Complex structure on the rational blowdown of sections in $E(4)$

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Abstract.

We show that there is a complex structure on the symplectic 4-manifold $W_{4,k}$ obtained from the elliptic surface E(4) by rationally blowing down k sections for $2 \leq k \leq 9$. And we interpret it via $\mathbb{Q}$-Gorenstein smoothing. This answers affirmatively to a question raised by R. Gompf.

§1. Introduction

Let $E(n)$ denote a simply connected, relatively minimal elliptic surface with topological Euler characteristic $c_2 = 12n > 0$ and with a section. The diffeomorphism type of $E(n)$ is unique, and $E(n)$ is symplectically isomorphic to the fiber sum of n copies of a rational elliptic surface $E(1)$. Recall that $E(1)$ is obtained from $\mathbb{P}^2$ by blowing up the nine base points of a pencil of cubics. It is known that $E(n)$ admits nine $(-n)$-curves as disjoint sections.

Consider the case $E(4)$, and rationally blow down k disjoint sections to obtain symplectic 4-manifolds $W_{4,k}$. Rational blowdown of k disjoint sections in $E(4)$ is the same as the normal connected sum of $E(4)$ with k copies of $\mathbb{P}^2$ identifying a conic in each $\mathbb{P}^2$ with a section in $E(4)$. The manifold $W_{4,1}$ does not admit any complex structure because it violates the Noether inequality, $p_g \leq \frac{1}{2}K^2 + 2$ for a minimal surface of general type: $p_g(W_{4,1}) = 3$ and $K^2_{W_{4,1}} = 1$.

In the paper [5], Gompf constructed a family of symplectic 4-manifolds by taking a fiber sum of other symplectic 4-manifolds, and he raised the following question.

Question. Is it possible to give a complex structure on $W_{4,k}$ for $2 \leq k \leq 9$?

The case of $k = 2$ was treated by using $\mathbb{Q}$-Gorenstein smoothing theory [10]. We denote the singular projective surface obtained by contracting k disjoint sections in $E(4)$ by $W'_{4,k}$. It is known that if there is a $\mathbb{Q}$-Gorenstein smoothing of $W'_{4,k}$, then a general fiber is an algebraic surface which is symplectically isomorphic to $W_{4,k}$. Recall the idea of the proof for the case of $k = 2$ in [10]: We consider $E(4)$ as a double cover of Hirzebruch surface $\mathbb{F}_4$ branched over an irreducible nonsingular curve D in the linear system $|4(C_0 + 4f)|$, where C_0 is the negative section and f is a fiber of $\mathbb{F}_4$. Since D does not intersect C_0, $W'_{4,2}$ is a double cover of the cone $\hat{\mathbb{F}}_4$ obtained by contracting C_0 in $\mathbb{F}_4$. Note that $\hat{\mathbb{F}}_4$ has a $\mathbb{Q}$-Gorenstein smoothing whose general fiber is $\mathbb{P}^2$. It is obtained by a pencil of hyperplane section of the cone of the Veronese surface imbedded in $\mathbb{P}^5$. Then $W'_{4,2}$ has a $\mathbb{Q}$-Gorenstein smoothing that is compatible with the $\mathbb{Q}$-Gorenstein smoothing of $\hat{\mathbb{F}}_4$. And the double covering structure extends to the $\mathbb{Q}$-Gorenstein smoothing.

In this paper, we will answer affirmatively to the above question. We will construct directly a $\mathbb{Q}$-Gorenstein smoothing of $W'_{4,k}$ for $2 \leq k \leq 9$.

Theorem. It is possible to give a complex structure on the rational blowdown of k sections in $E(4)$ for $2 \leq k \leq 9$.

The construction is as follows: A bidouble cover of $E(1)$ branched over three general fibers becomes a $E(4)$ with nine disjoint sections. Choose two smooth cubics D_1, D_2 in $\mathbb{P}^2$ such that D_1 and D_2 meet transversally at nine points $p_1, \ldots, p_9$. Let D_0 be a general member of the pencil induced by D_1 and D_2. Let V be a bidouble cover of $\mathbb{P}^2$ branched over D_1, D_2, and D_0. Then V has nine $1/4(1,1)$ singularities, and its minimal resolution is $E(4)$. By moving the cubic D_0, one can construct a one-parameter family of smooth cubics D_t such that D_t does not through $p_1, \ldots, p_k$ and D_t intersects D_1, D_2 transversally at $p_{k+1}, \ldots, p_9$ if $t \neq 0$. First, we note that $k \geq 2$, because if a cubic passes through 8 intersection points then it also passes through the 9-th point. This also explains that $W_{4,1}$ does not admit any complex structure. One-parameter family of smooth cubics D_t induces a one-parameter family $\mathcal{V}_k$ by using a bidouble cover of $\mathbb{P}^2$ branched over D_1, D_2, D_t. Then by using a simultaneous resolution, we have a one-parameter family $\mathcal{X}$ such that $X_0 = W'_{4,k}$ and a general fiber X_t is smooth with $K^2_{X_t} = k$. This family is a $\mathbb{Q}$-Gorenstein smoothing of $W'_{4,k}$.

In Section 3, we briefly review the theory of $\mathbb{Q}$-Gorenstein smoothing and interpret the result via the local-global exact sequence of smoothings.

§2.   Proof of Theorem

A bidouble cover is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ cover $\psi : V \to U$, with U a smooth surface. The building data for a bidouble cover consist of the following:

(1) smooth divisors D_1, D_2, D_3 in U having pairwise transverse intersections, and

(2) line bundles L_1, L_2, L_3 such that $2L_g = D_j + D_k$ for each permutation (g, j, k) of $(1, 2, 3)$.

We have a decomposition

$$\psi_* \mathcal{O}_V = \mathcal{O}_U \oplus \sum_{\chi \in G^*} L_\chi^{-1},$$

where L_χ is a line bundle and $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts on L_χ^{-1} via the character $\chi \in G^*$. Since we have three elements g_1, g_2, g_3 in G^*, there are three characters $\chi_i \in G^*$ for $i = 1, 2, 3$.

Suppose that D_1, D_2, and D_3 have no common intersection. Then V is also smooth, and

$$p_g(V) = p_g(U) + \sum_{i=1}^{3} h^0(U, \mathcal{O}_U(K_U + L_{\chi_i})),$$

$$\chi(\mathcal{O}_V) = 4\chi(\mathcal{O}_U) + \frac{1}{2} \sum_{i=1}^{3} L_{\chi_i}(K_U + L_{\chi_i}),$$

$$2K_V = \psi^*(2K_U + L_{\chi_1} + L_{\chi_2} + L_{\chi_3}).$$

For more information on bidouble covers, see [3], or [13], or [15].

Lemma 2.1. *If D_1, D_2, and D_3 have a common intersection point p in X then V has the singularity $1/4(1, 1)$.*

Proof. Since the problem is local, we may assume that $U = \mathbb{C}^2$, $D_1 : x_1 = 0$, $D_2 : x_2 = 0$, and $D_3 : x_1 + x_2 = 0$ where x_1, x_2 are coordinates of $\mathbb{C}^2$. Let

$$y_1^2 = x_1,\ y_2^2 = x_2,\ y_3^2 = x_1 + x_2.$$

Since $y_1^2 + y_2^2 = y_3^2$, the simple cover of $\mathbb{C}^2$ induced by $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has a A_1-singularity. And a bidouble cover is the quotient of the above simple cover by the involution multiplying all the three functions by -1. Therefore V has a singularity $1/4(1,1)$.   Q.E.D.

Let D_1, D_2 be two smooth cubics in $\mathbb{P}^2$ such that D_1 and D_2 meet transversally at nine points $p_1, \ldots, p_9$. Let D_0 be a general member of the pencil induced by D_1 and D_2. Let V be a bidouble cover of $\mathbb{P}^2$ branched over D_1, D_2, and D_0. Then V has nine $1/4(1,1)$ quotient singularities by Lemma 2.1, and $p_g(V) = 3, \chi(\mathcal{O}_V) = 4, K_V^2 = 9$. Its minimal resolution is $E(4)$.

By moving the cubic D_0, one can construct a one-parameter family of smooth cubics D_t such that D_t does not through $p_1, \ldots, p_k$ and D_t intersects D_1, D_2 transversally at $p_{k+1}, \ldots, p_9$ if $t \neq 0$. First, we note that $k \geq 2$, because if a cubic passes through 8 intersection points then it also passes through the 9-th point. One-parameter family of smooth cubics D_t induces a one-parameter family $\mathcal{V}_k$ by using a bidouble cover of $\mathbb{P}^2$ branched over D_1, D_2, D_t. Recall the definition of a $\mathbb{Q}$-Gorenstein smoothing (cf. [7]).

Definition. Let X be a normal projective surface with quotient singularities. Let $\mathcal{X} \to \Delta$ (or $\mathcal{X}/\Delta$) be a flat family of projective surfaces over a small disk Δ. The one-parameter family of surfaces $\mathcal{X} \to \Delta$ is called a $\mathbb{Q}$-*Gorenstein smoothing* of X if it satisfies the following three conditions;
(i) the general fiber X_t is a smooth projective surface,
(ii) the central fiber X_0 is X,
(iii) the canonical divisor $K_{\mathcal{X}/\Delta}$ is $\mathbb{Q}$-Cartier.

By using a simple cover $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and by simultaneous resolution of A_1-singularities, we have a one-parameter family $\mathcal{X}$ over a small disk Δ in $\mathbb{C}$ such that $X_0 = W'_{4,k}$ and a general fiber X_t is smooth with $K_{X_t}^2 = k$. This family is a $\mathbb{Q}$-Gorenstein smoothing of $W'_{4,k}$ because $2K_{\mathcal{X}/\Delta}$ is Cartier. It is well-known that a general fiber of a $\mathbb{Q}$-Gorenstein smoothing is the rational blowdown of k disjoint sections in $E(4)$. This gives the proof of Theorem.

Remark. One can also do by using a bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over D_1, D_2, and D_3 in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2))$. Then one

can show that there is a complex structure on the rational blowdown of k sections in $E(4)$ for $2 \leq k \leq 8$. Since a cubic in $\mathbb{P}^2$ can be transformed to a $(2,2)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up twice and blowing down once, two constructions of $\mathbb{Q}$-Gorenstein smoothing of $W'_{4,k}$ can be identified.

§3. $\mathbb{Q}$-Gorenstein smoothing

In this section we briefly review the theory of $\mathbb{Q}$-Gorenstein smoothing of projective surfaces with special quotient singularities. The $\mathbb{Q}$-Gorenstein smoothing theory can be used to construct surfaces of general type, but the methodology comes from 3-fold Mori theory (results of Kollár and Shepherd-Barron on $\mathbb{Q}$-Gorenstein smoothing and relative canonical models). The compactification theory of a moduli space of surfaces of general type was established during the last 20 years. It was originally suggested in [7] and it was established by Alexeev's proof for boundness [1] and by the Mori program for threefolds ([6] for details). It is natural to expect the existence of a surface with special quotient singularities in the boundary of a compact moduli space.

A quotient singularity which admits a $\mathbb{Q}$-Gorenstein smoothing is called a *singularity of class T*.

Proposition 3.1 ([7, 12, 19]). *Let $(X_0, 0)$ be a germ of two dimensional quotient singularity. If $(X_0, 0)$ admits a $\mathbb{Q}$-Gorenstein smoothing over the disk, then $(X_0, 0)$ is either a rational double point or a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna - 1)$ for some integers a, n, d with a and n relatively prime.*

Proposition 3.2 ([7, 12, 20]).   (1) *The quotient singularities*
$$\overset{-4}{\circ} \quad and \quad \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \cdots - \overset{-2}{\circ} - \overset{-3}{\circ} \quad are\ of\ class\ T.$$

(2) *If the singularity* $\overset{-b_1}{\circ} - \cdots - \overset{-b_r}{\circ}$ *is of class T, then so are*

$$\overset{-2}{\circ} - \overset{-b_1}{\circ} - \cdots - \overset{-b_{r-1}}{\circ} - \overset{-b_r-1}{\circ} \quad and \quad \overset{-b_1-1}{\circ} - \overset{-b_2}{\circ} - \cdots - \overset{-b_r}{\circ} - \overset{-2}{\circ}.$$

(3) *Every singularity of class T that is not a rational double point can be obtained by starting with one of the singularities described in (1) and iterating the steps described in (2).*

Let X be a normal projective surface with singularities of class T. Due to the result of Kollár and Shepherd-Barron [7], there is a $\mathbb{Q}$-Gorenstein smoothing locally for each singularity of class T on X (see Proposition 3.5). The natural question is whether this local $\mathbb{Q}$-Gorenstein smoothing can be extended over the global surface X or not. The answer can be obtained by figuring out the obstruction map

of the sheaves of deformation $T^i_X = Ext^i_X(\Omega^1_X, \mathcal{O}_X)$ for $i = 0, 1, 2$. For example, if X is a smooth surface, then T^0_X is the usual holomorphic tangent sheaf T_X and $T^1_X = T^2_X = 0$. By applying the standard result of deformations [11, 14] to a normal projective surface with quotient singularities, we get the following

Proposition 3.3 ([19], §4). *Let X be a normal projective surface with quotient singularities. Then*

(1)   *The first order deformation space of X is represented by the global Ext 1-group $\mathbb{T}^1_X = Ext^1_X(\Omega^1_X, \mathcal{O}_X)$.*

(2)   *The obstruction space lies in the global Ext 2-group $\mathbb{T}^2_X = Ext^2_X(\Omega^1_X, \mathcal{O}_X)$.*

Furthermore, by applying the general result of local-global spectral sequence of ext sheaves ([14], §3) to deformation theory of surfaces with quotient singularities so that $E^{p,q}_2 = H^p(T^q_X) \Rightarrow \mathbb{T}^{p+q}_X$, and by $H^j(T^i_X) = 0$ for $i, j \geq 1$, we also get

Proposition 3.4 ([12, 19]). *Let X be a normal projective surface with quotient singularities. Then*

(1)   *We have the exact sequence*

$$0 \to H^1(T^0_X) \to \mathbb{T}^1_X \to \ker[H^0(T^1_X) \to H^2(T^0_X)] \to 0$$

where $H^1(T^0_X)$ represents the first order deformations of X for which the singularities remain locally a product.

(2)   *If $H^2(T^0_X) = 0$, every local deformation of the singularities may be globalized.*

The vanishing $H^2(T^0_X) = 0$ can be obtained via the vanishing of $H^2(T_V(-\log E))$, where V is the minimal resolution of X and E is the reduced exceptional divisors. Note that every singularity of class T has a local $\mathbb{Q}$-Gorenstein smoothing by Proposition 3.5 below. With the help of the birational geometry of threefolds and their applications to deformations of surface singularities, the following proposition is obtained. Note that the cohomology $H^0(\tilde{T}^1_X)$ is given explicitly.

Proposition 3.5 ([7, 12]).   (1)   *Let $a, d, n > 0$ be integers with a, n relatively prime and consider a map $\pi : \mathcal{Y}/\mu_n \to \mathbb{C}^d$, where $\mathcal{Y} \subset \mathbb{C}^3 \times \mathbb{C}^d$ is the hypersurface of equation $uv - y^{dn} = \sum^{d-1}_{k=0} t_k y^{kn}$; $t_0, \ldots, t_{d-1}$ are linear coordinates over $\mathbb{C}^d$, μ_n acts on $\mathcal{Y}$ by*

$$\mu_n \ni \xi : (u, v, y, t_0, \ldots, t_{d-1}) \to (\xi u, \xi^{-1} v, \xi^a y, t_0, \ldots, t_{d-1})$$

and π is the factorization through the quotient of the projection $\mathcal{Y} \to \mathbb{C}^d$. Then π is a $\mathbb{Q}$-Gorenstein smoothing of the cyclic singularity germ $(X_0, 0)$ of type $\frac{1}{dn^2}(1, dna - 1)$. Moreover every $\mathbb{Q}$-Gorenstein smoothing of $(X_0, 0)$ is isomorphic to the pull-back of π under some germ of holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$.

(2) *Let X be a normal projective surface with singularities of class T. Then*

$$H^0(\tilde{T}_X^1) = \sum_{p \,\in\, singular\ points\ of\ X} \mathbb{C}_p^{\oplus d_p}$$

where p is a point of type $\frac{1}{d_p n^2}(1, d_p an - 1)$ with $(a, n) = 1$.

Theorem 3.1 ([8]). *Let X be a normal projective surface with singularities of class T. Let $\pi : V \to X$ be the minimal resolution and let E be the reduced exceptional divisors. Suppose that $H^2(T_V(-\log E)) = 0$. Then $H^2(T_X^0) = 0$ and there is a $\mathbb{Q}$-Gorenstein smoothing of X.*

As we see in Theorem 3.1 above, if $H^2(T_X^0) = 0$, then there is a $\mathbb{Q}$-Gorenstein smoothing of X. For example, a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$ was constructed in [8] by proving $H^2(T_X^0) = 0$. The vanishing $H^2(T_X^0) = 0$ is obtained by the careful study of the configuration of singular fibers in a rational elliptic surface.

Remark. All recent constructions of surfaces of general type with vanishing geometric genus via $\mathbb{Q}$-Gorenstein smoothing [8] [9] [16] [17] [18] have the vanishing $H^2(T_X^0) = 0$ by the same arguments in [8]. By upper semi-continuity, $H^2(X_t, T_{X_t}) = 0$ for a general fiber X_t of a $\mathbb{Q}$-Gorenstein smoothing. We note that

$$h^1(X_t, T_{X_t}) - h^2(X_t, T_{X_t}) = 10\chi(\mathcal{O}_{X_t}) - 2K_{X_t}^2.$$

If $H^2(X_t, T_{X_t}) = 0$ then the dimension of the deformation space of X_t is $h^1(X_t, T_{X_t}) = 10 - 2K_{X_t}^2$. It implies that there is no nontrivial deformation of X_t if $K_{X_t}^2 \geq 5$. In particular, it is not possible to construct a minimal surface of general type X_t with $p_g = 0, K^2 \geq 5$ via $\mathbb{Q}$-Gorenstein smoothing of a singular surface X with $H^2(T_X^0) = 0$.

But, in general, the cohomology $H^2(T_X^0)$ is not zero and it is a very difficult problem to determine whether there exists a $\mathbb{Q}$-Gorenstein smoothing of X. Hence, in the case that $H^2(T_X^0) \neq 0$, we need to develop another technique in order to investigate the existence of $\mathbb{Q}$-Gorenstein smoothing. Even though we do not know whether such a technique exists

in general, if X is a normal projective surface with singularities of class T which admits a cyclic group action with some additional properties, then we are able to show that it admits a $\mathbb{Q}$-Gorenstein smoothing. Explicitly, we get the following theorem.

Theorem 3.2 ([10]). *Let X be a normal projective surface with singularities of class T. Assume that a cyclic group G acts on X such that*

(1) $Y = X/G$ *is a normal projective surface with singularities of T,*

(2) $p_g(Y) = q(Y) = 0$,

(3) *Y has a $\mathbb{Q}$-Gorenstein smoothing,*

(4) *the map $\sigma : X \to Y$ induced by a cyclic covering is flat, and the branch locus D (resp. the ramification locus) of the map $\sigma : X \to Y$ is nonsingular curve lying outside the singular locus of Y (resp. of X), and*

(5) $H^1(Y, \mathcal{O}_Y(D)) = 0$.

Then there exists a $\mathbb{Q}$-Gorenstein smoothing of X that is compatible with a $\mathbb{Q}$-Gorenstein smoothing of Y. And the cyclic covering extends to the $\mathbb{Q}$-Gorenstein smoothing.

Example. Assume that $n \geq 5$ and let $\mathbb{F}_n$ be a Hirzebruch surface. Let C_0 be a negative section with $C_0^2 = -n$ and f be a fiber of $\mathbb{F}_n$. Choose a special irreducible (singular) curve D in the linear system $|4(C_0 + nf)|$ having a special intersection with one special fiber f: Note that $D \cdot f = 4$. We want D to intersect with f at two distinct points p and q that are not in C_0. Let $x = 0$ be the local equation of f and x, y be a coordinate at p (resp. at q). We require that the local equation of D at p (resp. at q) is $(y - x)(y + x) = 0$ (resp. $(y - x^{n-4})(y + x^{n-4}) = 0$). In [10], we show that there is a curve D satisfying the conditions above and being nonsingular at every point except at the two points p and q. Let $\sigma : \tilde{X}_n \to \mathbb{F}_n$ be a double cover branched over the curve D chosen above. Then $\tilde{X}_n$ is a singular elliptic surface with $p_g = n - 1$ and $\chi(\mathcal{O}_{\tilde{X}_n}) = n$ which has two rational double points by the local equations of D at p and q; one is of type A_1 and the other one is of type A_{2n-9}. Therefore its minimal resolution is also an elliptic surface $E(n)$. First we blow up at p and q in $\mathbb{F}_n$. Then we have an exceptional curve obtained by a blowing up at p which intersects with the proper transform of D transversally at two points, and we also have an exceptional curve obtained by a blowing up at q which intersects with the proper transform of D at one point, say q_1. Let $x = 0$ be the local equation of the (-1)-exceptional curve at q_1. Then the local equation of the proper transform of D at q_1 is $(y - x^{n-5})(y + x^{n-5}) = 0$. We blow up again at q_1. By the

continuation of blowing up at infinitely near points of q, we have the following configuration of smooth rational curves

$$
\begin{array}{ccccccc}
\overset{-n}{\underset{U_{n-3}}{\circ}} & - & \overset{-2}{\underset{U_{n-4}}{\circ}} & - & \overset{-2}{\underset{U_{n-5}}{\circ}} & - \cdots - & \overset{-2}{\underset{U_1}{\circ}} \\
\mid & & & & & & \mid \\
\underset{E_1,-1}{\circ} & & & & & & \underset{E_2,-1}{\circ}
\end{array}
$$

where the proper transform of D intersects with E_i, $i = 1, 2$ at two points transversally. We denote this surface by Z_n obtained by $(n-3)$ times blowing-ups of $\mathbb{F}_n$. Next, by Artin's criterion for contractibility [2], we can contract a configuration C_{n-2}, which is a linear chain of $\mathbb{P}^1$'s

$$
\overset{-n}{\underset{U_{n-3}}{\circ}} - \overset{-2}{\underset{U_{n-4}}{\circ}} - \overset{-2}{\underset{U_{n-5}}{\circ}} - \cdots - \overset{-2}{\underset{U_1}{\circ}},
$$

so that it produces a singular normal projective surface. We denote this surface by Y_n. We note that Δ is the proper transform of D in Z_n and that Y_n has a cyclic quotient singularity of type $\frac{1}{(n-2)^2}(1, n-3)$, which is a singularity of class T. In [10], it is proved that the singular surface Y_n admits a $\mathbb{Q}$-Gorenstein smoothing. Note that $\tilde{X}_n$ is a double covering of $\mathbb{F}_n$ branched over D, and the minimal resolution of two rational double points of type A_1 and A_{2n-9} in $\tilde{X}_n$ is $E(n)$, which is also a double cover of Z_n branched over the proper transform of D. Since the proper transform of D does not meet the contracted linear chain of $\mathbb{P}^1$'s, we have a double cover of Y_n branched over the image of the proper transform of D by the map ψ. We denote this surface by X_n. Then X_n is a singular surface obtained by contracting two disjoint configurations C_{n-2} from an elliptic surface $E(n)$ and it has two quotient singularities of class T, both are of type $\frac{1}{(n-2)^2}(1, n-3)$. Therefore we have the following commutative diagram of maps

$$
\begin{array}{ccccc}
\tilde{X}_n & \leftarrow & E(n) & \rightarrow & X_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{F}_n & \overset{\pi}{\leftarrow} & Z_n & \overset{\psi}{\rightarrow} & Y_n
\end{array}
$$

where all vertical maps are double covers. Then, it is shown in [10] that the singular surface X_n has a $\mathbb{Q}$-Gorenstein smoothing of two quotient singularities simultaneously by Theorem 3.2, and its smoothing is a Horikawa surface (cf. [4]).

Remark. Let $X = W'_{4,k}$ for $2 \le k \le 9$ in Introduction. Then $H^2(T_X^0) \ne 0$ because $H^2(T_{E(4)}) \ne 0$. We consider the local-global exact

sequence of smoothing in Proposition 3.4

$$H^0(T_X^1) = \oplus_{i=1}^k \mathbb{C}_i \to H^2(T_X^0)$$

where $\mathbb{C}_i = \mathbb{C}$. Then the dimension of the image of the above obstruction map is at most one because there is a $\mathbb{Q}$-Gorenstein smoothing of any $s \geq 2$ quotient singularities simultaneously by the argument in Section 2.

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