

## Generating functions of stable pair invariants via wall-crossings in derived categories

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### Abstract.

The notion of limit stability on Calabi–Yau 3-folds is introduced by the author to construct an approximation of Bridgeland–Douglas stability conditions at the large volume limit. It has also turned out that the wall-crossing phenomena of limit stable objects seem relevant to the rationality conjecture of the generating functions of Pandharipande–Thomas invariants. In this article, we shall make it clear how wall-crossing formula of the counting invariants of limit stable objects solves the above conjecture.

### §1. Introduction

A theory of curve counting on Calabi–Yau 3-folds is interesting in both algebraic geometry and string theory. Now there are three such theories, called Gromov–Witten (GW) theory, Donaldson–Thomas (DT) theory, and Pandharipande–Thomas (PT) theory. Conjecturally these theories are equivalent in terms of generating functions. To formulate this equivalence, however, we also need a conjectural rationality property of those functions for DT-theory and PT-theory, to formulate that equivalence. The purpose of this article is to interpret the rationality conjecture for PT-theory from the viewpoint of wall-crossing phenomena in derived categories of coherent sheaves.

#### 1.1. GW-DT-PT correspondences

First of all, let us recall the conjectural GW-DT-PT correspondences on curve counting theories. Suppose that  $X$  is a smooth projective

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Calabi–Yau 3-fold over  $\mathbb{C}$ , i.e. there is a nowhere vanishing holomorphic 3-form on  $X$ . For  $g \geq 0$  and  $\beta \in H_2(X, \mathbb{Z})$ , the *GW-invariant*  $N_{g,\beta}$  is defined by the integration of the virtual class,

$$N_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Q},$$

where  $\overline{M}_g(X, \beta)$  is the moduli stack of stable maps  $f: C \rightarrow X$  with  $g(C) = g$  and  $f_*[C] = \beta$ . The *GW-potential* is given by the following generating function,

$$Z_{\text{GW}} = \exp \left( \sum_{g,\beta \neq 0} N_{g,\beta} \lambda^{2g-2} v^\beta \right).$$

For  $n \in \mathbb{Z}$  and  $\beta \in H_2(X, \mathbb{Z})$ , let  $I_n(X, \beta)$  be the Hilbert scheme of 1-dimensional subschemes  $Z \subset X$  satisfying

$$[Z] = \beta, \quad \chi(\mathcal{O}_Z) = n.$$

The obstruction theory on  $I_n(X, \beta)$  is obtained by viewing it as a moduli space of ideal sheaves, and the *DT-invariant*  $I_{n,\beta}$  is defined by

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Z}.$$

The generating function of the *reduced DT-theory* is

$$Z'_{\text{DT}} = \sum_{n,\beta} I_{n,\beta} q^n v^\beta / \sum_n I_{n,0} q^n.$$

The theory of stable pairs and their counting invariants are introduced and studied by Pandharipande and Thomas [20], [21], [22] to give a geometric interpretation of the reduced DT-theory. By definition, a *stable pair* is data  $(F, s)$ ,

$$s: \mathcal{O}_X \longrightarrow F,$$

where  $F$  is a pure one dimensional sheaf on  $X$ , and  $s$  is a morphism with a zero dimensional cokernel. For  $\beta \in H_2(X, \mathbb{Z})$  and  $n \in \mathbb{Z}$ , the moduli space of stable pairs  $(F, s)$  with

$$[F] = \beta, \quad \chi(F) = n,$$

is constructed in [20], denoted by  $P_n(X, \beta)$ . The obstruction theory on  $P_n(X, \beta)$  is obtained by viewing stable pairs  $(F, s)$  as two term complexes,

$$(1) \quad \dots \longrightarrow 0 \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow 0 \longrightarrow \dots$$

The *PT-invariant*  $P_{n,\beta}$  is defined by

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Z}.$$

The corresponding generating function is

$$Z_{\text{PT}} = \sum_{n,\beta} P_{n,\beta} q^v v^\beta.$$

The functions  $Z_{\text{GW}}$ ,  $Z'_{\text{DT}}$  and  $Z_{\text{PT}}$  are conjecturally equal after suitable variable change. In order to state this, we need the following conjecture, called *rationality conjecture*.

**Conjecture 1.1.** [19, Conjecture 2], [20, Conjecture 3.2] *For a fixed  $\beta$ , the generating series*

$$I_\beta(q) = \sum_{n \in \mathbb{Z}} I_{n,\beta} q^n / \sum_{n \in \mathbb{Z}} I_{n,0} q^n, \quad P_\beta(q) = \sum_{n \in \mathbb{Z}} P_{n,\beta} q^n,$$

*are Laurent expansions of rational functions of  $q$ , invariant under  $q \leftrightarrow 1/q$ .*

The above conjecture is solved for  $I_\beta(q)$  when  $X$  is a toric local Calabi–Yau 3-fold [19], and for  $P_\beta(q)$  when  $\beta$  is an irreducible curve class [22]. Now we can state the conjectural GW-DT-PT-correspondences.

**Conjecture 1.2.** [19, Conjecture 3], [20, Conjecture 3.3] *After the variable change  $q = -e^{i\lambda}$ , we have*

$$Z_{\text{GW}} = Z'_{\text{DT}} = Z_{\text{PT}}.$$

*The variable change  $q = -e^{i\lambda}$  is well-defined by Conjecture 1.1.*

Note that ideal sheaves  $I \subset \mathcal{O}_X$  are objects in  $D^b(X)$ , where  $D^b(X)$  is the bounded derived category of coherent sheaves on  $X$ . We can also interpret stable pairs  $(F, s)$  as objects in  $D^b(X)$  by viewing them as two term complexes (1). As discussed in [20, Section 3], the equality  $Z'_{\text{DT}} = Z_{\text{PT}}$  should be interpreted as a wall-crossing formula for counting invariants in the category  $D^b(X)$ . The purpose of this article is to show that Conjecture 1.1 is also interpreted as a wall-crossing formula in  $D^b(X)$ , using the method of limit stability [24] together with Joyce’s works [9], [10], [11], [14].

### 1.2. Limit stability

The notion of limit stability on a Calabi–Yau 3-fold  $X$  is introduced in [24] to construct an approximation of Bridgeland–Douglas stability conditions [4], [6], [7] on  $D^b(X)$  at the large volume limit. It is a certain stability condition on the category of perverse coherent sheaves

$$\mathcal{A}^p \subset D^b(X),$$

in the sense of Bezrukavnikov [3] and Kashiwara [15]. (See Definition 3.2.) An element  $\sigma \in A(X)_{\mathbb{C}}$  determines  $\sigma$ -limit (semi)stable objects in  $\mathcal{A}^p$ , where  $A(X)_{\mathbb{C}}$  is the complexified ample cone,

$$A(X)_{\mathbb{C}} = \{B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is an ample class}\}.$$

It has also turned out in [24] that the objects (1) appear as  $\sigma$ -limit stable objects for some  $\sigma \in A(X)_{\mathbb{C}}$ , thus studying stable pairs and limit stable objects are closely related. The objects  $E$  given by (1) satisfy

$$(2) \quad (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E), \mathrm{ch}_3(E)) = (-1, 0, \beta, n), \quad \det E = \mathcal{O}_X,$$

for some  $\beta$  and  $n$ . Under the above observation, we have constructed in [24] the moduli space of  $\sigma$ -limit stable objects  $E \in \mathcal{A}^p$  satisfying (2) as an algebraic space of finite type, denoted by  $\mathcal{L}_n^\sigma(X, \beta)$ . Using that moduli space, the counting invariant of limit stable objects

$$(3) \quad L_{n,\beta}(\sigma) \in \mathbb{Z}$$

is also defined in [24] as a weighted Euler characteristic with respect to Behrend’s constructible function [2], and (3) coincides with the integration of the virtual class if  $\mathcal{L}_n^\sigma(X, \beta)$  is a projective variety. A particular choice of  $\sigma$  yields an equality  $L_{n,\beta}(\sigma) = P_{n,\beta}$ , however  $L_{n,\beta}(\sigma)$  becomes different from  $P_{n,\beta}$  if we deform  $\sigma$ . As discussed in [24, Section 4], a transformation formula of the invariants  $L_{n,\beta}(\sigma)$  under change of  $\sigma$  seems relevant to solving Conjecture 1.1 for PT-theory.

### 1.3. Main result

In this article, we shall proceed the above idea further, using D. Joyce’s works [9], [10], [11], [14] on counting invariants of semistable objects on abelian categories and their wall-crossing formulas. We will make it clear how such a formula for counting invariants of objects in  $\mathcal{A}^p$  implies Conjecture 1.1 for PT-theory. Unfortunately we are unable to solve Conjecture 1.1 at this moment, as Joyce’s theory is applied only for the motivic invariants (e.g. Euler characteristic) of the moduli spaces,

so they do not involve virtual classes. On the other hand, the invariant  $P_{n,\beta}$  coincides with the Euler characteristic of  $P_n(X, \beta)$  (up to sign),

$$P_{n,\beta}^{eu} := e(P_n(X, \beta)) \in \mathbb{Z},$$

if  $P_n(X, \beta)$  is non-singular. In general  $P_{n,\beta}$  is written as a weighted Euler characteristic with respect to Behrend’s constructible function [2], so  $P_{n,\beta}$  resembles  $P_{n,\beta}^{eu}$  in this sense. So instead of solving Conjecture 1.1, we shall show the motivic version of Conjecture 1.1, i.e. the rationality of the generating series,

$$P_{\beta}^{eu}(q) = \sum_{n \in \mathbb{Z}} P_{n,\beta}^{eu} q^n.$$

The limit stability does not work well to combine Joyce’s works, so we will introduce the notion of  $\mu_{\sigma}$ -limit stability for  $\sigma \in A(X)_{\mathbb{C}}$ , which is a coarse version of  $\sigma$ -limit stability. Then we will introduce the Joyce type invariants, (cf. Definition 4.1, Remark 4.2, )

$$L_{n,\beta}^{eu} \in \mathbb{Q}, \quad N_{n,\beta}^{eu} \in \mathbb{Q}.$$

Roughly speaking,  $L_{n,\beta}^{eu}$ , (resp.  $N_{n,\beta}^{eu}$ ) is the “Euler characteristic” of the moduli stack of  $\mu_{i\omega}$ -limit semistable objects  $E \in \mathcal{A}^p$  with  $\det E = \mathcal{O}_X$ , (resp. one dimensional  $\omega$ -Gieseker semistable sheaves  $F$ , ) satisfying

$$\text{ch}(E) = (-1, 0, \beta, n), \quad (\text{resp. } \text{ch}(F) = (0, 0, \beta, n).)$$

We will consider the generating series,

$$L_{\beta}^{eu}(q) = \sum_{n \in \mathbb{Z}} L_{n,\beta}^{eu} q^n, \quad N_{\beta}^{eu}(q) = \sum_{n \geq 0} n N_{n,\beta}^{eu} q^n.$$

It will turn out that  $L_{\beta}^{eu}(q)$  is a polynomial of  $q^{\pm 1}$ ,  $N_{\beta}^{eu}(q)$  is the Laurent expansion of a rational function of  $q$ , and they are invariant under  $q \leftrightarrow 1/q$ . (cf. Lemma 4.5, Lemma 4.6.) Somewhat surprisingly, Joyce’s wall-crossing formula yields the following equality of those generating functions.

**Theorem 1.3.** [Theorem 4.7] *We have the following equality of the generating series,*

$$(4) \quad \sum_{\beta \geq 0} P_{\beta}^{eu}(q) v^{\beta} = \left( \sum_{\beta \geq 0} L_{\beta}^{eu}(q) v^{\beta} \right) \cdot \exp \left( \sum_{\beta > 0} N_{\beta}^{eu}(q) v^{\beta} \right).$$

Here  $\beta > 0$  means that  $\beta$  is a numerical class of an effective one cycle on  $X$ . As a corollary, we have the following.

**Corollary 1.4.** [**Corollary 4.8**] *The generating series  $P_\beta^{ev}(q)$  is the Laurent expansion of a rational function of  $q$ , invariant under  $q \leftrightarrow 1/q$ .*

The series  $Z_{PT}$  also should have a decomposition such as (4). In Problem 4.18 we will address a certain technical problem on the Ringel–Hall Lie algebra of  $\mathcal{A}^p$ , which enables us to decompose  $Z_{PT}$  and solve Conjecture 1.1 for PT-theory. As a conclusion, we have obtained a conceptual understanding of the rationality conjecture and DT-PT correspondences in terms of wall-crossing phenomena in the derived category, and they have been reduced to showing a rather technical problem, namely a compatibility of Ringel–Hall Lie algebra structure of  $\mathcal{A}^p$  with taking virtual classes via Behrend’s constructible functions.

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#### 1.5. Convention

All the varieties and schemes are defined over  $\mathbb{C}$ . For a variety  $X$ , the category of coherent sheaves on  $X$  is denoted by  $\text{Coh}(X)$ . We say  $E \in \text{Coh}(X)$  is  $d$ -dimensional if  $\dim \text{Supp}(E) = d$ .

## §2. Review of Joyce’s work

This section is devoted to review Joyce’s works [9], [10], [11], [14] on counting invariants of semistable objects on abelian categories. We discuss in a general framework rather than working with the category of perverse coherent sheaves  $\mathcal{A}^p$ , which we will introduce in the next section.

### 2.1. Setting

We begin with a generality of (weak) stability conditions on abelian categories. Let  $\mathcal{A}$  be a  $\mathbb{C}$ -linear abelian category, and  $K(\mathcal{A})$  its Grothendieck group. We put the same assumption as in [14], i.e.  $\text{Hom}(E, F)$ ,  $\text{Ext}^1(E, F)$  for any  $E, F \in \mathcal{A}$  are finite dimensional  $\mathbb{C}$ -vector spaces, and compositions  $\text{Ext}^i(E, F) \times \text{Ext}^j(F, G) \rightarrow \text{Ext}^{i+j}(E, G)$  for  $i, j, i+j = 0, 1$  are bilinear. These conditions are satisfied in several good cases, i.e.

$\mathcal{A} = \text{mod } A$  for a finite dimensional algebra  $A$ , or  $\mathcal{A} = \text{Coh}(X)$  for a projective variety  $X$ . In the first case, the group  $K(\mathcal{A})$  is finitely generated, but this is not true in the latter case. So instead we fix a quotient space,

$$\mathcal{N}(\mathcal{A}) := K(\mathcal{A}) / \equiv,$$

for some equivalence relation  $\equiv$  such that a class  $[E] \in \mathcal{N}(\mathcal{A})$  is non-zero for any  $0 \neq E \in \mathcal{A}$ . For instance if  $\mathcal{A} = \text{Coh}(X)$ , an equivalence relation  $\equiv$  can be taken by

$$(5) \quad E_1 \equiv E_2 \stackrel{\text{def}}{\iff} \chi(E_1, F) = \chi(E_2, F) \quad \text{for any } F \in \mathcal{A},$$

where  $\chi(E, F)$  is defined by

$$(6) \quad \chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i(E, F).$$

Then  $\mathcal{N}(\mathcal{A})$  is embedded into  $H^*(X, \mathbb{Q})$ , and it is a finitely generated  $\mathbb{Z}$ -module. The *closed positive cone* and the *positive cone* of  $\mathcal{A}$  are defined by

$$\begin{aligned} \overline{C}(\mathcal{A}) &:= \text{im}(\mathcal{A} \rightarrow K(\mathcal{A}) \rightarrow \mathcal{N}(\mathcal{A})), \\ C(\mathcal{A}) &:= \overline{C}(\mathcal{A}) \setminus \{0\}, \end{aligned}$$

respectively. For a subcategory  $\mathcal{B} \subset \mathcal{A}$ , we shall use the notation  $C(\mathcal{B}) := \text{im}(\mathcal{B} \rightarrow C(\mathcal{A})) \subset C(\mathcal{A})$ , etc. For an object  $E \in \mathcal{A}$ , its class is denoted by  $[E] \in \overline{C}(\mathcal{A})$ , or we omit  $[\ ]$  if there is no confusion. Let  $(T, \succeq)$  be a totally ordered set.

**Definition 2.1.** A *weak stability function* is a map,

$$Z: C(\mathcal{A}) \longrightarrow T,$$

such that if  $E, F, G \in C(\mathcal{A})$  satisfies  $E = F + G$ , we have either

$$\begin{aligned} Z(F) \preceq Z(E) \preceq Z(G), \quad \text{or} \\ Z(F) \succeq Z(E) \succeq Z(G). \end{aligned}$$

A weak stability function is a *stability function* if, for  $E, F, G$  as above, we have either

$$\begin{aligned} Z(F) \prec Z(E) \prec Z(G), \quad \text{or} \\ Z(F) \succ Z(E) \succ Z(G), \quad \text{or} \\ Z(F) = Z(E) = Z(G). \end{aligned}$$

Given a weak stability function, we can define the set of (semi)stable objects.

**Definition 2.2.** Let  $Z: C(\mathcal{A}) \rightarrow T$  be a weak stability function. An object  $E \in \mathcal{A}$  is called  $Z$ -(semi)stable if for any nonzero subobject  $F \subset E$ , we have

$$Z(F) \prec Z(E/F), \quad (\text{resp. } Z(F) \preceq Z(E/F).)$$

The notion of (weak) stability conditions is defined as follows.

**Definition 2.3.** A (weak) stability function  $Z: C(\mathcal{A}) \rightarrow T$  is a (weak) stability condition if for any object  $E \in \mathcal{A}$ , there is a filtration

$$(7) \quad 0 = E_0 \subset E_1 \subset \dots \subset E_n = E,$$

such that each subquotient  $F_i = E_i/E_{i-1}$  is  $Z$ -semistable with

$$Z(F_1) \succ Z(F_2) \succ \dots \succ Z(F_n).$$

It is easy to see that the filtration (7) is unique up to an isomorphism, if exists. The filtration (7) is called a *Harder–Narasimhan filtration*. Here we give some examples.

**Example 2.4.** (i) For an abelian category  $\mathcal{A}$ , let  $W: \mathcal{N}(\mathcal{A}) \rightarrow \mathbb{C}$  be a group homomorphism such that for any  $E \in \mathcal{A} \setminus \{0\}$ , we have

$$W(E) \in \mathbb{H} := \{r \exp(i\pi\phi) \mid 0 < \phi \leq 1\}.$$

For instance if  $\mathcal{A} = \text{mod } A$  for a finite dimensional  $\mathbb{C}$ -algebra  $A$ , the positive cone  $C(\mathcal{A})$  is spanned by finite number of simple objects  $S_1, \dots, S_n \in \mathcal{A}$ , and such  $W$  is obtained by choosing the image of  $[S_i] \in C(\mathcal{A})$  for  $1 \leq i \leq n$  under  $W$ . We set  $(T, \succeq) = ((0, 1], \succeq)$ , and

$$Z: C(\mathcal{A}) \ni E \mapsto \frac{1}{\pi} \text{Im} \log Z(E) \in T.$$

Then  $Z$  is a stability condition on  $\mathcal{A}$ . This is Bridgeland’s approach of stability conditions [4].

(ii) Let  $X$  be a smooth projective surface and set  $\mathcal{A} = \text{Coh}(X)$ . Let  $\omega$  be an ample divisor on  $X$ . For  $E \in \text{Coh}(X)$  we set

$$\mu_\omega(E) = \begin{cases} \frac{c_1(E) \cdot \omega}{\text{rk}(E)} & \text{if } E \text{ is not torsion.} \\ \infty & \text{if } E \text{ is torsion.} \end{cases}$$

Then the map  $C(\mathcal{A}) \ni E \mapsto \mu_\omega(E) \in \mathbb{Q} \cup \{\infty\}$  is a weak stability condition on  $\mathcal{A}$ , but not a stability condition on  $\mathcal{A}$ .

**Remark 2.5.** Here we mention that a theory of stability conditions on triangulated categories is developed by Bridgeland [4], motivated by M. Douglas’s  $\Pi$ -stability [6], [7]. For a triangulated category  $\mathcal{D}$ , Bridgeland’s stability condition consists of  $(W, \mathcal{A})$ , where  $\mathcal{A} \subset \mathcal{D}$  is the heart of a bounded t-structure on  $\mathcal{D}$ , and  $W$  is a group homomorphism  $K(\mathcal{A}) \rightarrow \mathbb{C}$ , as in Example 2.4 (i). Especially  $W$  determines a stability condition on the abelian category  $\mathcal{A}$ . He then shows that the set of “good” stability conditions form a complex manifold  $\text{Stab}(\mathcal{D})$ . Although Bridgeland’s theory is quite powerful, we shall study in this paper more general notion of (weak) stability conditions, which is used in Joyce’s works.

**2.2. Ringel–Hall algebras**

In this subsection, we introduce the algebra  $\mathcal{H}(\mathcal{A})$  associated to an abelian category  $\mathcal{A}$ , whose details are seen in [10]. Let  $Z: C(\mathcal{A}) \rightarrow T$  be a weak stability function. At this moment, we put the following assumption.

- Assumption 2.6.**      •  $\mathcal{A}$  is noetherian and  $Z$ -artinian.
- The moduli stack of objects  $E \in \mathcal{A}$ , denoted by  $\mathfrak{D}\text{bj}(\mathcal{A})$ , is an Artin stack locally of finite type over  $\mathbb{C}$ .
  - For  $v \in \overline{C}(\mathcal{A})$ , let  $\mathfrak{M}^v(Z) \subset \mathfrak{D}\text{bj}(\mathcal{A})$  be the substack of  $Z$ -semistable objects  $E \in \mathcal{A}$  with  $[E] = v$ . Then  $\mathfrak{M}^v(Z)$  is an open substack of  $\mathfrak{D}\text{bj}(\mathcal{A})$ , and it is of finite type over  $\mathbb{C}$ .

Here we say  $\mathcal{A}$  is  $Z$ -artinian if there is no infinite sequence

$$\cdots \subset E_n \subset E_{n-1} \subset \cdots \subset E_1 \subset E_0,$$

such that  $E_{i+1} \neq E_i$  and  $Z(E_{i+1}) \succeq Z(E_i/E_{i+1})$  for any  $i$ . The first condition of Assumption 2.6 ensures the existence of Harder–Narasimhan filtrations, and hence  $Z$  is a weak stability condition, by the same argument as in [23, Theorem 2]. In order to state the second assumption, we need to know about the notion algebraic families of objects and morphisms in  $\mathcal{A}$ . This notion is obvious if  $\mathcal{A} = \text{Coh}(X)$  for a variety  $X$ , but in general we need some additional extra data, which is given in [9, Assumptions 7.1, 8.1]. For the introduction of Artin stacks, one can consult [17]. For instance, Assumption 2.6 is satisfied when  $\mathcal{A} = \text{mod } A$  for a finite dimensional  $\mathbb{C}$ -algebra  $A$ , and  $Z$  is given as in Example 2.4 (i).

For a variety  $Y$ , recall that the Grothendieck ring of varieties over  $Y$  is defined by

$$K_0(\text{Var} / Y) = \bigoplus_{(X, \rho)} \mathbb{Z}[(X, \rho)] / \sim,$$

where  $X$  is a variety with a morphism  $\rho: X \rightarrow Y$ , and equivalence relations are given by

$$[(X, \rho)] \sim [(X^\dagger, \rho|_{X^\dagger})] + [(X \setminus X^\dagger, \rho|_{X \setminus X^\dagger})],$$

where  $X^\dagger$  is a closed subvariety of  $X$ . Taking the fiber products over  $Y$ , there is a natural product on  $K_0(\text{Var}/Y)$ ,

$$(8) \quad [(X, \rho)] \cdot [(X', \rho')] = [(X \times_Y X', \rho \circ p)],$$

where  $p$  is the projection  $X \times_Y X' \rightarrow X$ .

In order to introduce  $\mathcal{H}(\mathcal{A})$ , let us introduce the notion of Grothendieck rings of Artin stacks.

**Definition 2.7.** [13] Let  $\mathcal{Y}$  be an Artin stack locally of finite type over  $\mathbb{C}$ . Define the  $\mathbb{Q}$ -vector space  $K_0(\text{St}/\mathcal{Y})$  to be

$$K_0(\text{St}/\mathcal{Y}) := \bigoplus_{(\mathcal{X}, \rho)} \mathbb{Q}[(\mathcal{X}, \rho)] / \sim,$$

where  $(\mathcal{X}, \rho)$  is a pair such that  $\mathcal{X}$  is an Artin  $\mathbb{C}$ -stack of finite type with affine geometric stabilizers, and  $\rho: \mathcal{X} \rightarrow \mathcal{Y}$  is a 1-morphism. The relations  $\sim$  are given by

$$[(\mathcal{X}, \rho)] \sim [(\mathcal{X}^\dagger, \rho|_{\mathcal{X}^\dagger})] + [(\mathcal{X} \setminus \mathcal{X}^\dagger, \rho|_{\mathcal{X} \setminus \mathcal{X}^\dagger})],$$

for closed substacks  $\mathcal{X}^\dagger \subset \mathcal{X}$ .

Again taking the fiber products over  $\mathcal{Y}$  gives a product  $\cdot$  on  $K_0(\text{St}/\mathcal{Y})$ ,

$$(9) \quad [(\mathcal{X}, \rho)] \cdot [(\mathcal{X}', \rho')] = [(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}', \rho \circ p)],$$

where  $p$  is the projection  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}' \rightarrow \mathcal{X}$ .

**Definition 2.8.** [10] Let  $\mathcal{A}$  be an abelian category satisfying the second condition of Assumption 2.6. We define the  $\mathbb{Q}$ -vector space  $\mathcal{H}(\mathcal{A})$  to be

$$\mathcal{H}(\mathcal{A}) := K_0(\text{St}/\mathfrak{D}\text{bj}(\mathcal{A})).$$

The vector space  $\mathcal{H}(\mathcal{A})$  is graded by  $v \in \overline{\mathcal{C}}(\mathcal{A})$ ,

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{v \in \overline{\mathcal{C}}(\mathcal{A})} \mathcal{H}^v(\mathcal{A}), \quad \mathcal{H}^v(\mathcal{A}) := K_0(\text{St}/\mathfrak{D}\text{bj}^v(\mathcal{A})),$$

where  $\mathfrak{D}\text{bj}^v(\mathcal{A})$  is the stack of objects  $E \in \mathcal{A}$  with  $[E] = v$ . There is an associative multiplication  $*$  on  $\mathcal{H}(\mathcal{A})$ , based on *Ringel–Hall algebras*, which differs from the product (9). Let  $\mathfrak{E}\mathfrak{r}(\mathcal{A})$  be the moduli stack of

exact sequences  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$  in  $\mathcal{A}$ . It is shown in [9, Theorem 8.2] that  $\mathfrak{E}\mathfrak{r}(\mathcal{A})$  is an Artin stack locally of finite type over  $\mathbb{C}$ . We have the following 1-morphisms,

$$p_i: \mathfrak{E}\mathfrak{r}(\mathcal{A}) \ni (0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0) \mapsto E_i \in \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}),$$

for  $i = 1, 2, 3$ . Take  $f_i = [(\mathcal{X}_i, \rho_i)] \in \mathcal{H}(\mathcal{A})$  for  $i = 1, 2$ . We have the following diagram,

$$\begin{array}{ccccc} (p_1, p_3)^*(\mathcal{X}_1 \times \mathcal{X}_2) & \xrightarrow{u} & \mathfrak{E}\mathfrak{r}(\mathcal{A}) & \xrightarrow{p_2} & \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \\ \downarrow & & \downarrow (p_1, p_3) & & \\ \mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{(\rho_1, \rho_2)} & \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \times \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) & & \end{array}$$

Here the left diagram is a Cartesian diagram.

**Definition 2.9.** We define the  $*$ -product  $f_1 * f_2$  by

$$f_1 * f_2 = [((p_1, p_3)^*(\mathcal{X}_1 \times \mathcal{X}_2), p_2 \circ u)].$$

It is shown in [10, Theorem 5.2] that  $*$  is associative and  $\mathcal{H}(\mathcal{A})$  is a  $\mathbb{Q}$ -algebra with identity  $[0 \hookrightarrow \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})]$ .

**Remark 2.10.** Our algebra  $\mathcal{H}(\mathcal{A})$  is denoted by  $\underline{\text{SF}}(\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$  in Joyce’s paper [10], and an element of  $\underline{\text{SF}}(\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$  is called a *stack function* on  $\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})$ . There is another version of Ringel–Hall type algebra discussed in [10], defined as the set of constructible functions on  $\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})$ , denoted by  $\text{CF}(\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$  in [10]. Although most of the readers might be more familiar with constructible functions than stack functions, we use the latter one since we want to apply [10, Theorem 6.12] which is formulated only for stack functions.

**2.3. Elements  $\delta^v(Z), \epsilon^v(Z)$**

Let  $Z: C(\mathcal{A}) \rightarrow T$  be a weak stability condition, satisfying Assumption 2.6. For an Artin substack  $i: \mathfrak{M} \hookrightarrow \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})$ , we write the element  $[(\mathfrak{M}, i)] \in \mathcal{H}(\mathcal{A})$  as  $[\mathfrak{M} \hookrightarrow \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})]$ .

**Definition 2.11.** For  $v \in C(\mathcal{A})$ , we define  $\delta^v(Z), \epsilon^v(Z) \in \mathcal{H}(\mathcal{A})$  to be

$$\begin{aligned} \delta^v(Z) &= [\mathfrak{M}^v(Z) \hookrightarrow \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})] \in \mathcal{H}^v(\mathcal{A}), \\ (10) \quad \epsilon^v(Z) &= \sum_{\substack{l \geq 1, v_i \in C(\mathcal{A}), \\ v_1 + \dots + v_l = v, \\ Z(v_i) = Z(v), 1 \leq i \leq l}} \frac{(-1)^{l-1}}{l} \delta^{v_1}(Z) * \dots * \delta^{v_l}(Z) \in \mathcal{H}^v(\mathcal{A}). \end{aligned}$$

Under Assumption 2.6, the sum (10) is a finite sum, (see [11, Proposition 4.9],) and hence  $\epsilon^v(Z)$  is an element of  $\mathcal{H}(\mathcal{A})$ . In [11, Theorem 8.7], Joyce shows that  $\epsilon^v(Z)$  is an element of a certain Lie subalgebra of  $\mathcal{H}(\mathcal{A})$ , called *Ringel–Hall Lie algebra*,

$$(11) \quad \epsilon^v(Z) \in \mathfrak{G}(\mathcal{A}) \subset \mathcal{H}(\mathcal{A}).$$

The Lie algebra  $\mathfrak{G}(\mathcal{A})$  is denoted by  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{D}\text{bj}(\mathcal{A}))$  in Joyce’s paper [10]. If we work over the Hall-type algebra  $\text{CF}(\mathfrak{D}\text{bj}(\mathcal{A}))$ , (see Remark 2.10,) the corresponding Lie algebra  $\text{CF}^{\text{ind}}(\mathfrak{D}\text{bj}(\mathcal{A}))$  is the set of constructible functions on  $\mathfrak{D}\text{bj}(\mathcal{A})$ , supported on indecomposable objects. One might expect, as an analogue for  $\mathcal{H}(\mathcal{A})$ , that an element  $[(\mathcal{X}, \rho)]$  is contained in  $\mathfrak{G}(\mathcal{A})$  if the image of  $\rho$  in  $\mathfrak{D}\text{bj}(\mathcal{A})$  is supported on indecomposable objects. However Joyce suggests that this definition is not the best analogue, and he introduces the notion of “virtual indecomposable objects”, and defines  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{D}\text{bj}(\mathcal{A}))$  in [10, Definition 5.13] as the set of stack functions supported on virtual indecomposable objects. We omit the precise definition of  $\mathfrak{G}(\mathcal{A})$  here, as we will not use this. The Lie algebra  $\mathfrak{G}(\mathcal{A})$  also has the decomposition,

$$\mathfrak{G}(\mathcal{A}) = \bigoplus_{v \in \overline{C}(\mathcal{A})} \mathfrak{G}^v(\mathcal{A}), \quad \mathfrak{G}^v(\mathcal{A}) := \mathcal{H}^v(\mathcal{A}) \cap \mathfrak{G}(\mathcal{A}),$$

and  $\epsilon^v(Z)$  is an element of  $\mathfrak{G}^v(\mathcal{A})$ . The conceptual meaning of the definition of  $\epsilon^v(Z)$  is that they are “logarithms” of  $\delta^v(Z)$ , i.e. for  $t \in T$ , we have formally

$$\sum_{v \in Z^{-1}(t)} \epsilon^v(Z) = \log \left( 1 + \sum_{v \in Z^{-1}(t)} \delta^v(Z) \right).$$

Also see [5] for more arguments on the elements  $\epsilon^v(Z)$ .

**2.4. Transformation of the elements  $\delta^v(Z)$ ,  $\epsilon^v(Z)$**

The descriptions of the variations of the elements  $\delta^v(Z)$ ,  $\epsilon^v(Z)$  under change of  $Z$  are investigated in [14]. Let us briefly recall the main idea of [14, Theorem 5.2] in this subsection. We first introduce the following definition.

**Definition 2.12.** Let  $Z, Z': C(\mathcal{A}) \rightarrow T$  be weak stability conditions and take  $v \in C(\mathcal{A})$ . We say  $Z'$  *dominates*  $Z$  with respect to  $v$  if for  $v_1, v_2 \in C_{\leq v}(\mathcal{A})$ ,  $Z(v_1) \succeq Z(v_2)$  implies  $Z'(v_1) \succeq Z'(v_2)$ . Here  $C_{\leq v}(\mathcal{A})$  is defined by

$$(12) \quad C_{\leq v}(\mathcal{A}) = \{v' \in C(\mathcal{A}) \mid \text{there is } v'' \in \overline{C}(\mathcal{A}) \text{ with } v' + v'' = v\}.$$

The first step is to show the following theorem.

**Theorem 2.13.** [14, Theorem 5.11] *For weak stability conditions  $Z, Z': C(\mathcal{A}) \rightarrow T$  satisfying Assumption 2.6, suppose that  $Z'$  dominates  $Z$  with respect to  $v$ . Then we have*

$$(13) \quad \delta^v(Z') = \sum_{\substack{l \geq 1, v_i \in C(\mathcal{A}), v_1 + \dots + v_l = v, \\ Z(v_i) \succ Z(v_{i-1}), Z'(v_i) = Z'(v), 1 \leq i \leq l}} \delta^{v_1}(Z) * \dots * \delta^{v_l}(Z).$$

The sum (13) may not be a finite sum, but it converges in the sense of [14, Definition 2.16]

*Proof.* We just explain the idea of the proof. For the full proof, see [14, Theorem 5.11]. For a  $Z'$ -semistable object  $E \in \mathcal{A}$  with  $[E] = v$ , there is a Harder–Narasimhan filtration with respect to  $Z$ , i.e. there is a unique filtration

$$(14) \quad 0 = E_0 \subset E_1 \subset \dots \subset E_l = E,$$

such that each  $F_i = E_i/E_{i-1}$  is  $Z$ -semistable with  $Z(F_i) \succ Z(F_{i-1})$ . Since  $Z'$  dominates  $Z$  with respect to  $v$ , and each class  $[F_i] \in C(\mathcal{A})$  is contained in  $C_{\leq v}(\mathcal{A})$ , we have  $Z'(F_i) \succeq Z'(F_{i-1})$ . Hence  $Z'$ -semistability of  $E$  implies  $Z'(F_i) = Z'(F_{i-1})$ .

Conversely for an object  $E \in \mathcal{A}$ , suppose that there is a filtration (14) such that  $F_i = E_i/E_{i-1}$  is  $Z$ -semistable with  $Z(F_i) \succ Z(F_{i-1})$  and  $Z'(F_i) = Z'(F_{i-1})$  for all  $i$ . Since  $Z'$  dominates  $Z$  with respect to  $v$ , the object  $F_i$  is also  $Z'$ -semistable, and hence  $E$  is  $Z'$ -semistable.

As a consequence, an object  $E \in \mathcal{A}$  with  $[E] = v$  is  $Z'$ -semistable if and only if there is a unique filtration (14) such that each  $F_i = E_i/E_{i-1}$  is  $Z$ -semistable and  $v_i = [F_i] \in C(\mathcal{A})$  for  $i = 1, \dots, l$  satisfy

$$(15) \quad \begin{aligned} v_1 + \dots + v_l &= v, \\ Z(v_1) &\succ Z(v_2) \succ \dots \succ Z(v_l), \\ Z'(v_1) &= Z'(v_2) = \dots = Z'(v_l). \end{aligned}$$

This observation is expressed as (13) in terms of the algebra  $\mathcal{H}(\mathcal{A})$ .  
Q.E.D.

We omit the definition of the convergence [14, Definition 2.16] here, as we will only treat the cases that the relevant sums have only finitely

many terms. The next step is to invert (13), and give the formula,

$$(16) \quad \delta^v(Z) = \sum_{\substack{l \geq 1, v_i \in C(\mathcal{A}), \\ v_1 + \dots + v_l = v, \\ Z'(v_i) = Z'(v), 1 \leq i \leq l, \\ Z(v_1 + \dots + v_i) \succ Z(v_{i+1} + \dots + v_l)}} (-1)^{l-1} \delta^{v_1}(Z') * \dots * \delta^{v_l}(Z').$$

The proof is provided in [14, Theorem 5.12]. The sum (16) may not converge in the sense of [14, Definition 2.16], but if we impose the assumption that the change from  $Z$  to  $Z'$  is *locally finite*, (we omit the definition of the local finiteness, see [14, Definition 5.1,]) then the sum (16) converges.

Finally for two weak stability conditions  $Z, Z'$ , consider the following situation ( $\spadesuit$ ).

- There are weak stability conditions  $Z = Z_1, Z_2, \dots, Z_m = Z', W_1, \dots, W_{m-1}$  satisfying Assumption 2.6, such that  $W_i$  dominates  $Z_i, Z_{i-1}$  w.r.t.  $v$ , and all changes from  $Z_i$  to  $W_i, W_{i-1}$  are locally finite.  $\dots$  ( $\spadesuit$ ).

Then in principle one can express  $\delta^v(Z')$  in terms of  $\delta^v(Z)$  in the algebra  $\mathcal{H}(\mathcal{A})$ , by applying the formulas (13), (16) successively. The transformation coefficients are determined purely combinatorial, and they are given as follows.

**Definition 2.14.** [14, Definition 4.2] Take  $v_1, \dots, v_l \in C(\mathcal{A})$  and weak stability conditions  $Z, Z': C(\mathcal{A}) \rightarrow T$ . If for each  $i = 1, \dots, l - 1$ , we have either (17) or (18),

- (17)  $Z(v_i) \preceq Z(v_{i+1})$  and  $Z'(v_1 + \dots + v_i) \succ Z'(v_{i+1} + \dots + v_l)$ ,
- (18)  $Z(v_i) \succ Z(v_{i+1})$  and  $Z'(v_1 + \dots + v_i) \preceq Z'(v_{i+1} + \dots + v_l)$ .

then define  $S(\{v_1, \dots, v_l\}, Z, Z')$  to be  $(-1)^r$ , where  $r$  is the number of  $i = 1, \dots, l - 1$  satisfying (17). Otherwise we define  $S(\{v_1, \dots, v_l\}, Z, Z') = 0$ .

We have the following formula.

**Theorem 2.15.** [14, Theorem 5.2] *Under the situation ( $\spadesuit$ ), we have*

$$(19) \quad \delta^v(Z') = \sum_{\substack{l \geq 1, v_i \in C(\mathcal{A}), \\ v_1 + \dots + v_l = v}} S(\{v_1, \dots, v_l\}, Z, Z') \delta^{v_1}(Z) * \dots * \delta^{v_l}(Z).$$

The sum (19) converges in the sense of [14, Definition 2.16].

**Remark 2.16.** Our condition “ $\ast'$  dominates  $\ast$  w.r.t.  $v$ ” in Definition 2.12 is weaker than Joyce’s condition “ $\ast'$  dominates  $\ast$ ” given in [14, Definition 3.16], and [14, Theorem 5.2] is formulated using the latter condition. However if we want to know (19) for a fixed  $v \in C(\mathcal{A})$ , it is enough to assume “ $\ast'$  dominates  $\ast$  w.r.t.  $v$ ” in ( $\spadesuit$ ), since all the  $v_i$  in the sum (19) are contained in  $C_{\leq v}(\mathcal{A})$ .

The relationship between  $\epsilon^v(Z')$  and  $\epsilon^v(Z)$  is deduced from (10), (19), and the inverse of (10),

$$(20) \quad \delta^v(Z) = \sum_{\substack{l \geq 1, v_i \in C(\mathcal{A}), v_1 + \dots + v_l = v, \\ Z(v_i) = Z(v), 1 \leq i \leq l}} \frac{1}{l!} \epsilon^{v_1}(Z) \ast \dots \ast \epsilon^{v_l}(Z).$$

The proof of (20) is provided in [11, Theorem 8.2]. The transformation coefficients are given as follows.

**Definition 2.17.** [14, Definition 4.4] For  $v_1, \dots, v_l \in C(\mathcal{A})$ , we define  $U(\{v_1, \dots, v_l\}, Z, Z') \in \mathbb{Q}$  to be

$$(21) \quad U(\{v_1, \dots, v_l\}, Z, Z') = \sum_{1 \leq m' \leq m \leq l} \sum_{\substack{\text{surjective } \psi: \{1, \dots, l\} \rightarrow \{1, \dots, m\}, \\ \text{surjective } \xi: \{1, \dots, m\} \rightarrow \{1, \dots, m'\}, \\ i \leq j \text{ implies } \psi(i) \leq \psi(j), \xi(i) \leq \xi(j), \\ \psi \text{ and } \xi \text{ satisfy } (\diamond)}} S(\{w_i\}_{i \in \xi^{-1}(a)}, Z, Z') \cdot \frac{(-1)^{m'}}{m'} \cdot \prod_{b=1}^m \frac{1}{|\psi^{-1}(b)|!}.$$

Here the condition  $(\diamond)$  is as follows.

- For  $1 \leq i, j \leq l$  with  $\psi(i) = \psi(j)$ , we have  $Z(v_i) = Z(v_j)$ , and for  $1 \leq i, j \leq m'$ , we have

$$Z' \left( \sum_{k \in \psi^{-1} \xi^{-1}(i)} v_k \right) = Z' \left( \sum_{k \in \psi^{-1} \xi^{-1}(j)} v_k \right). \quad \dots (\diamond)$$

Also  $w_i$  for  $1 \leq i \leq m$  is defined as

$$w_i = \sum_{j \in \psi^{-1}(i)} v_j \in C(\mathcal{A}).$$

**Theorem 2.18.** [14, Theorem 5.2] *In the situation ( $\spadesuit$ ), the following holds.*

$$(22) \quad \epsilon^v(Z') = \sum_{\substack{l \geq 1, v_i \in C(\mathcal{A}), \\ v_1 + \dots + v_l = v}} U(\{v_1, \dots, v_l\}, Z, Z') \epsilon^{v_1}(Z) \ast \dots \ast \epsilon^{v_l}(Z).$$

The sum (22) converges in the sense of [14, Definition 2.16].

**Remark 2.19.** It is possible to rewrite (22) by a  $\mathbb{Q}$ -linear combination of multiple commutators of  $\epsilon^{v_i}(Z)$  such as

$$[[\cdots [[\epsilon^{v_1}(Z), \epsilon^{v_2}(Z)], \epsilon^{v_3}(Z)], \cdots], \epsilon^{v_l}(Z)],$$

so (22) is an equality in  $\mathfrak{G}(\mathcal{A})$ , rather than in  $\mathcal{H}(\mathcal{A})$ . The proof of this fact is given in [14, Theorem 5.4].

**2.5. Motivic invariants of stacks**

As a final step, we integrate the elements  $\epsilon^v(Z) \in \mathfrak{G}^v(\mathcal{A})$  to give  $\mathbb{Q}$ -valued invariants, and establish the transformation formula of these invariants. Let us recall that a motivic invariant is a ring homomorphism

$$\Upsilon: K_0(\text{Var} / \text{Spec } \mathbb{C}) \longrightarrow \Lambda,$$

where  $\Lambda$  is a  $\mathbb{Q}$ -algebra and a ring structure on  $K_0(\text{Var} / \text{Spec } \mathbb{C})$  is given by (8). In order to simplify the arguments, we only consider the special case that  $\Lambda = \mathbb{Q}(t)$  and

$$\Upsilon([Y]) = \sum_i (-1)^i \dim H^i(Y, \mathbb{C}) t^i,$$

where  $Y$  is a smooth projective variety. Since  $K_0(\text{Var} / \text{Spec } \mathbb{C})$  is generated by  $[Y]$  for smooth projective varieties  $Y$ , the above data uniquely determines  $\Upsilon$ . In this situation, there is a unique extension of  $\Upsilon$ ,

$$\Upsilon': K_0(\text{St} / \text{Spec } \mathbb{C}) \longrightarrow \mathbb{Q}(t),$$

such that if  $G$  is a special algebraic group acting on  $Y$ , we have (cf. [13, Theorem 4.9])

$$\Upsilon'([Y/G]) = \Upsilon([Y]) / \Upsilon([G]).$$

Here an algebraic group is *special* if every principal  $G$ -bundle is locally trivial in Zariski topology. In what follows, we assume that  $\mathcal{A}$  satisfies the following condition.

( $\star$ ) : there is an anti-symmetric biadditive-pairing  $\chi: \mathcal{N}(\mathcal{A}) \times \mathcal{N}(\mathcal{A}) \rightarrow \mathbb{Z}$  such that for any  $E, F \in \mathcal{A}$ , we have

$$\begin{aligned} \chi(E, F) &= \dim \text{Hom}(E, F) - \dim \text{Ext}^1(E, F) + \\ &\quad \dim \text{Ext}^1(F, E) - \dim \text{Hom}(F, E). \end{aligned}$$

For instance if  $\mathcal{A} = \text{Coh}(X)$  for a smooth projective Calabi–Yau 3-fold  $X$ , the usual Euler pairing (6) descends to the pairing on  $\mathcal{N}(\mathcal{A})$ , which satisfies ( $\star$ ) by Serre duality. Using the pairing  $\chi$ , we can define the following Lie algebra.

**Definition 2.20.** For an abelian category  $\mathcal{A}$  satisfying  $(\star)$ , we define the Lie algebra  $\mathfrak{g}(\mathcal{A})$  to be the  $\mathbb{Q}$ -vector space,

$$\mathfrak{g}(\mathcal{A}) := \bigoplus_{v \in \mathcal{N}(\mathcal{A})} \mathbb{Q}c_v,$$

with its Lie-brackets given by  $[c_v, c_{v'}] = \chi(v, v')c_{v+v'}$ .

Let  $\Pi_v$  be the composition,

$$\Pi_v: \mathfrak{G}^v(\mathcal{A}) \subset \mathcal{H}(\mathcal{A}) \xrightarrow{\pi_*} K_0(\text{St}/\text{Spec } \mathbb{C}) \xrightarrow{\Upsilon'} \mathbb{Q}(t),$$

where the map  $\pi_*$  sends  $[(\mathcal{X}, \rho)]$  to  $[(\mathcal{X}, \pi \circ \rho)]$  and  $\pi: \mathfrak{Obj}(\mathcal{A}) \rightarrow \text{Spec } \mathbb{C}$  is the structure morphism. It is shown in [14, Section 6.2] that for  $\epsilon \in \mathfrak{G}^v(\mathcal{A})$ , the rational function  $\Pi_v(\epsilon) \in \mathbb{Q}(t)$  has a pole of order at most one at  $t = 1$ . Hence the following definition makes sense,

$$(23) \quad \Theta_v(\epsilon) = (t^2 - 1)\Pi_v(\epsilon)|_{t=1} \in \mathbb{Q}.$$

**Definition 2.21.** We define the invariant  $J^v(Z) \in \mathbb{Q}$  by

$$J^v(Z) = \Theta_v(\epsilon^v(Z)) \in \mathbb{Q}.$$

**Remark 2.22.** If all the  $Z$ -semistable objects  $E \in \mathcal{A}$  with  $[E] = v$  are in fact  $Z$ -stable, and their moduli problem is represented by a scheme, then  $\epsilon^v(Z)$  is written as  $[M^v(Z)/\mathbb{G}_m]$  for a scheme  $M^v(Z)$ . Here  $\mathbb{G}_m$  is acting on  $M^v(Z)$  trivially. In this case  $J^v(Z)$  equals to the Euler characteristic of  $M^v(Z)$ . Note that the factor  $\Upsilon([\mathbb{G}_m]) = t^2 - 1$  in (23) is required to cancel out the contribution of the stabilizer group  $\text{Aut}(E) \cong \mathbb{G}_m$ .

We have the following theorem.

**Theorem 2.23.** [10, Theorem 6.12] *The map,*

$$(24) \quad \Theta: \mathfrak{G}(\mathcal{A}) = \bigoplus_{v \in \overline{\mathcal{C}}(\mathcal{A})} \mathfrak{G}^v(\mathcal{A}) \ni \{\epsilon^v\}_v \longmapsto \sum_{v \in \overline{\mathcal{C}}(\mathcal{A})} \Theta_v(\epsilon^v)c_v \in \mathfrak{g}(\mathcal{A}),$$

*is a Lie algebra homomorphism.*

Since (22) is a relationship in the Lie algebra  $\mathfrak{G}(\mathcal{A})$ , we can obtain the relationship between  $J^v(Z')$  and  $J^v(Z)$  by applying  $\Theta$ . The result is as follows.

**Theorem 2.24.** [14, Theorem 6.28, Equation (130)] *In the situation of (♠), assume that there are only finitely many terms in (22). Applying  $\Theta$  to (22) yields the formula,*

$$(25) \quad J^v(Z') = \sum_{\substack{l \geq 1, v_i \in C(\mathcal{A}), \\ v_1 + \dots + v_l = v}} \sum_{\substack{\Gamma \text{ is a connected, simply connected oriented} \\ \text{graph with vertex } \{1, \dots, l\}, \bullet \xrightarrow{i} \bullet \text{ implies } i < j}} \frac{1}{2^{l-1}} U(\{v_1, \dots, v_l\}, Z, Z') \prod_{\bullet \xrightarrow{i} \bullet \text{ in } \Gamma} \chi(v_i, v_j) \prod_{i=1}^l J^{v_i}(Z).$$

**2.6. Generalization to quasi-abelian categories**

For our purpose it is useful to give a slight generalization of Theorem 2.24, especially we want to relax Assumption 2.6, as our abelian category  $\mathcal{A}^p$  which will be introduced in the next section does not satisfy that assumption. Let  $\mathcal{A}$  be an abelian category and  $Z: C(\mathcal{A}) \rightarrow T$  a weak stability condition. Here we do not assume Assumption 2.6. For  $t \in T$ , we set

$$\begin{aligned} \mathcal{A}_{Z \succeq t} &= \langle E \mid E \text{ is } Z\text{-semistable with } Z(E) \succeq t \rangle, \\ \mathcal{A}_{Z \prec t} &= \langle E \mid E \text{ is } Z\text{-semistable with } Z(E) \prec t \rangle. \end{aligned}$$

Here for a set of objects  $S$  in  $\mathcal{A}$ , we denote by  $\langle S \rangle \subset \mathcal{A}$  the smallest extension closed subcategory of  $\mathcal{A}$  which contains  $S$ . Equivalently, an object  $E \in \mathcal{A}$  is contained in  $\mathcal{A}_{Z \succeq t}$  (resp.  $\mathcal{A}_{Z \prec t}$ ) if and only if any  $Z$ -semistable factor  $F$  of  $E$  satisfies  $Z(F) \succeq t$ , (resp.  $Z(F) \prec t$ ). It can be shown that  $\mathcal{A}_{Z \succeq t}, \mathcal{A}_{Z \prec t}$  are quasi-abelian categories. See [4, Section 4] for the detail on quasi-abelian categories.

**Definition 2.25.** For objects  $E, F \in \mathcal{A}_{Z \prec t}$  and a morphism  $f: E \rightarrow F$ , it is called a *strict monomorphism* if  $f$  is injective in  $\mathcal{A}$  and  $\text{Coker}(f) \in \mathcal{A}_{Z \prec t}$ . Similarly  $f$  is called a *strict epimorphism* if  $f$  is surjective in  $\mathcal{A}$  and  $\text{Ker}(f) \in \mathcal{A}_{Z \prec t}$ .

For  $v \in C(\mathcal{A}_{Z \prec t})$ , we set

$$(26) \quad C_{\leq v}(\mathcal{A}_{Z \prec t}) = \{v' \in C(\mathcal{A}_{Z \prec t}) \mid v - v' \in \overline{C}(\mathcal{A}_{Z \prec t})\}.$$

For  $Z, t, v$  as above, we put the following assumption instead of Assumption 2.6.

- Assumption 2.26.**
- *The category  $\mathcal{A}_{Z \prec t}$  is noetherian and artinian with respect to strict monomorphisms.*
  - *There is an Artin stack locally of finite type  $\mathfrak{D}\text{bj}(\mathcal{A})$ , which parametrizes objects  $E \in \mathcal{A}$ .*

- For any  $v' \in C_{\leq v}(\mathcal{A}_{Z \prec t})$ , the substack  $\mathfrak{M}^{v'}(Z) \subset \mathfrak{D}\text{bj}(\mathcal{A})$  is an open substack, and it is of finite type.

We also modify the dominant conditions.

**Definition 2.27.** Let  $Z, Z': C(\mathcal{A}) \rightarrow T$  be weak stability conditions. For  $t \in T$  and  $v \in C(\mathcal{A}_{Z \prec t})$ , we say  $Z'$  dominates  $Z$  with respect to  $(v, t)$  if the following holds.

- We have  $\mathcal{A}_{Z \succeq t} = \mathcal{A}_{Z' \succeq t}$  and  $\mathcal{A}_{Z \prec t} = \mathcal{A}_{Z' \prec t}$ .
- For  $v_1, v_2 \in C_{\leq v}(\mathcal{A}_{Z \prec t})$ , if  $Z(v_1) \succeq Z(v_2)$  then  $Z'(v_1) \succeq Z'(v_2)$ .

For two weak stability conditions  $Z, Z'$ , we consider the following situation.

- There are weak stability conditions  $Z = Z_1, Z_2, \dots, Z_m = Z'$ ,  $W_1, \dots, W_{m-1}$  such that  $W_i$  dominates  $Z_i, Z_{i+1}$  w.r.t.  $(v, t)$ , all changes from  $Z_i$  to  $W_i, W_{i-1}$  are locally finite, and  $Z_i, W_i$  satisfy Assumption 2.26 for  $(v, t)$ .  $\dots(\spadesuit')$

We have the following generalization of Theorem 2.24.

**Theorem 2.28.** For weak stability conditions  $Z, Z'$  on  $\mathcal{A}$ ,  $t \in T$  and  $v \in C(\mathcal{A}_{Z \prec t})$ , suppose that the condition  $(\spadesuit')$  holds. Then the equation (22) with each  $v_i \in C(\mathcal{A}_{Z \prec t})$  holds. If there are only finitely many terms in (22) with  $v_i \in C(\mathcal{A}_{Z \prec t})$ , then (25) holds, with each  $v_i \in C(\mathcal{A}_{Z \prec t})$ .

*Proof.* First suppose that  $Z'$  dominates  $Z$  with respect to  $(v, t)$ , and check that (13) holds with each  $v_i \in C(\mathcal{A}_{Z \prec t})$ . Let  $E \in \mathcal{A}$  be  $Z'$ -semistable with  $[E] = v$ . As in the proof of Theorem 2.13, we have a unique filtration (14). Let  $F_i = E_i/E_{i-1}$ ,  $v_i = [F_i] \in C(\mathcal{A})$ . Since  $v \in C(\mathcal{A}_{Z \prec t}) = C(\mathcal{A}_{Z' \prec t})$ , we have  $E \in \mathcal{A}_{Z' \prec t} = \mathcal{A}_{Z \prec t}$ . Hence we have  $Z(v_i) \prec t$ , and  $v_i \in C(\mathcal{A}_{Z \prec t})$  follows.

Conversely given a filtration (14), suppose that each  $F_i$  is  $Z$ -semistable with  $v_i \in C(\mathcal{A}_{Z \prec t})$ . Then  $F_i \in \mathcal{A}_{Z \prec t} = \mathcal{A}_{Z' \prec t}$ , and hence  $F_i$  is also  $Z'$ -semistable as  $Z'$  dominates  $Z$  w.r.t.  $(v, t)$ .

As a summary, an object  $E \in \mathcal{A}_{Z \prec t}$  is  $Z$ -semistable if and only if there is a filtration (14), satisfying (15) with each  $v_i \in C(\mathcal{A}_{Z \prec t})$ . Then the same proof as in [14, Theorem 5.11] works and gives the formula (13) with each  $v_i \in C(\mathcal{A}_{Z \prec t})$ . Note that to state the formula (13), it is enough to assume that  $\mathfrak{M}^{v'}(Z) \subset \mathfrak{D}\text{bj}(\mathcal{A})$  is open and of finite type for any  $v' \in C_{\leq v}(\mathcal{A}_{Z \prec t})$ .

By the same idea, we can also show the formulas (16), (19), (20), (22) hold with each  $v_i \in C(\mathcal{A}_{Z \prec t})$ . We leave the readers to follow Joyce's work and that the same proofs are applied in this case. Q.E.D.

**§3. Limit stability and  $\mu$ -limit stability**

In this section, we recall the notion of limit stability on a Calabi–Yau 3-fold  $X$  introduced in [24], and also introduce the notion of  $\mu$ -limit stability. Below we always assume that  $X$  is a projective complex 3-fold with a trivial canonical class, i.e.  $X$  is a Calabi–Yau 3-fold. We denote by  $D^b(X)$  the bounded derived category of  $\text{Coh}(X)$ , and  $K(X)$  the Grothendieck group of  $\text{Coh}(X)$ . The *numerical Grothendieck group* of  $\text{Coh}(X)$  is given by

$$\mathcal{N}(X) = K(X) / \equiv,$$

where the numerical equivalence relation  $\equiv$  is given by (5). Note that if  $\mathcal{A} \subset D^b(X)$  is the heart of a bounded t-structure on  $D^b(X)$ , then the group  $\mathcal{N}(\mathcal{A}) = K(\mathcal{A}) / \equiv$ , where  $\equiv$  is (5) as above, coincides with  $\mathcal{N}(X)$ . So we always regard  $C(\mathcal{A})$  as a subset of  $\mathcal{N}(X)$ .

Let us fix notation of the numerical classes of curves on  $X$ . An element  $\beta \in H^4(X, \mathbb{Z})$  is called an *effective class* if there is a one dimensional subscheme  $C \subset X$  such that  $\beta$  is the Poincaré dual of the fundamental cycle of  $C$ . We set  $C(X), \overline{C}(X)$  as

$$\begin{aligned} C(X) &:= \{ \beta \in H^4(X, \mathbb{Z}) \mid \beta \text{ is an effective class } \}, \\ \overline{C}(X) &:= C(X) \cup \{0\}. \end{aligned}$$

**3.1. Definition of limit stability**

The limit stability introduced in [24] is a stability condition on the category of perverse coherent sheaves  $\mathcal{A}^p$  in the sense of Bezrukavnikov [3] and Kashiwara [15], and it is also one of the polynomial stability conditions introduced by Bayer [1] independently. In order to introduce  $\mathcal{A}^p$ , let us define the subcategories  $(\text{Coh}_{\leq 1}(X), \text{Coh}_{\geq 2}(X))$  of  $\text{Coh}(X)$ , as follows.

**Definition 3.1.** We define the pair of subcategories,  $(\text{Coh}_{\leq 1}(X), \text{Coh}_{\geq 2}(X))$ , to be

$$\begin{aligned} \text{Coh}_{\leq 1}(X) &:= \{ E \in \text{Coh}(X) \mid \dim \text{Supp}(E) \leq 1 \}, \\ \text{Coh}_{\geq 2}(X) &:= \{ E \in \text{Coh}(X) \mid \text{Hom}(\text{Coh}_{\leq 1}(X), E) = 0 \}. \end{aligned}$$

The category  $\mathcal{A}^p$  is defined as follows.

**Definition 3.2.** We define the subcategory  $\mathcal{A}^p \subset D^b(X)$  to be

$$\mathcal{A}^p := \left\{ E \in D^b(X) : \begin{array}{l} \mathcal{H}^{-1}(E) \in \text{Coh}_{\geq 2}(X), \mathcal{H}^0(E) \in \text{Coh}_{\leq 1}(X), \\ \text{and } \mathcal{H}^i(E) = 0 \text{ for } i \neq -1, 0. \end{array} \right\}.$$

It is easy to see that  $(\text{Coh}_{\leq 1}(X), \text{Coh}_{\geq 2}(X))$  determines a torsion theory on  $\text{Coh}(X)$ , and  $\mathcal{A}^p$  is the corresponding tilting. (cf. [24, Definition 2.9, Lemma 2.10].) Therefore  $\mathcal{A}^p$  is the heart of a bounded t-structure on  $D^b(X)$ , in particular  $\mathcal{A}^p$  is an abelian category.

Next recall that the *complexified ample cone* is defined by

$$A(X)_{\mathbb{C}} = \{B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is an ample class}\}.$$

Given  $\sigma = B + i\omega \in A(X)_{\mathbb{C}}$ , one can define the map  $Z_{\sigma}: K(X) \rightarrow \mathbb{C}$ ,

$$Z_{\sigma}: K(X) \ni E \longmapsto - \int e^{-(B+i\omega)} \text{ch}(E) \sqrt{\text{td}_X} \in \mathbb{C}.$$

Explicitly we have

$$(27) \quad Z_{\sigma}(E) = \left(-v_3^B(E) + \frac{1}{2}\omega^2 v_1^B(E)\right) + \left(\omega v_2^B(E) - \frac{1}{6}\omega^3 v_0^B(E)\right) i,$$

where  $v_i^B \in H^{2i}(X, \mathbb{R})$  for  $0 \leq i \leq 3$  are given by

$$e^{-B} \text{ch}(E) \sqrt{\text{td}_X} = (v_0^B(E), v_1^B(E), v_2^B(E), v_3^B(E)) \in H^{\text{even}}(X, \mathbb{R}).$$

For  $\sigma_m = B + im\omega$  with  $m \in \mathbb{R}$ , one can show the following: for each non-zero object  $E \in \mathcal{A}^p$ , one has

$$(28) \quad Z_{\sigma_m}(E) \in \left\{ r \exp(i\pi\phi) : r > 0, \frac{1}{4} < \phi < \frac{5}{4} \right\},$$

for  $m \gg 0$ . (See [24, Lemma 2.20].) Hence the *phase* of  $E$  is well-defined for  $m \gg 0$  as follows,

$$\phi_{\sigma_m}(E) = \frac{1}{\pi} \text{Im} \log Z_{\sigma_m}(E) \in \left(\frac{1}{4}, \frac{5}{4}\right).$$

**Definition 3.3.** [14, Definition 2.21] An object  $E \in \mathcal{A}^p$  is  $\sigma$ -*limit (semi)stable* if for any non-zero subobject  $F \subset E$  in  $\mathcal{A}^p$ , we have

$$(29) \quad \phi_{\sigma_m}(F) < \phi_{\sigma_m}(E), \quad (\text{resp. } \phi_{\sigma_m}(F) \leq \phi_{\sigma_m}(E),)$$

for  $m \gg 0$ .

Let us interpret the above stability in terms of Definition 2.1. Let  $T$  be the one variable function field  $\mathbb{R}(m)$ . We define the total order on  $\mathbb{R}(m)$  to be

$$f(m) \succeq g(m) \stackrel{\text{def}}{\iff} f(m) \geq g(m) \text{ for } m \gg 0.$$

Note that we have

$$(30) \quad \text{Im } e^{-\pi i/4} Z_{\sigma_m}(E) > 0,$$

for  $m \gg 0$  since (28) holds. In particular (30) is non-zero as a polynomial of  $m$ , thus the following map is well-defined.

$$Z_{\sigma}^T : C(\mathcal{A}^p) \ni E \longmapsto -\frac{\text{Re } e^{-\pi i/4} Z_{\sigma_m}(E)}{\text{Im } e^{-\pi i/4} Z_{\sigma_m}(E)} \in T.$$

**Lemma 3.4.** *The map  $Z_{\sigma}^T$  is a stability condition on  $\mathcal{A}^p$ , and an object  $E \in \mathcal{A}^p$  is  $Z_{\sigma}^T$ -(semi)stable if and only if  $E$  is  $\sigma$ -limit (semi)stable.*

*Proof.* Since (30) holds, it is obvious that for  $v_1, v_2 \in C(\mathcal{A}_p)$ , the inequality  $\phi_{\sigma_m}(v_1) \leq \phi_{\sigma_m}(v_2)$  holds for  $m \gg 0$  if and only if  $Z_{\sigma}^T(v_1) \preceq Z_{\sigma}^T(v_2)$  holds in  $T$ . Hence  $Z_{\sigma}^T$  is a stability function, and the latter statement also follows. The existence of Harder–Narasimhan filtrations for limit stability is proved in [24, Theorem 2.28], and hence  $Z_{\sigma}^T$  is a stability condition. Q.E.D.

### 3.2. $\mu$ -limit stability

In this subsection, we introduce a weak stability condition on  $\mathcal{A}^p$ , which we call  $\mu$ -limit stability. Let us introduce the following notation.

**Definition 3.5.** Let  $f = \sum_{i=0}^d a_i(\sigma)m^i$  be a polynomial such that each coefficient  $a_i(\sigma)$  is a  $\mathbb{R}$ -valued function on  $A(X)_{\mathbb{C}}$ , and  $a_d(\sigma) \neq 0$ . We define  $f^{\dagger} = a_d(\sigma)m^d$ .

By the formula (27),  $\text{Re } Z_{\sigma_m}(E)$  and  $\text{Im } Z_{\sigma_m}(E)$  are written as polynomials of  $m$  whose coefficients are  $\mathbb{R}$ -valued functions on  $A(X)_{\mathbb{C}}$ . Thus the following makes sense,

$$Z_{\sigma_m}^{\dagger}(E) := (\text{Re } Z_{\sigma_m}(E))^{\dagger} + (\text{Im } Z_{\sigma_m}(E))^{\dagger} i.$$

The same argument as in [24, Lemma 2.20] shows that

$$Z_{\sigma_m}^{\dagger}(E) \in \left\{ r \exp(i\pi\phi) : r > 0, \frac{1}{4} < \phi < \frac{5}{4} \right\},$$

for  $m \gg 0$ . Hence as before we can define the following map,

$$Z_{\mu_{\sigma}} : C(\mathcal{A}^p) \ni E \longmapsto -\frac{\text{Re } e^{-\pi i/4} Z_{\sigma_m}^{\dagger}(E)}{\text{Im } e^{-\pi i/4} Z_{\sigma_m}^{\dagger}(E)} \in T.$$

We have the following lemma.

**Lemma 3.6.** *The map  $Z_{\mu_{\sigma}}$  is a weak stability function on  $\mathcal{A}^p$ .*

*Proof.* Since the operation  $f \mapsto f^\dagger$  is just taking the initial term of the polynomials, we have the implication

$$(31) \quad Z_\sigma^T(E) \succ Z_\sigma^T(F) \quad \Rightarrow \quad Z_{\mu_\sigma}(E) \succeq Z_{\mu_\sigma}(F),$$

for  $E, F \in C(\mathcal{A}^p)$ . Hence  $Z_{\mu_\sigma}$  is a weak stability function. Q.E.D.

It is easy to see that for  $0 \neq E \in \mathcal{A}^p$ , one has  $Z_{\mu_\sigma}(E) \rightarrow 1$  or  $-1$  as  $m \rightarrow \infty$ . To see that  $Z_{\mu_\sigma}$  is a weak stability condition, we introduce a pair of subcategories in  $\mathcal{A}^p$ . (cf. [24, subsection 2.3].)

**Definition 3.7.** We define  $(\mathcal{A}_1^p, \mathcal{A}_{1/2}^p)$  to be

$$\begin{aligned} \mathcal{A}_1^p &= \{E \in \mathcal{A}^p \mid \dim \text{Supp } \mathcal{H}^0(E) = 0 \text{ and } \mathcal{H}^{-1}(E) \text{ is a torsion sheaf}\}, \\ \mathcal{A}_{1/2}^p &= \{E \in \mathcal{A}^p \mid \text{Hom}(F, E) = 0 \text{ for any } F \in \mathcal{A}_1^p\}. \end{aligned}$$

It is shown in [24, Lemma 2.16] that  $(\mathcal{A}_1^p, \mathcal{A}_{1/2}^p)$  determines a torsion theory on  $\mathcal{A}^p$ . We shall use the same notation of “strict monomorphisms”, “strict epimorphisms” in  $\mathcal{A}_i^p$  as in Definition 2.25, i.e. we replace  $\mathcal{A}_{Z \prec t}$  by  $\mathcal{A}_i^p$  to define them. We have the following.

**Lemma 3.8.** *An object  $E \in \mathcal{A}^p$  is  $Z_{\mu_\sigma}$ -semistable with  $Z_{\mu_\sigma}(E) \rightarrow -1$  (resp.  $1$ ) as  $m \rightarrow \infty$  if and only if  $E \in \mathcal{A}_{1/2}^p$ , (resp.  $\mathcal{A}_1^p$ ), and for any strict monomorphism  $0 \neq F \hookrightarrow E$  in  $\mathcal{A}_{1/2}^p$ , (resp.  $\mathcal{A}_1^p$ ), one has  $Z_{\mu_\sigma}(F) \preceq Z_{\mu_\sigma}(E/F)$ .*

*Proof.* The proof is same as in [24, Lemma 2.26] for the limit stability, by noting that

$$Z_{\mu_\sigma}(E) \rightarrow -1, \quad (\text{resp. } Z_{\mu_\sigma}(E) \rightarrow 1,)$$

for  $E \in \mathcal{A}_{1/2}^p$ , (resp.  $E \in \mathcal{A}_1^p$ ), and  $m \rightarrow \infty$ . Q.E.D.

We also have the following.

**Lemma 3.9.** *The weak stability function  $Z_{\mu_\sigma}$  is a weak stability condition.*

*Proof.* The existence of Harder–Narasimhan filtrations follows from the same argument for the limit stability. The proof of [24, Theorem 2.28] also works in this case, by noting Lemma 3.8. Q.E.D.

We say  $E \in \mathcal{A}^p$  is  $\mu_\sigma$ -limit (semi)stable if it is (semi)stable in  $Z_{\mu_\sigma}$ -weak stability. To explain this notation, let us recall that the (usual)  $\mu$ -stability is defined by cutting off the lower degree terms of the reduced Hilbert polynomials. In this sense, our  $\mu_\sigma$ -limit stability resembles to

$\mu$ -stability, as we also cut off the lower degree terms of the polynomial  $Z_{\sigma_m}(\ast)$ . By (31), we have the following implications,

$$\begin{aligned} \mu_\sigma\text{-limit stable} &\Rightarrow \sigma\text{-limit stable} \\ &\Rightarrow \sigma\text{-limit semistable} \Rightarrow \mu_\sigma\text{-limit semistable} . \end{aligned}$$

**Remark 3.10.** For  $0 \neq E \in \mathcal{A}^p$ , let  $\phi_{\sigma_m}^\dagger(E)$  be

$$\phi_{\sigma_m}^\dagger(E) = \frac{1}{\pi} \operatorname{Im} \log Z_{\sigma_m}^\dagger(E) \in \left( \frac{1}{4}, \frac{5}{4} \right).$$

Then obviously an object  $E \in \mathcal{A}^p$  is  $\mu_\sigma$ -limit (semi)stable if and only if for any non-zero subobject  $F \subset E$ , one has

$$\phi_{\sigma_m}^\dagger(F) < \phi_{\sigma_m}^\dagger(E/F), \quad (\text{resp. } \phi_{\sigma_m}^\dagger(F) \leq \phi_{\sigma_m}^\dagger(E/F),)$$

for  $m \gg 0$ .

**Remark 3.11.** Let us take  $F \in \operatorname{Coh}_{\leq 1}(X)$ . In this case we have  $Z_{\sigma_m}^\dagger(F) = Z_{\sigma_m}(F)$ , and  $\operatorname{Coh}_{\leq 1}(X) \subset \mathcal{A}^p$  is closed under taking subobjects and quotients. So  $F$  is  $\mu_\sigma$ -limit (semi)stable if and only if it is  $\sigma$ -limit (semi)stable. On the other hand for  $\sigma = B + i\omega \in A(X)_\mathbb{C}$ , let  $\mu_\sigma(F) \in \mathbb{Q}$  be

$$(32) \quad \mu_\sigma(F) = \frac{\operatorname{ch}_3(F) - B \operatorname{ch}_2(F)}{\omega \operatorname{ch}_2(F)} \in \mathbb{Q}.$$

As in [24, Example 2.24 (ii)], the object  $F$  is  $\sigma$ -limit (semi)stable if and only if for any subsheaf  $0 \neq F' \subset F$  we have  $\mu_\sigma(F') < \mu_\sigma(F)$ , (resp.  $\mu_\sigma(F') \leq \mu_\sigma(F)$ ), i.e.  $F$  is a  $(B, \omega)$ -twisted (semi)stable sheaf. If  $B = k\omega$  for  $k \in \mathbb{R}$ , then  $F$  is  $\sigma$ -limit (semi)stable if and only if  $F$  is a  $\omega$ -Gieseker (semi)stable sheaf, and this notion does not depend on  $k$ .

**Remark 3.12.** By Lemma 3.8, it is obvious that

$$(33) \quad \mathcal{A}_{Z_{\mu_\sigma} < 0}^p = \mathcal{A}_{1/2}^p, \quad \mathcal{A}_{Z_{\mu_\sigma} \geq 0}^p = \mathcal{A}_1^p,$$

in the notation of subsection 2.6. Since  $\mathcal{A}_{1/2}^p$  and  $\mathcal{A}_1^p$  are of finite length with respect to strict monomorphisms and strict epimorphisms, (cf. [24, Lemma 2.19],) the categories  $\mathcal{A}_{Z_{\mu_\sigma} < 0}^p$  and  $\mathcal{A}_{Z_{\mu_\sigma} \geq 0}^p$  also have such properties.

### 3.3. Characterization of $\mu$ -limit semistable objects

Take  $v \in C(\mathcal{A}^p)$  satisfying

$$(34) \quad (\text{ch}_0(v), \text{ch}_1(v), \text{ch}_2(v), \text{ch}_3(v)) = (-1, 0, \beta, n),$$

for some  $\beta \in H^4(X, \mathbb{Q})$  and  $n \in H^6(X, \mathbb{Q}) \cong \mathbb{Q}$ . In this subsection we give a characterization of  $\mu$ -limit semistable objects  $E \in \mathcal{A}^p$  of numerical type  $v$ . i.e.  $[E] = v \in C(\mathcal{A}^p)$ . Note that such objects satisfy  $Z_{\mu_\sigma}(E) \rightarrow -1$  as  $m \rightarrow \infty$ , and hence we have

$$E \in \mathcal{A}_{1/2}^p, \quad v \in C(\mathcal{A}_{1/2}^p).$$

Also if such  $E$  exists, the classes  $\beta, n$  are contained in  $\overline{C}(X), H^6(X, \mathbb{Z}) \cong \mathbb{Z}$  respectively, by [24, Remark 3.3]. We have the following proposition, whose corresponding result for limit stability is seen in [24, Section 3].

**Proposition 3.13.** *Take  $\sigma = B + i\omega \in A(X)_{\mathbb{C}}$ . For an object  $E \in \mathcal{A}_{1/2}^p$  of numerical type  $v$ , it is  $\mu_\sigma$ -limit (semi)stable if and only if the following conditions hold.*

(a) *For any pure one dimensional sheaf  $G \neq 0$  which admits a strict epimorphism  $E \twoheadrightarrow G$  in  $\mathcal{A}_{1/2}^p$ , one has*

$$(35) \quad \mu_\sigma(G) > -\frac{3B\omega^2}{\omega^3}. \quad \left( \text{resp. } \mu_\sigma(G) \geq -\frac{3B\omega^2}{\omega^3}. \right)$$

(b) *For any pure one dimensional sheaf  $F \neq 0$  which admits a strict monomorphism  $F \hookrightarrow E$  in  $\mathcal{A}_{1/2}^p$ , one has*

$$(36) \quad \mu_\sigma(F) < -\frac{3B\omega^2}{\omega^3}. \quad \left( \text{resp. } \mu_\sigma(F) \leq -\frac{3B\omega^2}{\omega^3}. \right)$$

*Proof.* By Lemma 3.8 and applying the same argument as in [24, Lemma 3.4], an object  $E \in \mathcal{A}_{1/2}^p$  of numerical type  $v$  is  $Z_{\mu_\sigma}$ -limit semistable if and only if

(a') *For any pure one dimensional sheaf  $G \neq 0$  which admits an exact sequence  $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  in  $\mathcal{A}_{1/2}^p$ , one has  $Z_{\mu_\sigma}(F) \preceq Z_{\mu_\sigma}(G)$ .*

(b') *For any pure one dimensional sheaf  $F \neq 0$  which admits an exact sequence  $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  in  $\mathcal{A}_{1/2}^p$ , one has  $Z_{\mu_\sigma}(F) \preceq Z_{\mu_\sigma}(G)$ .*

By Lemma 3.14 below, the conditions (a'), (b') are equivalent to (a), (b) respectively. Q.E.D.

**Lemma 3.14.** *Take  $v_1, v_2 \in C(\mathcal{A}_{1/2}^p)$  with  $\text{ch}(v_1) = (-1, 0, \beta_1, n_1)$ ,  $\text{ch}(v_2) = (0, 0, \beta_2, n_2)$ , and  $\beta_2 \neq 0$ . Then*

$$Z_{\mu_\sigma}(v_1) \preceq Z_{\mu_\sigma}(v_2), \quad (\text{resp. } Z_{\mu_\sigma}(v_1) \succeq Z_{\mu_\sigma}(v_2),)$$

if and only if

$$(37) \quad \mu_\sigma(v_2) \geq -\frac{3B\omega^2}{\omega^3}, \quad (\text{resp. } \mu_\sigma(v_2) \leq -\frac{3B\omega^2}{\omega^3}).$$

If  $\sigma = k\omega + i\omega$  for  $k \in \mathbb{R}$ , (37) is equivalent to

$$(38) \quad k \geq -\frac{1}{2}\mu_{i\omega}(v_2), \quad (\text{resp. } k \leq -\frac{1}{2}\mu_{i\omega}(v_2)).$$

*Proof.* An easy computation shows,

$$(39) \quad \begin{aligned} Z_{\sigma_m}^\dagger(v_1) &= \frac{1}{2}m^2\omega^2B + \frac{1}{6}m^3\omega^3i, \\ Z_{\sigma_m}^\dagger(v_2) &= -n_2 + B\beta_2 + m\omega\beta_2i. \end{aligned}$$

Since  $\omega\beta_2 > 0$ , we have

$$\begin{aligned} Z_{\mu_\sigma}(v_1) \leq Z_{\mu_\sigma}(v_2) &\Leftrightarrow -\frac{m^2\omega^2B/2}{m^3\omega^3/6} \leq \frac{\mu_\sigma(v_2)}{m} \\ &\Leftrightarrow \mu_\sigma(v_2) \geq -\frac{3\omega^2B}{\omega^3}. \end{aligned}$$

If  $\sigma = k\omega + i\omega$  with  $k \in \mathbb{R}$ , then  $\mu_\sigma(v_2) = \mu_{i\omega}(v_2) - k$  and  $-3B\omega^2/\omega^3 = -3k$ . Hence (37) is equivalent to (38). Q.E.D.

### 3.4. Moduli theory of $\mu$ -limit semistable objects

In this subsection, we establish a moduli theory of  $\mu$ -limit semistable objects. In [24, Theorem 1.1], a moduli theory of  $\sigma$ -limit stable objects is studied, and the resulting moduli space is an algebraic subspace of Inaba's algebraic space [8]. Since we need a moduli theory not only for stable objects but also for semistable objects, the resulting space should not be an algebraic space in general, but an Artin stack. So instead of working with Inaba's algebraic space, we use Lieblich's algebraic stack of objects  $E \in D^b(X)$ , satisfying the condition,

$$(40) \quad \text{Ext}_X^i(E, E) = 0, \quad \text{for all } i < 0,$$

which we denote by  $\mathfrak{M}$ . More precisely, the stack  $\mathfrak{M}$  is defined by the 2-functor,

$$\mathfrak{M}: (\text{Sch}/\mathbb{C}) \longrightarrow (\text{groupoid}),$$

which takes a  $\mathbb{C}$ -scheme  $S$  to the groupoid  $\mathfrak{M}(S)$ , whose objects consist of relatively perfect object  $\mathcal{E} \in D^b(X \times S)$  such that  $\mathcal{E}_s$  satisfies (40) for any closed point  $s \in S$ . Lieblich [18] shows the following.

**Theorem 3.15.** ([18]) *The 2-functor  $\mathfrak{M}$  is an Artin stack locally of finite type.*

For  $v \in C(\mathcal{A}_{1/2}^p)$  as in (34), we consider a moduli problem of  $\mu$ -limit semistable objects of numerical type  $v' \in C_{\leq v}(\mathcal{A}_{1/2}^p)$ , where  $C_{\leq v}(\mathcal{A}_{1/2}^p)$  is given in (26). First we show the following.

**Lemma 3.16.** *For any  $v' \in C_{\leq v}(\mathcal{A}_{1/2}^p)$ , we have one of the following.*

- *There is  $\beta' \in C(X)$  and  $n' \in \mathbb{Z}$  such that  $\text{ch}(v') = (-1, 0, \beta', n')$ .*
- *We have  $\text{ch}(v') = (-1, 0, 0, 0)$ .*
- *There is  $\beta' \in C(X)$  and  $n' \in \mathbb{Z}$  such that  $\text{ch}(v') = (0, 0, \beta', n')$ .*

*Proof.* For  $v' \in C_{\leq v}(\mathcal{A}_{1/2}^p)$ , let  $v'' = v - v' \in C(\mathcal{A}_{1/2}^p)$ . Since  $\text{ch}_0(v) = -1$ , we have  $\text{ch}_0(v') = 0$  or  $\text{ch}_0(v') = -1$ . Suppose that  $\text{ch}_0(v') = 0$  and take  $E \in \mathcal{A}_{1/2}^p$  with  $[E] = v'$ . Then  $\mathcal{H}^{-1}(E)$  must be torsion, and hence  $\mathcal{H}^{-1}(E) = 0$  since  $E \in \mathcal{A}_{1/2}^p$ . Therefore  $E$  is a non-zero one dimensional sheaf, thus  $\text{ch}(v') = (0, 0, \beta', n')$  for some  $\beta' \in C(X)$  and  $n' \in \mathbb{Z}$ .

In the latter case, we have  $\text{ch}_0(v'') = 0$  thus  $\text{ch}_0(v'') = (0, 0, \beta'', n'')$  for some  $\beta'' \in C(X)$  and  $n'' \in \mathbb{Z}$ . Therefore  $\text{ch}_0(v') = (-1, 0, \beta', n')$  for some  $\beta' \in \overline{C}(X)$  and  $n' \in \mathbb{Z}$ . If  $\beta' = 0$ , then  $\mathcal{H}^{-1}(E)$  is a line bundle and  $\mathcal{H}^0(E)$  is a zero dimensional sheaf, by [24, Lemma 3.2]. Thus  $E$  is isomorphic to a direct sum of  $\mathcal{H}^{-1}(E)[1]$  and  $\mathcal{H}^0(E)$ , which contradicts to  $E \in \mathcal{A}_{1/2}^p$  unless  $n' = \dim \mathcal{H}^0(E) = 0$ . Q.E.D.

For  $\sigma = B + i\omega \in A(X)_{\mathbb{C}}$  and  $v' \in C_{\leq v}(\mathcal{A}_{1/2}^p)$ , let us consider the following (abstract) stacks,

$$\mathfrak{M}^{v'}(Z_{\mu_\sigma}) \subset \mathfrak{D}\text{bj}(\mathcal{A}^p) \subset \mathfrak{M},$$

where  $\mathfrak{D}\text{bj}(\mathcal{A}^p)$  is the stack of objects  $E \in \mathcal{A}^p$ , and  $\mathfrak{M}^{v'}(Z_{\mu_\sigma})$  is the stack of  $\mu_\sigma$ -limit semistable objects  $E \in \mathcal{A}_{1/2}^p$  of numerical type  $v'$ . We have the following.

**Proposition 3.17.** *The substacks  $\mathfrak{D}\text{bj}(\mathcal{A}^p)$  and  $\mathfrak{M}^{v'}(Z_{\mu_\sigma})$  are open substacks of  $\mathfrak{M}$ , and hence they are Artin stacks locally of finite type. Moreover  $\mathfrak{M}^{v'}(Z_{\mu_\sigma})$  is of finite type.*

*Proof.* The openness of  $\mathfrak{D}\text{bj}(\mathcal{A}^p) \subset \mathfrak{M}$  follows from [24, Lemma 3.14]. Let us take  $v' \in C_{\leq v}(\mathcal{A}_{1/2}^p)$ . Suppose first that  $\text{ch}(v') = (0, 0, \beta', n')$  for  $\beta' \in C(X)$  and  $n' \in \mathbb{Z}$ . Then any  $\mu_\sigma$ -limit semistable object of numerical type  $v'$  is a  $(B, \omega)$ -twisted semistable sheaf. (cf. Remark 3.11.) Then

it is well-known that  $\mathfrak{M}^{v'}(Z_{\mu_\sigma})$  is open in  $\mathfrak{M}$ , and it is of finite type. (cf. [25, Proposition 3.9].)

Next suppose that  $\text{ch}(v') = (-1, 0, \beta', n')$ . In this case the claim for  $\mathfrak{M}^{v'}(Z_{\mu_\sigma})$  follows from the straightforward adaptation of the argument of [24, Section 3]. In fact using Lemma 3.13, we can show the boundedness of  $\mu_\sigma$ -limit semistable objects of numerical type  $v'$ , and destabilizing objects in a family of objects in  $\mathcal{A}_{1/2}^P$ , along with the same arguments of [24, Proposition 3.13, Lemma 3.15]. Then the same proof as in [24, Theorem 3.20] works to show the openness of  $\mathfrak{M}^{v'}(Z_{\mu_\sigma}) \subset \mathfrak{M}$ . The boundedness of relevant  $\mu_\sigma$ -limit semistable objects implies that  $\mathfrak{M}^{v'}(Z_{\mu_\sigma})$  is of finite type. Q.E.D.

### 3.5. Stable pairs and $\mu$ -limit semistable objects

The notion of stable pairs and their counting invariants are introduced by Pandharipande and Thomas [20] to interpret the reduced Donaldson–Thomas theory geometrically. In [24, Section 4], the relationship between PT-invariants and counting invariants of limit stable objects are discussed. In this subsection we state the similar result for  $\mu$ -limit semistable objects. Since the proofs are straightforward adaptation of the arguments in [24, Section 4], we again leave the readers to check the detail. First let us recall the definition of stable pairs.

**Definition 3.18.** A *stable pair* on a Calabi–Yau 3-fold  $X$  is data  $(F, s)$ , where  $F$  is a pure one dimensional sheaf on  $X$ , and  $s$  is a morphism

$$s: \mathcal{O}_X \longrightarrow F,$$

whose cokernel is a zero dimensional sheaf.

To simplify the notation, we also include the pair  $(F = 0, s = 0)$  in the definition of stable pairs. For a stable pair  $(F, s)$ , we have the associated two term complex,

$$(41) \quad I^\bullet = (\mathcal{O}_X \xrightarrow{s} F) \in D^b(X),$$

where  $F$  is located in degree zero. Note that the object  $I^\bullet$  satisfies

$$I^\bullet \in \mathcal{A}_{1/2}^P, \quad \det I^\bullet = \mathcal{O}_X, \\ \text{ch}(I^\bullet) = (-1, 0, \beta, n),$$

for  $\beta = \text{ch}_2(F)$ ,  $n = \text{ch}_3(F)$ . By abuse of notation, we also call an object (41) as a stable pair. In [20], the moduli space of stable pairs is constructed as a projective variety, and denoted by  $P_n(X, \beta)$ ,

$$P_n(X, \beta) := \{(F, s) \mid (F, s) \text{ is a stable pair, } (\text{ch}_2(F), \text{ch}_3(F)) = (\beta, n)\}.$$

Let  $\mathfrak{Obj}_0(\mathcal{A}^p)$  be the closed fiber at the point  $[\mathcal{O}_X] \in \text{Pic}(X)$  of the following morphism,

$$\det: \mathfrak{Obj}(\mathcal{A}^p) \ni E \longmapsto \det E \in \text{Pic}(X).$$

**Definition 3.19.** For  $\beta \in \overline{C}(X)$ ,  $n \in \mathbb{Z}$ , and  $\sigma \in A(X)_{\mathbb{C}}$ , define  $\mathfrak{L}_n^{\mu\sigma}(X, \beta)$  to be

$$(42) \quad \mathfrak{L}_n^{\mu\sigma}(X, \beta) := \mathfrak{M}^v(Z_{\mu\sigma}) \cap \mathfrak{Obj}_0(\mathcal{A}^p),$$

where  $v \in C(\mathcal{A}_{1/2}^p)$  satisfies  $\text{ch}(E) = (-1, 0, \beta, n)$ .

Note that  $\mathfrak{L}_n^{\mu\sigma}(X, \beta)$  is the moduli stack of  $\mu_\sigma$ -limit semistable objects  $E \in \mathcal{A}^p$  with  $\det E = \mathcal{O}_X$  and  $[E] = v$ . We shall compare  $\mathfrak{L}_n^{\mu\sigma}(X, \beta)$  and  $P_n(X, \beta)$ , when  $\sigma$  is written as  $\sigma = k\omega + i\omega$  with  $k \in \mathbb{R}$ . For  $\beta \in \overline{C}(X)$ , we set

$$(43) \quad \begin{aligned} \overline{C}_{\leq\beta}(X) &:= \{\beta' \in \overline{C}(X) \mid \beta - \beta' \in \overline{C}(X)\}, \\ C_{\leq\beta}(X) &:= \overline{C}_{\leq\beta}(X) \setminus \{0\}. \end{aligned}$$

**Definition 3.20.** For  $\beta \in \overline{C}(X)$ , we define  $m(\beta)$  as follows. If  $\beta = 0$ , we set  $m(\beta) = 0$ . Otherwise  $m(\beta)$  is

$$m(\beta) := \min\{\text{ch}_3(\mathcal{O}_C) \mid C \subset X \text{ satisfies } \dim C = 1, [C] \in \overline{C}_{\leq\beta}(X)\}.$$

It is well-known that  $\overline{C}_{\leq\beta}(X)$  is a finite set and  $m(\beta) > -\infty$ , whose proofs are seen in [24, Lemma 3.9, Lemma 3.10]. Thus Definition 3.20 makes sense. For  $\beta \in C(X)$  and  $n \in \mathbb{Z}$ , we define  $\mu_{n,\beta} \in \mathbb{Q}$  to be

$$(44) \quad \mu_{n,\beta} := \max \left\{ \frac{n - m(\beta - \beta')}{\omega\beta'} : \beta' \in C_{\leq\beta}(X) \right\}.$$

The following is  $\mu$ -stability version of [24, Theorem 4.7].

**Theorem 3.21.** Let  $\sigma = k\omega + i\omega$  for  $k \in \mathbb{R}$ . We have

$$(45) \quad \mathfrak{L}_n^{\mu\sigma}(X, \beta) \cong [P_n(X, \beta)/\mathbb{G}_m], \quad \text{if } k < -\mu_{n,\beta}/2,$$

$$(46) \quad \mathfrak{L}_n^{\mu\sigma}(X, \beta) \cong [P_{-n}(X, \beta)/\mathbb{G}_m], \quad \text{if } k > \mu_{-n,\beta}/2.$$

Here  $\mathbb{G}_m$  is acting on  $P_{\pm n}(X, \beta)$  trivially.

*Proof.* The same proof as in [24, Theorem 4.7] shows that, if  $k < -\mu_{n,\beta}/2$ , then  $E \in \mathcal{A}_{1/2}^p$  is  $\mu_\sigma$ -limit semistable if and only if  $E$  is isomorphic to a stable pair (41). Note that [24, Lemma 4.6] is crucial in [24, Theorem 4.7], and in our case Proposition 3.13 and (38) are applied

instead of [24, Lemma 4.6]. Thus the  $\mathbb{C}$ -valued points of  $\mathcal{L}_n^{\mu\sigma}(X, \beta)$  and  $P_n(X, \beta)$  are identified.

The existence of a universal stable pair on  $X \times P_n(X, \beta)$  (cf. [20, Section 2]) yields a 1-morphism  $P_n(X, \beta) \rightarrow \mathcal{L}_n^{\mu\sigma}(X, \beta)$ , which descends to

$$(47) \quad [P_n(X, \beta)/\mathbb{G}_m] \longrightarrow \mathcal{L}_n^{\mu\sigma}(X, \beta).$$

Since any  $E \in P_n(X, \beta)$  is  $\tau$ -limit stable for some  $\tau$  by [24, Theorem 4.7], we have  $\text{Hom}(E, E) = \mathbb{C}$  and  $\text{Aut}(E) = \mathbb{G}_m$ . Therefore (47) is an equivalence of groupoids on  $\mathbb{C}$ -valued points. As proved in [20, Theorem 2.7], the stack  $[P_n(X, \beta)/\mathbb{G}_m]$  is considered as an open substack of  $\mathfrak{D}\text{bj}_0(\mathcal{A}^p)$ . By Proposition 3.17,  $\mathcal{L}_n^{\mu\sigma}(X, \beta)$  is also open in  $\mathfrak{D}\text{bj}_0(\mathcal{A}^p)$ , and hence (47) gives an isomorphism of Artin stacks. The isomorphism (46) is also similarly proved. Q.E.D.

#### §4. Generating functions of stable pair invariants

In this section, we combine the arguments in the previous sections to show the rationality of the generating functions of stable pair invariants. As in the previous section,  $X$  is a projective Calabi–Yau 3-fold,  $\mathcal{A}^p \subset D^b(X)$  is the heart of a perverse t-structure on  $D^b(X)$ .

##### 4.1. Counting invariants of $\mu$ -limit stable objects

In this subsection, we construct counting invariants of  $\mu$ -limit semi-stable objects. Take  $v \in C(\mathcal{A}_{1/2}^p)$  which satisfies (34) with  $\beta \in \overline{C}(X)$  and  $n \in \mathbb{Z}$ . As in subsection 2.3, there is a Ringel–Hall Lie-algebra  $\mathfrak{G}(\mathcal{A}^p) \subset \mathcal{H}(\mathcal{A}^p)$  and the elements,

$$\epsilon^{v'}(Z_{\mu\sigma}) \in \mathfrak{G}^{v'}(\mathcal{A}^p) \subset \mathcal{H}(\mathcal{A}^p),$$

for any  $v' \in C_{\leq v}(\mathcal{A}_{1/2}^p)$  and  $\sigma \in A(X)_{\mathbb{C}}$ . Here we have used Proposition 3.17 which ensures the existence of  $\mathcal{H}(\mathcal{A}^p)$  and  $\epsilon^{v'}(Z_{\mu\sigma})$ . We also use the following map on  $\mathcal{H}(\mathcal{A}^p)$  to construct the counting invariants,

$$(48) \quad \Xi: \mathfrak{G}(\mathcal{A}^p) \ni f \longmapsto f \cdot [\mathfrak{D}\text{bj}_0(\mathcal{A}^p) \hookrightarrow \mathfrak{D}\text{bj}(\mathcal{A}^p)] \in \mathfrak{G}(\mathcal{A}^p).$$

The product  $\cdot$  is given by (9). In the following, we use the notation of subsection 2.5.

**Definition 4.1.** For  $\beta \in \overline{C}(X)$  and  $n \in \mathbb{Z}$ , we define <sup>1</sup>  $P_{n,\beta}^{eu} \in \mathbb{Z}$ ,  $L_{n,\beta}^{eu}(\sigma) \in \mathbb{Q}$  and  $N_{n,\beta}^{eu}(\sigma) \in \mathbb{Q}$  to be

$$\begin{aligned}
 P_{n,\beta}^{eu} &:= e(P_n(X, \beta)), \\
 L_{n,\beta}^{eu}(\sigma) &:= \Theta_v \Xi \epsilon^v(Z_{\mu_\sigma}), \quad \text{where } \text{ch}(v) = (-1, 0, \beta, n), \\
 N_{n,\beta}^{eu}(\sigma) &:= \Theta_{v'} \epsilon^{v'}(Z_{\mu_\sigma}) = J^{v'}(Z_{\mu_\sigma}), \quad \text{where } \text{ch}(v') = (0, 0, \beta, n).
 \end{aligned}$$

Here  $e(*)$  is the topological Euler characteristic.

For simplicity we set

$$L_{n,\beta}^{eu} := L_{n,\beta}^{eu}(i\omega), \quad N_{n,\beta}^{eu} := N_{n,\beta}^{eu}(i\omega).$$

**Remark 4.2.** Suppose that  $\sigma = k\omega + i\omega$  for  $k \in \mathbb{R}$ . Then we have

$$N_{n,\beta}^{eu}(\sigma) = N_{n,\beta}^{eu},$$

by noting Remark 3.11. Also if  $k < -\mu_{n,\beta}/2$ , then Theorem 3.21 and Remark 2.22 imply

$$L_{n,\beta}^{eu}(\sigma) = P_{n,\beta}^{eu}.$$

Let us recall that for a fixed  $\beta$ , the moduli space  $P_n(X, \beta)$  is empty for a sufficiently negative  $n$ . (See [20].) So we can take  $N(\beta) \in \mathbb{Z}$  such that

$$(49) \quad P_{n,\beta}^{eu} = 0 \quad \text{for } n < N(\beta).$$

In particular the series  $P_\beta^{eu}(q)$  is a Laurent series of  $q$ .

### 4.2. Generating functions of counting invariants of $\mu$ -limit semistable objects

In this subsection, we study the generating functions of the invariants given in Definition 4.1. Below we fix an ample divisor  $\omega$  on  $X$  and only consider the case  $\sigma = k\omega + i\omega$  for  $k \in \mathbb{R}$ . For  $\sigma = k\omega + i\omega$ , we set  $\sigma^\vee = -k\omega + i\omega$ . We have the following symmetry for the invariants  $L_{n,\beta}^{eu}(\sigma)$  and  $N_{n,\beta}^{eu}$ .

**Lemma 4.3.** (i) We have the equalities,

$$(50) \quad L_{n,\beta}^{eu}(\sigma) = L_{-n,\beta}^{eu}(\sigma^\vee), \quad N_{n,\beta}^{eu} = N_{-n,\beta}^{eu}.$$

(ii) For  $d := \omega \cdot \beta$ , we have  $N_{n+d,\beta}^{eu} = N_{n,\beta}^{eu}$  for any  $n \in \mathbb{Z}$ .

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<sup>1</sup>The subscript  $*^{eu}$  means “Euler characteristic” of the moduli spaces.

*Proof.* (i) Let  $\mathbb{D}: D^b(X) \rightarrow D^b(X)^{\text{op}}$  be the dualizing functor,

$$\mathbb{D}(E) = \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X[2]).$$

The functor  $\mathbb{D}$  induces an isomorphism of rings,

$$(51) \quad \mathbb{D}: \mathcal{H}(\mathcal{A}^p) \xrightarrow{\sim} \mathcal{H}(\mathbb{D}(\mathcal{A}^p))^{\text{op}}.$$

On the other hand, the functor  $\mathbb{D}$  preserves the subcategory  $\mathcal{A}_{1/2}^p \subset D^b(X)$  by [24, Lemma 2.18]. Moreover the same argument as in [24, Lemma 2.27] shows that an object  $E \in \mathcal{A}_{1/2}^p$  is  $\mu_\sigma$ -limit semistable if and only if  $\mathbb{D}(E) \in \mathcal{A}_{1/2}^p$  is  $\mu_{\sigma^\vee}$ -limit semistable. Hence the map (51) takes  $\delta^{v'}(Z_{\mu_\sigma})$  to  $\delta^{v'^\vee}(Z_{\mu_{\sigma^\vee}})$  for any  $v' \in C_{\leq v}(\mathcal{A}_{1/2}^p)$ . Here if  $v'$  is given by  $\text{ch}(v') = (r, 0, \beta', n')$  for  $r = 0$  or  $-1$ , then  $v'^\vee$  is given by  $\text{ch}(v'^\vee) = (r, 0, \beta', -n')$ . Hence (50) follows by the definitions of  $L_{n,\beta}^{eu}(\sigma)$  and  $N_{n,\beta}^{eu}$ .

(ii) Let us take an ample line bundle  $\mathcal{L} \in \text{Pic}(X)$  with  $c_1(\mathcal{L}) = \omega$ . The equivalence  $\otimes \mathcal{L}: \mathcal{A}^p \rightarrow \mathcal{A}^p$  induces an isomorphism of algebras,

$$(52) \quad \otimes \mathcal{L}: \mathcal{H}(\mathcal{A}^p) \longrightarrow \mathcal{H}(\mathcal{A}^p).$$

On the other hand, it is easy to see that an object  $E \in \mathcal{A}_{1/2}^p$  is  $\mu_\sigma$ -limit semistable if and only if  $E \otimes \mathcal{L}$  is  $\mu_\sigma$ -limit semistable. Thus the map (52) takes  $\delta^{v'}(Z_{\mu_\sigma})$  to  $\delta^{v''}(Z_{\mu_\sigma})$ , where  $\text{ch}(v') = (0, 0, \beta, n)$  and  $\text{ch}(v'') = (0, 0, \beta, n + d)$ . Hence we can conclude  $N_{n+d,\beta}^{eu} = N_{n,\beta}^{eu}$ . Q.E.D.

Next we show the following finiteness result.

**Lemma 4.4.** *For a fixed  $\beta \in \overline{C}(X)$  and  $\sigma = k\omega + i\omega$ , the set*

$$(53) \quad \{n \in \mathbb{Z} \mid L_{n,\beta}^{eu}(\sigma) \neq 0\}$$

*is a finite set.*

*Proof.* If  $\beta = 0$ , then  $L_{n,\beta}^{eu}(\sigma) = 0$  unless  $n = 0$  by Lemma 3.16. Suppose that  $\beta \neq 0$ , and take an integer  $N(\beta)$  as in (49). Assume that  $L_{n,\beta}^{eu}(\sigma) \neq 0$  for  $n < N(\beta)$ . Then at least the moduli stack  $\mathcal{L}_n^\sigma(X, \beta)$  is non-empty, and hence we must have  $k \geq -\frac{1}{2}\mu_{n,\beta}$  by Theorem 3.21. By the definition of  $\mu_{n,\beta}$ , there is  $\beta' \in C_{\leq \beta}(X)$  such that

$$k \geq -\frac{n - m(\beta - \beta')}{2\omega\beta'}.$$

Hence we have either

$$n \geq -2k(\omega\beta') + m(\beta - \beta'), \quad \text{or} \quad n \geq N(\beta).$$

Thus the set (53) is bounded below. Since  $L_{n,\beta}^{eu}(\sigma) = L_{-n,\beta}^{eu}(\sigma^\vee)$  by Lemma 4.3, the set (53) is also bounded above. Q.E.D.

**Lemma 4.5.** For a fixed  $\beta \in \overline{C}(X)$ , the generating series

$$(54) \quad L_\beta^{eu}(q) = \sum_{n \in \mathbb{Z}} L_{n,\beta}^{eu} q^n$$

is a polynomial of  $q^{\pm 1}$ , and hence a rational function of  $q$ , invariant under  $q \leftrightarrow 1/q$ .

*Proof.* By Lemma 4.4, the series  $L_\beta^{eu}(q)$  is a polynomial of  $q^{\pm 1}$ . Since

$$L_{n,\beta}^{eu} = L_{n,\beta}^{eu}(i\omega) = L_{-n,\beta}^{eu}(i\omega^\vee) = L_{-n,\beta}^{eu}(i\omega) = L_{-n,\beta}^{eu},$$

by Lemma 4.3, the polynomial  $L_\beta^{eu}(q)$  is invariant under  $q \leftrightarrow 1/q$ . Q.E.D.

**Lemma 4.6.** The generating series

$$N_\beta^{eu}(q) = \sum_{n \geq 0} n N_{n,\beta}^{eu} q^n,$$

is the Laurent expansion of a rational function of  $q$ , invariant under  $q \leftrightarrow 1/q$ .

*Proof.* Let  $d \in \mathbb{Z}$  be as in Lemma 4.3. Applying Lemma 4.3, we have

$$\begin{aligned} N_\beta^{eu}(q) &= \sum_{j=0}^{d-1} \sum_{m \geq 0} (dm + j) N_{j,\beta}^{eu} q^{dm+j} \\ &= \frac{1}{2} \sum_{j=0}^{d-1} \sum_{m \geq 0} \{ (dm + j) N_{j,\beta}^{eu} q^{dm+j} + (dm + d - j) N_{d-j,\beta}^{eu} q^{dm+d-j} \} \\ &= \sum_{j=0}^{d-1} \frac{N_{j,\beta}^{eu}}{2} \sum_{m \geq 0} \{ (dm + j) q^{dm+j} + (dm + d - j) q^{dm+d-j} \}. \end{aligned}$$

We can calculate as

$$(55) \quad \begin{aligned} &\sum_{m \geq 0} \{ (dm + j) q^{dm+j} + (dm + d - j) q^{dm+d-j} \} \\ &= \frac{(d-j)(q^{j+d} + q^{d-j}) + j(q^j + q^{2d-j})}{(1 - q^d)^2}. \end{aligned}$$

Then the assertion follows since (55) is a rational function of  $q$ , invariant under  $q \leftrightarrow 1/q$ . Q.E.D.

For a fixed  $\beta \in \overline{C}(X)$ , we consider the following generating series,

$$P_\beta^{eu}(q) = \sum_{n \in \mathbb{Z}} P_{n,\beta}^{eu} q^n.$$

Now we state our main theorem in this paper.

**Theorem 4.7.** *We have the following equality of the generating series,*

$$(56) \quad \sum_{\beta \geq 0} P_\beta^{eu}(q)v^\beta = \left( \sum_{\beta \geq 0} L_\beta^{eu}(q)v^\beta \right) \cdot \exp \left( \sum_{\beta > 0} N_\beta^{eu}(q)v^\beta \right).$$

Here  $\beta \geq 0, \beta > 0$  mean  $\beta \in \overline{C}(X), \beta \in C(X)$  respectively. Combining Theorem 4.7 with Lemma 4.5 and Lemma 4.6, we obtain the following.

**Corollary 4.8.** *The generating series  $P_\beta^{eu}(q)$  is the Laurent expansion of a rational function of  $q$ , invariant under  $q \leftrightarrow 1/q$ .*

The proof of Theorem 4.7 will be given in subsection 4.5 below.

**4.3. Transformation of the invariants  $L_{n,\beta}^{eu}(\sigma)$**

In this subsection, we investigate the transformation formula of our invariants  $L_{n,\beta}^{eu}(\sigma)$  under change of  $\sigma = k\omega + i\omega$ . For  $\beta \in C(X)$ , we set  $S(\beta) \subset \mathbb{R}$  as

$$S(\beta) := \left\{ \frac{m}{2\omega\beta'} \mid \beta' \in C_{\leq \beta}(X), m \in \mathbb{Z} \right\} \subset \mathbb{R}.$$

Note that  $S(\beta)$  is a discrete subset in  $\mathbb{R}$  because  $C_{\leq \beta}(X)$  is a finite set. For  $k_0 \in S(\beta)$ , let  $C_\pm \subset \mathbb{R} \setminus S(\beta)$  be the connected components such that  $C_- \subset \mathbb{R}_{< k_0}$  and  $C_+ \subset \mathbb{R}_{> k_0}$ . We take  $k_\pm \in C_\pm$  and set  $\sigma_* = k_*\omega + i\omega$  for  $* = \pm, 0$ . We have the following.

**Lemma 4.9.** *Take  $v \in C(\mathcal{A}_{1/2}^p)$  as in (34) and  $0 \in T = \mathbb{R}(m)$ . The weak stability condition  $Z_{\mu_{\sigma_0}}$  dominates  $Z_{\mu_{\sigma_\pm}}$  with respect to  $(v, 0)$ .*

*Proof.* Take  $v_1, v_2 \in C_{\leq v}(\mathcal{A}_{1/2}^p)$ , and suppose that  $Z_{\mu_{\sigma_\pm}}(v_1) \preceq Z_{\mu_{\sigma_\pm}}(v_2)$ . We want to show that

$$(57) \quad Z_{\mu_{\sigma_0}}(v_1) \preceq Z_{\mu_{\sigma_0}}(v_2).$$

By Lemma 3.16, we have  $\text{ch}(v_i) = (r_i, 0, \beta_i, n_i)$  with  $r_i = 0$  or  $-1$ . If  $r_1 = r_2 = -1$ , it is easy to see that  $Z_{\mu_\sigma}(v_1) = Z_{\mu_\sigma}(v_2)$  for any  $\sigma$ . If  $r_1 = r_2 = 0$ , (57) follows easily from (39). If  $r_1 \neq r_2$ , (57) follows from Lemma 3.14. Then the assertion follows by noting Remark 3.12. Q.E.D.

**Remark 4.10.** Lemma 4.9 is not true for the limit stability. This is the reason why we use  $\mu$ -limit stability rather than the limit stability.

For simplicity, we fix  $k \in \mathbb{R}_{<0}$  and write

$$(58) \quad Z = Z_{\mu_\sigma} \text{ for } \sigma = k\omega + i\omega \text{ with } k < 0, \quad Z' = Z_{\mu_{i\omega}}.$$

Applying the results in the previous sections, we obtain the following proposition.

**Proposition 4.11.** For  $v \in C(\mathcal{A}_{1/2}^p)$  as in (34), and  $Z, Z'$  as in (58) w.r.t.  $(v, 0)$ , we have the following.

$$(59) \quad \epsilon^v(Z') = \sum_{\substack{l \geq 1, \\ v_i \in C(\mathcal{A}_{1/2}^p), \\ v_1 + \dots + v_l = v}} U(\{v_1, \dots, v_l\}, Z, Z') \epsilon^{v_1}(Z) * \dots * \epsilon^{v_l}(Z).$$

The sum (59) has only finitely many non-zero terms.

*Proof.* By Proposition 3.17 and Remark 3.12,  $Z, Z'$  satisfy Assumption 2.26 w.r.t.  $(v, 0)$ . Furthermore by Lemma 4.9, the condition  $(\spadesuit')$  is also satisfied with respect to  $(v, 0)$ , except the local finiteness condition which is satisfied if we knew that (59) has only finitely many non-zero terms. The finiteness of (59) will be shown in Proposition 4.14 (iii) below. Therefore (59) follows from Theorem 2.28. Q.E.D.

In the formula (59),  $\{v_i\}_{i=1}^l \subset C(\mathcal{A}_{1/2}^p)$  satisfy the following by Lemma 3.16.

$$(60) \quad \begin{aligned} &\text{There is a unique } 1 \leq e \leq l \text{ such that } \text{ch}(v_i) = (0, 0, \beta_i, n_i) \\ &\text{for } i \neq e \text{ with } \beta_i \in C(X), n_i \in \mathbb{Z}, \text{ and } \text{ch}(v_e) = (-1, 0, \beta_e, n_e) \\ &\text{with } \beta_e \in \overline{C}(X), n_e \in \mathbb{Z}. \end{aligned}$$

Let us see that (59) is a finite sum. For simplicity we write

$$\mu_i := \mu_{i\omega}(v_i) = \frac{n_i}{\beta_i\omega} \in \mathbb{Q},$$

if  $v_i \in C(\mathcal{A}_{1/2}^p)$  satisfies  $\text{ch}(v_i) = (0, 0, \beta_i, n_i)$ .

**Lemma 4.12.** Take  $v_1, \dots, v_l \in C(\mathcal{A}_{1/2}^P)$  such that

$$\text{ch}(v_i) = (0, 0, \beta_i, n_i),$$

for all  $i$  with  $\beta_i \in C(X)$  and  $n_i \in \mathbb{Z}$ . For  $Z, Z'$  as in (58), we have

$$S(\{v_1, \dots, v_l\}, Z, Z') = \begin{cases} 1, & l = 1, \\ 0, & l \geq 2. \end{cases}$$

*Proof.* Note that we have

$$(61) \quad Z(v_i) \preceq Z(v_j) \iff \mu_i \leq \mu_j \iff Z'(v_i) \preceq Z'(v_j).$$

Then one can check that the same proof as in [9, Theorem 4.5] is applied.

Q.E.D.

**Lemma 4.13.** Take  $v_1, \dots, v_l \in C(\mathcal{A}_{1/2}^P)$  as in (60). For  $Z, Z'$  as in (58),  $S(\{v_1, \dots, v_l\}, Z, Z')$  is non-zero only if

$$(62) \quad 0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_{e-1} \leq -2k > \mu_{e+1} > \mu_{e+2} > \dots > \mu_l \geq 0.$$

Moreover in this case, we have  $S(\{v_1, \dots, v_l\}, Z, Z') = (-1)^{e-1}$ .

*Proof.* Note that for  $v_i, v_j$  with  $i, j \neq e$ , the same implications (61) hold. Also for  $i \neq e$ , we have

$$\begin{aligned} Z(v_i) \preceq Z(v_e) &\iff \mu_i \leq -2k, \\ Z'(v_i) \preceq Z'(v_e) &\iff \mu_i \leq 0. \end{aligned}$$

Suppose that  $S(\{v_1, \dots, v_l\}, Z, Z') \neq 0$ . We say  $1 < i < l$  is of type A (resp. B) if the following holds,

$$Z(v_{i-1}) \succ Z(v_i) \preceq Z(v_{i+1}), \quad (\text{resp. } Z(v_{i-1}) \preceq Z(v_i) \succ Z(v_{i+1})).$$

If  $1 < i \leq e - 1$  is of type A, we have

$$\begin{aligned} Z'(v_1 + \dots + v_{i-1}) &\preceq Z'(v_i + \dots + v_l), \\ Z'(v_1 + \dots + v_i) &\succ Z'(v_{i+1} + \dots + v_l). \end{aligned}$$

Hence we have

$$\begin{aligned} \mu_{i\omega}(v_1 + \dots + v_{i-1}) &\leq 0, \\ \mu_{i\omega}(v_1 + \dots + v_{i-1} + v_i) &> 0, \end{aligned}$$

which implies  $\mu_i > 0$ . Similarly if  $i$  is of type B, we have  $\mu_i < 0$ .

Suppose that there is  $1 \leq i \leq e - 1$  of type A or B, and take the smallest such  $i$ . We assume  $i$  is of type A, and hence  $\mu_i > 0$ . We have

$$Z(v_1) \succ \cdots \succ Z(v_{i-1}) \succ Z(v_i) \preceq Z(v_{i+1}) \cdots,$$

thus  $\mu_1 > \cdots > \mu_i > 0$  holds. On the other hand, we have  $Z'(v_1) \preceq Z'(v_2 + \cdots + v_l)$ , thus  $\mu_1 \leq 0$ . This is a contradiction, so there is no  $1 \leq i \leq e - 1$  of type A. Similarly there is no  $1 \leq i \leq e - 1$  of type B.

By the above argument, one of (63) or (64) holds.

$$(63) \quad Z(v_1) \succ Z(v_2) \succ \cdots \succ Z(v_{e-1}) \succ Z(v_e),$$

$$(64) \quad Z(v_1) \preceq Z(v_2) \preceq \cdots \preceq Z(v_{e-1}) \preceq Z(v_e).$$

Assume by a contradiction that (63) holds. Then  $Z'(v_1) \preceq Z'(v_2 + \cdots + v_l)$ , thus (63) implies

$$(65) \quad 0 \geq \mu_1 > \mu_2 > \cdots > \mu_{e-1} > -2k.$$

The inequality (65) does not occur since we took  $k < 0$ . Therefore we must have (64).

A similar argument for  $v_{e+1}, \dots, v_l$  shows

$$(66) \quad Z(v_1) \preceq \cdots \preceq Z(v_e) \succ Z(v_{e+1}) \succ \cdots \succ Z(v_l).$$

Obviously (66) together with  $S(\{v_1, \dots, v_l\}, Z, Z') \neq 0$  imply (62).

Q.E.D.

**Proposition 4.14.** (i) Take  $v_1, \dots, v_l \in C(\mathcal{A}_{1/2}^p)$  as in (60), and let  $Z, Z'$  be as in (58). Then  $U(\{v_1, \dots, v_l\}, Z, Z')$  is non-zero only if

$$(67) \quad 0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{e-1} \leq -2k \geq \mu_{e+1} \geq \cdots \geq \mu_l \geq 0.$$

(ii) In the same situation of (i), suppose that the following holds,

$$U(\{v_1, \dots, v_l\}, Z, Z') \prod_{i \neq e} n_i \neq 0.$$

Then we have

$$(68) \quad U(\{v_1, \dots, v_l\}, Z, Z') = \sum_{1 \leq m \leq l} \sum_{\substack{\text{surjective } \psi: \{1, \dots, l\} \rightarrow \{1, \dots, m\}, \\ i \leq j \text{ implies } \psi(i) \leq \psi(j), \text{ and satisfies (69)}}} (-1)^{\psi(e)-1} \prod_{b=1}^m \frac{1}{|\psi^{-1}(b)|!}.$$

Here  $\psi$  satisfies the following.

(69) For  $i, j < e$  with  $\psi(i) = \psi(j)$ , we have  $\mu_i = \mu_j$ ,  
 if  $\psi(i) = \psi(e)$  then  $\mu_i = -2k$ , and for  $e < i, j$ ,  
 we have  $\psi(i) = \psi(j)$  if and only if  $\mu_i = \mu_j$ .

(iii) The formula (59) is a finite sum.

*Proof.* (i) Suppose that  $U(\{v_1, \dots, v_l\}, Z, Z') \neq 0$  and take

(70)  $\psi: \{1, \dots, l\} \rightarrow \{1, \dots, m\}, \quad \xi: \{1, \dots, m\} \rightarrow \{1, \dots, m'\},$

as in (21). Let us set  $c = \xi\psi(e) \in \{1, \dots, m'\}$  and take  $a \in \{1, \dots, m'\}$  with  $a \neq c$ . By Lemma 4.12, the set  $\xi^{-1}(a)$  consists of one element, say  $b \in \{1, \dots, m\}$ . Then by the definition of  $U(\{v_1, \dots, v_l\}, Z, Z')$ , we have  $\mu_i = \mu_j$  for  $i, j \in \psi^{-1}(b)$  and

(71) 
$$Z'(\sum_{i \in \psi^{-1}(b)} v_i) = Z'(\sum_{i \in \psi^{-1}\xi^{-1}(c)} v_i).$$

The condition (71) implies  $\mu_{i\omega}(\sum_{i \in \psi^{-1}(b)} v_i) = 0$ , and hence  $\mu_i = 0$  for any  $i \in \psi^{-1}(b)$ , i.e.  $\mu_i = 0$  for any  $i \notin \psi^{-1}\xi^{-1}(c)$ . By Lemma 4.13, we must have (67).

(ii) Suppose that

(72) 
$$U(\{v_1, \dots, v_l\}, Z, Z') \prod_{i \neq e} n_i \neq 0,$$

and take  $\psi: \{1, \dots, l\} \rightarrow \{1, \dots, m\}, \xi: \{1, \dots, m\} \rightarrow \{1, \dots, m'\}$  as in (i). Then the proof of (i) shows that (72) is non-zero only if  $m' = 1$ . Then (68) follows from the definition of  $U(\{v_1, \dots, v_l\}, Z, Z')$  and Lemma 4.13.

(iii) Since  $v_1, \dots, v_l$  satisfy (60), the number  $l$  is bounded, and there is only finite number of possibilities for  $\beta_i = \text{ch}_2(v_i)$ . Hence we may fix  $l$  and  $\beta_1, \dots, \beta_l$ . Then the values  $n_i = \text{ch}_3(v_i)$  have only finite number of possibilities by (67). Q.E.D.

Now we have the wall-crossing formula of the invariants  $L_{n,\beta}^{eu}(\sigma)$ .

**Proposition 4.15.** For  $\sigma = k\omega + i\omega$  with  $k < 0$ ,  $\beta \in \overline{C}(X)$  and  $n \in \mathbb{Z}$ , we have the following formula,

$$L_{n,\beta}^{eu} = \sum_{\substack{l \geq 1, 1 \leq e \leq l, \beta_i \in C(X) \text{ for } i \neq e, \beta_e \in \overline{C}(X), n_i \in \mathbb{Z}, \\ \beta_1 + \dots + \beta_l = \beta, n_1 + \dots + n_l = n, \mu_i = n_i / \beta_i \omega \text{ satisfy} \\ 0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_{e-1} \leq -2k \geq \mu_{e+1} \geq \dots \geq \mu_l > 0}} \sum_{\substack{1 \leq m \leq l, \text{ surjective} \\ \psi: \{1, \dots, l\} \rightarrow \{1, \dots, m\}, \\ i \leq j \text{ implies } \psi(i) \leq \psi(j), \\ \text{and satisfies (69)}}$$

$$(73) \quad \left(-\frac{1}{2}\right)^{l-1} \prod_{b=1}^m \frac{(-1)^{\psi(e)-e}}{|\psi^{-1}(b)|!} \prod_{i \neq e} n_i N_{n_i, \beta_i}^{eu} L_{n_e, \beta_e}^{eu}(\sigma).$$

*Proof.* Let  $v \in C(\mathcal{A}_{1/2}^p)$  be as in (34), and  $Z, Z'$  be as in (58). Applying  $\Xi$  given in (48) to (59), we obtain

$$\Xi \epsilon^v(Z') = \sum_{l \geq 1, 1 \leq e \leq l} \sum_{\substack{v_i \in C(\mathcal{A}_{1/2}^p), v_1 + \dots + v_l = v, \\ \text{cho}(v_i) = 0 \text{ for } i \neq e, \text{cho}(v_e) = -1}} U(\{v_1, \dots, v_l\}, Z, Z')$$

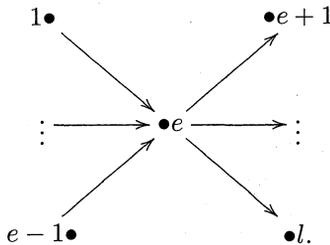
$$(74) \quad \epsilon^{v_1}(Z) * \dots * \Xi \epsilon^{v_e}(Z) * \dots * \epsilon^{v_l}(Z).$$

Note that for  $i \neq e$ , the element  $\epsilon^{v_i}(Z) \in \mathcal{H}(\mathcal{A}^p)$  is supported on  $\mathcal{O}bj_0(\mathcal{A}^p) \subset \mathcal{O}bj(\mathcal{A}^p)$ , and hence  $\Xi(\epsilon^{v_i}(Z) * \epsilon) = \epsilon^{v_i}(Z) * \Xi(\epsilon)$  follows for any  $\epsilon \in \mathcal{H}(\mathcal{A}^p)$ . Thus (74) follows from (59). Also we note that (74) is a finite sum by Proposition 4.14 (iii). Hence applying  $\Theta$  given in (24) and using the same argument as in Theorem 2.24, we obtain

$$L_{n,\beta}^{eu} = \sum_{\substack{l \geq 1, v_i \in C(\mathcal{A}_{1/2}^p), \Gamma \text{ is a connected, simply connected oriented} \\ v_1 + \dots + v_l = v \text{ graph with vertex } \{1, \dots, l\}, \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \text{ implies } i < j}}$$

$$(75) \quad U(\{v_1, \dots, v_l\}, Z, Z') \frac{1}{2^{l-1}} \prod_{\substack{i \rightarrow j \\ \text{in } \Gamma}} \chi(v_i, v_j) \prod_{i \neq e} N_{n_i, \beta_i}^{eu} L_{n_e, \beta_e}^{eu}(\sigma),$$

where  $v_1, \dots, v_l$  satisfy (60) and we have used the notation of (60). By Riemann–Roch theorem, we have  $\chi(v_i, v_j) = 0$  for  $i, j \neq e$ ,  $\chi(v_i, v_e) = n_i$  and  $\chi(v_e, v_i) = -n_i$ . Hence a term in the sum (75) is non-zero only if the oriented graph  $\Gamma$  is of the following form,



Hence applying Proposition 4.14 (ii), we obtain the formula (73).

Q.E.D.

**4.4. Relationship between  $L_{n,\beta}^{eu}$  and  $P_{n,\beta}^{eu}$**

We next establish a relationship between  $L_{n,\beta}^{eu}$  and  $P_{n,\beta}^{eu}$ . Let us take  $N(\beta) \in \mathbb{Z}$  as in (49). We choose  $k < 0$  so that

$$(76) \quad k < -\frac{1}{2}(n - N(\beta')), \quad k < -\frac{1}{2}\mu_{n,\beta'} \quad \text{for any } \beta' \in C_{\leq \beta}(X).$$

In this particular choice of  $k$ , we have the following formula.

**Proposition 4.16.** *If  $k$  satisfies (76), then (73) implies the following.*

$$(77) \quad L_{n,\beta}^{eu} = \sum_{\substack{l \geq 1, 1 \leq e \leq l, 0 \leq t \leq e-1, 0 \leq s \leq l-e, \\ 0 = m_0 < m_1 < \dots < m_t = e-1, \\ e = m'_0 < m'_1 < \dots < m'_s = l}} \sum_{\substack{\beta_i \in C(X) \text{ for } i \neq e, \beta_e \in \overline{C}(X), n_i \in \mathbb{Z}, \\ \beta_1 + \dots + \beta_l = \beta, n_1 + \dots + n_l = n, \mu_i = n_i / \omega \beta_i \\ \text{satisfy } 0 < \mu_1 = \dots = \mu_{m_1} < \mu_{m_1+1} = \dots, \\ 0 < \mu_1 = \dots = \mu_{m'_s-1+1} < \mu_{m'_s-1} = \dots}} \left(-\frac{1}{2}\right)^{l-1} \prod_{i=1}^t \frac{1}{(m_i - m_{i-1})!} \prod_{i=1}^s \frac{1}{(m'_i - m'_{i-1})!} \prod_{i \neq e} n_i N_{n_i, \beta_i}^{eu} P_{n_e, \beta_e}^{eu}.$$

*Proof.* First note that all the  $n_i$  in the formula (73) are positive except  $i = e$ . Thus we have  $n_e \leq n$ , and hence

$$k < -\mu_{n,\beta_e} / 2 \leq -\mu_{n_e, \beta_e} / 2.$$

Therefore in the formula (73), we have

$$L_{n_e, \beta_e}^{eu}(\sigma) = P_{n_e, \beta_e}^{eu},$$

by Remark 4.2. Thus we may assume  $n_e \geq N(\beta_e)$  in the formula (73). Then the condition  $\mu_i \leq -2k$  in (73) is automatically satisfied, since

$$0 < \mu_i \leq n_i \leq n - n_e \leq n - N(\beta_e) < -2k,$$

by our choice of  $k$ . Hence we can eliminate the condition  $\mu_i \leq -2k$  in (73), and obtain the formula,

$$(78) \quad L_{n,\beta}^{eu} = \sum_{\substack{l \geq 1, 1 \leq e \leq l, \beta_i \in C(X) \text{ for } i \neq e, \beta_e \in \overline{C}(X), n_i \in \mathbb{Z}, \\ \beta_1 + \dots + \beta_l = \beta, n_1 + \dots + n_l = n, \mu_i = n_i / \beta_i \omega \text{ satisfy } \\ 0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_{e-1}, 0 < \mu_1 \leq \mu_{l-1} \leq \dots \leq \mu_{e+1}}} \sum_{\substack{1 \leq m \leq l, \text{ surjective} \\ \psi: \{1, \dots, l\} \rightarrow \{1, \dots, m\}, \\ i \leq j \text{ implies } \psi(i) \leq \psi(j), \\ \text{and satisfies (69)}}} \left(-\frac{1}{2}\right)^{l-1} \prod_{b=1}^m \frac{(-1)^{\psi(e)-e}}{|\psi^{-1}(b)|!} \prod_{i \neq e} n_i N_{n_i, \beta_i}^{eu} P_{n_e, \beta_e}^{eu}.$$

We rearrange the sum (78) by first choosing partitions  $0 = m_0 < m_1 < \dots < m_l = e - 1$ ,  $e = m'_0 < m'_1 < \dots < m'_s = l$  and then choosing  $\beta_i, n_i$  so that  $0 < \mu_1 = \dots = \mu_{m_1} < \mu_{m_1+1} = \dots$  and  $0 < \mu_l = \dots = \mu_{m'_{s-1}+1} < \mu_{m'_{s-1}} = \dots$  are satisfied. Noting that  $\psi$  satisfies (69), we obtain

$$\begin{aligned}
 L_{n,\beta}^{eu} = & \sum_{\substack{l \geq 1, 1 \leq e \leq l, \\ 0 \leq t \leq e-1, 0 \leq s \leq l-e, \\ 0 = m_0 < m_1 < \dots < m_t = e-1, \\ e = m'_0 < m'_1 < \dots < m'_s = l}} \sum_{\substack{\beta_i \in C(X) \text{ for } i \neq e, \beta_e \in \overline{C}(X), n_i \in \mathbb{Z}, \\ \beta_1 + \dots + \beta_l = \beta, n_1 + \dots + n_l = n, \mu_i = n_i / \omega \beta_i \\ \text{satisfy } 0 < \mu_1 = \dots = \mu_{m_1} < \mu_{m_1+1} = \dots, \\ 0 < \mu_l = \dots = \mu_{m'_{s-1}+1} < \mu_{m'_{s-1}} = \dots}} \\
 (79) \quad & \left( -\frac{1}{2} \right)^{l-1} \prod_{i=1}^t \left( \sum_{\substack{l_i = m_i - m_{i-1}, 1 \leq m \leq l_i, \\ \text{surjective } \psi: \{1, \dots, l_i\} \rightarrow \{1, \dots, m\}, \\ i \leq j \text{ implies } \psi(i) \leq \psi(j)}} \prod_{b=1}^m \frac{(-1)^{l_i - m}}{|\psi^{-1}(b)|!} \right) \\
 & \prod_{i=1}^s \frac{1}{(m'_i - m'_{i-1})!} \prod_{i \neq e} n_i N_{n_i, \beta_i}^{eu} P_{n_e, \beta_e}^{eu}.
 \end{aligned}$$

Then (77) follows from Lemma 4.17 below. Q.E.D.

**Lemma 4.17.** *For a fixed  $l$ , we have*

$$(80) \quad \sum_{\substack{1 \leq m \leq l, \text{ surjective } \psi: \{1, \dots, l\} \rightarrow \{1, \dots, m\}, \\ i \leq j \text{ implies } \psi(i) \leq \psi(j)}} \prod_{b=1}^m \frac{(-1)^{l-m}}{|\psi^{-1}(b)|!} = \frac{1}{l!}.$$

*Proof.* The proof is elementary, and this is a special case of [14, Proposition 4.9]. Q.E.D.

**4.5. Proof of Theorem 4.7**

We finally give a proof of Theorem 4.7.

*Proof.* For a fixed data  $l \geq 1, 1 \leq e \leq l, \beta_i \in C(X) (i \neq e)$  and  $\beta_e \in \overline{C}(X)$ , we set

$$\begin{aligned}
 F_e(\beta_1, \dots, \beta_l) = & \sum_{n \in \mathbb{Z}} \sum_{\substack{0 \leq t \leq e-1, 0 \leq s \leq l-e, \\ 0 = m_0 < m_1 < \dots < m_t = e-1, \\ e = m'_0 < m'_1 < \dots < m'_s = l.}} \sum_{\substack{n_i \in \mathbb{Z}, n_1 + \dots + n_l = n, \\ \mu_i = n_i / \omega \beta_i \text{ satisfy} \\ 0 < \mu_1 = \dots = \mu_{m_1} < \mu_{m_1+1} = \dots, \\ 0 < \mu_l = \dots = \mu_{m'_{s-1}+1} < \mu_{m'_{s-1}} = \dots}} \\
 (81) \quad & \prod_{i=1}^t \frac{1}{(m_i - m_{i-1})!} \prod_{i=1}^s \frac{1}{(m'_i - m'_{i-1})!} \prod_{i \neq e} n_i N_{n_i, \beta_i}^{eu} P_{n_e, \beta_e}^{eu} q^n.
 \end{aligned}$$

Let us fix  $\beta \in \overline{C}(X)$ . By the formula (77), we have

$$\begin{aligned}
 & \sum_{n \in \mathbb{Z}} L_{n, \beta}^{eu} q^n \\
 &= \sum_{l \geq 1, 1 \leq e \leq l} \left(-\frac{1}{2}\right)^{l-1} \sum_{\substack{\beta_i \in C(X) \text{ for } i \neq e, \\ \beta_e \in \overline{C}(X), \\ \beta_1 + \dots + \beta_l = \beta}} F_e(\beta_1, \dots, \beta_e, \dots, \beta_l) \\
 &= \sum_{l \geq 1, 1 \leq e \leq l} \sum_{\substack{\kappa_1: I_1 \rightarrow C(X), \kappa_2: I_2 \rightarrow C(X), \\ \beta_e \in \overline{C}(X), \\ \sum_{i \in I_1} \kappa_1(i) + \sum_{i \in I_2} \kappa_2(i) + \beta_e = \beta}} \sum_{\substack{\lambda_1: \{1, \dots, e-1\} \xrightarrow{\sim} I_1, \\ \lambda_2: \{e+1, \dots, l\} \xrightarrow{\sim} I_2}} \\
 & \quad \frac{1}{(e-1)!(l-e)!} \left(-\frac{1}{2}\right)^{l-1} \\
 (82) \quad & F_e(\kappa_1 \lambda_1(1), \dots, \kappa_1 \lambda_1(e-1), \beta_e, \kappa_2 \lambda_2(e+1), \dots, \kappa_2 \lambda_2(l)).
 \end{aligned}$$

Here  $I_1$  and  $I_2$  are finite sets with  $|I_1| = e - 1$ ,  $|I_2| = l - e$ . Let us fix data  $l \geq 1$ ,  $1 \leq e \leq l$ ,  $\kappa_1: I_1 \rightarrow C(X)$ ,  $\kappa_2: I_2 \rightarrow C(X)$  and  $\beta_e \in \overline{C}(X)$ , and consider the last sum of (82). If we also fix bijections  $\lambda'_1: \{1, \dots, e - 1\} \rightarrow I_1$  and  $\lambda'_2: \{e + 1, \dots, l\} \rightarrow I_2$ , then the choices of  $\lambda_1, \lambda_2$  in (82) correspond to the elements of the symmetric groups  $\gamma \in \mathfrak{S}_{e-1}, \gamma' \in \mathfrak{S}_{l-e}$  respectively. Here an element  $\gamma' \in \mathfrak{S}_{l-e}$  is regarded as a permutation on  $\{e + 1, \dots, l\}$ . Let us rewrite  $\beta_i = \kappa_1 \lambda'_1(i)$  for  $1 \leq i \leq e - 1$  and  $\beta_i = \kappa_2 \lambda'_2(i)$  for  $e + 1 \leq i \leq l$ . Then we have

$$\begin{aligned}
 & \sum_{\substack{\lambda_1: \{1, \dots, e-1\} \xrightarrow{\sim} I_1, \\ \lambda_2: \{e+1, \dots, l\} \xrightarrow{\sim} I_2}} F_e(\kappa_1 \lambda_1(1), \dots, \beta_e, \dots, \kappa_2 \lambda_2(l)) \\
 &= \sum_{\substack{\gamma \in \mathfrak{S}_{e-1} \\ \gamma' \in \mathfrak{S}_{l-e}}} F_e(\beta_{\gamma(1)}, \dots, \beta_{\gamma(e-1)}, \beta_e, \beta_{\gamma'(e+1)}, \dots, \beta_{\gamma'(l)}) \\
 (83) \quad &= \sum_{\substack{0 \leq t \leq e-1, 0 \leq s \leq l-e, \\ 0 = m_0 < m_1 < \dots < m_t = e-1 \\ e = m'_0 < m'_1 < \dots < m'_s = l, \\ \gamma \in \mathfrak{S}_{e-1} \ \gamma' \in \mathfrak{S}_{l-e}}} \prod_{i=1}^t \frac{1}{(m_i - m_{i-1})!} \prod_{i=1}^s \frac{1}{(m'_i - m'_{i-1})!} G_{\gamma, \gamma'}
 \end{aligned}$$

where  $G_{\gamma, \gamma'}$  is given by

$$G_{\gamma, \gamma'} = \sum_{n \in \mathbb{Z}} \sum_{\substack{n_i \in \mathbb{Z}, n_1 + \dots + n_l = n, \mu_i = n_i / \omega \beta_i \text{ satisfy} \\ 0 < \mu_{\gamma(1)} = \dots = \mu_{\gamma(m_1)} < \mu_{\gamma(m_1+1)} = \dots, \\ 0 < \mu_{\gamma'(1)} = \dots = \mu_{\gamma'(m'_s-1+1)} < \mu_{\gamma'(m'_s-1)} = \dots}} \prod_{i \neq e} n_i N_{n_i, \beta_i}^{eu} P_{n_e, \beta_e}^{eu} q^n.$$

Note that for  $\gamma^\# \in \prod_{i=1}^t \mathfrak{S}_{m_i - m_{i-1}} \subset \mathfrak{S}_{e-1}$  and  $\gamma^\flat \in \prod_{i=1}^s \mathfrak{S}_{m'_i - m'_{i-1}} \subset \mathfrak{S}_{l-e}$ , we have

$$G_{\gamma, \gamma'} = G_{\gamma^\# \gamma^\flat, \gamma' \gamma^\flat}.$$

Since we have

$$\prod_{i=1}^t |\mathfrak{S}_{m_i - m_{i-1}}| = \prod_{i=1}^t (m_i - m_{i-1})!, \quad \prod_{i=1}^s |\mathfrak{S}_{m'_i - m'_{i-1}}| = \prod_{i=1}^s (m'_i - m'_{i-1})!,$$

(83) is written as

$$(84) \quad (83) = \sum_{\substack{0 \leq t \leq e-1, 0 \leq s \leq l-e, \\ 0 = m_0 < m_1 < \dots < m_t = e-1, \\ e = m'_0 < m'_1 < \dots < m'_s = l}} \sum_{\substack{\gamma \in \mathfrak{S}_{e-1}, \gamma(i) < \gamma(i') \text{ if} \\ i, i' \in [m_j + 1, m_{j+1}] \text{ for some } j \\ \gamma' \in \mathfrak{S}_{l-e}, \gamma'(i) < \gamma'(i') \text{ if} \\ i, i' \in [m'_j + 1, m'_{j+1}] \text{ for some } j}} G_{\gamma, \gamma'}.$$

On the other hand, if we are given  $n_i \in \mathbb{Z}_{>0}$  for  $i \neq e$  and  $n_e \in \mathbb{Z}$  with  $n_1 + \dots + n_l = n$ , there are unique  $\gamma \in \mathfrak{S}_{e-1}$ ,  $\gamma' \in \mathfrak{S}_{l-e}$  and partitions  $0 = m_0 < m_1 < \dots < m_t = e - 1$ ,  $e = m'_0 < m'_1 < \dots < m'_s = l$  such that  $\gamma(i) < \gamma(i')$  for  $i, i' \in [m_j + 1, m_{j+1}]$ ,  $\gamma'(i) < \gamma'(i')$  for  $i, i' \in [m'_j + 1, m'_{j+1}]$ , and  $\mu_i = n_i / \omega \beta_i$  satisfy

$$\begin{aligned} 0 < \mu_{\gamma(1)} = \dots = \mu_{\gamma(m_1)} < \mu_{\gamma(m_1+1)} = \dots, \\ 0 < \mu_{\gamma'(1)} = \dots = \mu_{\gamma'(m'_s-1+1)} < \mu_{\gamma'(m'_s-1)} = \dots. \end{aligned}$$

Therefore (84) is written as

$$\begin{aligned} (84) &= \sum_{n \in \mathbb{Z}} \sum_{\substack{n_1 + \dots + n_l = n, i \neq e \\ n_i \in \mathbb{Z}_{>0} \text{ for } i \neq e}} \prod_{i \neq e} n_i N_{n_i, \beta_i}^{eu} P_{n_e, \beta_e}^{eu} q^n \\ (85) &= \prod_{i \neq e} N_{\beta_i}^{eu}(q) \cdot P_{\beta_e}^{eu}(q). \end{aligned}$$

Noting

$$\sum_{1 \leq e \leq l} \frac{1}{(e-1)!(l-e)!2^{l-1}} = \frac{1}{(l-1)!},$$

we obtain

$$(86) \quad (82) = \sum_{l \geq 1} \sum_{\substack{\beta_i \in C(X) \text{ for } i \neq l, \\ \beta_1 + \dots + \beta_l = \beta}} \frac{(-1)^{l-1}}{(l-1)!} \prod_{i \neq l} N_{\beta_i}^{eu}(q) \cdot P_{\beta_l}^{eu}(q).$$

The formula (86) implies (56) as desired. Q.E.D.

**4.6. Problem of incorporating virtual classes to Joyce’s work**

Since invariants defined in Definition 4.1 are interpreted as Euler characteristics of moduli stacks, they are unlikely to be unchanged under deformations of  $X$ . In order to construct invariants which are unchanged under deformations, we need to construct virtual moduli cycles on the moduli spaces and integrate them. The resulting invariants are Euler characteristics of the moduli spaces (up to sign) if the moduli spaces are non-singular, but in general they differ from Euler characteristics. Thus in order to solve Conjecture 1.1, we have to construct invariants involving virtual classes and establish the formulas like (25). At this moment we are unable to overcome this problem. However if we could involve virtual classes with Joyce’s theory, then Conjecture 1.1 for PT-theory follows along with the same argument in this paper. To state this, let us recall that the integrations of virtual classes are also realized as weighted sums of certain constructible functions introduced by Behrend [2]. He shows that, for any scheme  $M$ , there is a canonical constructible function  $\chi_M: M \rightarrow \mathbb{Z}$  such that  $\chi_M = (-1)^{\dim M}$  if  $M$  is non-singular, and if  $M$  carries a symmetric perfect obstruction theory, we have

$$\sharp^{\text{vir}} M = \sum_{n \in \mathbb{Z}} ne(\chi_M^{-1}(n)).$$

Under the situation in this section, we shall address the following question. <sup>2</sup>

**Problem 4.18.** *Does there exist a map*

$$\Theta': \mathfrak{G}(\mathcal{A}^p) \longrightarrow \mathfrak{g}(\mathcal{A}^p),$$

*such that the following conditions hold?*

- *For  $v \in C(\mathcal{A}^p)$ , suppose that  $\mathfrak{M}^v(Z_{\mu_\sigma})$  is written as  $[M/\mathbb{G}_m]$  for a scheme  $M$ . Then*

$$\Theta'(\epsilon^v(Z_{\mu_\sigma})) = \sum_{n \in \mathbb{Z}} -n\Theta([[\chi_M^{-1}(n)/\mathbb{G}_m] \hookrightarrow \mathfrak{D}\text{bj}(\mathcal{A}^p)]).$$

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<sup>2</sup>The formulation of Problem 4.18 is taught to the author by D. Joyce.

- For  $v_1, v_2 \in C(\mathcal{A}^p)$ , we have

$$(87) \quad [\Theta'(\epsilon^{v_1}(Z)), \Theta'(\epsilon^{v_2}(Z))] = (-1)^{\chi(v_1, v_2)} \Theta'[\epsilon^{v_1}(Z), \epsilon^{v_2}(Z)].$$

There should be sign change in (87), because  $\chi_M = (-1)^{\dim M}$  on a smooth variety  $M$ . We are unable to solve Problem 4.18 at this moment, but the techniques given in this paper yield the following.

**Theorem 4.19.** *Suppose that Problem 4.18 is true. Then Conjecture 1.1 is true for PT-theory.*

*Proof.* It is enough to work over the invariants, defined by  $\Theta'$ . As a modification of Definition 4.1, let us define  $L_{n,\beta}(\sigma)$ ,  $N_{n,\beta}(\sigma)$  to be

$$\begin{aligned} L_{n,\beta}(\sigma) &:= \Theta' \Xi \epsilon^v(Z_{\mu_\sigma}), & \text{where } \text{ch}(v) &= (-1, 0, \beta, n), \\ N_{n,\beta}(\sigma) &:= \Theta' \epsilon^v(Z_{\mu_\sigma}), & \text{where } \text{ch}(v) &= (0, 0, \beta, n). \end{aligned}$$

Then (87) yields a similar wall-crossing formula for  $L_{n,\beta}(\sigma)$ , and

$$L_{n,\beta}(\sigma) = (-1)^{\dim \text{Pic}(X)-1} P_{n,\beta},$$

for  $\sigma = k\omega + i\omega$  with  $k < -\mu_{n,\beta}/2$ . Therefore the same proof as in Theorem 4.7 works, and we have the similar expansion of the generating series  $Z_{\text{PT}}$  as in (56). Then Conjecture 1.1 for  $P_\beta(q)$  follows as a corollary. Q.E.D.

At this moment, the author does not know how to solve Problem 4.18. It seems that the recent work of Kontsevich and Soibelman [16] gives a crucial idea of this problem.

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