.H_1).h = \bar{\psi}^* b.(h^2 + c_1 h) = 0$ by the Chern relation. $b.\xi = 0$ since b is disjoint from ξ.

For the extremal ray ℓ, $\ell.p = 0$ is clear. $\ell.h = 1$ since h is the class of relative $\mathcal{O}(1)$. $\ell.\xi = 0$ since ℓ is in Z which is disjoint from ξ.

For the fiber class γ, $\gamma.p = 0 = \gamma.h$ since γ can be made disjoint from p and h. Also $\gamma.\xi = 1$ since ξ is the class of relative $\mathcal{O}(1)$. Q.E.D.

So far the discussions extend easily to the general ordinary flops of splitting type. The special choice of type $(k+2,k)$ has the consequence that "X is Calabi–Yau in the base direction". Recall that

$$c_1(X) = (r+2)\xi + (c_1(F) + c_1(F') + c_1(S)). \tag{4.12}$$

Our special choice leads to $c_1(X)|_Z = 0$.

4.2. The hypergeometric I function

For a projective bundle of splitting type $P = \mathbb{P}_B(V)$ with $c(V) = \prod_i (1+\lambda_i)$, we associate the hypergeometric I factor for each $\beta \in NE(P)$ as

$$I_\beta^{P/B}(z^{-1}) = \prod_{i=1}^{\mathrm{rk} V} \frac{1}{\prod_{m=0}^{\beta.(h+\lambda_i)} (h + \lambda_i + mz)} \tag{4.13}$$

with $h = c_1(\mathcal{O}_{P/B}(1))$.

The product in $m \in \mathbb{Z}$ is directed in the sense that

$$\prod_{m=0}^{s} \equiv \prod_{m=0+}^{s+} := \prod_{m=-\infty}^{s} \Big/ \prod_{m=-\infty}^{0}. \tag{4.14}$$

Thus for each i with $\beta.(h+\lambda_i) \le -1$, the corresponding subfactor is understood as in the numerator. The subfactor is 1 if $\beta.(h+\lambda_i) = 0$ since there is no such m.

If $\beta.(h+\lambda_i) \le -1$ for all i, then $I_\beta^{P/B} = 0$ since it contains the Chern polynomial factor $\prod_i (h+\lambda_i) = 0$. Notice that $I_\beta^{P/B}$ is regarded as a cohomology valued Laurent formal power series in z^{-1}. That is, $I_\beta^{P/B}$ has the upper bounded degree in z.

The above I factor is a relative object since it takes care of the fiber direction only. If we have a tower of projective bundles $P = P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 = S$ of splitting type in each step $p_j : P_j \to P_{j-1}$, then we form the I factor

$$I_\beta^{P/S} := I_\beta^{P_m/P_{m-1}} I_\beta^{P_{m-1}/P_{m-2}} \cdots I_\beta^{P_1/P_0} \tag{4.15}$$

for each $\beta \in NE(P)$. The P_j/P_{j-1} factor depends only on $p_{(j+1)*} \circ \cdots \circ p_{m*}\beta \in NE(P_j)$, and the factor is identified with its pull back on P.

The (relative) I function over $\beta_S \in NE(S)$ is the fiber generating series

$$I_{\beta_S}^{P/S} := \sum_{\beta \mapsto \beta_S} I_\beta^{P/S} q^{\beta - \beta_S}. \tag{4.16}$$

Remark 4.1. The I function can be introduced for varieties with large group actions like toric varieties or Grassmannian/flag varieties. In these cases, I comes from the localization data in the equivariant cohomology. The above discussion can be extended to bundles of splitting type with fibers being toric or flag manifolds.

To encode the combinatorial data needed for the Gromov–Witten invariants on P, in general we need to consider the weighted J function on S with weight on each $J_{\beta_S}^S$ to be $I_{\beta_S}^{P/S}$.

In the special case when $I^S = I^{S/\mathrm{pt}}$ is defined, say for S being a projective space, we can also consider the (absolute) I function

$$I^P = \sum_{\beta \in NE(P)} I_\beta^P q^\beta \tag{4.17}$$

where $I_\beta^P = I_\beta^{P/\mathrm{pt}}$, by regarding the base to be a point. For $\mathbb{P}^1$ flops with base $S = \mathbb{P}^1$ as in the main theorem, both $X \to Z \to S \to \mathrm{pt}$ and $X' \to Z' \to S \to \mathrm{pt}$ are in this special case. In particular, we have

Lemma 4.5. Let $f : X \dashrightarrow X'$ be a flop of type $(k+2, k)$. Then for $\beta = sb + d\ell \in NE(X)$, that is, $s, d \geq 0$,

$$I_{s,d} := I_\beta^X = \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{-d}^{0} (\xi - h + mz) \prod_{-d+ks}^{0} (\xi - h + kp + mz)}{\prod_0^d (h+mz) \prod_0^{d-(k+2)s} (h - (k+2)p + mz)}. \tag{4.18}$$

For $\beta' = sb + d\ell' \in NE(X')$, that is, $s \geq 0$, $d \geq -ks$,
(4.19)

$$I'_{s,d} := \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{-d}^0 (\xi' - h' + mz) \prod_{-d-(k+2)s}^0 (\xi' - h' - (k+2)p + mz)}{\prod_0^d (h' + mz) \prod_0^{d+ks} (h' + kp + mz)}.$$

If z and all divisors are of degree one, then $I_{s,d}$ (resp. $I'_{s,d}$) is homogeneous of degree $-\beta.c_1(X)$ (resp. $-\beta'.c_1(X')$), which is zero if β (resp. β') has no γ (resp. γ') component.

Proof. The rational expression follows directly from

$$I^X_\beta = I^{X/Z}_\beta I^{Z/S}_\beta I^S_\beta.$$

By (4.13), it has degree $-\beta.((r+1)h + c_1(F)) - \beta.((r+2)\xi + c_1(N)) - \beta.c_1(S)$. Since $c_1(N) = c_1(F') - (r+1)h$, the degree is $-\beta.c_1(X) = -(r+2)\beta.\xi$, which is zero in our special case and special choice of β. Q.E.D.

The products in the above formulae are not always polynomials. To put them in the correct position, we introduce the following notion:

Definition 4.1. A class $\beta \in N_1(X)$ is called $\mathfrak{F}$-effective (or in the *unstable range*) if both $\beta \in NE(X)$ and $\mathfrak{F}\beta \in NE(X')$.

It is easy to get the following lemma.

Lemma 4.6. A class $\beta = sb + d\ell$ is $\mathfrak{F}$-effective if and only if $s \geq 0$ and $0 \leq d \leq ks$. In this unstable range, the I factors can be written as rational expressions of polynomial products as
(4.20)

$$I_{s,d} = \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{-d}^0 (\xi - h + mz) \prod_{d-(k+2)s}^0 (h - (k+2)p + mz)}{\prod_0^d (h + mz) \prod_0^{-d+ks} (\xi - h + kp + mz)},$$

(4.21)

$$I'_{s,-d} = \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{-d}^0 (h' + mz) \prod_{d-(k+2)s}^0 (\xi' - h' - (k+2)p + mz)}{\prod_0^d (\xi' - h' + mz) \prod_0^{-d+ks} (h' + kp + mz)}.$$

In the stable (= not unstable) range, (4.18) and (4.19) give the corresponding rational expressions of polynomial products except in the initial range $ks < d < (k+2)s$ where the product $\prod_0^{d-(k+2)s}(h-(k+2)p+mz)$ in (4.18) should appear in the numerator as $\prod_{d-(k+2)s}^{0}$.

Definition 4.2. We call the initial range $ks < d < (k+2)s$ the *gap range.* It is only defined on the X side. There is no gap range on the X' side.

Since $\deg I_{s,d} = 0$, we may expand it into series in z^{-1} as

$$I_{s,d} = I_{s,d;0} + \frac{I_{s,d;1}}{z} + \frac{I_{s,d;2}}{z^2} + \frac{I_{s,d;3}}{z^3} + \frac{I_{s,d;4}}{z^4}. \tag{4.22}$$

Here $\deg I_{s,d;j} = j$. Hence the higher terms vanish by dimension reason. We will also use the notations

$$\tilde{I}_s := \sum_d I_{s,d}\, q^{d\ell}, \qquad I_{s;j} := \sum_d I_{s,d;j}\, q^{d\ell}. \tag{4.23}$$

And thus $\tilde{I}_s = \sum_j I_{s;j}/z^j$.

For our purpose, we have to study $I_{s;j}$ in details for $j = 0, 1, 2$ and for all $s \geq 0$ in order to perform the change of variables needed in next section.

Lemma 4.7. In the stable range, $I_{s,d} = O(z^{-2})$ and $I'_{s,d} = O(z^{-2})$. In fact, in the gap range $ks < d < (k+2)s$, we even have $I_{s,d} = O(z^{-3})$.

In the unstable range, $I_{s,d} = O(z^{-2})$ except for $d = 0$ where $I_{0,0} = 1$ and $I_{s,0} = O(z^{-1})$ for $s > 0$. The same holds for $I'_{s,d}$.

Proof. This follows easily from (4.18) and (4.20) by noticing that $\deg I = 0$ and each polynomial product in the numerator will loss one z degree corresponding to $m = 0$. For example, in the gap range, there are three products in the numerator while if $d = 0$ the product $\prod_{-d}^{0}$ will disappear and should not be counted. Q.E.D.

More precisely,

Lemma 4.8. For $d \geq 1$, all $I_{0,d}$, $I'_{0,d}$ are in the stable range and

$$\begin{aligned} \tilde{I}_0 &= 1 + \sum_{d\geq 1} \frac{1}{d^2} \frac{(\xi - h)(\xi - h + kp)}{z^2} q^{d\ell} + O(z^{-3}), \\ \tilde{I}'_0 &= 1 + \sum_{d\geq 1} \frac{1}{d^2} \frac{(\xi' - h')(\xi' - h' - (k+2)p)}{z^2} q^{d\ell'} + O(z^{-3}). \end{aligned} \tag{4.24}$$

Remark 4.2. For the local model of simple $\mathbb{P}^r$ flops with extremal curve class $d\ell$, $d \geq 1$, it is well known that

$$I_{0,d} = \frac{(-1)^{(r+1)(d+1)}}{(h+dz)^{r+1}}. \tag{4.25}$$

From (4.18), $\tilde{I}_0$ reduces to it by setting $p=0$ and $\xi = 0$.

Lemma 4.9. For $s \geq 1$,

(4.26)
$$I_{s,0} = (-1)^{s-1}\frac{((k+2)s-1)!}{(s!)^2(ks)!}\frac{(h-(k+2)p)}{z}\times$$
$$\Big[1 - \frac{1}{z}\Big((h-(k+2)p)\sum_{j=1}^{(k+2)s-1}\frac{1}{j} + 2p\sum_{j=1}^{s}\frac{1}{j} + (\xi - h + kp)\sum_{j=1}^{ks}\frac{1}{j}\Big)\Big]$$
$$+O(z^{-3}).$$

Similarly,

(4.27)
$$I'_{s,0} = (-1)^{s-1}\frac{((k+2)s-1)!}{(s!)^2(ks)!}\frac{(\xi'-h'-(k+2)p)}{z}\times$$
$$\Big[1 - \frac{1}{z}\Big((\xi'-h'-(k+2)p)\sum_{j=1}^{(k+2)s-1}\frac{1}{j} + 2p\sum_{j=1}^{s}\frac{1}{j} + (h'+kp)\sum_{j=1}^{ks}\frac{1}{j}\Big)\Big]$$
$$+O(z^{-3}).$$

Lemma 4.10. In the unstable range with $d \neq 0$,

(4.28)
$$I_{s,d} = (-1)^s\frac{(-d+(k+2)s-1)!}{(s!)^2 d(-d+ks)!}\frac{(\xi-h)(h-(k+2)p)}{z^2} + O(z^{-3}),$$
$$I'_{s,-d} = (-1)^s\frac{(-d+(k+2)s-1)!}{(s!)^2 d(-d+ks)!}\frac{h'(\xi'-h'-(k+2)p)}{z^2} + O(z^{-3}).$$

Lemma 4.11. In the stable range, namely $I_{s,d}$ for $d \geq ks+1$ and $I'_{s,d}$ for $d \geq 1$,

(4.29)
$$I_{s,d} = (-1)^s\frac{(d-ks-1)!}{(s!)^2 d(d-(k+2)s)!}\frac{(\xi-h)(\xi-h+kp)}{z^2} + O(z^{-3}),$$
$$I'_{s,d} = (-1)^s\frac{(d+(k+2)s-1)!}{(s!)^2 d(d+ks)!}\frac{(\xi'-h')(\xi'-h'-(k+2)p)}{z^2} + O(z^{-3}).$$

On X, $I_{s,d;2} = 0$ precisely in the gap range. Moreover, in the stable range, the leading scalar coefficient of $I_{s,d}$ equals

$$(4.30) \qquad (-1)^s \frac{(d-(ks+1))(d-(ks+2))\cdots(d-((k+2)s-1))}{(s!)^2 d},$$

and the leading scalar coefficient of $I'_{s,d}$ equals

$$(4.31) \quad (-1)^s \frac{(d+((k+2)s-1))(d+((k+2)s-2))\cdots(d+(ks+1))}{(s!)^2 d}.$$

These two formal expressions are symmetric under $d \mapsto -d$.

For later use, we define

$$(4.32) \qquad G(s,d) := \frac{1}{(s!)^2} \prod_{j=ks+1}^{(k+2)s-1} (d+j).$$

4.3. From I to the geometric J function by mirror map

The one point J function is defined to be

$$(4.33) \qquad J_\beta = e_{1*}^{vir} \frac{1}{z(z-\psi)} = \frac{J_{\beta,2}}{z^2} + \frac{J_{\beta,3}}{z^3} + \cdots$$

where $e_1 : M_{0,1}(X,\beta) \to X$ is the evaluation map. Thus $J_{\beta,2+k}$ calculate the k-th descendent invariants through the Poincaré pairing

$$(4.34) \qquad \langle \tau_k \alpha \rangle_\beta = (J_{\beta;2+k}.\alpha)^X.$$

We are only interested in the case $k=0$ in this section, so we need only to study the z^{-2} term: $J_2 = \sum_\beta J_{\beta;2}\, q^\beta$. We formally set $J_0 = 1$, $J_1 = 0$ and $J = \sum_{i\geq 0} J_i/z^i$.

The mirror theorem states that J and I are related by some change of variables. We also write $I = \sum_{i\geq 0} I_i/z^i$. Here, $I_0 = 1$ and $\deg I_i = i$. In this case, the change of variables is particularly easy to describe. Let D_i's be a cohomology basis of divisor classes with dual curve class basis β_i's. Let also $D = \sum_i t_i D_i$ be a general divisor with coordinates t_i's. We use the following formal identification

$$(4.35) \qquad q^{\beta_i} = e^{t_i}.$$

Then the mirror theorem says that

$$(4.36) \qquad e^{D/z} J = e^{D/z} I$$

after the change of variables (mirror map)

$$M : t_i \mapsto t_i + (\beta_i . I_1) \tag{4.37}$$

on the J side. First of all, this makes sense as I_1 is a divisor-valued power series in $q^{\beta_i} = e^{t_i}$. More importantly this equates the z^{-1} term on both sides. Indeed

$$D = \sum_i t_i D_i \mapsto \sum t_i D_i + (\beta_i . I_1) D_i = D + I_1. \tag{4.38}$$

After the mirror map on $e^{D/z}J$ and by removing the common $e^{D/z}$, we get

$$e^{I_1/z} J^M = I. \tag{4.39}$$

And then

$$\begin{aligned} & 1 + \frac{J_2^M}{z^2} + \frac{J_3^M}{z^3} + \cdots \\ & = \Big(1 + \frac{I_1}{z} + \frac{I_2}{z^2} + \cdots\Big)\Big(1 - \frac{I_1}{z} + \frac{1}{2}\frac{I_1^2}{z^2} - \cdots\Big) \\ & = 1 + \frac{1}{z^2}\Big(I_2 - \frac{1}{2} I_1^2\Big) + \cdots . \end{aligned} \tag{4.40}$$

We may write (4.37) as

$$M : q^{\beta_i} \mapsto q^{\beta_i} e^{(\beta_i . I_1)}. \tag{4.41}$$

Then the z^{-2} term in (4.40) takes the form

$$\sum_\beta J_{\beta;2}\, q^\beta e^{(\beta . I_1)} = \sum_\beta I_{\beta;2}\, q^\beta - \frac{1}{2} I_1^2. \tag{4.42}$$

We will use it to compare $J_2.hp$ and $J'_2.\mathfrak{F}hp = J'_2.(\xi' - h')p$.

4.4. Analytic continuations

More precisely, we are going to prove the following theorem.

Theorem 4.2. For $s \neq 0$, we have analytic continuations under $\mathfrak{F}$:

$$\langle hp \rangle_{sb}^X \cong \langle (\xi' - h')p \rangle_{sb}^{X'}. \tag{4.43}$$

Proof. By Lemma 4.9, we have

$$I_1 = g(q)(h - (k+2)p), \qquad I'_1 = g(q)(\xi' - h' - (k+2)p) \tag{4.44}$$

where

$$
g(q) = g(q^b) := \sum_{s\geq 1}(-1)^{s-1}\frac{((k+2)s-1)!}{(s!)^2(ks)!}\, q^{sb} \tag{4.45}
$$
$$
= (k+1)q^b - \frac{(2k+1)(2k+2)(2k+3)}{4}q^{2b} + \cdots
$$

depends only on q^b, hence is constant in $\mathcal{F}$ and $\mathcal{F}I_1 = I_1'$. Notice that in terms of $G(s,d)$, we have $g(q^b) = \sum_{s\geq 1}(-1)^{s-1}G(s,0)\, q^{sb}$.

Moreover for any $d \in \mathbb{Z}$ (or more accurately $s \geq 0$ and $d \geq 0$ for X and $d \geq -ks$ for X'), (4.42) implies that

(4.46)

$$
\sum_{s\geq 0} J_{s,d;2}\, q^{sb} e^{(d-(k+2)s)g(q)} = \sum_{s\geq 0} I_{s,d;2}\, q^{sb} - \delta_{d,0}\frac{1}{2}g^2(q)(h-(k+2)p)^2,
$$
$$
\sum_{s\geq 0} J'_{s,d;2}\, q^{sb} e^{(-d-(k+2)s)g(q)} = \sum_{s\geq 0} I'_{s,d;2}\, q^{sb} - \delta_{d,0}\frac{1}{2}g^2(q)(\xi'-h'-(k+2)p)^2.
$$

For simplicity, we denote

$$
\hat{I} = (I.hp)^X, \qquad \hat{J} = (J.hp)^X, \tag{4.47}
$$
$$
\hat{I}' = (I'.(\xi'-h')p)^{X'}, \qquad \hat{J}' = (J'.(\xi'-h')p)^{X'}.
$$

Then $\hat{J}$ and $\hat{I}$ are related by

$$
\sum_{s\geq 0} \hat{J}_{s,d;2}\, q^{sb} e^{(d-(k+2)s)g(q)} = \sum_{s\geq 0} \hat{I}_{s,d;2}\, q^{sb}, \tag{4.48}
$$
$$
\sum_{s\geq 0} \hat{J}'_{s,d;2}\, q^{sb} e^{(-d-(k+2)s)g(q)} = \sum_{s\geq 0} \hat{I}'_{s,d;2}\, q^{sb} + \delta_{d,0}\frac{1}{2}g^2(q),
$$

since $(h-(k+2)p)p = 0$ and $(\xi'-h'-(k+2)p)^2(\xi'-h')p = \xi'^3p - 3\xi'^2h'p = -1$.

For $\beta = sb + d\ell$ in the unstable range $(0 \leq d \leq ks)$, by Lemma 4.9 and Lemma 4.10, we have $\hat{I}_{s,d;2} = 0$.

On the other hand if $\beta = sb + d\ell$ is in the stable range $d \geq ks+1$, then by Lemma 4.11, we may formally define

$$
\hat{I}'_{s,-d} := -\hat{I}_{s,d} \tag{4.49}
$$

since

$$
(\xi - h)(\xi - h)hp = \xi^2 hp = 1, \tag{4.50}
$$
$$
(\xi'-h')(\xi'-h'-(k+2)p)(\xi'-h')p = \xi'^3p - 3\xi'^2h'p = -1.
$$

Then the two recursive relations in (4.48) can be merged into a single relation on the X' side for $d \in \mathbb{Z}$.

Lemma 4.12. Let $w_s(d) := \hat{J}'_{s,d;2}$ for $d \neq 0$. Then we have $w_0(d) = \hat{J}'_{0,d;2} = \hat{I}'_{0,d,2} = -1/d^2$. For $s \geq 1$, the function $w_s(d)$ is a polynomial in d of degree $2(s-1)$.

Proof. The first statement is clear. For the second statement we fix a $d \neq 0$. We start by writing out the first few relations explicitly to explain the general structure. Up to q^{2b}, we compute

(4.51)
$$e^{-(d+(k+2)s)g(q)} = 1 - (d+(k+2)s)G(1,0)\,q^b + (d+(k+2)s)\Big(G(2,0) + \frac{G(1,0)^2}{2}(d+(k+2)s)\Big)\,q^{2b} + \cdots.$$

Then by looking at the q^b terms in (4.48), we get

$$w_1(d) - G(1,0)dw_0(d) = \hat{I}'_{1,d;2}. \tag{4.52}$$

By Lemma 4.11, we find $\hat{I}'_{s,d;2} = (-1)^{s-1}G(s,d)/d$ (which is NOT a polynomial in d) and this reads as

$$w_1(d) + G(1,0)d\frac{1}{d^2} = \frac{G(1,d)}{d} = \frac{d+G(1,0)}{d}. \tag{4.53}$$

Thus

$$w_1(d) = 1, \quad \forall\, d \neq 0. \tag{4.54}$$

By looking at the q^{2b} terms, we get

(4.55)
$$w_2(d) - (d+(k+2))G(1,0)w_1(d) + d\Big(G(2,0) + \frac{G(1,0)^2}{2}d\Big)w_0(d) = \hat{I}'_{2,d;2} = -\frac{G(2,d)}{d}.$$

Since $w_1(d)$ is a polynomial ($= 1$ indeed), the only trouble to forbid $w_2(d)$ to be a polynomial comes from $dG(2,0)w_0(d) = -G(2,0)/d$.

However, this is exactly cancelled out from the singular part from the right hand side. This proves that $w_2(d)$ is a polynomial in d. Since $G(s,d)$ has degree $(k+2)s - 1 - ks = 2s - 1$, it is clear that $w_2(d)$ has degree $2 \times 2 - 1 - 1 = 2$ in d.

By induction, suppose that the statement is proved up to $s-1$. Then by looking at the coefficients of the q^{sb} terms, we get

$$\begin{aligned} w_s(d) - (d+(k+2)(s-1))G(1,0)w_{s-1}(d) + \cdots \\ + (-1)^s d(G(s,0)+dp(d))w_0(d) \\ = \hat{I}'_{s,d;2} = (-1)^{s-1}\frac{G(s,d)}{d} \end{aligned} \tag{4.56}$$

for some polynomial $p(d)$. Here the coefficient of $w_0(d)$ is the degree q^{sb} term in

$$\begin{aligned} e^{-dg(q)} &= 1 - dg + \frac{d^2g^2}{2!} - \cdots \\ &= 1 - d\sum_{k\geq 1}(-1)^{k-1}G(k,0)\,q^{kb} + d^2(\cdots). \end{aligned} \tag{4.57}$$

Again since $w_0(d) = -1/d^2$, the only singular term $(-1)^{s-1}G(s,0)/d$ in (4.56) cancels out from both sides. Hence $w_s(d)$ is a polynomial in d. It is also clear that the highest d-degree term comes from $G(s,d)/d$ which is

$$\frac{d^{2(s-1)}}{(s!)^2}. \tag{4.58}$$

This completes the proof of the lemma. Q.E.D.

The remaining proof is based on the following idea: Consider the formal expression

$$E(q) := \sum_{d=-\infty}^{\infty} q^d = \cdots + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots. \tag{4.59}$$

If we denote $f(q) = q/(1-q)$, then the analytic continuation

$$f(q) + f(q^{-1}) = -1 \tag{4.60}$$

simply means the formal assignment $E(q) = 0$. For any polynomial $w(d)$, we have

$$\sum_{d=-\infty}^{\infty} w(d)\,q^d = w(\delta)E(q) \tag{4.61}$$

where $\delta = q\partial/\partial q$. Thus we may establish the analytic continuation result on $\hat{J}_{s;2}$ if we show that $\hat{J}'_{s,d;2}$ is polynomial in d for all $d \in \mathbb{Z}$.

By the lemma, such a polynomial $w_s(d)$ is obtained for $d \neq 0$. Hence it remains to show that $\hat{J}'_{s,0;2} = w_s(0)$.

By (4.48), we have for $d = 0$,

$$\sum_{s\geq 1} \hat{J}'_{s,0;2}\, q^{sb} e^{-(k+2)sg} = \sum_{s\geq 1} \hat{I}'_{s,0;2}\, q^{sb} + \frac{1}{2}g^2, \tag{4.62}$$

while for $d \neq 0$ it is

$$\sum_{s\geq 0} \hat{J}'_{s,d;2}\, q^{sb} e^{(-d-(k+2)s)g} = \sum_{s\geq 0} \hat{I}'_{s,d;2}\, q^{sb}. \tag{4.63}$$

In order to compare these two relations, let $\mathbb{I}_{s,d}$ be the regular (polynomial) part of $\hat{I}'_{s,d;2}$. Namely

$$\mathbb{I}_{s,d} := I'_{s,d;2} - (-1)^{s-1}\frac{G(s,0)}{d}. \tag{4.64}$$

Then (4.63) may be rewritten as

$$\sum_{s\geq 1} \hat{J}'_{s,d;2}\, q^{sb} e^{(-d-(k+2)s)g} - \frac{1}{d^2}e^{-dg} = \sum_{s\geq 1} \mathbb{I}_{s,d}\, q^{sb} + \frac{g}{d} - \frac{1}{d^2}. \tag{4.65}$$

Since

$$\frac{1}{d^2}(e^{-dg} - 1 + dg) = \frac{1}{2}g^2 + dQ \tag{4.66}$$

for some $Q \in \mathbb{Q}[d][\![q^b]\!]$, (4.65) may be arranged to

$$\sum_{s\geq 1} \hat{J}'_{s,d;2}\, q^{sb} e^{(-d-(k+2)s)g} = \sum_{s\geq 1} \mathbb{I}_{s,d}\, q^{sb} + \frac{1}{2}g^2 + dQ. \tag{4.67}$$

Now by (4.27) and noticing that $(\xi'-h'-(k+2)p)(h'+kp)(\xi'-h')p = 1$, we get

$$\hat{I}'_{s,0;2} = (-1)^{s-1}\frac{((k+2)s-1)!}{(s!)^2(ks)!}\sum_{j=ks+1}^{(k+2)s-1}\frac{1}{j}. \tag{4.68}$$

This is precisely the degree zero term (constant term) of

$$\hat{I}'_{s,d;2} = (-1)^{s-1}\frac{G(s,d)}{d} = (-1)^{s-1}\frac{1}{(s!)^2 d}\prod_{j=ks+1}^{(k+2)s-1}(d+j), \tag{4.69}$$

hence also the constant term of the polynomial $\mathbb{I}_{s,d}$.

Now it is clear that the relation (4.67) specializes to (4.62) in the special value $d = 0$. Hence $\hat{J}'_{s,d;2} = w_s(d)$ must specializes to $\hat{J}'_{s,0;2}$ when $d = 0$.

This is exactly what we want and the proof of the theorem is complete. Q.E.D.

4.5. The remaining symmetric case $k = -1$

In the case $k = -1$, the flop is of type

$$(S, F, F') = (\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(-1)). \tag{4.70}$$

Thus most of the formulae involved will be symmetric in X and X'.

The proof on analytic continuations proceeds in the same way with the following modifications. First of all, since both the canonical liftings of b in Z and Z' have the negative normal bundle $\mathcal{O}(-1)$, we have that $\beta = sb + d\ell \in NE(X)$ if and only if $s, d \geq 0$ and $\beta' = sb + d\ell' \in NE(X')$ if and only if $s, d \geq 0$ as well. Since $\mathcal{F}(sb + d\ell) = sb - d\ell'$, $\beta = sb + d\ell$ is in the unstable range if and only if $d = 0$. That is, for any fixed s, there is essentially no unstable range.

The I function can be determined by the same formula with $k = -1$. In fact, $I = 1 + O(z^{-2})$:

Lemma 4.13. For $s, d \geq 0$,

(4.71)

$$I_{s,d} := I^X_\beta = \frac{1}{\prod_0^s (p + mz)^2} \frac{\prod_{-d}^0 (\xi - h + mz) \prod_{-d-s}^0 (\xi - h - p + mz)}{\prod_0^d (h + mz) \prod_0^{d-s} (h + p + mz)}.$$

For $s > 0$ and $d = 0$,

$$I_{s,0} = \frac{1}{s^2} \frac{(h - p)(\xi - h - p)}{z^2} + O(z^{-3}). \tag{4.72}$$

For $1 \leq d \leq s - 1$, $I_{s,d} = O(z^{-3})$.
For $d \geq s$ or $d \geq 1$ if $s = 0$,

$$I_{s,d} = (-1)^s \frac{(d + (s - 1))!}{(s!)^2 d(d - s)!} \frac{(\xi - h)(\xi - h - p)}{z^2} + O(z^{-3}). \tag{4.73}$$

In particular, no mirror map is needed and we have $J_{s,d} = I_{s,d}$ for all $s, d \geq 0$.

The above lemma applies to X' as well. As before we set

$$\hat{J} = (J.hp)^X, \quad \hat{J}' = (J'.(\xi' - h')p)^{X'}. \tag{4.74}$$

By straightforward intersection calculations, we get

Lemma 4.14. For $s > 0$ and $d = 0$,

$$\hat{J}_{s,0;2} = 0, \qquad \hat{J}'_{s,0;2} = \frac{1}{s^2}. \tag{4.75}$$

For $1 \le d \le s-1$, $\hat{J}_{s,d;2} = 0 = \hat{J}'_{s,d;2}$.
For $d \ge s$ or $d \ge 1$ if $s = 0$,

$$\hat{J}_{s,d;2} = (-1)^s \frac{(d+(s-1))!}{(s!)^2 d(d-s)!}, \qquad \hat{J}'_{s,d;2} = (-1)^{s-1} \frac{(d+(s-1))!}{(s!)^2 d(d-s)!}. \tag{4.76}$$

For each $s \ge 1$ we define the even *polynomial* function

$$w_s(d) := (-1)^{s-1} \frac{(d+(s-1))(d+(s-2))\cdots(d-(s-1))}{(s!)^2 d}. \tag{4.77}$$

Notice that $w_s(d)$ is a polynomial in d of degree $2(s-1)$ since the numerator contains the factor d. It has zeros at $d = \pm 1, \cdots, \pm(s-1)$ and $w_s(0) = 1/s^2$.

We will prove that for $s > 0$

$$\mathcal{F}\hat{J}_{s;2} \cong \hat{J}'_{s;2} \tag{4.78}$$

up to analytic continuations.

By Lemma 4.14, for $d \ge 1$ we may formally define

$$\hat{J}'_{s,-d;2} = -\hat{J}_{s,d;2}. \tag{4.79}$$

Notice that for $d = 0$ we have $\hat{J}'_{s,0;2} = 1/s^2 = w_s(0)$. Thus

$$\hat{J}'_{s,d;2} = w_s(d) \tag{4.80}$$

for all $d \in \mathbb{Z}$. By (4.61), this leads to the result on analytic continuations.

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