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Projective surfaces with many nodes

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Abstract.

We prove that a smooth projective complex surface X, not necessarily minimal, contains $h^{1,1}(X) - 1$ disjoint (-2)-curves if and only if X is isomorphic to a relatively minimal ruled rational surface $\mathbf{F}_2$ or $\mathbf{P}^2$ or a fake projective plane.

We also describe smooth projective complex surfaces X with $h^{1,1}(X) - 2$ disjoint (-2)-curves.

§1. Introduction

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

A smooth rational curve on a surface with self-intersection -2 is called a (-2)-curve or a nodal curve as it may be contracted to give a nodal singularity (conical double point). For a smooth surface X, we denote by $\mu(X)$ the maximum of the cardinality of a set of disjoint (-2)-curves on X. Hodge index theorem implies that

$$\mu(X) \leq \rho(X) - 1 \leq h^{1,1}(X) - 1,$$

in particular, X contains at most $h^{1,1}(X) - 1$ disjoint nodal curves, where $\rho(X)$ denotes the Picard number and $h^{1,1}(X)$ the (1,1)-th Hodge number of X.

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A result of I. Dolgachev, M. Mendes Lopes, and R. Pardini gives a classification of smooth projective complex surfaces X with $q(X) = p_g(X) = 0$ containing $\rho(X) - 1$ disjoint nodal curves ([6], Theorem 3.3 and Proposition 4.1).

Theorem 1.1. [6] *Let X be a smooth projective surface, not necessarily minimal, with $q(X) = p_g(X) = 0$. Then $\mu(X) = h^{1,1}(X) - 1$ if and only if X is isomorphic to the minimal rational ruled surface $\mathbf{F}_2$ or the complex projective plane $\mathbf{P}^2$ or a fake projective plane.*

Note that $\rho(X) = h^{1,1}(X)$ for a smooth projective surface X with $p_g(X) = 0$. The Hirzebruch surface $\mathbf{F}_2$ contains one nodal curve, while $\mathbf{P}^2$ or a fake projective plane contains none. The latter two cases were not mentioned in [6], as the authors focused on the case with $\mu(X) > 0$.

A $\mathbb{Q}$-homology projective plane is a normal projective surface having the same $\mathbb{Q}$-homology groups as $\mathbf{P}^2$. If a $\mathbb{Q}$-homology projective plane has rational singularities only, then both the surface and its resolution have $p_g = q = 0$. Theorem 1.1 also gives the following classification of $\mathbb{Q}$-homology projective planes with nodes only.

Corollary 1.2. *Let S be a $\mathbb{Q}$-homology projective plane. Assume that all singularities of S are nodes. Then S is isomorphic to $\mathbf{P}^2$ or a fake projective plane or a cone in $\mathbf{P}^3$ over a conic curve.*

In this paper we first show that the condition "$q(X) = p_g(X) = 0$" in Theorem 1.1 is not necessary.

Theorem 1.3. *Let X be a smooth projective surface, not necessarily minimal. Then $\mu(X) = h^{1,1}(X) - 1$ if and only if X is isomorphic to $\mathbf{F}_2$ or $\mathbf{P}^2$ or a fake projective plane.*

Next, we describe smooth projective complex surfaces X with $\mu(X) = h^{1,1}(X) - 2$.

Theorem 1.4. *Let X be a smooth projective surface, not necessarily minimal. Assume that $\mu(X) = h^{1,1}(X) - 2$. Then X belongs to one of the following cases:*

(1) *nef K_X :*
   - (i) *a bi-elliptic surface, i.e. a minimal surface of Kodaira dimension 0 with $q = 1$, $p_g = 0$, $h^{1,1} = 2$;*
   - (ii) *a minimal surface of Kodaira dimension 1 with $q = 1$, $p_g = 0$, $h^{1,1} = 2$;*
   - (iii) *an Enriques surface with 8 disjoint nodal curves;*
   - (iv) *a minimal surface of Kodaira dimension 1 with $q = p_g = 0$ whose elliptic fibration has 2 reducible fibres of type I_0^* whose end components give the 8 disjoint nodal curves;*

(v) *a ball quotient with* $q = 0$, $p_g = 1$, *i.e. a minimal surface of general type with* $q = 0$, $p_g = 1$, $h^{1,1} = 2$;

(vi) *a minimal surface of general type with* $q = p_g = 0$, $K^2 = 1, 2, 4, 6, 7, 8$ *containing* $8 - K^2$ *disjoint nodal curves*;

(2) *non-nef* K_X :

(i) *the blowup of a fake projective plane at one point or at two infinitely near points*;

(ii) *a relatively minimal irrational ruled surface; or its blowup at two infinitely near points on each of* $k \geq 1$ *fibres so that each of the k fibres becomes a string of* 3 *rational curves* (-2)—(-1)—(-2);

(iii) *a rational ruled surface* $\mathbf{F}_e$, $e \neq 2$;

(iv) *the blowup of* $\mathbf{F}_e$ *at two infinitely near points on each of* $k \geq 1$ *fibres so that each of the k fibres becomes a string of* 3 *rational curves* (-2)—(-1)—(-2);

(v) *the blowup of* $\mathbf{F}_2$ *at two infinitely near points away from the negative section so that one fibre becomes a string of* 3 *rational curves* (-1)—(-2)—(-1);

(vi) *the blowup of* $\mathbf{F}_2$ *at one point away from the negative section; or equivalently the blowup of* $\mathbf{F}_1$ *at one point on the negative section.*

We remark that all cases of Theorem 1.4 are supported by an example except the case (1-vi) with $K^2 = 1$, or 7.

For the case (1-ii), such surfaces can be obtained by taking a quotient $(E \times C)/G$ of the product of an elliptic curve E and a hyperelliptic curve C of genus $g(C) \geq 2$ by a group G of order 2 acting on E as a translation by a point of order 2 and on C as the hyperelliptic involution. (If $g(C) = 1$ we get a bi-elliptic surface.)

For (1-iii), such Enriques surfaces were completely classified in [13]. See also [11] and [9] for explicit examples.

For (1-iv), the Jacobian fibration of such a surface is a rational elliptic surface Y with two singular fibres of type I_0^*. (The Jacobian fibration of an elliptic fibration has singular fibres of the same type as the original fibration (cf. [5]).) In other words, such surfaces are torsors of Y, i.e., can be obtained by performing logarithmic transformations on Y. If the orders of logarithmic transformations are $(2, 2)$, then the resulting surface is an Enriques surface belonging to the case (1-iii). Such a rational elliptic surface Y can be constructed in many ways, e.g., by blowing up the base points of a specific cubic pencil on $\mathbf{P}^2$ ([5]) or by taking a minimal resolution of a $\mathbb{Z}/2$-quotient of the product of an elliptic curve E and $\mathbf{P}^1$ where the group acts on E as the inversion and on $\mathbf{P}^1$ as an involution. It is easy to see that any such rational elliptic surface Y is a special case of (2-iv). (Consider a free pencil $|N_1+2C+N_2|$ on Y where C is a section meeting two simple components N_1 and N_2 of the two reducible fibres.)

For the case (1-vi) with $K^2 = 2, 4, 6$, examples can be found in [2]. See Remark 4.2.

Notation

$\mathbf{F}_e$:=Proj$(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(e))$, $e \geq 0$, a rational ruled surface (Hirzebruch surface)
$\mathbf{P}^n$: the complex projective n-space
$\rho(Y)$: the Picard number of a variety Y
K_Y: the canonical class of Y
$b_i(Y)$: the i-th Betti number of Y
$e(Y)$: the topological Euler number of Y
$c_i(X)$: the i-th Chern class of X
$e_{orb}(S)$: the orbifold Euler number of a surface S with quotient singularities only
$h^{i,j}(X)$: the (i,j)-th Hodge number of a smooth variety X
$q(X) := \dim H^1(X, \mathcal{O}_X)$ the irregularity of a surface X
$p_g(X) := \dim H^2(X, \mathcal{O}_X)$ the geometric genus of a surface X
$|G|$: the order of a finite group G
$(-m)$-curve: a smooth rational curve on a surface with self-intersection $-m$
$\mu(Z)$: the maximum of the cardinality of a set of disjoint (-2)-curves on a smooth surface Z

§2. The orbifold Bogomolov–Miyaoka–Yau inequality

A singularity p of a normal surface S is called a quotient singularity if the germ at p is analytically isomorphic to the germ of $\mathbb{C}^2/G_p$ at the image of the origin $O \in \mathbb{C}^2$ for some nontrivial finite subgroup G_p of $GL_2(\mathbb{C})$ not containing quasi-reflections. Brieskorn classified all such finite subgroups of $GL(2, \mathbb{C})$ in [4]. We call G_p the local fundamental group of the singularity p.

Let S be a normal projective surface with quotient singularities and

$$f : S' \to S$$

be a minimal resolution of S. For each quotient singular point $p \in S$, there is a string of smooth rational curves E_j such that

$$f^{-1}(p) = \cup_{j=1}^{l} E_j.$$

It is well-known that quotient singularities are log-terminal singularities. Thus one can compare canonical classes $K_{S'}$ and K_S, and write

$$K_{S'} \underset{num}{\equiv} f^*K_S - \sum \mathcal{D}_p$$

where $\mathcal{D}_p = \sum_{j=1}^{l}(a_jE_j)$ is an effective $\mathbb{Q}$-divisor supported on $f^{-1}(p) = \cup_{j=1}^{l} E_j$ and $0 \le a_j < 1$. It implies that

$$K_S^2 = K_{S'}^2 - \sum_{p \in Sing(S)} \mathcal{D}_p^2.$$

The coefficients $a_1, \dots, a_l$ of $\mathcal{D}_p$ are uniquely determined by the system of linear equations

$$\mathcal{D}_p \cdot E_i = -K_{S'} \cdot E_i = 2 + E_i^2 \quad (1 \le i \le l).$$

In particular, $\mathcal{D}_p = 0$ if and only if p is a rational double point.

Also we recall the orbifold Euler characteristic

$$e_{orb}(S) := e(S) - \sum_{p \in Sing(S)} \left(1 - \frac{1}{|G_p|}\right)$$

where $e(S)$ is the topological Euler number of S, and $|G_p|$ the order of the local fundamental group G_p of p.

The following theorem is called the orbifold Bogomolov–Miyaoka–Yau inequality.

Theorem 2.5 ([15], [14], [10], [12]). *Let S be a normal projective surface with quotient singularities such that K_S is nef. Then*

$$K_S^2 \le 3e_{orb}(S).$$

In particular,

$$0 \le e_{orb}(S).$$

The second (weaker) inequality holds true even if $-K_S$ is nef.

Theorem 2.6 ([8]). *Let S be a normal projective surface with quotient singularities such that $-K_S$ is nef. Then*

$$0 \le e_{orb}(S).$$

The following corollary is well-known (e.g. [7], Corollary 3.4) and immediately follows from Theorems 2.5 and 2.6.

Corollary 2.7. *A $\mathbb{Q}$-homology projective plane with quotient singularities only has at most* 5 *singular points.*

§3. Proof of Theorem 1.3

The "if"-part is trivial.

Let X be a smooth projective surface, not necessarily minimal, with $h^{1,1}(X)-1$ disjoint nodal curves. We shall show that X is isomorphic to $\mathbf{F}_2$ or $\mathbf{P}^2$ or a fake projective plane. Let

$$f : X \to S$$

be the contraction morphism of the $h^{1,1}(X)-1$ disjoint nodal curves.

Note first that $\rho(S)=1$. Thus K_S is nef or anti-ample.

Assume that K_S is nef.
Then we can apply the orbifold Bogomolov–Miyaoka–Yau inequality (Theorem 2.5). Note that $K_S^2 = K_X^2$. From Noether formula

$$K_X^2 = 12\{1-q(X)+p_g(X)\} - \{2-4q(X)+h^{1,1}(X)+2p_g(X)\}.$$

Also we have

$$e_{orb}(S) = e(S) - \frac{h^{1,1}(X)-1}{2}.$$

Since $e(S) = e(X) - (h^{1,1}(X)-1) = 3-4q(X)+2p_g(X)$, Theorem 2.5 implies that

$$12(1-q+p_g) - (2-4q+h^{1,1}+2p_g) \le 3\Big(3-4q+2p_g - \frac{h^{1,1}-1}{2}\Big),$$

i.e.,

$$4q+4p_g+\frac{h^{1,1}}{2} \le \frac{1}{2},$$

hence

$$q(X)=p_g(X)=0, \quad h^{1,1}(X)=1.$$

In particular, $b_2(X)=1$, $e(X)=3$, $K_X^2=9$. Note that $K_X=f^*K_S$ is nef, hence X is not rational. Thus, by classification theory of complex surfaces (see [3]), X must be a fake projective plane, i.e. a smooth surface of general type with $q=p_g=0$, $K^2=9$.

Assume that $-K_S$ is ample.
Then $-K_X = -f^*K_S$ is nef and non-zero, hence X has Kodaira dimension $\kappa(X)=-\infty$. Suppose X is not rational. Then there is a morphism $g : X \to C$ onto a curve of genus ≥ 1, with general fibres isomorphic to $\mathbf{P}^1$. Since a curve of genus ≥ 1 cannot be covered by a rational curve, we see that all nodal curves of X are contained in a union of fibres. This implies that S has Picard number ≥ 2, a contradiction. Thus X is rational. Now by Theorem 3.3 of [6], $X \cong \mathbf{F}_2$ or $\mathbf{P}^2$.

Here we give an alternative proof. Since we assume that X is rational, S is a $\mathbb{Q}$-homology projective plane with nodes only. Let k be the number of nodes on S. Then $k \le 5$ by Corollary 2.7. Note that $b_2(X) = 1 + k$, so $K_X^2 = 9 - k$. Let L be the sublattice of the cohomology lattice of X generated by the class of K_X and the classes of the k nodal curves. Then L is of finite index in the cohomology lattice that is unimodular, hence $|\det(L)|$ is a square integer. Note that $|\det(L)| = (9-k)2^k$. If $k \le 5$, then it is a square integer only if $k = 0$ or 1. If $k = 0$, then $X \cong \mathbf{P}^2$. If $k = 1$, then $K_X^2 = 8$ and $\rho(X) = 2$, hence $X \cong \mathbf{F}_2$.
This completes the proof of Theorem 1.3.

Remark 3.8. *Proposition 4.1 of* [6] *was also proved by using the orbifold Bogomolov–Miyaoka–Yau inequality. Our proof is just a slight refinement of their argument.*

§4. Proof of Theorem 1.4

For a smooth surface Z, we denote by $\mu(Z)$ the maximum of the cardinality of a set of disjoint (-2)-curves of Z. The following useful lemma is due to M. Mendes Lopes and R. Pardini.

Lemma 4.9. *Let X be a smooth surface with Kodaira dimension $\kappa(X) \ge 0$. Let $\phi : X \to Y$ be the map to the minimal model, and let $r := \rho(X) - \rho(Y)$. Then*

$$\mu(X) \le \mu(Y) + \frac{r}{2}.$$

Proof. The proof is essentially contained in the proof of Proposition 4.1 of [6].

Use induction on r.

When $r = 0$, it is trivial.

Assume $r > 0$. Write

$$K_X = \phi^* K_Y + E$$

and let $C_1, \ldots, C_{\mu(X)}$ be disjoint (-2)-curves on X. For each i there are 2 possibilities:

(1) C_i is exceptional for f, hence $(\phi^* K_Y) C_i = 0$.
(2) C_i is not exceptional for ϕ. Then since $K_X C_i = 0$ and K_Y is nef, we see that $(\phi^* K_Y) C_i = 0$, $E C_i = 0$, hence C_i is disjoint from the support of E.

Let E_1 be an irreducible (-1)-curve of X and let X_1 be the surface obtained by blowing down E_1. If E_1 does not intersect any of the C_i's, then the C_i's give $\mu(X)$ disjoint (-2)-curves on X_1, hence $\mu(X) \le \mu(X_1)$ and the statement follows by induction, i.e.,

$$\mu(X) \le \mu(X_1) \le \mu(Y) + \frac{r-1}{2}.$$

So assume that $E_1C_1 > 0$. By the above remark, this implies that C_1 is exceptional for ϕ. In particular, we have $C_1E_1 = 1$. Notice that $E_1C_i = 0$ for every $i > 1$. Indeed, if, say, $E_1C_2 = 1$, then the images of C_1 and C_2 in X_1 are (-1)-curves that intersect, contradicting the assumption that $\kappa(X_1) \ge 0$. Hence the image of C_1 in X_1 is a (-1)-curve that can be contracted to get a surface X_2 with $\mu(X) - 1$ disjoint (-2)-curves, and again we get the result by induction, i.e.,

$$\mu(X) - 1 \le \mu(X_2) \le \mu(Y) + \frac{r-2}{2}.$$

Q.E.D.

Now we prove Theorem 1.4. Let

$$f : X \to S$$

be the contraction morphism of the disjoint nodal curves $C_1, \dots, C_{\mu(X)}$, where $\mu(X) = h^{1,1}(X) - 2$.

Note first that $K_X = f^*K_S$. Thus K_X is nef if and only if K_S is nef.

Assume that K_X is nef.
Then we again apply the orbifold Bogomolov–Miyaoka–Yau inequality (Theorem 2.5). In this case we have

$$K_S^2 = K_X^2 = 12\{1 - q(X) + p_g(X)\} - \{2 - 4q(X) + h^{1,1}(X) + 2p_g(X)\},$$

$$e_{orb}(S) = e(S) - \frac{h^{1,1}(X) - 2}{2} = 4 - 4q(X) + 2p_g(X) - \frac{h^{1,1}(X) - 2}{2}.$$

Thus Theorem 2.5 implies that

$$12(1 - q + p_g) - (2 - 4q + h^{1,1} + 2p_g) \le 3\Big(4 - 4q + 2p_g - \frac{h^{1,1} - 2}{2}\Big),$$

and hence,

$$4q + 4p_g + \frac{h^{1,1}}{2} \le 5.$$

This inequality has the following solutions:

- (A) $q(X) = 1$, $p_g(X) = 0$, $h^{1,1}(X) = 2$;

- (B) $q(X) = 0$, $p_g(X) = 1$, $h^{1,1}(X) = 2$;
- (C) $q(X) = p_g(X) = 0$, $2 \leq h^{1,1}(X) \leq 10$.

Assume the case (A). In this case, $e(X) = 0$ and $K_X^2 = 0$. Since X is a minimal surface of Kodaira dimension $\kappa(X) \geq 0$, the classification theory of complex surfaces (cf. [3]) shows that X belongs to the case (1-i) or (1-ii).

Assume the case (B). In this case, $e(X) = 6$ and $K_X^2 = 18$. Hence, X is of general type. Since $3e(X) = K_X^2$, it is a ball quotient. This gives the case (1-v).

Assume the case (C). In this case, $e(X) = h^{1,1}(X) + 2$ and $K_X^2 = 10 - h^{1,1}(X)$.

If $h^{1,1}(X) = 10$, then $e(X) = 12$ and $K_X^2 = 0$. Since X is a minimal surface of Kodaira dimension $\kappa(X) \geq 0$, X is an Enriques surface or a minimal surface of Kodaira dimension $\kappa(X) = 1$. This gives the case (1-iii) and (1-iv). In the latter case, a fibre of the elliptic fibration of X is a rational multiple of K_X, hence the 8 nodal curves must be contained in fibres of the elliptic fibration. By the formula for computing the topological Euler number of a fibration (cf. [3], Chap. III) this is possible only if the reducible fibres are two fibres of type I_0^* and the eight nodal curves are the end-components of these fibres.

If $2 \leq h^{1,1}(X) \leq 9$, then $8 \geq K_X^2 \geq 1$. Since X is a minimal surface of Kodaira dimension $\kappa(X) \geq 0$, X is of general type. By Theorem 1.3, any minimal surface X of general type with $p_g(X) = 0$ and $K_X^2 = 8$ cannot contain a (-2)-curve, hence belongs to the case (1-vi).

We claim that $K_X^2 \neq 3, 5$. This can be proved by a lattice theoretic argument. Let L be the cohomology lattice $H^2(X, \mathbb{Z})/(\text{torsion})$, which is an odd unimodular lattice of signature $(1, h^{1,1}(X) - 1)$. Let M be the sublattice of L generated by the classes of the nodal curves $C_1, \dots, C_{\mu(X)}$ where $\mu(X) = h^{1,1}(X) - 2$. Consider the homomorphism of quadratic forms of finite abelian groups

$$\tau : M/2M \to L/2L.$$

Note that

$$M/2M \cong (\mathbb{Z}/2\mathbb{Z})^{\mu(X)}$$

is totally isotropic, and

$$L/2L \cong (\mathbb{Z}/2\mathbb{Z})^{\mu(X)+2}.$$

Assume that $K_X^2 = 3$. Then $\mu(X) = 5$, so the kernel $\ker(\tau)$ must have length ≥ 2. If $\sum_{j=1}^{k} C_{i_j} (\text{mod } 2M) \in \ker(\tau)$, then

$$\sum_{j=1}^{k} C_{i_j} = 2D + \text{torsion}$$

for some divisor D. Since $D \cdot K_X = 0$, D^2 is an even integer. This implies that k is a multiple of 4. This means that any non-trivial element of $\ker(\tau)$ is a sum of 4 members of $C_1, \ldots, C_5$. But this is impossible since $\ker(\tau)$ has length ≥ 2. Assume that $K_X^2 = 5$. Then $\mu(X) = 3$, so the kernel $\ker(\tau)$ must have length ≥ 1. But no linear combination of C_1, C_2, C_3 gives a non-trivial element of $\ker(\tau)$. This completes the case (1-vi).

Assume that K_X is not nef and $\kappa(X) \geq 0$.
In this case X is not minimal. Consider the map $\phi : X \to Y$ to the minimal model, and let

$$r := \rho(X) - \rho(Y).$$

By Lemma 4.9,

$$h^{1,1}(Y) + r - 2 = h^{1,1}(X) - 2 = \mu(X) \leq \mu(Y) + \frac{r}{2} \leq h^{1,1}(Y) - 1 + \frac{r}{2},$$

hence

$$r \leq 2.$$

If $r = 1$, then the above inequality shows that

$$h^{1,1}(Y) - 1 = \mu(X) \leq \mu(Y) + \frac{1}{2},$$

hence $\mu(Y) = h^{1,1}(Y) - 1$. So by Theorem 1.3 Y is a fake projective plane and $\mu(X) = h^{1,1}(Y) - 1 = 0$.
If $r = 2$, then the above inequality shows that

$$h^{1,1}(Y) = \mu(X) \leq \mu(Y) + 1 \leq h^{1,1}(Y),$$

hence $\mu(Y) = h^{1,1}(Y) - 1$. So by Theorem 1.3 Y is a fake projective plane and $\mu(X) = h^{1,1}(Y) = 1$. This gives the case (2-i).

Assume that $\kappa(X) = -\infty$ and X is irrational.
In this case X is an irrational ruled surface. If X is relatively minimal, then $\mu(X) = h^{1,1}(X) - 2 = 0$. Assume that $\mu(X) = h^{1,1}(X) - 2 > 0$. The (-2)-curves must be contained in the union of fibers of the Albanese

fibration a_X on X. Let $\phi : X \to Y$ be the map to a relatively minimal irrational ruled surface. Then

$$\mu(X) = h^{1,1}(X) - 2 = h^{1,1}(Y) + \rho(X) - \rho(Y) - 2 = \rho(X) - \rho(Y),$$

i.e. the number of disjoint nodal curves on X is the same as the the number of blowups from Y to X. This is possible only if the number of nodal curves contained in each reducible fibre of a_X is the same as the number of blowups on the corresponding fibre of Y. The only possibility is that each reducible fibre of a_X is a string of three smooth rational curves (−2)—(−1)—(−2) obtained by blowing up twice. This gives the case (2-ii).

Assume that $\kappa(X) = -\infty$ and X is rational.
This case has been classified in [6], Theorem 3.3 and Remark 3. This gives the cases (2-iii), (2-iv), (2-v), (2-vi).

This completes the proof of Theorem 1.4.

Remark 4.10. *(1) There are examples of the case (1-vi) with $K^2 = 6, 4, 2$, as given in* [2]*. In the paper they give a complete classification of the surfaces Y occurring as the minimal resolution of a surface $Z := (C_1 \times C_2)/G$, where G is a finite group with an unmixed action on a product of smooth projective curves $C_1 \times C_2$ of respective genera ≥ 2, and such that (i) Z has only rational double points as singularities, (ii) $q(Y) = p_g(Y) = 0$. In particular they show that Z has only nodes as singularities, and the number of nodes is even and equal to $t := 8 - K_Z^2$ (see Corollary 5.3, ibid). Furthermore, they give examples with $t = 2, 4, 6$. The case $t = 0$, i.e., G acts freely on $C_1 \times C_2$, was completely classified in* [1].

The cases $t = 6, 4$ can also be obtained as the quotient of a minimal surface of general type with $K^2 = 8$ and $p_g = 0$ by an action of $(\mathbb{Z}/2\mathbb{Z})^2$, or by an action of $\mathbb{Z}/2\mathbb{Z}$, where each non-trivial involution has isolated fixed points only. This was confirmed by Ingrid Bauer.

(2) We do not know the existence of the case (1-vi) with $K^2 = 1$, i.e. a Godeaux surface with 7 disjoint nodal curves. However, there is a possible construction of such an example. If one can find a minimal surface of general type with $K^2 = 8$ and $p_g = 0$ admitting an action of $(\mathbb{Z}/2\mathbb{Z})^3$, each of the 7 involutions having isolated fixed points only, then the quotient has the minimal resolution with $K^2 = 1$, $p_g = 0$, and 7 disjoint nodal curves.

(3) We do not know the existence of the case (1-vi) with $K^2 = 7$.

Remark 4.11. *The case (2-i) gives counterexamples to Proposition 4.1 of* [6]*. Indeed the authors, though their proof was correct, overlooked* *the case of fake projective planes for the minimal case, and consequently the case of blowups of fake projective planes for the non-minimal case as they used induction on the number of blowups from the minimal model. Thus the first statement of their proposition holds true except for the case where Y is a fake projective plane, and the second statement except for the case where Y is the blowup of a fake projective plane at one point or at two infinitely near points.*

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