Advanced Studies in Pure Mathematics 58, 2010
Algebraic and Arithmetic Structures of Moduli Spaces (Sapporo 2007)
pp. 403–459

An action of a Lie algebra on the homology groups of moduli spaces of stable sheaves

Kōta Yoshioka

Abstract.

We construct an action of a Lie algebra on the homology groups of moduli spaces of stable sheaves on $K3$ surfaces under some technical conditions. This is a generalization of Nakajima's construction of $\mathfrak{sl}_2$-action on the homology groups [N6]. In particular, for an A, D, E-type configulation of (-2)-curves, we shall give a collection of moduli spaces such that the associated Lie algebra acts on their homology groups.

§0. Introduction

Let X be a smooth projective surface defined over $\mathbb{C}$ and H an ample divisor on X. Assume that X is a $K3$ surface. Let $M_H(v)$ be the moduli space of H-stable sheaves E with the Mukai vector $v(E) = v$ (cf. (1.5)). In [Y2], we studied a special kind of Fourier–Mukai transform called (-2)-reflection. For this purpose, we introduced the Brill–Noether locus on the moduli space and studied its properties. Similar results are obtained by Markman [Mr]. We fix a vector bundle G on X. A stable sheaf E_0 is said to be exceptional, if $\operatorname{Ext}^1(E_0, E_0) = 0$. Then $v(E_0)$ is a (-2)-vector, that is, $\langle v(E_0)^2 \rangle = -2$. We assume that the twisted degree $\deg_G(E_0) := \deg(G^\vee \otimes E_0) = 0$. Let $v \in H^*(X, \mathbb{Z})$ be a Mukai vector such that

$$(0.1) \qquad \deg_G(E) = \min\{\deg_G(E') > 0 | E' \in K(X)\}$$

for $E \in M_H(v)$. Let

$$M_H(v)_{E_0,n} := \{E \in M_H(v) | \dim \operatorname{Hom}(E_0, E) = n\}$$

Received July 28, 2008.
Revised August 28, 2009.
2000 *Mathematics Subject Classification.* 14D20.
Supported by the Grant-in-Aid for Scientific Research (B-18340010, S-19104002).

be *the Brill–Noether locus* with respect to E_0. Under the condition (0.1), we showed that $M_H(v)_{E_0,n}$ is a Grassmannian bundle over a smooth manifold such that the relative cotangent bundle is isomorphic to the normal bundle $N_{M_H(v)_{E_0,n}/M_H(v)}$. Similar Grassmannian structure appears in Nakajima's quiver varieties [N2]. By usnig this structure, he constructed a Lie algebra action on the (Borel–Moore) homology groups of quiver varieties. Based on our description of the Brill–Noether locus, recently Nakajima [N6] constructed an $\mathfrak{sl}_2$-action on the homology groups of moduli spaces $\bigoplus_v H_*(M_H(v), \mathbb{C})$, where v runs a suitable set of Mukai vectors satisfing minimality condition (0.1).

In this note, under the same condition, we shall generalize Nakajima's result. Thus we shall construct a Lie algebra action on the homology groups of moduli spaces of stable sheaves (Theorem 2.1): For a collection of exceptional sheaves E_i, $i = 1, 2, \dots, s$ which satisfy some technical conditions, we shall construct operators h_i, e_i, f_i, $i = 1, 2, \dots, s$ and show that they satisfy the commutation relations for Chevalley generators. In particular, we show that $[e_i, f_i] = h_i$ and $[e_i, f_j] = 0$, $i \neq j$. Since the first relation is proved by Nakajima, we only need to show the second one. For this purpose, we introduce the notion of universal extension (resp. division) with respect to E_i, $i = 1, 2, \dots, s$ (see, sect. 1.3). This is our main idea and the other arguments are included in Nakajima's papers. Since the action is defined by algebraic correspondences, we also have an action on the rational Chow groups. In Section 3, we give some examples of the actions of Lie algebras.

Replacing E_0 by a purely 1-dimensional exceptional sheaf and the minimality condition by $\chi(E) = 1$, our construction also works for moduli spaces of purely 1-dimensional stable sheaves. In particular, we shall construct an action of the affine Lie algebra associated to a singular fiber of an elliptic surface. On an elliptic surface, purely 1-dimensional sheaves are related to torsion free sheaves of relative degree 0 via the relative Fourier–Mukai transform. Moreover purely 1-dimensional sheaves are related to the enumerative geometry of curves on X (cf. [YZ]). Thus the moduli spaces of purely 1-dimensional stable sheaves are important objects to study. For a rational elliptic surface X, it is observed in [MNWV] that the Euler characteristics of the moduli spaces are $W(E_8^{(1)})$-invariant, where $W(E_8^{(1)})$ is the Weyl group associated to the $E_8^{(1)}$-lattice $K_X^\perp \subset H^2(X, \mathbb{Z})$. An explanation is given in terms of the monodromy action, that is, we use the invariance of the homology groups of the moduli spaces under the deformation of X. Our construction of the Lie algebra gives another explanation of this invariance. These are treated in Section 4. In Section 5, we give a remark on the case of G-equivariant sheaves. We also treat the moduli of stable perverse coherent sheaves on a resolution of a rational double point. Many examples of the action of affine Lie algebra seem to be related by suitable Fourier–Mukai transforms. We shall study the relations elsewhere.

§1. Moduli of stable sheaves of minimal degree

Notation.

Let X be a smooth projective surface. Let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X and $K(X)$ the Grothendieck group of X. In this paper, we use the Borel–Moore homology groups. For an algebraic set M, $H_*(M, \mathbb{C})$ denotes the Borel–Moore homology group of M. If M is compact, then $H_*(M, \mathbb{C})$ coincides with the usual singular homology group of M.

Let $\mathbf{D}(X) := \mathbf{D}^b(\mathrm{Coh}(X))$ be the bounded derived category of $\mathrm{Coh}(X)$. For complexes $\mathbb{E}, \mathbb{F} \in \mathbf{D}(X)$, we set

$$\mathrm{Ext}^i(\mathbb{E}, \mathbb{F}) := \mathrm{Hom}_{\mathbf{D}(X)}(\mathbb{E}, \mathbb{F}[i]).$$

We usually denote $\mathrm{Ext}^0(\mathbb{E}, \mathbb{F})$ by $\mathrm{Hom}(\mathbb{E}, \mathbb{F})$. For a morphism $\phi : \mathbb{E} \to \mathbb{F}$, $[\mathbb{E} \to \mathbb{F}]$ denotes the mapping cone of a representative of ϕ. If $H^i([\mathbb{E} \to \mathbb{F}]) = 0$ for all i, then we write $\mathbb{E} \cong \mathbb{F}$. We usually denote $\mathrm{Ext}^i([\mathbb{E}_1 \to \mathbb{E}_2], \mathbb{F})$ (resp. $\mathrm{Ext}^i(\mathbb{F}, [\mathbb{E}_1 \to \mathbb{E}_2])$) by $\mathrm{Ext}^i(\mathbb{E}_1 \to \mathbb{E}_2, \mathbb{F})$ (resp. $\mathrm{Ext}^i(\mathbb{F}, \mathbb{E}_1 \to \mathbb{E}_2)$).

Let H be an ample divisor on X and G an element of $K(X)$ with $\mathrm{rk}\, G > 0$. For a coherent sheaf E on X, we set $\deg_G(E) := \deg(G^\vee \otimes E)$ and $\chi_G(E) := \chi(G^\vee \otimes E)$.

1.1. Technical lemmas

In this subsection, we introduce some technical conditions (1.1), (1.2), (1.3) and under these conditions we give some technical lemmas. These will play important roles for our construction of the action.

Definition 1.1. A purely 1-dimensional sheaf E is μ-stable, if the scheme-theoretic support $\mathrm{Div}(E)$ of E is reduced and irreducible.

We fix an ample divisor H on X. Let G be an element of $K(X)$ with $\mathrm{rk}\, G > 0$. In this note, we treat μ-semi-stable sheaves E (with respect to H) such that

$$\deg_G(E) = \min\{\deg_G(E') > 0 \mid E' \in K(X)\}. \tag{1.1}$$

This is a fairly strong condition for E, but such E behave very well.

Lemma 1.1. *Let G be an element of $K(X)$ with $\operatorname{rk} G > 0$ and E_i, $i = 1, 2, \dots, s$ be (mutually different) μ-stable vector bundles with $\deg_G(E_i) = 0$. Let E be a μ-semi-stable sheaf satisfying* (1.1).

(1) *Then E is μ-stable.*

(2) *Every non-trivial extension*

$$0 \to E_1 \to F \to E \to 0$$

defines a μ-stable sheaf.

(3) *Let V_i be subspaces of $\operatorname{Hom}(E_i, E)$, $i = 1, 2, \dots, s$. Then $\phi : \bigoplus_{i=1}^{s} V_i \otimes E_i \to E$ is injective or surjective in codimension 1. Moreover,*

(3-1) *if $\phi : \bigoplus_{i=1}^{s} V_i \otimes E_i \to E$ is injective, then the cokernel is μ-stable,*

(3-2) *if $\phi : \bigoplus_{i=1}^{s} V_i \otimes E_i \to E$ is surjective in codimension 1, then $\ker \phi$ is μ-stable. In particular*

$$D(E) := \mathcal{E}xt^1(\bigoplus_{i=1}^{s} V_i \otimes E_i \to E, \mathcal{O}_X)$$

is μ-stable.

Since $\deg_G(E)/\operatorname{rk}(E) = \operatorname{rk}(G)(\deg(E)/\operatorname{rk} E - \deg(G)/\operatorname{rk} G)$, the μ-stability can be defined by using the G-twisted slope $\deg_G(E)/\operatorname{rk}(E)$. By using the following lemmas, the proof of [Y2, Lem. 2.1] implies our lemma. So we only give a proof of (1), (3). We first note the following easy lemmas.

Lemma 1.2. *A purely 1-dimensional sheaf E with* (1.1) *is μ-stable.*

Lemma 1.3. *Let r, d, x be positive integers. Let y be an integer such that $y \in d\mathbb{Z}$. If $0 < y/x < d/r$, then $y \geq d$ and $x > r$.*

Proof of Lemma 1.1 (1), (3). Let E' be a subsheaf of E with $\deg_G(E)/\operatorname{rk} E = \deg_G(E')/\operatorname{rk} E'$. Then $1 \geq \deg_G(E)/\deg_G(E') = \operatorname{rk} E/\operatorname{rk} E' \geq 1$. Hence $\operatorname{rk} E' = \operatorname{rk} E$ and $\deg_G(E') = \deg_G(E)$, which implies that E is μ-stable. Thus (1) holds. We shall prove (3). We first assume that $\operatorname{rk} E > 0$. By the μ-stability of E, we have

$$0 \leq \frac{\deg_G(\operatorname{im} \phi)}{\operatorname{rk}(\operatorname{im} \phi)} \leq \frac{\deg_G(E)}{\operatorname{rk} E}.$$

By Lemma 1.3, we have

(i) $\deg_G(\operatorname{im} \phi) = 0$ or

(ii) $\deg_G(\operatorname{im} \phi)/\operatorname{rk}(\operatorname{im} \phi) = \deg_G(E)/\operatorname{rk} E$.

In the first case, $\deg_G(\ker \phi) = 0$. Assume that $\ker \phi \neq 0$. Let F be a μ-stable locally free subsheaf of $\ker \phi$ with $\deg_G(F) = 0$. Then there is a non-zero homomorphism $F \to E_i$, which is isomorphic. Hence $\operatorname{Hom}(E_i, \ker \phi) \neq 0$, which is a contradiction. Therefore $\ker \phi = 0$. We shall show that $E' := \operatorname{coker} \phi$ is μ-stable. We note that E' does not have a 0-dimensional subsheaf and $\deg_G(E') = \deg_G(E)$. We first assume that $\operatorname{rk} E' > 0$. If E' is not μ-stable, then (1) implies that E' is not μ-semi-stable. Then there is a quotient $E' \to F$ with $\deg_G(F)/\operatorname{rk} F < \deg_G(E')/\operatorname{rk} E'$. By Lemma 1.3, $\deg_G(F) \leq 0$, which implies that $\deg_G(F)/\operatorname{rk} F < \deg_G(E)/\operatorname{rk} E$. This is a contradiction. Therefore E' is μ-stable. If $\operatorname{rk} E' = 0$, then E' is of pure dimension 1. Then Lemma 1.2 implies that E' is μ-stable.

We next treat the second case. In this case, ϕ is surjective in codimension 1. We shall show that $\ker \phi$ is μ-stable. Assume that there is a locally free subsheaf F of $\ker \phi$ with

$$\frac{\deg_G(F)}{\operatorname{rk} F} > \frac{\deg_G(\ker \phi)}{\operatorname{rk}(\ker \phi)} = -\frac{\deg_G(E)}{\operatorname{rk}(\ker \phi)}.$$

Then we get that $\deg_G(F) \leq 0$. If $\deg_G(F) = 0$, then $\operatorname{Hom}(F, E_i) \neq 0$ for an i. Since E_i and F are μ-stable sheaves with the same slope, non-trivial homomorphism $F \to E_i$ is isomorphic in codimension 1. Since F is locally free, we conclude that $F \cong E_i$. Then $\operatorname{Hom}(E_i, F) \neq 0$, which is a contradiction. Hence $\deg_G(F) < 0$, which means that $0 < -\deg_G(F)/\operatorname{rk} F < \deg_G(E)/\operatorname{rk}(\ker \phi)$, Then Lemma 1.3 implies that $-\deg_G(F) \geq \deg_G(E)$ and $\operatorname{rk} F > \operatorname{rk}(\ker \phi)$, which is a contradiction. Therefore $\ker \phi$ is μ-stable.

If $\operatorname{rk} E = 0$, then since E is μ-stable, we get $\phi = 0$ or ϕ is surjective in codimension 1. Then by the same arguments as above, we see that $\ker \phi$ is μ-stable. Q.E.D.

Besides the condition for μ-semi-stable sheaves (1.1), we also introduce similar conditions and lemmas for Gieseker (twisted) semi-stabilities.

Definition 1.2. Let G be an element of $K(X)$ with $\operatorname{rk} G > 0$. A torsion free sheaf E is G-twisted stable, if

$$\frac{\chi_G(F(nH))}{\operatorname{rk} F} < \frac{\chi_G(E(nH))}{\operatorname{rk} E}, n \gg 0$$

for all proper subsheaves $F (\neq 0)$ of E.

As in the proof of Lemma 1.1, we also have the following assertions.

Lemma 1.4. *Let G be an element of $K(X)$ with $\operatorname{rk} G > 0$ and E_i, $i = 1, 2, \ldots, s$, be (mutually different) G-twisted stable sheaves with*

$\deg_G(E_i) = \chi_G(E_i) = 0$. *Let E be a G-twisted stable torsion free sheaf with* $\deg_G(E) = 0$ *and*

$$\chi_G(E) = \min\{\chi_G(E') > 0 | \, E' \in \mathrm{Coh}(X), \deg_G(E') = 0\} \tag{1.2}$$

or $E = \mathbb{C}_P$, $P \in X$ with (1.2).

(1) *Then every non-trivial extension*

$$0 \to E_1 \to F \to E \to 0$$

defines a G-twisted stable sheaf.

(2) *Let V_i be a subspace of* $\mathrm{Hom}(E_i, E)$. *Then $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is injective or surjective. Moreover,*

(2-1) *if $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is injective, then the cokernel is a G-twisted stable torsion free sheaf or $\mathbb{C}_P, P \in X$,*

(2-2) *if $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is surjective, then* $\ker \phi$ *is G-twisted stable.*

Lemma 1.5. *Let G be an element of $K(X)$ with* $\mathrm{rk}\, G > 0$ *and E_i, $i = 1, 2, \dots, s$, be (mutually different) G-twisted stable sheaves with* $\deg_G(E_i) = \chi_G(E_i) = 0$. *Let E be a G-twisted stable torsion free sheaf with* $\deg_G(E) = 0$ *and*

$$\chi_G(E) = \max\{\chi_G(E') < 0 | \, E' \in \mathrm{Coh}(X), \deg_G(E') = 0\}. \tag{1.3}$$

(1) *Then every non-trivial extension*

$$0 \to E \to F \to E_1 \to 0$$

defines a G-twisted stable sheaf.

(2) *Let V_i be a subspace of* $\mathrm{Hom}(E, E_i)$. *Then $\phi : E \to \bigoplus_{i=1}^s V_i^\vee \otimes E_i$ is injective or surjective. Moreover,*

(2-1) *if $\phi : E \to \bigoplus_{i=1}^s V_i^\vee \otimes E_i$ is injective, then the cokernel is a G-twisted stable torsion free sheaf or $\mathbb{C}_P$, $P \in X$*

(2-2) *if $\phi : E \to \bigoplus_{i=1}^s V_i^\vee \otimes E_i$ is surjective, then* $\ker \phi$ *is G-twisted stable.*

1.2. Basic properties of stable sheaves of minimal degree

Assume that K_X is numerically trivial. We define a bilinear form $\langle \ , \ \rangle$ on $H^*(X, \mathbb{Q}) := \bigoplus_{i=0}^2 H^{2i}(X, \mathbb{Q})$ by

$$\langle x, y \rangle := \int_X x_1 \wedge y_1 - x_0 \wedge y_2 - x_2 \wedge y_0 \tag{1.4}$$

where $x_i \in H^{2i}(X, \mathbb{Q})$ (resp. $y_i \in H^{2i}(X, \mathbb{Q})$) is the $2i$-th component of x (resp. y).

For an object $\mathbb{E} \in \mathbf{D}(X)$, we define the Mukai vector of $\mathbb{E}$ by

$$(1.5)\qquad \begin{aligned} v(\mathbb{E}) &= \sum_i (-1)^i v(H^i(\mathbb{E})) \\ &= \sum_i (-1)^i \operatorname{ch}(H^i(\mathbb{E}))\sqrt{\operatorname{td}_X} \in H^*(X,\mathbb{Q}), \end{aligned}$$

where td_X is the todd class of X. We have a map $v : \mathbf{D}(X) \to H^*(X,\mathbb{Q})$. We call an element of $v(\mathbf{D}(X))$ a Mukai vector. For $\mathbb{E}, \mathbb{F} \in \mathbf{D}(X)$, we define the Riemann–Roch number by

$$\chi(\mathbb{E},\mathbb{F}) := \sum_i (-1)^i \dim \operatorname{Ext}^i(\mathbb{E},\mathbb{F}).$$

Then the Riemann–Roch theorem says the following.

Proposition 1.6.

$$\chi(\mathbb{E},\mathbb{F}) = -\langle v(\mathbb{E}), v(\mathbb{F})\rangle.$$

By a similar way, we also define the rank $\operatorname{rk}\mathbb{E}$ and other invariants. We fix an element $G \in K(X)$ with $\operatorname{rk} G > 0$. For an object $\mathbb{E} \in \mathbf{D}(X)$ such that $\deg_G(\mathbb{E})$ satisfies (1.1), we define a stability condition.

Definition 1.3. Let $\mathbb{E} \in \mathbf{D}(X)$ be an object such that $\deg_G(\mathbb{E})$ satisfies (1.1). Then $\mathbb{E}$ is stable, if

$$(1.6)\qquad H^i(\mathbb{E} \overset{\mathbb{L}}{\otimes} \mathbb{C}_P) = 0, i \neq -1, 0$$

for all $P \in X$ and one of the following conditions holds:

(i) $H^i(\mathbb{E}) = 0$, $i \neq 0$ and $H^0(\mathbb{E})$ is a stable sheaf.
(ii) $H^i(\mathbb{E}) = 0$, $i \neq -1, 0$, $H^{-1}(\mathbb{E})^\vee := \mathcal{H}om_{\mathcal{O}_X}(H^{-1}(\mathbb{E}), \mathcal{O}_X)$ is a stable sheaf and $H^0(\mathbb{E})$ is a 0-dimensional sheaf.

Remark 1.1. (1) The condition (1.6) implies that there is a complex $C_{-1} \to C_0$ of locally free sheaves which is quasi-isomorphic to $\mathbb{E}$. In particular, if $H^{-1}(\mathbb{E}) = 0$, then $H^0(\mathbb{E})$ does not contain a 0-dimensional subsheaf.

(2) If $\operatorname{rk}\mathbb{E} < 0$, then $H^i(D(\mathbb{E})) = 0$, $i \neq 1$ and $H^1(D(\mathbb{E}))$ is a stable sheaf, where $D(\mathbb{E}) := \mathbf{R}\mathcal{H}om(\mathbb{E}, \mathcal{O}_X)$ is the dual of $\mathbb{E}$. Since we want to treat two cases simultaniously, we use $\mathbb{E}$ instead of using $D(\mathbb{E})$.

Lemma 1.7. *Let* $\mathbb{E}$ *be an object of* $\mathbf{D}(X)$.

(1) $\deg_G(\mathbb{E})$ *satisfies* (1.1) *if and only if* $\deg_{G^\vee}(D(\mathbb{E})[1])$ *satisfies* (1.1).

(2) $\mathbb{E}$ *is stable if and only if* $D(\mathbb{E})[1]$ *is stable.*

Proof. (1) Since $\deg_{G^\vee}(D(\mathbb{E})[1]) = \deg_G(\mathbb{E})$, we get (1).

(2) Obviously (1.6) for $\mathbb{E}$ is equivalent to that for $D(\mathbb{E})[1]$. If $\operatorname{rk}\mathbb{E} \neq 0$, then (i) for $\mathbb{E}$ is equivalent to (ii) for $D(\mathbb{E})[1]$ and (ii) for $\mathbb{E}$ is equivalent to (i) for $D(\mathbb{E})[1]$. If $\operatorname{rk}\mathbb{E} = 0$, then (i) for $\mathbb{E}$ is equivalent to (i) for $D(\mathbb{E})[1]$ and (ii) does not occur. Therefore $\mathbb{E}$ is stable if and only if $D(\mathbb{E})[1]$ is stable. Q.E.D.

Definition 1.4. For a Mukai vector $v \in H^*(X, \mathbb{Q})$ with the property (1.1), let $M_H(v)$ be the moduli space of (quasi-isomorphism classes of) stable complexes $\mathbb{E}$ with $v(\mathbb{E}) = v$.

If $\operatorname{rk} v < 0$, then by Remark 1.1, $M_H(v)$ has a scheme structure. The Zariski tangent space of $M_H(v)$ at $\mathbb{E}$ is $\operatorname{Ext}^1(\mathbb{E}, \mathbb{E})$ and the obstruction for the infinitesimal liftings belongs to the kernel of the trace map

$$\operatorname{tr} : \operatorname{Ext}^2(\mathbb{E}, \mathbb{E}) \to H^2(X, \mathcal{O}_X).$$

In this paper, we require the following condition.

Condition 1. The trace map

$$\operatorname{tr} : \operatorname{Ext}^2(\mathbb{E}, \mathbb{E}) \to H^2(X, \mathcal{O}_X)$$

is isomorphic.

By Lemma 1.1 and Condition 1, we get the following assertions.

Lemma 1.8. *Assume that* $v \in H^*(X, \mathbb{Q})$ *satisfies* (1.1).

(i) *If* $M_H(v) \neq \emptyset$, *then* $\dim M_H(v) = \langle v^2 \rangle + 1 + p_g$. *In particular, if there is a stable complex* $\mathbb{E}$ *with* $v(\mathbb{E}) = v$, *then* $\langle v(\mathbb{E})^2 \rangle \geq -(p_g + 1)$.

(ii) *Assume that* X *is a K3 surface. Then there is a stable complex* $\mathbb{E}$ *with* $v(\mathbb{E}) = v$ *if and only if* $\langle v^2 \rangle \geq -2$.

Fot the proof of (ii), we also use [Y2, Thm. 0.2].

Let $S := \{E_1, E_2, \dots, E_n\}$ be a finite set of μ-stable vector bundles such that $\deg_G(E_i) = 0$, $1 \leq i \leq n$. We require the following condition.

Condition 2.

$$\operatorname{Ext}^1(E_i, E_i) = 0,\ E_i \otimes K_X \cong E_i,\ E_i \in S.$$

Let $\mathcal{S}$ be a subcategory of $\operatorname{Coh}(X)$ consisting of semi-stable sheaves F whose Jordan-Hölder grading is $\bigoplus_i E_i^{\oplus n_i}$.

Lemma 1.9. $\operatorname{Hom}(\mathbb{E}, F) = 0$ *and* $\operatorname{Hom}(F[1], \mathbb{E}) = 0$ *for* $F \in \mathcal{S}$.

Proof. We use the spectral sequence

$$E_2^{p,q} = \bigoplus_{q'+q''=q} \mathrm{Ext}^p(H^{-q'}(*), H^{q''}(**)) \\ \Longrightarrow E_\infty^{p+q} = \mathrm{Ext}^{p+q}(*, **).$$

Since $H^i(\mathbb{E}) = 0$, $i \neq -1, 0$, $\mathrm{Hom}(\mathbb{E}, F) = \mathrm{Hom}(H^0(\mathbb{E}), F)$. If $\mathrm{rk}\,\mathbb{E} \geq 0$, then $H^0(\mathbb{E})$ is a stable sheaf of positive G-twisted degree. Hence $\mathrm{Hom}(H^i(\mathbb{E}), F) = 0$. If $\mathrm{rk}\,\mathbb{E} < 0$, then $H^0(\mathbb{E})$ is a 0-dimension sheaf. Hence $\mathrm{Hom}(H^0(\mathbb{E}), F) = 0$. Therefore the first claim holds. Since $\mathrm{Hom}(F[1], \mathbb{E}) = \mathrm{Hom}(F, H^{-1}(\mathbb{E}))$, we also get the second claim. Q.E.D.

1.3. A universal division and a universal extension

Definition 1.5. Let $\mathbb{E}$ be a stable complex in Definition 1.3. An exact triangle

$$F \to \mathbb{E} \to \widetilde{\mathbb{E}} \to F[1]$$

is a universal division of $\mathbb{E}$ with respect to $\{E_1, E_2, \dots, E_n\}$, if $F \in \mathcal{S}$ and $\widetilde{\mathbb{E}}$ is a stable complex such that $\mathrm{Hom}(E_i, \widetilde{\mathbb{E}}) = 0$, $1 \leq i \leq n$.

For an exact triangle

$$F' \to \mathbb{E} \to \mathbb{E}' \to F'[1]$$

with $F' \in \mathcal{S}$, we have an exact sequence

$$\mathrm{Hom}(F'[1], \widetilde{\mathbb{E}}) \to \mathrm{Hom}(\mathbb{E}', \widetilde{\mathbb{E}}) \to \mathrm{Hom}(\mathbb{E}, \widetilde{\mathbb{E}}) \to \mathrm{Hom}(F', \widetilde{\mathbb{E}}).$$

By our assumption and Lemma 1.9, $\mathrm{Hom}(\mathbb{E}', \widetilde{\mathbb{E}}) \to \mathrm{Hom}(\mathbb{E}, \widetilde{\mathbb{E}})$ is an isomorphism. Hence we have a unique morphism $\mathbb{E}' \to \widetilde{\mathbb{E}}$ in $\mathbf{D}(X)$ which induces a commutative diagram of exact triangles (in $\mathbf{D}(X)$):

$$\begin{tikzcd}
F' \arrow[r] \arrow[d] & \mathbb{E} \arrow[r] \arrow[d, equal] & \mathbb{E}' \arrow[r] \arrow[d] & F'[1] \arrow[d] \\
F \arrow[r] & \mathbb{E} \arrow[r] & \widetilde{\mathbb{E}} \arrow[r] & F[1].
\end{tikzcd}$$

In particular, a universal division of $\mathbb{E}$ is unique (up to isomorphism in $\mathbf{D}(X)$). Since $\mathrm{Hom}(\widetilde{\mathbb{E}}, \widetilde{\mathbb{E}}) \cong \mathbb{C}$ and $\mathrm{Hom}(F, \widetilde{\mathbb{E}}) = 0$, we get

$$\mathrm{Hom}(\mathbb{E}, \widetilde{\mathbb{E}}) \cong \mathbb{C}. \tag{1.7}$$

Since $E_i \otimes K_X \cong E_i$, we see that $\mathrm{Hom}(F \otimes K_X^\vee, \widetilde{\mathbb{E}}) = \mathrm{Hom}(F \otimes K_X^\vee[1], \widetilde{\mathbb{E}}) = 0$. Hence we also get that

$$\mathrm{Hom}(\mathbb{E}, \widetilde{\mathbb{E}} \otimes K_X) \cong \mathrm{Hom}(\widetilde{\mathbb{E}}, \widetilde{\mathbb{E}} \otimes K_X) \cong H^0(X, K_X). \tag{1.8}$$

Definition 1.6. Let $\mathbb{E}$ be a stable complex in Definition 1.3. An exact triangle

$$F \to \widehat{\mathbb{E}} \to \mathbb{E} \to F[1]$$

is a universal extension of $\mathbb{E}$ with respect to $\{E_1, E_2, \dots, E_n\}$, if $F \in \mathcal{S}$ and $\widehat{\mathbb{E}}$ is a stable complex such that $\operatorname{Ext}^1(\widehat{\mathbb{E}}, E_i) = 0$, $1 \le i \le n$.

For an exact triangle

$$F' \to \mathbb{E}' \to \mathbb{E} \to F'[1]$$

with $F' \in \mathcal{S}$, we have an exact sequence

$$\operatorname{Hom}(\widehat{\mathbb{E}}, F') \to \operatorname{Hom}(\widehat{\mathbb{E}}, \mathbb{E}') \to \operatorname{Hom}(\widehat{\mathbb{E}}, \mathbb{E}) \to \operatorname{Ext}^1(\widehat{\mathbb{E}}, F').$$

By our assumption and Lemma 1.9, $\operatorname{Hom}(\widehat{\mathbb{E}}, \mathbb{E}') \to \operatorname{Hom}(\widehat{\mathbb{E}}, \mathbb{E})$ is an isomorphism. Hence we have a unique morphism $\widehat{\mathbb{E}} \to \mathbb{E}'$ which induces a commutative diagram of exact triangles

$$\begin{array}{ccccccc}
F' & \longrightarrow & \mathbb{E}' & \longrightarrow & \mathbb{E} & \longrightarrow & F'[1] \\
\uparrow & & \uparrow & & \| & & \uparrow \\
F & \longrightarrow & \widehat{\mathbb{E}} & \longrightarrow & \mathbb{E} & \longrightarrow & F[1].
\end{array}$$

In particular, a universal extension of $\mathbb{E}$ is unique. For a universal extension, we also see that

$$\begin{aligned}
&\operatorname{Hom}(\widehat{\mathbb{E}}, \mathbb{E}) \cong \mathbb{C}, \\
&\operatorname{Hom}(\widehat{\mathbb{E}}, \mathbb{E} \otimes K_X) \cong \operatorname{Hom}(\widehat{\mathbb{E}}, \widehat{\mathbb{E}} \otimes K_X) \cong H^0(X, K_X).
\end{aligned} \tag{1.9}$$

1.3.1. *Condition for the existence*

Lemma 1.10. (i) *If the matrix $(-\chi(E_i, E_j)_{i,j=1}^n)$ is negative definite, then a universal extension and a universal division exist for $\mathbb{E}$.*

(ii) *Assume that the matrix $(-\chi(E_i, E_j)_{i,j=1}^n)$ is negative semi-definite of affine type. Let $\delta := \sum_i a_i v(E_i)$ satisfy $\langle \delta, v(E_i) \rangle = 0$ for all $E_i \in S$. If $\langle v(\mathbb{E}), \delta \rangle \neq 0$, then a universal extension or a universal division exist for $\mathbb{E}$.*

Proof.

Claim 1.1. For a non-zero morphism $\psi : E_{n_1} \to \mathbb{E}$, $\mathbb{E}^{(1)} := [E_{n_1} \to \mathbb{E}]$ is also stable.

Proof of Claim 1.1: For a non-zero morphism $\psi : E_{n_1} \to \mathbb{E}$, we have an exact sequence

$$\operatorname{Hom}(E_{n_1}, \mathbb{E}[-1]) \to \operatorname{Hom}(E_{n_1}, \mathbb{E}^{(1)}[-1]) \to \mathbb{C} \xrightarrow{\psi} \operatorname{Hom}(E_{n_1}, \mathbb{E}).$$

By Lemma 1.9, $\operatorname{Hom}(E_{n_1}, \mathbb{E}[-1]) = 0$. Hence we get $\operatorname{Hom}(E_{n_1}, \mathbb{E}^{(1)}[-1]) = 0$. We note that $\mathbb{E}^{(1)}$ satisfies (1.6) and we have the following exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{-1}(\mathbb{E}) & \longrightarrow & H^{-1}(\mathbb{E}^{(1)}) & \longrightarrow & \\ E_{n_1} & \longrightarrow & H^0(\mathbb{E}) & \longrightarrow & H^0(\mathbb{E}^{(1)}) & \longrightarrow & 0. \end{array}$$

Then we get $0 = \operatorname{Hom}(E_{n_1}, \mathbb{E}^{(1)}[-1]) \cong \operatorname{Hom}(E_{n_1}, H^{-1}(\mathbb{E}^{(1)}))$. If $H^{-1}(\mathbb{E}) = 0$, then $E_{n_1} \to H^0(\mathbb{E})$ is a non-zero homomorphism. By Lemma 1.1, $\mathbb{E}^{(1)}$ is stable. Assume that $H^{-1}(\mathbb{E}) \neq 0$. Then $D(\mathbb{E})[1] = H^1(D(\mathbb{E}))$ is a stable sheaf (cf. Remark 1.1). Hence $D(\mathbb{E}^{(1)})[1] = H^1(D(\mathbb{E}^{(1)}))$ and we have an exact sequence

$$0 \to E_{n_1}^{\vee} \to H^1(D(\mathbb{E}^{(1)})) \to H^1(D(\mathbb{E})) \to 0.$$

Since $\deg_{G^{\vee}}(D(\mathbb{E})[1]) = \deg_G(\mathbb{E})$ and

$$\begin{aligned} \operatorname{Hom}(H^1(D(\mathbb{E}^{(1)})), E_{n_1}^{\vee}) &= \operatorname{Hom}(D(\mathbb{E}^{(1)})[1], E_{n_1}^{\vee}) \\ &= \operatorname{Hom}(E_{n_1}, \mathbb{E}^{(1)}[-1]) = 0, \end{aligned}$$

Lemma 1.1 implies that $H^1(D(\mathbb{E}^{(1)}))$ is a stable sheaf. Therefore $\mathbb{E}^{(1)}$ is a stable complex. Thus the claim holds.

If there is a non-zero morphism $E_{n_2} \to \mathbb{E}^{(1)}$, then we set $\mathbb{E}^{(2)} := [E_{n_2} \to \mathbb{E}^{(1)}]$. Then by applying the Octahedral axiom to $\mathbb{E} \to \mathbb{E}^{(1)} \to \mathbb{E}^{(2)}$, we have an exact triangle

$$F^2 \to \mathbb{E} \to \mathbb{E}^{(2)} \to F^2[1],$$

where F^2 fits in an exact sequence

$$0 \to E_{n_1} \to F^2 \to E_{n_2} \to 0.$$

Continueing this procedure, we get a sequence of stable complexes

$$\mathbb{E} = \mathbb{E}^{(0)}, \mathbb{E}^{(1)}, \dots, \mathbb{E}^{(s)}, \dots,$$

where $\mathbb{E}^{(s)}$ fits in an exact triangle

$$F^s \to \mathbb{E} \to \mathbb{E}^{(s)} \to F^s[1],$$

$F^s \in \mathcal{S}$. Since $v(\mathbb{E}^{(s)}) = v(\mathbb{E}^{(0)}) - \sum_i v(E_{n_i})$, if S generate a negative definite lattice or $\langle \delta, v(\mathbb{E}) \rangle > 0$, then $\langle v(\mathbb{E}^{(s)})^2 \rangle < -(1+p_g)$ for some s. By Lemma 1.8, this is impossible. Hence $\operatorname{Hom}(E_i, \mathbb{E}^{(s)}) = 0$, $1 \le i \le n$ for some s.

For a non-zero morphism $\psi : \mathbb{E} \to E_{n_0}[1]$, we set $\mathbb{E}^{(-1)}[1] := [\mathbb{E} \to E_{n_0}[1]]$. Then $\mathbb{E}^{(-1)}$ fits in an exact triangle:

$$E_{n_0} \to \mathbb{E}^{(-1)} \to \mathbb{E} \to E_{n_0}[1].$$

Claim 1.2. $\mathbb{E}^{(-1)}$ is a stable complex.

Proof of Claim 1.2: For a non-zero morphism $\psi : \mathbb{E} \to E_{n_0}[1]$, we have an exact sequence

$$\operatorname{Hom}(\mathbb{E}[1], E_{n_0}[1]) \to \operatorname{Hom}(\mathbb{E}^{(-1)}[1], E_{n_0}[1]) \to \mathbb{C} \xrightarrow{\psi} \operatorname{Hom}(\mathbb{E}, E_{n_0}[1]).$$

By Lemma 1.9, $\operatorname{Hom}(\mathbb{E}[1], E_{n_0}[1]) = 0$, and hence $\operatorname{Hom}(\mathbb{E}^{(-1)}[1], E_{n_0}[1]) = 0$. By our assumption, $\mathbb{E}^{(-1)}$ satisfies (1.6) and $H^i(\mathbb{E}^{(-1)})$, $i = -1, 0$ fits in the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{-1}(\mathbb{E}^{(-1)}) & \longrightarrow & H^{-1}(\mathbb{E}) & \longrightarrow & \\ E_{n_0} & \longrightarrow & H^0(\mathbb{E}^{(-1)}) & \longrightarrow & H^0(\mathbb{E}) & \longrightarrow & 0. \end{array}$$

If $H^{-1}(\mathbb{E}) = 0$, then since

$$\operatorname{Hom}(H^0(\mathbb{E}^{(-1)}), E_{n_0}) = \operatorname{Hom}(\mathbb{E}^{(-1)}[1], E_{n_0}[1]) = 0,$$

Lemma 1.1 (2) implies that $H^0(\mathbb{E}^{(-1)})$ is stable. Assume that $H^{-1}(\mathbb{E}) \neq 0$. If $H^{-1}(\mathbb{E}) \to E_{n_0}$ is a zero map, then since $H^0(\mathbb{E})$ is 0-dimensional and E_{n_0} is locally free, we get $\operatorname{Ext}^1(H^0(\mathbb{E}), E_{n_0}) = 0$. Hence the second line splits, which is a contradiction. Thus $\xi : H^{-1}(\mathbb{E}) \to E_{n_0}$ is non trivial. Then by applying Lemma 1.1 (3) to $\xi^\vee : E_{n_0}^\vee \to H^{-1}(\mathbb{E})^\vee$, we see that (1) $\xi^\vee$ is injective except finite subset of X and $\operatorname{coker}(\xi^\vee)$ is μ-stable torsion free sheaf, or (2) $\xi^\vee$ is injective except a divisor of X and $\operatorname{coker}(\xi^\vee)$ is μ-stable purely 1-dimensional sheaf, or (3) $\xi^\vee$ is surjective in codimension 1 and $\ker \xi^\vee$ is a μ-stable sheaf. In the case of (1), $H^0(\mathbb{E}^{(-1)})$ is 0-dimensional and $H^{-1}(\mathbb{E}^{(-1)})$ is a μ-stable sheaf. If the case (2) occur, then $H^0(\mathbb{E}^{(-1)})$ is a μ-stable 1-dimensional sheaf and $H^{-1}(\mathbb{E}^{(-1)}) = 0$. In the last case, $H^0(\mathbb{E}^{(-1)})$ is a μ-stable torsion free sheaf and $H^{-1}(\mathbb{E}^{(-1)}) = 0$. Therefore $\mathbb{E}^{(-1)}$ is a stable complex and we complete the proof of the claim.

If there is a non-zero homomorphism $\mathbb{E}^{(-1)} \to E_{n_{-1}}[1]$, we set $\mathbb{E}^{(-2)}[1] := [\mathbb{E}^{(-1)} \to E_{n_{-1}}[1]]$. Continueing this procedure, we get a sequence of stable complexes

$$\dots, \mathbb{E}^{(-t)}, \dots, \mathbb{E}^{(-1)}, \mathbb{E}^{(0)}.$$

Since $v(\mathbb{E}^{(-t)}) = v(\mathbb{E}^{(0)}) + \sum_i v(E_{n_i})$, if S generate a negative definite lattice or $\langle \delta, v(\mathbb{E}) \rangle < 0$, then we see that $\operatorname{Hom}(E_i, \mathbb{E}^{(-t)}) = 0$, $1 \le i \le n$ for some $-t$. Therefore Lemma 1.10 holds. Q.E.D.

Lemma 1.11. *Assume that S satisfies the condition of (i) or (ii) in Lemma 1.10. If there is an exact triangle*

$$F \to \mathbb{E} \to \mathbb{E}' \to F[1]$$

such that $\mathbb{E}, \mathbb{E}'$ are stable complexes and $F \in \mathcal{S}$, then we have

$$\operatorname{Hom}(\mathbb{E}, \mathbb{E}') \cong \mathbb{C}, \ \operatorname{Hom}(\mathbb{E}, \mathbb{E}' \otimes K_X) \cong H^0(X, K_X).$$

Proof. We only show the first assertion. We assume that there is a universal division $\widetilde{\mathbb{E}}$ of $\mathbb{E}'$. By applying the Octahedral axiom to $\mathbb{E} \to \mathbb{E}' \to \widetilde{\mathbb{E}}$, we have an exact triangle

$$F' \to \mathbb{E}' \to \widetilde{\mathbb{E}} \to F'[1],$$

where $F' \in \mathcal{S}$. By the exact sequence

$$\operatorname{Hom}(\mathbb{E}, F') \to \operatorname{Hom}(\mathbb{E}, \mathbb{E}') \to \operatorname{Hom}(\mathbb{E}, \widetilde{\mathbb{E}}) = \mathbb{C}$$

and Lemma 1.9, we get our claim. If there is a universal extension $\widehat{\mathbb{E}}$, we also see that $\operatorname{Hom}(\mathbb{E}, \mathbb{E}') \cong \mathbb{C}$. Q.E.D.

1.4. Coherent systems

We set

$$\mathfrak{P}_{E_i}^{(n)}(v) := \{(\mathbb{E}, U) | \mathbb{E} \in M_H(v), U \subset \operatorname{Hom}(E_i, \mathbb{E}), \dim U = n\}.$$

$\mathfrak{P}_{E_i}^{(n)}(v)$ is the moduli space of coherent systems. For the construction of $\mathfrak{P}_{E_i}^{(n)}(v)$, see Section 7.1. The Zariski tangent space of $\mathfrak{P}_{E_i}^{(n)}(v)$ at $(\mathbb{E}, U)$ is

$$\operatorname{coker}(\phi : \operatorname{End}(U \otimes E_i) \to \operatorname{Ext}^1(U \otimes E_i \to \mathbb{E}, \mathbb{E}))$$

and the obstruction for the infinitesimal deformation belongs to the kernel of

$$\tau : \operatorname{Ext}^2(U \otimes E_i \to \mathbb{E}, \mathbb{E}) \to \operatorname{Ext}^2(\mathbb{E}, \mathbb{E}) \xrightarrow{\operatorname{tr}} H^2(X, \mathcal{O}_X).$$

For brevity, we usually denote $\operatorname{coker}\phi$ by $\operatorname{Ext}^1(U\otimes E_i \to \mathbb{E}, \mathbb{E})/\operatorname{End}(U\otimes E_i)$. We also use similar conventions later.

By Lemma 1.11 and the Serre duality, $\ker\tau = 0$. Thus $\mathfrak{P}_{E_i}^{(n)}(v)$ is a smooth scheme with

$$\begin{aligned}\dim \mathfrak{P}_{E_i}^{(n)}(v) &= \dim \operatorname{Ext}^1(U\otimes E_i \to \mathbb{E}, \mathbb{E})/\operatorname{End}(U\otimes E_i)\\ &= \langle v - nv_i, v\rangle - n^2 + (1+p_g)\\ &= \frac{1}{2}(\dim M_H(v) + \dim M_H(v - nv_i)).\end{aligned}$$

For $(\mathbb{E}, U) \in \mathfrak{P}_{E_i}^{(n)}(v)$, $\mathbb{E}$ and $[U\otimes E_i \to \mathbb{E}]$ are stable. Hence we have morphisms $\pi : \mathfrak{P}_{E_i}^{(n)}(v) \to M_H(v)$ and $\varpi : \mathfrak{P}_{E_i}^{(n)}(v) \to M_H(v - nv_i)$.

Remark 1.2. If $\operatorname{rk}\mathbb{E} < 0$, then $E' := H^1(D(\mathbb{E})) \in M_H(-v^\vee)$ and $\operatorname{Hom}(E_i, \mathbb{E}) \cong \operatorname{Hom}(E', E_i^\vee[1])$. Hence

$$\mathfrak{P}_{E_i}^{(n)}(v) = \{(E', U) | E' \in M_H(-v^\vee), U \subset \operatorname{Ext}^1(E_i, E'), \dim U = n\}.$$

Remark 1.3. We set $\mathbb{F} := [U\otimes E_i \to \mathbb{E}]$. Since $\operatorname{Hom}(\mathbb{E}[1], E_i[1]) = \operatorname{Hom}(E_i \otimes U, E_i[1]) = 0$, by the exact triangle

$$U\otimes E_i \to \mathbb{E} \to \mathbb{F} \to U\otimes E_i[1],$$

we have an exact sequence

$$0 \to U^\vee \to \operatorname{Hom}(\mathbb{F}, E_i[1]) \to \operatorname{Hom}(\mathbb{E}, E_i[1]) \to 0.$$

Thus we have

$$\mathfrak{P}_{E_i}^{(n)}(v) := \left\{ (\mathbb{F}, U^\vee) \,\middle|\, \begin{matrix} \mathbb{F} \in M_H(v - nv_i), \dim U = n \\ U^\vee \subset \operatorname{Hom}(\mathbb{F}, E_i[1]) \end{matrix} \right\}. \tag{1.10}$$

We set $F_i := U\otimes E_i$ and $\mathbb{F} := [F_i \to \mathbb{E}]$. Then we have the following exact and commutative diagram:

(1.11)

$$\begin{tikzcd} & \operatorname{Hom}(F_i, F_i) \arrow[d] & & \\ \operatorname{Hom}(\mathbb{F}, \mathbb{F}) \arrow[r] \arrow[d] & \operatorname{Ext}^1(\mathbb{F}, F_i) \arrow[r] \arrow[d] & \operatorname{Ext}^1(\mathbb{F}, \mathbb{E}) \arrow[r] \arrow[d] & \operatorname{Ext}^1(\mathbb{F}, \mathbb{F}) \arrow[d] \\ \operatorname{Hom}(\mathbb{E}, \mathbb{F}) \arrow[r] & \operatorname{Ext}^1(\mathbb{E}, F_i) \arrow[r] & \operatorname{Ext}^1(\mathbb{E}, \mathbb{E}) \arrow[r] & \operatorname{Ext}^1(\mathbb{E}, \mathbb{F}). \end{tikzcd}$$

By Lemma 1.11, we see that $\operatorname{Ext}^1(\mathbb{E}, F_i) \to \operatorname{Ext}^1(\mathbb{E}, \mathbb{E})$ is injective, which implies that

$$\operatorname{Ext}^1(F_i \to \mathbb{E}, \mathbb{E}) / \operatorname{End}(F_i) \to \operatorname{Ext}^1(F_i \to \mathbb{E}, F_i \to \mathbb{E}) \oplus \operatorname{Ext}^1(\mathbb{E}, \mathbb{E})$$

is injective. Therefore $\pi \times \varpi : \mathfrak{P}_{E_i}^{(n)}(v) \to M_H(v) \times M_H(v - nv_i)$ is a closed immersion.

Definition 1.7.

$$M_H(v)_{E_i,n} := \{\mathbb{E} \in M_H(v) | \dim \operatorname{Hom}(E_i, \mathbb{E}) = n\}.$$

Then $\pi_*(\mathfrak{P}_{E_i}^{(n)}(v)) = \cup_{k \geq n} M_H(v)_{E_i,k}$.

§2. An action of a Lie algebra

We define a lattice

$$L(S) := \left(\sum_{i=1}^{n} \mathbb{Z} v(E_i), -\langle \ , \ \rangle \right).$$

Let $\mathfrak{g}$ be the Lie algebra associated to $L(S)$, that is, the Cartan matrix of $\mathfrak{g}$ is $(-\langle v(E_i), v(E_j) \rangle_{i,j=1}^{n})$. In the same way as in [N2] and [N6], we shall construct an action of $\mathfrak{g}$ on $\bigoplus_v H_*(M_H(v), \mathbb{C})$, where v runs a suitable set of Mukai vectors with (1.1).

The fundamental class of $\mathfrak{P}_{E_i}^{(n)}$ defines an operator $f_{v_i}^{(n)}$:

$$\begin{array}{ccc} H_*(M_H(v - nv_i), \mathbb{C}) & \to & H_*(M_H(v), \mathbb{C}) \\ x & \mapsto & p_{2*}(p_1^*(x) \cap [\mathfrak{P}_{E_i}^{(n)}(v)]) \end{array}$$

where p_1, p_2 are the first and the second projections of $M_H(v - nv_i) \times M_H(v)$. We also define the operator $e_{v_i}^{(n)}$:

$$\begin{array}{ccc} H_*(M_H(v), \mathbb{C}) & \to & H_*(M_H(v - nv_i), \mathbb{C}) \\ x & \mapsto & (-1)^{nr(v)} p_{1*}(p_2^*(x) \cap [\mathfrak{P}_{E_i}^{(n)}(v)]) \end{array}$$

where $r(v) = \frac{1}{2}(\dim M_H(v - v_i) - \dim M_H(v)) = -\langle v_i, v \rangle - 1$. We set $e_{v_i} := e_{v_i}^{(1)}$ and $f_{v_i} := f_{v_i}^{(1)}$. We also set

$$h_{v_i | H_*(M_H(v), \mathbb{C})} = \langle v_i, v \rangle \operatorname{id}_{H_*(M_H(v), \mathbb{C})}.$$

Theorem 2.1. *For a fixed element $G \in K(X)$ with $\operatorname{rk} G > 0$, we consider Mukai vectors v satisfying* (1.1)*. Assume that Condition 1 holds for any element $E \in M_H(v)$. Assume that S satsifies Condition 2 and* *the assumptions in (i) or (ii) of Lemma 1.10. Then* $e_{v_i}, f_{v_j}, h_{v_k}$ *satisfy the following relations:*

$$[h_{v_i}, e_{v_j}] = -\langle v_i, v_j \rangle e_{v_j} \tag{2.1}$$

$$[h_{v_i}, f_{v_j}] = \langle v_i, v_j \rangle f_{v_j} \tag{2.2}$$

$$[e_{v_i}, f_{v_j}] = \delta_{i,j} h_{v_i}, \tag{2.3}$$

$$\mathrm{ad}(e_{v_i})^{1+\langle v_i, v_j \rangle}(e_{v_j}) = \mathrm{ad}(f_{v_i})^{1+\langle v_i, v_j \rangle}(f_{v_j}) = 0,\ i \neq j, \tag{2.4}$$

where ad *means the adjoint action* $\mathrm{ad}(x)(y) := [x, y] = xy - yx$.

Since $\langle (v \pm nv_i)^2 \rangle < -(1 + p_g)$ for $n \gg 0$, e_{v_i} and f_{v_i} are locally nilpotent. Therefore we get an integral representation of $\mathfrak{g}$.

2.1. Proof of Theorem 2.1

The proof is similar to [N2] and [N6]. We first note that the Serre relations (2.4) follows from the other relations and $\langle (v \pm nv_i)^2 \rangle < -(1 + p_g)$ for $n \gg 0$: Let L be the subspace of

$$\mathrm{Hom}(\oplus_{k \in \mathbb{Z}} H_*(M_H(v + kv_i)), \oplus_{k \in \mathbb{Z}} H_*(M_H(v + v_j + kv_i)))$$

generated by $\mathrm{ad}(e_{v_i})^n(e_{v_j})$, $n \geq 0$. Then $\mathfrak{sl}_2$ generated by $e_{v_i}, f_{v_i}, h_{v_i}$ acts on L. Since $\langle (v \pm nv_i)^2 \rangle < -(1 + p_g)$ for $n \gg 0$, L is of finite dimension. By the theory of the $\mathfrak{sl}_2$-representation, we get

$$\mathrm{ad}(e_{v_i})^{1+\langle v_i, v_j \rangle}(e_{v_j}) = 0.$$

The proof of the other relation is the same.

Hence we only need to show relations (2.1), (2.2) and (2.3). The proofs of (2.1), (2.2) are easy. We shall prove (2.3). For this purpose, we shall study the convolution products:

$$\begin{aligned} &p_{13*}\left(p_{12}^*\left[\omega(\mathfrak{P}_{E_i}^{(n_i)}(v))\right] \cap p_{23}^*\left[\mathfrak{P}_{E_j}^{(n_j)}(v)\right]\right), \\ &q_{13*}\left(q_{12}^*\left[\mathfrak{P}_{E_j}^{(n_j)}(v - n_i v_i)\right] \cap q_{23}^*\left[\omega(\mathfrak{P}_{E_i}^{(n_i)}(v - n_i v_i))\right]\right), \end{aligned} \tag{2.5}$$

where $n_i, n_j \in \mathbb{Z}_{>0}$, p_{ij} and q_{ij} are the projections to the products of the i-th and the j-th factors in

$$M_H(v - n_i v_i) \times M_H(v) \times M_H(v - n_j v_j),$$
$$M_H(v - n_i v_i) \times M_H(v - n_i v_i - n_j v_j) \times M_H(v - n_j v_j)$$

respectively, and ω is the exchange of the factor. The both products have degree $\frac{1}{2}(\dim M_H(v - n_i v_i) + \dim M_H(v - n_j v_j))$.

(I) We first study the case where $i \neq j$.

Lemma 2.2. *We have an isomophism over $M_H(v-n_iv_i)\times M_H(v-n_jv_j)$:*

$$(2.6)\quad p_{12}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v)))\cap p_{23}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v))\to q_{12}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v-n_iv_i))\cap q_{23}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v-n_jv_j))).$$

Proof. We set $F_i:=U_i\otimes E_i$ and $F_j:=U_j\otimes E_j$. Let $\mathbb{E}_1$ be an element of $M_H(v-n_jv_j)$. For $F_i\to\mathbb{E}_1$ and $[F_i\to\mathbb{E}_1]\to F_j[1]$, we set

$$\begin{aligned}\mathbb{E}_2:=&[[F_i\to\mathbb{E}_1]\to F_j[1]][-1]\in M_H(v-n_iv_i),\\ \mathbb{E}:=&[\mathbb{E}_1\to F_j[1]][-1]\in M_H(v).\end{aligned}$$

Applying the Octahedral axiom to $\mathbb{E}_1\to[F_i\to\mathbb{E}_1]\to F_j[1]$, we have a commutative diagram of exact triangles:

$$(2.7)\quad \begin{tikzcd} & F_i \arrow[r, equal] \arrow[d] & F_i \arrow[d] & \\ F_j \arrow[r] \arrow[d, equal] & \mathbb{E} \arrow[r] \arrow[d] & \mathbb{E}_1 \arrow[r] \arrow[d] & F_j[1] \arrow[d, equal] \\ F_j \arrow[r] & \mathbb{E}_2 \arrow[r] \arrow[d] & {[F_i\to\mathbb{E}_1]} \arrow[r] \arrow[d] & F_j[1] \\ & F_i[1] \arrow[r, equal] & F_i[1]. & \end{tikzcd}$$

Hence $\mathbb{E}_1\cong[F_j\to\mathbb{E}]$, $\mathbb{E}_2\cong[F_i\to\mathbb{E}]$ and $[F_i\to\mathbb{E}_1]\cong[F_i\oplus F_j\to\mathbb{E}]$. Conversely for $\mathbb{E}:=[\mathbb{E}_1\to F_j[1]][-1]$ and $\mathbb{E}_2:=[F_i\to\mathbb{E}]$, we get the commutative diagram of exact triangles (2.7). Since the correspondence is functorial, we have a desired isomorphism

$$p_{12}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v)))\cap p_{23}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v))\to q_{12}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v-n_iv_i))\cap q_{23}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v-n_jv_j))).$$

Q.E.D.

Lemma 2.3. (i)

$$p_{12}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v)))\cap p_{23}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v))\to M_H(v-n_iv_i)\times M_H(v-n_jv_j)$$

is injective.

(ii)

$$q_{12}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v-n_iv_i))\cap q_{23}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v-n_jv_j))) \\ \to M_H(v-n_iv_i)\times M_H(v-n_jv_j)$$

is injective.

Proof. We shall prove (i). The proof of (ii) is similar. Assume that we have isomorphisms in the derived category:

$$[F_1 \to \mathbb{E}] \cong [F_1 \to \mathbb{E}'],$$
$$[F_2 \to \mathbb{E}] \cong [F_2 \to \mathbb{E}'],$$

where $\mathbb{E}, \mathbb{E}' \in M_H(v)$ and $F_i = U_i \otimes E_i$, $i = 1, 2$. We shall show that there is an isomorphism $\phi : \mathbb{E} \to \mathbb{E}'$ which is compatible with the morphisms $F_i \to \mathbb{E}$, $F_i \to \mathbb{E}'$ ($i = 1, 2$). Applying $\mathrm{Hom}(\mathbb{E}',\ \ )$ to the exact triangles

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d, equal] & \mathbb{E} \arrow[r] \arrow[d] & {[F_1 \to \mathbb{E}]} \arrow[r] \arrow[d] & F_1[1] \arrow[d, equal] \\
F_1 \arrow[r] & {[F_2 \to \mathbb{E}]} \arrow[r] & {[F_1 \oplus F_2 \to \mathbb{E}]} \arrow[r] & F_1[1],
\end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{E}', F_1) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\mathbb{E}', \mathbb{E}) \arrow[r] \arrow[d] & {} \\
\mathrm{Hom}(\mathbb{E}', F_1) \arrow[r] & \mathrm{Hom}(\mathbb{E}', F_2 \to \mathbb{E}) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{E}', F_1 \to \mathbb{E}) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\mathbb{E}', F_1) \arrow[d, equal] \\
\mathrm{Hom}(\mathbb{E}', F_1 \oplus F_2 \to \mathbb{E}) \arrow[r] & \mathrm{Ext}^1(\mathbb{E}', F_1).
\end{tikzcd}$$

Since $\mathrm{Hom}(\mathbb{E}', F_1) = 0$, Lemma 1.11 implies that

$$\mathrm{Hom}(\mathbb{E}', F_2 \to \mathbb{E}) \to \mathrm{Hom}(\mathbb{E}', F_1 \oplus F_2 \to \mathbb{E})$$

is an isomorphism. Hence $\mathrm{Hom}(\mathbb{E}', \mathbb{E}) \to \mathrm{Hom}(\mathbb{E}', F_1 \to \mathbb{E}) \cong \mathbb{C}$ is also an isomorphism. We also have an isomorphism

$$\mathrm{Hom}(\mathbb{E}, \mathbb{E}') \to \mathrm{Hom}(\mathbb{E}, F_1 \to \mathbb{E}').$$

Then the claim easily follow from these isomorphisms. Hence

$$p_{12}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v))) \cap p_{23}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v)) \to M_H(v-n_iv_i)\times M_H(v-n_jv_j)$$

is injective. Q.E.D.

Lemma 2.4. *If $i \neq j$, then $p_{12}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v)))$ and $p_{23}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v))$ intersect transversely and $q_{12}^{-1}(\mathfrak{P}_{E_j}^{(n_j)}(v-n_i v_i))$ and $q_{23}^{-1}(\omega(\mathfrak{P}_{E_i}^{(n_i)}(v-n_j v_j)))$ intersect transversely.*

Proof. We set $F_i := U_i \otimes E_i$ and $F_j := U_j \otimes E_j$. We shall show that the map of the tangent spaces

$$\begin{aligned}(2.8)\qquad &\operatorname{Ext}^1(F_i \to \mathbb{E}, \mathbb{E})/\operatorname{End}(F_i) \oplus \operatorname{Ext}^1(F_j \to \mathbb{E}, \mathbb{E})/\operatorname{End}(F_j)\\ &\to \operatorname{Ext}^1(\mathbb{E}, \mathbb{E})\end{aligned}$$

is surjective and

$$\begin{aligned}(2.9)\qquad \operatorname{Ext}^1(F_i \oplus F_j \to \mathbb{E}, F_j \to \mathbb{E})/\operatorname{End}(F_i)&\\ \oplus \operatorname{Ext}^1(F_i \oplus F_j \to \mathbb{E}, F_i \to \mathbb{E})/\operatorname{End}(F_j)&\\ \to \operatorname{Ext}^1(F_i \oplus F_j \to \mathbb{E}, F_i \oplus F_j \to \mathbb{E})&\end{aligned}$$

is surjective.

We shall only prove (2.8). By (1.11) and Lemma 1.11, it is sufficient to show that the natural homomorphism

$$\operatorname{Ext}^1(F_i \to \mathbb{E}, \mathbb{E}) \to \operatorname{Ext}^1(\mathbb{E}, \mathbb{E}) \to \operatorname{Ext}^1(F_j, \mathbb{E})$$

is surjective. Since $\operatorname{Ext}^2(F_i \oplus F_j \to \mathbb{E}, \mathbb{E}) \cong \operatorname{Ext}^2(F_i \to \mathbb{E}, \mathbb{E}) \cong H^2(X, \mathcal{O}_X)$, the exact triangle

$$F_j \to [F_i \to \mathbb{E}] \to [F_i \oplus F_j \to \mathbb{E}] \to F_j[1]$$

implies that this homomorphism is surjective. Q.E.D.

By Lemmas 2.2, 2.3, 2.4, we obtain that

$$\begin{aligned}(2.10)\qquad &p_{13*}\left(p_{12}^*\left[\omega(\mathfrak{P}_{E_i}^{(n_i)}(v))\right] \cap p_{23}^*\left[\mathfrak{P}_{E_j}^{(n_j)}(v)\right]\right)\\ =&q_{13*}\left(q_{12}^*\left[\mathfrak{P}_{E_j}^{(n_j)}(v-n_i v_i)\right] \cap q_{23}^*\left[\omega\mathfrak{P}_{E_i}^{(n_i)}(v-n_i v_i)\right]\right).\end{aligned}$$

Hence we get

$$[e_{v_i}^{(n_i)}, f_{v_j}^{(n_j)}] = 0,\ i \neq j.$$

In particular, (2.3) holds for $i \neq j$.

(II) We next treat the case where $i = j$. This case was treated by Nakajima [N6]. For convenience of the reader, we write a self-contained proof. We assume that $n_i = 1$. If $i = j$, then $p_{12}^{-1}(\omega(\mathfrak{P}_{E_i}^{(1)}))$ and $p_{23}^{-1}(\mathfrak{P}_{E_j}^{(1)})$ intersect transversely outside $p_{13}^{-1}(\Delta_{M_H(v-v_i)})$, and $q_{12}^{-1}(\mathfrak{P}_{E_i}^{(1)}(v-v_i))$ and $q_{23}^{-1}(\omega\mathfrak{P}_{E_i}^{(1)}(v-v_i))$ intersect transversely outside $q_{13}^{-1}(\Delta_{M_H(v-v_i)})$. Then we see that

$$\begin{aligned} &p_{13*}\left(p_{12}^*\left[\omega(\mathfrak{P}_{E_i}^{(1)}(v))\right] \cap p_{23}^*\left[\mathfrak{P}_{E_i}^{(1)}(v)\right]\right) \\ =&q_{13*}\left(q_{12}^*\left[\mathfrak{P}_{E_i}^{(1)}(v-v_i)\right] \cap q_{23}^*\left[\omega\mathfrak{P}_{E_i}^{(1)}(v-v_i)\right]\right) \end{aligned} \tag{2.11}$$

outside $\Delta_{M_H(v-v_i)}$. Thus

$$\begin{aligned} &p_{13*}\left(p_{12}^*\left[\omega(\mathfrak{P}_{E_i}^{(1)}(v))\right] \cap p_{23}^*\left[\mathfrak{P}_{E_i}^{(1)}(v)\right]\right) \\ =&q_{13*}\left(q_{12}^*\left[\mathfrak{P}_{E_i}^{(1)}(v-v_i)\right] \cap q_{23}^*\left[\omega\mathfrak{P}_{E_i}^{(1)}(v-v_i)\right]\right) + c\Delta_{M_H(v-v_i)} \end{aligned} \tag{2.12}$$

for some integer c. In order to compute c, we may restrict to a suitable open neighbourhood of the generic point of $\Delta_{M_H(v-v_i)}$. We set $w := v - v_i$.

(II-1) Assume that $-\langle v_i, w\rangle \geq 0$. We set

$$\begin{aligned} M_H(w)' :=&M_H(w)_{E_i,-\langle v_i,w\rangle}, \\ M_H(w-v_i)' :=&M_H(w-v_i) \setminus \pi(\mathfrak{P}_{E_i}^{(-\langle v_i,w\rangle)}(w-v_i)). \end{aligned}$$

Then $\mathfrak{P}_{E_i}^{(1)}(w)' := \pi^{-1}(M_H(w)')$ is a projective bundle over $M_H(w)'$ and $\mathfrak{P}_{E_i}^{(1)}(w)' \to M_H(w-v_i)'$ is a closed immersion. We have a fiber product diagram:

$$\begin{array}{ccc} \mathfrak{P}_{E_i}^{(1)}(w)' & \longrightarrow & q_{23}^{-1}(\omega(\mathfrak{P}_{E_i}^{(1)}(w)')) \\ \downarrow & & \downarrow \\ q_{12}^{-1}(\mathfrak{P}_{E_i}^{(1)}(w)') & \longrightarrow & M_H(w)' \times M_H(w-v_i)' \times M_H(w)'. \end{array}$$

By the excess intersection theory, we get that

$$q_{12}^*\left[\mathfrak{P}_{E_i}^{(1)}(w)'\right] \cap q_{23}^*\left[\omega(\mathfrak{P}_{E_i}^{(1)}(w)')\right] = c_{top}(N_{\mathfrak{P}_{E_i}^{(1)}(w)'/M_H(w-v_i)'}).$$

We take $\mathbb{E} \in M_H(w)$ with $\operatorname{Ext}^1(E_i, \mathbb{E}) = 0$. We set $V := \operatorname{Hom}(E_i, \mathbb{E})$. Let $\mathbb{P} := \mathbb{P}(V^\vee)$ be the fiber of π. Then

$$\begin{aligned} &q_{13*}\left(q_{12}^*\left[\mathfrak{P}_{E_i}^{(1)}(w)'\right] \cap q_{23}^*\left[\omega(\mathfrak{P}_{E_i}^{(1)}(w)')\right]\right) \\ &\qquad = \left(\int_{\mathbb{P}} c_{top}(N_{\mathfrak{P}_{E_i}^{(1)}(w)'/M_H(w-v_i)'})\right) \Delta_{M_H(w)'}. \end{aligned}$$

We have a family of non-trivial homomorphisms:

$$\mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i \to \mathcal{O}_{\mathbb{P}} \boxtimes \mathbb{E}.$$

We set

$$\mathcal{E} := [\mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i \to \mathcal{O}_{\mathbb{P}} \boxtimes \mathbb{E}].$$

We have an exact sequence

$$\begin{aligned} &\operatorname{Ext}^1_{p_{\mathbb{P}}}(\mathcal{E}, \mathcal{O}_{\mathbb{P}} \boxtimes \mathbb{E}) \longrightarrow \operatorname{Ext}^1_{p_{\mathbb{P}}}(\mathcal{E}, \mathcal{E}) \longrightarrow \operatorname{Ext}^2_{p_{\mathbb{P}}}(\mathcal{E}, \mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i) \\ &\operatorname{Ext}^2_{p_{\mathbb{P}}}(\mathcal{E}, \mathcal{O}_{\mathbb{P}} \boxtimes \mathbb{E}) \longrightarrow \operatorname{Ext}^2_{p_{\mathbb{P}}}(\mathcal{E}, \mathcal{E}). \end{aligned}$$

The restriction of the normal bundle $(N_{\mathfrak{P}^{(1)}_{E_i}(w)'/M_H(w-v_i)'})_{|\mathbb{P}}$ is

$$\operatorname{Ext}^2_{p_{\mathbb{P}}}(\mathcal{E}, \mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i) = \operatorname{Hom}_{p_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i, \mathcal{E})^{\vee}.$$

By the exact triangle

$$\mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i \to \mathcal{O}_{\mathbb{P}} \boxtimes \mathbb{E} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i[1],$$

we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to V \otimes \mathcal{O}_{\mathbb{P}}(1) \to \operatorname{Hom}_{p_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i, \mathcal{E}) \to 0.$$

Hence $\operatorname{Hom}_{p_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-1) \boxtimes E_i, \mathcal{E})^{\vee} = \Omega^1_{\mathbb{P}}$. Therefore

$$\int_{\mathbb{P}} c_{top}(N_{\mathfrak{P}^{(1)}_{E_i}(w)'/M_H(w-v_i)'}) = (-1)^{\dim \mathbb{P}}(\dim \mathbb{P}+1) = (-1)^{-\langle v_i, w\rangle} \langle v_i, w\rangle.$$

Since $\mathfrak{P}^{(1)}_{E_i}(v)'$ does not meet $M_H(v) \times M_H(w)'$,

$$p_{13*}\left(p_{12}^*\left[\omega(\mathfrak{P}^{(1)}_{E_i}(v))\right] \cap p_{23}^*\left[\mathfrak{P}^{(1)}_{E_i}(v)\right]\right) = 0$$

on $M_H(w)' \times M_H(w)'$. Hence we see that

(2.13) $[e_{v_i}, f_{v_i}]_{|H_*(M_H(w),\mathbb{C})} = \langle v_i, w\rangle \operatorname{id}_{H_*(M_H(w),\mathbb{C})} = h_{v_i|H_*(M_H(w),\mathbb{C})}.$

(II-2) Assume that $\langle v_i, w\rangle \geq 0$. We set $M_H(v)' := M_H(v) \setminus \pi(\mathfrak{P}^{(2)}_{E_i})$ and $M_H(w)' := M_H(w)_{E_i,0}$. For $\mathbb{E} \in M_H(v)'$, we set $V := \operatorname{Ext}^1(\mathbb{E}, E_i)$. We have a family of exact triangles:

$$\mathcal{O}_{\mathbb{P}} \boxtimes E_i \to \mathcal{E}' \to \mathcal{O}_{\mathbb{P}}(-1) \boxtimes \mathbb{E} \to \mathcal{O}_{\mathbb{P}} \boxtimes E_i[1].$$

The restriction of the normal bundle $(N_{\mathfrak{P}^{(1)}_{E_i}(w)'/M_H(v)'})_{|\mathbb{P}}$ is

$$\operatorname{Ext}^1_{p_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}} \boxtimes E_i, \mathcal{E}') = \operatorname{Ext}^1_{p_{\mathbb{P}}}(\mathcal{E}', \mathcal{O}_{\mathbb{P}} \boxtimes E_i)^{\vee}.$$

We have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} = \mathrm{Hom}_{p_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}} \boxtimes E_i, \mathcal{O}_{\mathbb{P}} \boxtimes E_i) \to \\ \mathrm{Ext}^1_{p_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-1) \boxtimes \mathbb{E}, \mathcal{O}_{\mathbb{P}} \boxtimes E_i) \to \mathrm{Ext}^1_{p_{\mathbb{P}}}(\mathcal{E}', \mathcal{O}_{\mathbb{P}} \boxtimes E_i) \to 0.$$

Hence $\mathrm{Ext}^1_{p_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}} \boxtimes E_i, \mathcal{E}') = \Omega^1_{\mathbb{P}}$. Therefore

$$\int_{\mathbb{P}} c_{top}(N_{\mathfrak{P}^{(1)}_{E_i}(w)'/M_H(v)'}) = (-1)^{\dim \mathbb{P}}(\dim \mathbb{P} + 1) = -(-1)^{-\langle v_i, w\rangle}\langle v_i, w\rangle.$$

By using this equality, we see that (2.13) also holds.

2.2. The case where the twisted degree is zero

Let G be an element of $K(X)$.

Definition 2.1. Let $\mathbb{E} \in \mathbf{D}(X)$ be an object such that $\deg_G(\mathbb{E}) = 0$ and

$$\chi_G(\mathbb{E}) = \min\{\chi_G(E') > 0 | E' \in \mathrm{Coh}(X), \deg_G(E') = 0\}.$$

$\mathbb{E}$ is G-twisted stable, if

(i) $H^i(\mathbb{E}) = 0$, $i \neq 0$ and $H^0(\mathbb{E})$ is G-twisted stable, or
(ii) $H^i(\mathbb{E}) = 0$, $i \neq -1$ and $H^{-1}(\mathbb{E})$ is G-twisted stable.

Let $M_H^G(v)$ be the moduli space of G-twisted stable complex $\mathbb{E}$ with $v(\mathbb{E}) = v$.

Remark 2.1. If (ii) holds, then

$$\chi(H^{-1}(\mathbb{E})) = \max\{\chi_G(E') < 0 | E' \in \mathrm{Coh}(X), \deg_G(E') = 0\}.$$

Let E_i, $i = 1, \ldots, n$ be a collection of G-twisted stable vector bundles with $\deg_G(E_i) = \chi_G(E_i) = 0$ and $\langle v(E_i)^2\rangle = -2$. Assume that E_i satisfies Condition 2. By using Lemmas 1.4 and 1.5, we also obtain the same assertions as in Lemma 1.10. Hence we also get an action of the Lie algebra associated to $E_i, i = 1, \ldots, n$.

§3. Examples

3.1. Stable sheaves on a $K3$ surface

3.1.1. *Example 1.* Let X be a $K3$ surface and H an ample divisor on X. Let G be a μ-semi-stable vector bundle with respect to H such that $\langle v(G)^2\rangle = 0$. Assume that $G = \bigoplus_{i=0}^n E_i^{\oplus a_i}$, where E_i is a G-twisted stable vector bundle such that

$$\begin{aligned} \frac{\deg(E_i)}{\operatorname{rk} E_i} &= \frac{\deg(G)}{\operatorname{rk} G}, \\ \frac{\chi_G(E_i)}{\operatorname{rk} E_i} &= \frac{\chi_G(G)}{\operatorname{rk} G} = 0. \end{aligned} \tag{3.1}$$

By [O-Y, Thm. 0.1], $v(E_0), v(E_1), \dots, v(E_n)$ generate a lattice of affine type. We may assume that $a_0 = 1$. We set

$$l := \min\{\deg_G(E) > 0 | E \in \operatorname{Coh}(X)\}.$$

We set $v_i := v(E_i)$, $i = 0, 1, \dots, n$. Let $\mathfrak{g}$ be the affine Lie algebra associated with v_i, $i = 0, 1, \dots, n$ and $\overline{\mathfrak{g}}$ the finite Lie algebra associated with v_i, $i = 1, \dots, n$. Let $\overline{\mathfrak{h}}$ be the Cartan subalgebra of $\overline{\mathfrak{g}}$. For a root α, $\overline{\mathfrak{g}}_\alpha$ denotes the root space of α. $\theta := \sum_{i=1}^n a_i v_i$ denotes the highest root of $\overline{\mathfrak{g}}$. Then $\mathfrak{g}$ has the following standard expression:

$$\mathfrak{g} = \mathbb{C}[t, t^{-1}] \otimes \overline{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

The Chevalley generator of $\mathfrak{g}$ is

$$e_{v_i} = 1 \otimes \overline{e}_{v_i}, \quad f_{v_i} = 1 \otimes \overline{e}_{-v_i}, \quad h_{v_i} = 1 \otimes \overline{h}_{v_i} \quad 1 \le i \le n, \tag{3.2}$$

$$e_{v_0} = t \otimes \overline{e}_{-\theta}, \quad f_{v_0} = t^{-1} \otimes \overline{e}_\theta, \quad h_{v_0} = -\sum_{i=1}^n a_i h_{v_i} + c, \tag{3.3}$$

and d, where $\overline{e}_\alpha \in \overline{\mathfrak{g}}_\alpha$, $\overline{h}_{v_i} \in \mathfrak{h}$ and (3.2) are the Chevalley generator of $\overline{\mathfrak{g}}$. Hence we get

$$c = \sum_{i=0}^n a_i h_{v_i}.$$

The action of d on $H_*(M_H(v), \mathbb{C})$ is defined as follows: We take $w \in H^*(X, \mathbb{Q})$ such that $\langle w, v(E_i)\rangle = \delta_{i,0}$, $i = 0, 1, \dots, n$ and set

$$d_{|H_*(M_H(v),\mathbb{C})} := \langle w, v\rangle \operatorname{id}_{H_*(M_H(v),\mathbb{C})}.$$

Then we have the desired properties:

$$\begin{aligned} [d, e_{v_i}] &= \delta_{i,0} e_{v_i}, \\ [d, f_{v_i}] &= -\delta_{i,0} f_{v_i}. \end{aligned}$$

Proposition 3.1. *Assume that E_i are μ-stable for all i. Then we have an action of $\mathfrak{g}$ on $\bigoplus_{v \in V} H_*(M_H(v), \mathbb{C})$ such that the center c acts as a scalar multiplication $\langle v, v(G)\rangle$, where*

$$V := \{v \in H^*(X, \mathbb{Z}) | \deg_G(v) = l\}.$$

We shall give an example of Proposition 3.1. Let $C := (-a_{i,j})_{i,j=0}^{n}$ be a Cartan matrix of affine type and $\delta := (a_0, a_1, \dots, a_n)$, $a_i \in \mathbb{Z}_{>0}$ the primitive vector with $\delta C = 0$. Let (X, H) be a polarized $K3$ surface such that

(i) $\operatorname{Pic}(X) = \bigoplus_{i=0}^{n} \mathbb{Z}\xi_i$, $(\xi_i, \xi_j) = -a_{i,j} + 2ra$ and
(ii) $H = \sum_{i=0}^{n} a_i \xi_i$, where $r, a \in \mathbb{Z}_{>0}$.

For an existence of (X, H), see [O-Y, sect. 3]. We set $v_i := r + \xi_i + a\rho$, where $r \in H^0(X, \mathbb{Z}) = \mathbb{Z}$, $\xi_i \in \operatorname{NS}(X)$, $\rho \in H^4(X, \mathbb{Z})$ and $\int_X \rho = 1$. Then

(i) $(\langle v_i, v_j \rangle)_{i,j=0}^{n} = -C$,
(ii) $\deg(v_i) = (\xi_i, H) = 2ra(\sum_{i=0}^{n} a_i)$ and
(iii) $v := \sum_i a_i v_i$ is a primitive isotropic Mukai vector.

Lemma 3.2. *There is a vector bundle E_i with $v(E_i) = v_i$ which is μ-stable with respect to H.*

Proof. Since $\langle v_i^2 \rangle = -2$, Proposition 7.1 in appendix implies that there is a semi-stable sheaf E_i with $v(E_i) = v_i$. For a coherent sheaf F, we set $c_1(F) := \sum_i x_i \xi_i$. Then $\deg(F) = (\sum_i x_i) 2ra(\sum_i a_i)$. Since $\operatorname{rk} E_i = r$ and $\deg E_i = 2ra(\sum_i a_i)$ for all i, if

$$\frac{\deg(F)}{\operatorname{rk} F} = \frac{\deg(E_i)}{\operatorname{rk} E_i} = 2a(\sum_i a_i),$$

then $\operatorname{rk} F = (\sum_i x_i) r \geq \operatorname{rk} E_i$. Therefore E_i is μ-stable. Q.E.D.

We set $G := \bigoplus_{i=0}^{n} E_i^{\oplus a_i}$. Then (i), (ii) and (iii) imply that (3.1) holds. We set $w := (r(\sum_i x_i) - 1) + \sum_i x_i \xi_i + b\rho$, $x_i, b \in \mathbb{Z}$. Then

$$\deg_G(w) = \min\{\deg_G(E) > 0 | E \in \operatorname{Coh}(X)\},$$

where $G \in M_H(v)$. Hence applying Proposition 3.1, we have an action of $\mathfrak{g}$ on $\bigoplus_{w \in W} H_*(M_H(w), \mathbb{C})$, where

$$W := \left\{ w = \sum_i x_i v_i - 1 + b\rho \,\middle|\, x_i, b \in \mathbb{Z} \right\}.$$

3.1.2. *Example 2.* Let G be a vector bundle such that $\operatorname{rk} G = (H^2)$ and $c_1(G) = H$. For a Mukai vector $v := (1 + (D, H)) - D + a\rho$, $D \in \operatorname{NS}(X)$, $a \in \mathbb{Z}$, we get

$$\begin{aligned} \deg_G(v) &= (H, H)(-D, H) - (1 + (D, H))(-H, H) \\ &= (H, H) \\ &= \min\{\deg_G(E') > 0 | E' \in \operatorname{Coh}(X)\}. \end{aligned} \tag{3.4}$$

Let $C_1, C_2, \dots, C_n$ be irreducible (-2)-curves on X. We set $v_i := (C_i, H) - C_i$.

Lemma 3.3. *There is a stable vector bundle E_i with $v(E_i) = v_i$. Moreover if $H = nH'$ and $(C_i, H') < 2(n-1)(H'^2)$, then E_i is μ-stable.*

Proof. By Proposition 7.1, there is a semi-stable sheaf E_i with $v(E_i) = v_i$. We shall show that E_i is stable. Let $\bigoplus_{j=1}^s E_{i,j}$ be the Jordan–Hölder grading of E_i with respect to the Gieseker stability. We set $v(E_{i,j}) := r_j - D_j + a_j\rho$, $r_j \in \mathbb{Z}$, $D_j \in \mathrm{NS}(X)$, $a_j \in \mathbb{Z}$. Then $(D_j, H)/r_j = 1$ and $a_j/r_j = 0$. Hence $(D_j, H) > 0$ and $\langle v(E_{i,j})^2 \rangle = (D_j^2) \geq -2$, which implies that D_j is effective. By our assumption on C_i, $s = 1$. Thus E_i is stable. Assume that $H = nH'$ and $(C_i, H') < 2(n-1)(H'^2)$. Let $\bigoplus_{j=1}^s E_{i,j}$ be the Jordan–Hölder grading of E_i with respect to the μ-stability. We set $v(E_{i,j}) := r_j - D_j + a_j\rho$. Then $r_j = (D_j, H) = n(D_j, H')$, and hence $(D_j, H') > 0$. By the stability of $E_{i,j}$, $\langle v(E_{i,j})^2 \rangle = (D_j^2) - 2r_ja_j \geq -2$. By the Hodge index theorem, $(D_j, H')^2 \geq (D_j^2)(H'^2)$. If $a_j > 0$, then we see that $(C_i, H') > (D_j, H') \geq 2(n-1)(H'^2)$. Therefore $a_j \leq 0$. Since $\sum_j a_j = 0$, $a_j = 0$ for all j. Since E_i is stable, $s = 1$. Thus E_i is μ-stable. Q.E.D.

Proposition 3.4. *Assume that there are μ-stable sheaves E_i with $v(E_i) = v_i$. Then we have an action of the Lie algebra $\mathfrak{g}$ associated to C_i, $i = 1, 2, \dots, n$ on $\bigoplus_{v \in V} H_*(M_H(v), \mathbb{C})$, where*

$$V := \{v = (1 + (D, H)) - D + a\rho | D \in \mathrm{Pic}(X), a \in \mathbb{Z}\}.$$

We shall give an example. Let $\pi : X \to \mathbb{P}^1$ be an elliptic $K3$ surface with a section C_0. Let $C_1, \dots, C_n$ be smooth (-2)-curves on fibers of π. We set $v_i := (C_i, H) - C_i$, $i = 0, 1, \dots, n$. Then $(\langle v_i, v_j \rangle_{i,j}) = ((C_i, C_j)_{i,j})$. We assume that $(C_i, C_j) \leq 1$. Hence we get an action of the Lie algebra generated by $C_i, 0 \leq i \leq n$ on $\bigoplus_{v \in V} H_*(M_H(v), \mathbb{C})$, where

$$V := \{v = (1 + (D, H)) - D + a\rho | D \in \mathrm{NS}(X), a \in \mathbb{Z}\}.$$

3.1.3. *Example 3.* We give examples of Subsection 2.2. Let $G = \bigoplus_{i=0}^n E_i^{\oplus a_i}$ be a G-twisted semi-stable sheaf with isotropic Mukai vector and E_i, $i = 0, 1, ..., n$, G-twisted stable vector bundles with (3.1). We set $v_i := v(E_i)$. Then we see that

$$\begin{aligned} &\{\chi_G(w) | w \in v(\mathbf{D}(X)), \deg_G(w) = 0\} \\ =&\{\chi_G(w) | w = \sum_i x_i v_i + y\rho\} \\ =&\mathbb{Z}\langle v(G), \rho \rangle. \end{aligned}$$

Hence we have an action of $\mathfrak{g}$ on $\bigoplus_{w \in W} H_*(M_H^G(w), \mathbb{C})$, where

$$W := \{w = \sum_i x_i v_i + \rho | x_i \in \mathbb{Z}\}.$$

We give another example of the action. Let $\pi : X \to \mathbb{P}^1$ be the elliptic $K3$ surface in Example 2. Let G be an element of $K(X)$ with $v(G) = (H, f) - f$. We set $v_D := (H, C_0 + D) - (C_0 + D)$. Then $\deg_G(v) = 0$ and $\chi_G(v) = -(1 + (D, f))$. We assume that E_i are G-twisted stable (or more strongly μ-stable). Let $\mathfrak{g}'$ be the Lie algebra generated by $C_1, \dots, C_n$. By the remarks in Subsection 2.2, we can construct an action of $\mathfrak{g}'$ on $\bigoplus_{D \in \mathcal{D}} H_*(M_H^G(v_D), \mathbb{C})$, where

$$\mathcal{D} := \{D \in \mathrm{NS}(X) |\ D \text{ is an effective divisor with } (D, f) = 0\}.$$

3.2. Stable sheaves on an Enriques surface

Let X be an Enriques surface and $\pi : Y \to X$ be the covering $K3$ surface of X. Assume that X contains a smooth (-2) curve C. Let C' be a connected component of $\pi^{-1}(C)$. Let H' be an ample divisor on Y and set $H := \pi_*(H')$. Then H is an ample divisor on X with $(H, C) = 2(H', C')$. We take a semi-stable sheaf E' on Y with $v(E') = (H', C') - C'$. E' is a rigid vector bundle. If H' is sufficiently ample, then Lemma 3.3 implies that E' is μ-stable.

Proposition 3.5. *We set $E := \pi_*(E')$. Then E is a μ-stable vector bundle with the Mukai vector $(H, C) - C$ which satisfies $E \otimes K_X \cong E$ and*

$$\begin{cases} \mathrm{Hom}(E, E) = \mathbb{C} \\ \mathrm{Ext}^1(E, E) = 0 \\ \mathrm{Ext}^2(E, E) = \mathbb{C}. \end{cases}$$

If there is a configuration of (-2)-curves, then as in the $K3$ surface case, we have an action of the Lie algebra associated to (-2)-curves on $\bigoplus_{v \in V} H_*(M_H(v), \mathbb{C})$, where

$$V := \{v = (1 + (D, H)) + D + a\rho | D \in \mathrm{NS}(X), a - 1/2 \in \mathbb{Z}\}.$$

§4. Actions associated to purely 1-dimensional exceptional sheaves

4.1. Purely 1-dimensional sheaves

In this section, we shall consider Lie algebra actions associated to purely 1-dimensional exceptional sheaves such as line bundles on (-2)-curves. Unfortunately we cannot construct the action for the moduli spaces of stable torsion free sheaves in general. Instead, we can construct it for the moduli spaces of purely 1-dimensional sheaves. In some cases, the moduli spaces of stable torsion free sheaves are deformation equivalent to moduli spaces of purely 1-dimensional sheaves. In this sense, we have an action for the moduli spaces of stable torsion free sheaves. This will be explained in 4.3. We also explain a partial result on the moduli spaces of stable torsion free sheaves in 4.4.

Let (X, H) be a pair of a smooth projective surface X and an ample divisor H on X.

Definition 4.1. [Y4] Let G be an element of $K(X)$ with $\operatorname{rk} G > 0$. A purely 1-dimensional sheaf E is G-twisted stable, if

$$\frac{\chi_G(F)}{(c_1(F), H)} < \frac{\chi_G(E)}{(c_1(E), H)}$$

for all proper subsheaves $F(\neq 0)$ of E.

We have the following result whose proof is similar to Lemma 1.1.

Lemma 4.1. *Let G be an element of $K(X)$ with $\operatorname{rk} G > 0$ and E_i, $i = 1, 2, \ldots, s$, be purely 1-dimensional G-twisted stable sheaves with $\chi_G(E_i) = 0$. Let E be a purely 1-dimensional G-twisted stable sheaf with*

$$\chi_G(E) = \min\{\chi_G(E') > 0 | E' \in \operatorname{Coh}(X), \operatorname{rk} E' = 0\} \tag{4.1}$$

or $E = \mathbb{C}_P$, $P \in X$ with the condition (4.1).

(1) *Then every non-trivial extension*

$$0 \to E_1 \to F \to E \to 0$$

defines a G-twisted stable sheaf.

(2) *Let V_i be a subspace of $\operatorname{Hom}(E_i, E)$. Then $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is injective or surjective. Moreover,*

- (2-1) *if $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is injective, then the cokernel is a G-twisted stable purely 1-dimensional sheaf or $\mathbb{C}_P$, $P \in X$,*
- (2-2) *if $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is surjective, then $\ker \phi$ is G-twisted stable.*

Lemma 4.2. *Let G be an element of $K(X)$ with $\operatorname{rk} G > 0$ and E_i, $i = 1, 2, \ldots, s$, be purely 1-dimensional G-twisted stable sheaves with $\chi_G(E_i) = 0$. Let E be a purely 1-dimensional G-twisted stable sheaf with*

$$\chi_G(E) = \max\{\chi_G(E') < 0 | E' \in \operatorname{Coh}(X), \operatorname{rk} E' = 0\}. \tag{4.2}$$

(1) *Then every non-trivial extension*

$$0 \to E \to F \to E_1 \to 0$$

defines a G-twisted stable sheaf.

(2) *Let V_i be a subspace of* $\operatorname{Hom}(E, E_i)$*. Then $\phi : E \to \bigoplus_{i=1}^{s} V_i^{\vee} \otimes E_i$ is injective or surjective. Moreover,*

(2-1) *if $\phi : E \to \bigoplus_{i=1}^{s} V_i^{\vee} \otimes E_i$ is injective, then the cokernel is a G-twisted stable purely 1-dimensional sheaf or $\mathbb{C}_P$, $P \in X$,*

(2-2) *if $\phi : E \to \bigoplus_{i=1}^{s} V_i^{\vee} \otimes E_i$ is surjective, then* $\ker \phi$ *is G-twisted stable.*

Remark 4.1. We set

$$d := \min\{\chi_G(E') > 0 | E' \in \operatorname{Coh}(X), \operatorname{rk} E' = 0\}.$$

For a purely 1-dimensional sheaf E with $\chi_G(E) = d$, E is G-twisted stable if and only if $\chi_G(F) \le 0$ for all proper subsheaves F of E. Thus the G-twisted stability does not depend on the choice of H.

Definition 4.2. For a complex $\mathbb{E}$ with $\operatorname{rk}(\mathbb{E}) = 0$, we set

$$v(\mathbb{E}) := (c_1(\mathbb{E}), \chi(\mathbb{E})) \in H^2(X, \mathbb{Z}) \times \mathbb{Z}.$$

We define a pairing of $v_i := (\xi_i, a_i) \in H^2(X, \mathbb{Z}) \times \mathbb{Z}$, $i = 1, 2$ by

$$\langle v_1, v_2 \rangle := (\xi_1, \xi_2) \in \mathbb{Z}.$$

Then the Riemann–Roch theorem says that

$$\chi(\mathbb{E}, \mathbb{F}) = -\langle v(\mathbb{E}), v(\mathbb{F}) \rangle$$

for $\mathbb{E}, \mathbb{F} \in \mathbf{D}(X)$ with $\operatorname{rk}(\mathbb{E}) = \operatorname{rk}(\mathbb{F}) = 0$. We set $\rho := v(\mathbb{C}_P) = (0, 1)$.

Definition 4.3. Let $\mathbb{E} \in \mathbf{D}(X)$ be an object such that $\operatorname{rk}(\mathbb{E}) = 0$ and

$$\chi_G(\mathbb{E}) = \min\{\chi_G(\mathbb{E}') > 0 | \mathbb{E}' \in \mathbf{D}(X), \operatorname{rk}(\mathbb{E}') = 0\}.$$

$\mathbb{E}$ is G-twisted stable, if

(i) $H^i(\mathbb{E}) = 0$, $i \ne 0$ and $H^0(\mathbb{E})$ is G-twisted stable, or
(ii) $H^i(\mathbb{E}) = 0$, $i \ne -1$ and $H^{-1}(\mathbb{E})$ is G-twisted stable.

Let $M_H^G(v)$ be the moduli space of G-twisted stable complexes $\mathbb{E}$ with $v(\mathbb{E}) = v$.

Let E_i, $i = 1, 2, \dots, n$ be G-twisted stable purely 1-dimensional sheaves such that $\chi_G(E_i) = 0$, $E_i \otimes K_X \cong E_i$ and $\langle v(E_i)^2 \rangle = -2$. We set $v_i := v(E_i)$. Let $\mathfrak{g}$ be the Lie algebra associated to E_i, $i = 1, \dots, n$. By using Lemma 4.1, 4.2, we get the following similar results to the results in Section 2.

Proposition 4.3. *For any $\mathbb{E} \in M_H^G(v+\sum_i x_i v_i)$, $x_i \in \mathbb{Z}$, we assume that*

$$\chi_G(\mathbb{E}) = \min\{\chi_G(\mathbb{E}') > 0 | \mathbb{E}' \in \mathbf{D}(X), \operatorname{rk} \mathbb{E}' = 0\}$$

and $\mathbb{E}$ satisfies Condition 1. Then we have an action of $\mathfrak{g}$ on $\bigoplus_{x_i \in \mathbb{Z}} H_*(M_H^G(v + \sum_i x_i v_i), \mathbb{C})$.

Let C be an irreducible (-2)-curve on X. If $G = \mathcal{O}_X$, then $\mathcal{O}_C(-1)$ is a stable sheaf with $\chi(\mathcal{O}_C(-1)) = 0$. Then we can apply Proposition 4.3.

Lemma 4.4. *Let X be a 9 point blow-up of $\mathbb{P}^2$ and assume that $|-K_X|$ contains a reducible curve $Y = \sum_{i=0}^n a_i C_i$, where C_i are smooth (-2)-curves. Then every G-twisted stable purely 1-dimensional sheaf E with $(c_1(E), K_X) < 0$ satsifies Condition 1.*

Proof. Assume that there is a non-zero map $\psi : E \to E(K_X) = E(-Y)$. By the homomorphism $\mathcal{O}_X(-Y) \to \mathcal{O}_X$, we have a homomorphism $E(-Y) \to E$, which is isomorphic on $\operatorname{Supp}(E) \setminus Y \neq \emptyset$. If $E \to E(-Y) \to E$ is a zero map, then $F := \psi(E)(-K_X)$ satisfies $\operatorname{Supp}(F(K_X)) \subset Y$ and

$$\frac{\chi_G(E)}{(c_1(E), H)} < \frac{\chi_G(F(K_X))}{(c_1(F), H)}.$$

On the other hand, since F is a proper subsheaf of E, we have

$$\frac{\chi_G(F)}{(c_1(F), H)} < \frac{\chi_G(E)}{(c_1(E), H)}.$$

Since $(C_i, K_X) = 0$, we get $(c_1(F), K_X) = 0$. This means that $\chi_G(F(K_X)) = \chi_G(F)$. Then we get

$$\frac{\chi_G(E)}{(c_1(E), H)} < \frac{\chi_G(E)}{(c_1(E), H)}.$$

This is a contradiction. Therefore $E \to E(-Y) \to E$ is a non-zero map. Then by using the stability of E and $(\operatorname{Div}(E), Y) > 0$, we get a contradiction. Hence we conclude that $\operatorname{Hom}(E, E(K_X)) = 0$. Q.E.D.

Corollary 4.5. *Under the assumption in Lemma 4.4, we have an action of the affine Lie algebra associated to C_i, $0 \leq i \leq n$ on $\bigoplus_{D \in \mathcal{D}} H_*(M_H((D,1)),\mathbb{C})$, where*

$$\mathcal{D} := \{D \in \mathrm{NS}(X) | (D, K_X) < 0\}.$$

Proposition 4.6. *Let C_i, $i = 0,1,\dots,n$ be a configuration of smooth (-2)-curves of ADE or affine type such that K_X is trivial in a neighbouhood of $\cup_i C_i$ and $(C_i, C_j) \leq 1$ for $i \neq j$. Let $D := \sum_{i=0}^n b_i C_i$, $b_i > 0$ be an effective divisor such that $(D^2) = -2$ and m an integer. Then there is a G-twisted stable sheaf E with $(c_1(E), \chi(E)) = (D, m)$ for a general (H, G).*

Proof. If $n = 0$, then $D = C_0$ and obviously the claim holds. Hence we may assume $n > 0$ and $\cup_i C_i$ is connected. We set $v_i := v(\mathcal{O}_{C_i}(-1))$. We first show that $M_H(\rho+\sum_i b_i v_i) \neq \emptyset$. Assume that there is a stable sheaf E such that $\mathrm{Supp}(E) \subset \cup_i C_i$. Since K_X is trivial in a neighbourhood of $\cup_i C_i$, $\mathrm{Ext}^2(E,E) \cong \mathrm{Hom}(E, E \otimes K_X)^\vee = \mathbb{C}$. Hence we see that $(c_1(E)^2) \geq -2$ and the equality holds when $\mathrm{Ext}^1(E,E) = 0$. In particular $M_H(\rho+\sum_i b_i v_i)$ is smooth. Let R_{v_i} be the (-2)-reflection defined by v_i and W the Weyl group generated by R_{v_i}, $i = 0,1,\dots,n$. Then by the action of W, we have an isomorphism $M_H(\rho + \sum_i b_i v_i) \to M_H(\rho + v_j)$ for some j. Obviously $M_H(\rho + v_j) = \{\mathcal{O}_{C_j}\}$. Therefore $M_H(\rho + \sum_i b_i v_i) \neq \emptyset$.

We shall treat the general cases. Since K_X is trivial in a neighbouhood of $\cup_i C_i$, by using [Y4, Prop. 2.7], we see that the non-emptyness of $M_H^G(v)$ does not depend on the choice of a general (H, G). There is a divisor C such that $(C, D) = 1$. Indeed we take an element $w \in W$ such that $w(D) = C_i$ for some i. Then $1 = (C_j, C_i) = (w(C_j), D)$ for a j. Let E be a stable sheaf with $(c_1(E), \chi(E)) = (D, 1)$. Then $E(nC)$ is a $\mathcal{O}_X(nC)$-twisted stable sheaf with $\chi(E(nC)) = 1 + n$. Therefore our claim holds for general cases.

Q.E.D.

Example 4.1. Let Y be a germ of a rational double point and $\pi : X \to Y$ the minimal resolution. Let H be a π-ample divisor on X. Let C_i, $i = 1,2,\dots,n$ be irreducible components of the exceptional divisor. We set $v_i := v(\mathcal{O}_{C_i}(-1))$. Let $\mathfrak{g}$ be the Lie algebra associated to C_i. We note that $K_X \cong \mathcal{O}_X$. For a coherent sheaf E on X with a compact support, we can define the stability with respect to H. For a stable sheaf E with $v(E) = \rho+\sum_i n_i v_i$, $\dim \mathrm{Ext}^1(E,E) = \langle(\rho+\sum_i n_i v_i)^2\rangle+2$. Hence we get

$$\dim \operatorname{Ext}^1(E,E) = \begin{cases} 2, & \text{if } v = \rho, \\ 0, & \text{if } \langle (\sum_i n_i v_i)^2 \rangle = -2. \end{cases}$$

If $v = \rho$, then all stable sheaves are of the form $\mathbb{C}_P, P \in X$. Hence $M_H(\rho)$ has a coarse moduli space which is isomorphic to X. Hence $M_H(\rho)$ is smooth. If $\langle (\sum_i n_i v_i)^2 \rangle = -2$, then the proof of Proposition 4.6 implies that $M_H(\rho + \sum_i n_i v_i)$ is not empty and consists of a stable sheaf on the exceptional divisors. Then we have an action of $\mathfrak{g}$ on $\bigoplus_{n_i \in \mathbb{Z}} H_*(M_H(\rho+\sum_i n_i v_i), \mathbb{C})$. Indeed the submodule consisting of the middle degree homology groups is isomorphic to $\mathfrak{g}$. For the structure of $M_H(\rho + \sum_i n_i v_i)$, we get the following: Let $D = \sum_i n_i C_i$ be an effective divisor with $(D^2) = -2$. Then $M_H(\rho + \sum_i n_i v_i) = \{\mathcal{O}_D\}$ and $M_H(\rho - \sum_i n_i v_i) = \{\mathcal{O}_D(D)\}$.

Proof of the claim: We note that $\chi(\mathcal{O}_D) = -(D^2)/2 = 1$. If there is a quotient $\mathcal{O}_D \to \mathcal{O}_{D'}$, then since $({D'}^2) < 0$, we have $\chi(\mathcal{O}_{D'}) = -({D'}^2)/2 \geq 1$. Therefore $\mathcal{O}_D$ is stable. We note that $\mathcal{O}_D(D)$ is the derived dual of $\mathcal{O}_D$. By using this fact, we can easily see the stability of $\mathcal{O}_D(D)$.

Example 4.2. Let C be a germ of a curve at P and $\pi : X \to C$ an elliptic surface with a section σ. Let H be a π ample divisor on X. Assume that $\pi^{-1}(P)$ is reducible and consists of smooth (-2)-curves C_i, $i = 0, 1, \dots, n$: $\pi^{-1}(P) = \sum_{i=0}^n a_i C_i$. We may assume that $a_0 = 1$ and $(\sigma, C_0) = 1$. We set $v_i := v(\mathcal{O}_{C_i}(-1))$. Then we see that $M_H(\rho + \sum_i n_i v_i)$ is smooth with

$$\dim M_H(\rho + \sum_i n_i v_i) = \langle (\sum_i n_i v_i)^2 \rangle + 2$$

and $M_H(\rho+\sum_i n_i v_i) \cong X$, if $\langle (\sum_i n_i v_i)^2 \rangle = 0$. We also have an action of affine Lie algebra $\mathfrak{g}$ associated to v_i on $\bigoplus_{n_i \in \mathbb{Z}} H_*(M_H(\rho + \sum_i n_i v_i), \mathbb{C})$. Indeed, if $(C_i, C_j) \leq 1$ for all $i \neq j$, then the result obviously holds. If $(C_i, C_j) = 2$, then we can directly check the commutation relation (2.3). We set $\delta := v(\mathcal{O}_{\pi^{-1}(P)}) = \sum_{i=0}^n a_i v_i$. If $\sum_i n_i v_i = m\delta$, $m \in \mathbb{Z}$, then under an identification $M_H(\rho + m\delta) \cong X$, we have an isomorphism

$$H_2(M_H(\rho + m\delta), \mathbb{C}) \cong \mathbb{C}[\sigma] \oplus \bigoplus_{i=1}^n \mathbb{C}[C_i].$$

Let

$$\mathfrak{g} = \mathbb{C}[t, t^{-1}] \otimes \overline{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

be the standard expression of the affine Lie algebra, where c is the center of $\mathfrak{g}$. Then we have an exact sequence of $\mathfrak{g}$-modules:

$$0 \to \mathbb{C}[t,t^{-1}] \otimes \overline{\mathfrak{g}} \to \bigoplus_{n_i \in \mathbb{Z}} H_{mid}(M_H(\rho + \sum_i n_i v_i), \mathbb{C}) \to \mathbb{C}[t,t^{-1}] \to 0,$$

where $H_{mid}(\star)$ is the middle degree homology group of $\star$ and $\mathbb{C}[t,t^{-1}]$ is the $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = \mathbb{C}d$-module.

4.2. Moduli of stable sheaves on elliptic surfaces

We first collect some basic facts on the moduli spaces of stable sheaves on elliptic surfaces X. If K_X is not numerically trivial, then we do not have a good invariant of a torsion free sheaf E which is a suitable generalization of the Mukai vector. In these cases, we shall use $\gamma(E) := (\mathrm{rk}(E), c_1(E), \chi(E)) \in H^*(X,\mathbb{Z})$ as an invariant of E. If $\mathrm{rk}\, E = 0$, then $\gamma(E) = (0, c_1(E), \chi(E))$ is the same as the Mukai vector $v(E)$ defined in Definition 4.2. We denote the moduli space of G-twisted stable sheaves E on X with $\gamma(E) = (r, \xi, \chi)$ by $M_H^G(r,\xi,\chi)$. We also denote the moduli of G-twisted semi-stable sheaves by $\overline{M}_H^G(r,\xi,\chi)$.

Let $\pi : X \to C$ be an elliptic surface. Let f be a smooth fiber and L a nef and big divisor on X. Since $(L,f) > 0$, replacing L by $L + nf$, $n > 0$, we may assume that $(L, C') > 0$ unless C' is a (-2)-curve in a fiber of π. Let G be a locally free sheaf on X such that $\mathrm{rk}\, G = r$ and $(c_1(G), f) = d$ with $\gcd(r,d) = 1$. We first study the stability condition, when the polarization is sufficiently close to f.

Lemma 4.7. *For $(\xi,\chi) \in \mathrm{NS}(X) \times \mathbb{Z}$ with $r\chi - (c_1(G),\xi) > 0$, we take $(n,\varepsilon) \in \mathbb{Z} \times \mathrm{NS}(X) \otimes \mathbb{Q}$ such that $n \gg 0$ and ε is an ample $\mathbb{Q}$-divisor with $|\varepsilon| \ll 1$. Let E be a purely 1-dimensional sheaf with $v(E) = (\xi,\chi)$.*

(i) *E is G-twisted stable with respect to $L + nf + \varepsilon$ if and only if for any proper subsheaf F of E, one of the following holds*
(a)

$$\frac{\chi_G(E)}{(c_1(E),f)} > \frac{\chi_G(F)}{(c_1(F),f)}.$$

(b)

$$\frac{\chi_G(E)}{(c_1(E),f)} = \frac{\chi_G(F)}{(c_1(F),f)}, \quad \frac{\chi_G(E)}{(c_1(E),L)} > \frac{\chi_G(F)}{(c_1(F),L)}.$$

(c)

$$\frac{\chi_G(E)}{(c_1(E),f)} = \frac{\chi_G(F)}{(c_1(F),f)}, \frac{\chi_G(E)}{(c_1(E),L)} = \frac{\chi_G(F)}{(c_1(F),L)},$$
$$\frac{\chi_G(E)}{(c_1(E),\varepsilon)} > \frac{\chi_G(F)}{(c_1(F),\varepsilon)}.$$

(ii) *Moreover if we assume that $L + nf$ is ample and* $\gcd((c_1(E),f),(c_1(E),L),\chi_G(E)) = 1$, *then there is no properly G-twisted semi-stable sheaf E with respect to $L+nf$. In particular, E is G-twisted stable with respect to $L+nf$ if and only if E is $G(\eta)$-twisted stable with respect to $L+nf+\varepsilon'$, where η, ε' are sufficiently small $\mathbb{Q}$-divisors.*

Remark 4.2. If $(c_1(E),f) = 0$ or $(c_1(F),f) = 0$ in the condition (a), we regard the inequality as $(c_1(F),f)\chi_G(E) - (c_1(E),F)\chi_G(F) > 0$. Similar conventions are used for the conditions (b) and (c).

Proof. We note that $\chi_G(E) = r\chi - (c_1(G),\xi) > 0$. If

$$\chi_G(E)(c_1(F),f) - \chi_G(F)(c_1(E),f) < 0,$$

then we see that

$$\begin{aligned}&\chi_G(E)(c_1(F),L+nf+\varepsilon) - \chi_G(F)(c_1(E),L+nf+\varepsilon)\\ \le& n(\chi_G(E)(c_1(F),f) - \chi_G(F)(c_1(E),f)) + \chi_G(E)(c_1(E),L+\varepsilon).\end{aligned}$$

If $\chi_G(E)(c_1(F),f) - \chi_G(F)(c_1(E),f) \ge 0$ and $\chi_G(F) \ge 0$, then

$$\begin{aligned}&\chi_G(E)(c_1(F),L+nf+\varepsilon) - \chi_G(F)(c_1(E),L+nf+\varepsilon)\\ \ge& n(\chi_G(E)(c_1(F),f) - \chi_G(F)(c_1(E),f))\\ &+ \frac{\chi_G(E)}{(c_1(E),f)}((c_1(E),f)(c_1(F),L+\varepsilon) - (c_1(F),f)(c_1(E),L+\varepsilon))\\ \ge& n(\chi_G(E)(c_1(F),f) - \chi_G(F)(c_1(E),f)) - \chi_G(E)(c_1(E),L+\varepsilon).\end{aligned}$$

By using these inequalities, we can show claim (i). Moreover, if $L+nf$ is ample, then the equalities

$$\begin{aligned}\chi_G(E)(c_1(F),f) - \chi_G(F)(c_1(E),f) &= 0,\\ \chi_G(E)(c_1(F),L) - \chi_G(F)(c_1(E),L) &= 0\end{aligned}$$

imply that $(c_1(F),f) = (c_1(F),L) = \chi_G(F) = 0$ or $(c_1(E/F),f) = (c_1(E/F),L) = \chi_G(E/F) = 0$. By the ampleness of $L+nf$, we get $F = 0$ or $E/F = 0$. Therefore the claim (ii) holds. Q.E.D.

Remark 4.3. Obviously the choice of n and ε depends on (ξ, χ). Lemma 4.7 says that we have the moduli space of coherent sheaves satisfying conditions (a), (b), or (c).

Under some conditions, we can interpret Lemma 4.7 in the following way.

Lemma 4.8. *Let E be a purely 1-dimensional sheaf with $\chi_G(E) > 0$ and $(c_1(E), f) > 0$.*

(1) *Assume that* $\gcd(\chi_G(E), (c_1(E), f)) = 1$. *Then E satisfies (a), (b) or (c) in Lemma 4.7 if and only if the following three conditions hold:*
  (i) *E does not have a non-trivial subsheaf F with $(c_1(F), f) = 0$ and $\chi_G(F) > 0$.*
  (ii) *E does not have a non-trivial quotient F with $(c_1(F), f) = 0$ and $\chi_G(F) \leq 0$.*
  (iii) *For any subsheaf F of E,*
$$(c_1(F), f)\chi_G(E) \geq (c_1(E), f)\chi_G(F).$$

(2) *Assume that* $\gcd(\chi_G(E), (c_1(E), f), (c_1(E), L)) = 1$. *Then E satisfies (a), (b) or (c) in Lemma 4.7 if and only if the following three conditions hold:*
  (i) *E does not have a non-trivial subsheaf F with $(c_1(F), f) = (c_1(F), L) = 0$ and $\chi_G(F) > 0$.*
  (ii) *E does not have a non-trivial quotient F with $(c_1(F), f) = (c_1(F), L) = 0$ and $\chi_G(F) \leq 0$.*
  (iii) *For any subsheaf F of E,*
$$(c_1(F), L + nf)\chi_G(E) \geq (c_1(E), L + nf)\chi_G(F), n \gg 0.$$

Proof. We only prove (1). Assume that E satisfies (a), (b) or (c). Let F be a non-trivial subsheaf of E with $(c_1(F), f) = 0$. Then $0 = (c_1(F), f)\chi_G(E) \leq (c_1(E), f)\chi_G(F)$ implies (i). Let $\phi : E \to F$ be a non-trivial quotient of E with $(c_1(F), f) = 0$. Since $(c_1(E), f) = (c_1(\ker \phi), f)$, we get $\chi_G(E) \geq \chi(\ker \phi)$. Thus $\chi_G(F) \geq 0$. If the equality holds, then $\chi_G(E) = \chi(\ker \phi)$. In this case, (b) or (c) imply that $\ker \phi = E$, which is a contradiction. Therefore (ii) also holds. Since (iii) also holds, E satisfies (i), (ii) and (iii).

Conversely we assume that E satisfies (i), (ii) and (iii). Let F be a subsheaf of E. If (a) does not hold, then (iii) and $\gcd(\chi_G(E), (c_1(E), f)) = 1$ imply that $(\chi_G(F), (c_1(F), f)) = (\chi_G(E), (c_1(E), f))$ or $(0, 0)$. For the first case, E/F satisfies $\chi_G(E/F) = 0$ and $(c_1(E/F), f) = 0$. By (ii), $E/F = 0$. For the second case, (b) or (c) holds. Q.E.D.

Lemma 4.9. *Let $\pi : X \to C$ be an elliptic surface and f a fiber of π. If $(D, f) = 1$, then $\mathbb{E} \in M_H(0, D, 1)$ satisfies Condition 1.*

For the proof, see [Y4, Prop. 3.18]. Let $\pi^{-1}(p) = \sum_{i=0}^{n} a_i C_i$ be a singular fiber of π such that $(C_i, C_j) \leq 1$. We may assume that $(C_0, \sigma) = a_0 = 1$.

Lemma 4.10. *There is a G-twisted stable sheaf E_0 with $c_1(E_0) = (r-1)f + C_0$ and $\chi_G(E_0) = 0$.*

Proof. By Proposition 4.6, there is a $G(\varepsilon)$-twisted stable sheaf E_0 with $c_1(E_0) = (r-1)f + C_0$ and $\chi_G(E_0) = 0$, where $\varepsilon \in \mathrm{NS}(X) \otimes \mathbb{Q}$ is sufficiently small. Then E_0 is G-twisted semi-stable. Assume that E_0 is S-equivalent to $\bigoplus_i F_i$, where F_i are G-twisted stable sheaves with $\chi_G(F_i) = 0$. Since $\mathrm{Supp}(F_i)$ does not contain σ, $(c_1(F_i), \sigma) \geq 0$. Since $\chi_G(F_i) = r\chi(F_i) - d(\sigma, c_1(F_i))$, there is an integer i_0 such that $(\sigma, c_1(F_{i_0})) = r$ and $(\sigma, c_1(F_i)) = 0$ for $i \neq i_0$. Thus $\mathrm{Supp}(F_i)$, $i \neq i_0$ do not contain C_0, which implies that $(c_1(F_i), C_0) \geq 0$, $i \neq i_0$. Then we see that $(c_1(F_{i_0})^2) \leq (c_1(E_0)^2) + ((\sum_{i \neq i_0} c_1(F_i))^2) < -2$. This is a contradiction. Therefore E_0 is G-twisted stable. Q.E.D.

Lemma 4.11. *Let E_0 be the G-twisted stable sheaf with $c_1(E_0) = (r-1)f + C_0$ and $\chi_G(E_0) = 0$. We set $E_i := \mathcal{O}_{C_i}(-1)$, $i > 0$. Let E be a properly G-twisted semi-stable sheaf with $\gamma(E) = (0, rf, d)$ and $\mathrm{Supp}(E) = \cup_{i=0}^{n} C_i$. Then E is S-equivalent to $\bigoplus_i E_i^{\oplus a_i}$.*

Proof. Assume that E is S-equivalent to $\bigoplus_j F_j$, where F_j are G-twisted stable sheaves with $\chi_G(F_j) = 0$. If $\langle v(F_j)^2 \rangle = 0$, then $c_1(F_j) = r_j f$, $r_j \in \mathbb{Z}_{>0}$. Since $\gcd(r, d) = 1$, we see that $\chi_G(F_j) \neq 0$, which is a contradiction. Thus $\langle v(F_j)^2 \rangle = -2$ for all j. Then we can choose an integer i such that $\chi(E_i, F_j) = -(c_1(F_j), C_i) > 0$, which implies that $\mathrm{Hom}(E_i, F_j) \neq 0$ or $\mathrm{Hom}(F_j, E_i) \neq 0$. By the stability of E_i and F_j, we see that $E_i \cong F_j$. Therefore E is S-equivalent to $\bigoplus_i E_i^{\oplus b_i}$. Then we see that $b_i = a_i$, which implies the claim. Q.E.D.

We take a sufficiently small $\mathbb{Q}$-divisor η such that $(\sigma, \eta) = (f, \eta) = 0$ and $\chi_{G(\eta)}(E_i) < 0$ for $i > 0$. Then in the same way as in [O-Y], we see that $Y := M_H^{G(\eta)}(0, rf, d)$ is a resolution of $\overline{M}_H^G(0, rf, d)$ at $\bigoplus_i E_i^{\oplus a_i}$ and the exceptional divisors are

$$C_i' := \{E \in Y \mid \mathrm{Ext}^2(E, E_i) \neq 0\} \cong \mathbb{P}^1, i > 0$$

and $(C_i', C_j') = (C_i, C_j)$.

Let $\mathfrak{g}$ (resp. $\overline{\mathfrak{g}}$) be the affine Lie algebra associated to E_i, $i \geq 0$ (resp. the finite Lie algebra associated to E_i, $i \geq 1$).

Proposition 4.12. *Let $\pi : X \to C$ be an elliptic surface with a section σ. Assume that (1) $l = 1$, or (2) X is rational or of type $K3$. We set $L := \sigma + (1 - (\sigma^2))f$. Let G be a locally free sheaf on X such that $\operatorname{rk} G = r$ and $c_1(G) = d\sigma$ with $\gcd(r, d) = 1$. Then $\overline{\mathfrak{g}}$ acts on $\bigoplus_{(D,k)\in V} H_*(M^G_{L+nf+\varepsilon}(0, l\sigma + D, k), \mathbb{C})$, where*

$$V := \left\{ (D,k) \;\middle|\; \begin{array}{l} D \text{ is an effective divisor on fibers with} \\ (l\sigma + D)^2 + p_g + 1 \geq 0,\ k \in \mathbb{Z}, \\ \chi_G := rk - (c_1(G), l\sigma + D) > 0, \\ \gcd(l, (D, \sigma), \chi_G) = 1 \end{array} \right\}.$$

Moreover we also have an action of $\mathfrak{g}$, if $\gcd(l, \chi_G) = 1$.

Proof. We first note that Condition 1 holds, under (1) or (2) by the proof of [Y4, Prop. 3.18] and Lemma 4.4. We also note that $(c_1(E) \pm c_1(E_i), \sigma) = (c_1(E), \sigma)$ for $i > 0$. In our case, (4.1) does not hold, but if $\gcd(\chi_G(E), (c_1(E), f), (c_1(E), \sigma)) = 1$, then by using Lemma 4.8, we see that the statements in Lemma 4.1 still hold, where E_1 in Lemma 4.1 corresponds to $E_i, i > 0$. Hence we get our claim for $\overline{\mathfrak{g}}$. Moreover if $\gcd(\chi_G(E), (c_1(E), f)) = 1$, then we can apply the results in Lemma 4.1 for E_0. Therefore our claim also holds for $\mathfrak{g}$. Q.E.D.

Corollary 4.13. *Under the same notations as above, the Poincaré polynomial $P(M^G_{L+nf+\varepsilon}(0, l\sigma + D, k))$ is $W(\mathfrak{g})$-invariant:*

$$P(M^G_{L+nf+\varepsilon}(0, w(l\sigma + D), k)) = P(M^G_{L+nf+\varepsilon}(0, l\sigma + D, k)),\ w \in W(\mathfrak{g}),$$

where $W(\mathfrak{g})$ is the Weyl group of $\mathfrak{g}$.

Let X be a rational elliptic surface with a singular fiber of type $E_8^{(1)}$. As we shall see in Subsection 4.3, $M^G_{L+nf+\varepsilon}(0, l\sigma + D, k)$ is related to a moduli space of torsion free sheaves. In [MNWV], [Y1] and [Iq], it is observed that the Eular characteristic of $M^G_{L+nf+\varepsilon}(0, l\sigma + D, k)$ is Weyl group invariant. Proposition 4.12 gives an explanation of this invariance.

4.3. Moduli of stable sheaves on rational elliptic surfaces

Let $\pi : X \to \mathbb{P}^1$ be a rational elliptic surface with a section σ. Then there is a family of elliptic surfaces $\widetilde{\pi} : \mathcal{X} \to \mathbb{P}^1_T$ over a smooth curve T such that

(i) $\mathcal{X}_{t_0} \cong X$, $t_0 \in T$,
(ii) there is a section $\widetilde{\sigma}$ of $\widetilde{\pi}$ with $\widetilde{\sigma}_{t_0} = \sigma$ and
(iii) for a general point $t \in T$, $\mathcal{X}_t$ is a nodal elliptic surface, that is, all singular fibers are irreducible nodal curves.

Let T_0 be the open subset of T consisting of nodal elliptic surfaces. Replacing T by a suitable covering of T, we may assume that $\mathrm{Pic}(\mathcal{X}/T) \cong R^2\phi_*(\mathbb{Z})$ is a trivial local system, where $\phi : \mathcal{X} \to T$ is the projection. Hence there is a relatively ample $\mathbb{Q}$-divisor $\mathcal{H}$ on $\mathcal{X}$. Moreover, by adding $m\sigma + nf$, we may assume that $\mathcal{H} = nf + m\sigma + \varepsilon$, $\varepsilon \in (\mathbb{Q}\sigma + \mathbb{Q}f)^\perp$, $n \gg m \gg |(\varepsilon^2)|$, where we use the identification $R^2\phi_*\mathbb{Z} \cong H^2(X, \mathbb{Z})$. For positive integers r, d with $\gcd(r, d) = 1$, we take a vector bundle G of rank r and $c_1(G) = d\sigma$ on $\mathcal{X}$. We set $\gamma := (r, \xi, \chi) \in R^*\phi_*\mathbb{Z}$. Then we have a family of moduli spaces of semi-stable sheaves $\psi : \overline{M}^G_{(\mathcal{X},\mathcal{H})/T}(\gamma) \to T$, which is smooth on the locus of stable sheaves. For the existence of stable sheaves, see Appendix 7.3.

From now on, we assume that $\gamma = (0, \xi, \chi)$, where $\xi = l\sigma + kf + D$, $l > 0$, $\gcd(l, (\xi, \sigma), r\chi - (c_1(G), \xi)) = 1$ and $(D, f) = (D, \sigma) = 0$. We take a sufficiently small $\mathbb{Q}$-divisor η such that $\chi_{G(\eta)}(\mathcal{O}_{C_i}(-1)) < 0$ for all $i > 0$. We set $\mathcal{Y} := M^{G(\eta)}_{(\mathcal{X},\mathcal{H})/T}(0, r\sigma, d)$. Then $\mathcal{Y}_t$, $t \in T$ are smooth projective surfaces isomorphic to $\mathcal{X}_t$. Hence $\mathcal{Y} \to T$ is a smooth morphism. We have an isomorphism $\mathcal{Y} \times_T T_0 \cong \mathcal{X} \times_T T_0$ over T_0 (cf. [Y4]). Let $\mathcal{H}'$ be a relatively ample $\mathbb{Q}$-divisor on $\mathcal{Y}$ whose restriction to $\mathcal{Y} \times_T T_0$ corresponds to a divisor $m\sigma + nf + \varepsilon'$ on $\mathcal{X} \times_T T_0$, where $\varepsilon' \in (\mathbb{Q}\sigma + \mathbb{Q}f)^\perp$ is sufficiently small. By Lemma 4.7, we have an isomorphism

$$\xi : M^G_{(\mathcal{X}\times_T T_0, \mathcal{H}_{|\phi^{-1}(T_0)})/T_0}(\gamma) \to M^{G(-\varepsilon')}_{(\mathcal{X}\times_T T_0, m\sigma + nf)/T_0}(\gamma).$$

By our assumption, there is a universal family $\mathcal{E}$ on $\mathcal{X} \times_T \mathcal{Y}$. We consider a family of Fourier–Mukai transforms $\Phi^{\mathcal{E}^\vee}_{\mathcal{X}\to\mathcal{Y}} : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y})$. If $r\chi - (c_1(G), \xi) > 0$, then $\Phi^{\mathcal{E}^\vee}_{\mathcal{X}\to\mathcal{Y}} \circ \xi$ induces a birational map

$$\zeta : M^G_{(\mathcal{X},\mathcal{H})/T}(\gamma) \cdots \to M^{G'^\vee}_{(\mathcal{Y},\mathcal{H}')/T}(\gamma')$$

which is an isomorphism over T_0, where $G' := \Phi^{\mathcal{E}^\vee[1]}_{\mathcal{X}\to\mathcal{Y}}(\mathcal{O}_\sigma)$ and $\gamma' := \Phi^{\mathcal{E}^\vee[1]}_{\mathcal{X}\to\mathcal{Y}}(\gamma)$ (see [Y4, Thm. 3.13, Rem. 3.1]). Let $\mathcal{Z}$ be the graph of this birational correspondence. Then the cycle $[\mathcal{Z}]_{t_0}$ induces an isomorphism of the homology groups

$$H_*(M^G_{\mathcal{H}_{t_0}}(\gamma), \mathbb{Z}) \to H_*(M^{G'^\vee}_{\mathcal{H}'_{t_0}}(\gamma'), \mathbb{Z})$$

via the convolution product. Let E_i, $i = 0, 1, \dots, n$ be G-twisted stable sheaves on X in Section 4.2. We set $Y := \mathcal{Y}_{t_0}$ and

$$\begin{aligned} \rho :=& \Phi^{\mathcal{E}^\vee_{t_0}}_{\mathcal{X}_{t_0}\to\mathcal{Y}_{t_0}}(\mathbb{C}_x) \in K(Y), x \in X \\ u_i :=& \Phi^{\mathcal{E}^\vee_{t_0}}_{\mathcal{X}_{t_0}\to\mathcal{Y}_{t_0}}(E_i) \in K(Y), i = 0, 1, \dots, n. \end{aligned}$$

Then $\sum_{i=0}^{n} a_i u_i = \Phi_{\mathcal{X}_{t_0}\to\mathcal{Y}_{t_0}}^{\mathcal{E}_{t_0}^{\vee}}(\bigoplus_{i=0}^{n} E_i^{\oplus a_i}) = \mathbb{C}_y, y \in Y$. By Proposition 4.12, we get the following:

Proposition 4.14. *We have an action of* $\overline{\mathfrak{g}}$ *on the homology groups*

$$\bigoplus_{n_i,k} H_*(M_H^{G'^{\vee}}(\gamma(lG' + \sum_i n_i u_i + k\rho))), n_i \in \mathbb{Q},$$

where $\sum_i n_i u_i \in K(Y)$, *H is sufficiently close to* f, $l\chi_G(\mathcal{O}_\sigma) + kr > 0$ *and* $\gcd(l, rn_0, kr) = 1$. *Moreover if* $\gcd(l, kr) = 1$, *then we have an action of* $\mathfrak{g}$.

Proof. We note that $(c_1(\sum_i n_i u_i), \sigma) = n_0 r$ and $\chi_G(l\mathcal{O}_\sigma + k\mathbb{C}_x) \equiv kr \mod l$. Hence the claim holds. Q.E.D.

4.4. Moduli of stable vector bundles on an ADE-type configuration.

In this subsection, we explain a relation with a paper by Nakajima [N2]. Let X be a smooth projective surface containing an ADE-type configuration of smooth rational curves C_i, $i = 1, 2, \dots, n$, that is, the intersection matrix $((C_i, C_j)_{i,j})$ is of type ADE. Assume that there is a nef and big divisor H such that $(C_i, H) = 0$ for all C_i.

For $\xi \in \mathrm{NS}(X)$ and $d \geq 0$, we set

$$B_{(\xi,d)} := \left\{ x \in \oplus_{i=1}^{n} \mathbb{Z}C_i \,|\, (x^2) + 2(\xi, x) + d \geq 0 \right\}.$$

Since $\bigoplus_{i=1}^{n} \mathbb{Z}C_i$ is negative definite, $B_{(\xi,d)}$ is a finite set. Let r be a positive integer such that $2r > (x^2) + 2(\xi, x) + d$ for all $x \in B_{(\xi,d)}$. Assume that there is an integer χ_0 such that $d = (\xi^2) + 2r\chi_0 - r(K_X, \xi) + (r^2 + 1)\chi(\mathcal{O}_X)$.

Definition 4.4. Let $M_H(r, \xi + y, \chi)^\mu$, $y \in \bigoplus_{i=1}^{n} \mathbb{Z}C_i, \chi \in \mathbb{Z}$ be the moduli space of μ-stable sheaves E with respect to H such that $\gamma(E) = (r, \xi + y, \chi)$.

$M_H(r, \xi + y, \chi)^\mu$ is contained in a moduli space of μ-stable sheaves with respect to an ample divisor H' which is sufficiently close to H. If $\gcd(r, (\xi, H)) = 1$, then $M_H(r, \xi + y, \chi)^\mu$ is projective. We assume that Condition 1 holds for all $E \in M_H(r, \xi + y, \chi)^\mu$, $y \in \bigoplus_{i=1}^{n} \mathbb{Z}C_i, \chi \in \mathbb{Z}$. Then $M_H(r, \xi + y, \chi)^\mu$, $y \in \bigoplus_{i=1}^{n} \mathbb{Z}C_i, \chi \in \mathbb{Z}$ is a smooth scheme of dimension $(y^2) + 2(\xi, y) + d + 2r(\chi_0 - \chi) + q$, $q = \dim H^1(X, \mathcal{O}_X)$, if it is not empty.

Lemma 4.15. (i) $M_H(r, \xi + x, \chi_0)^\mu$, $x \in B_{(\xi,d)}$ *consists of locally free sheaves.*

(ii) $H^1(C_i, E_{|C_i}) = 0$ *for all* $E \in M_H(r, \xi + x, \chi_0)^\mu$, $x \in B_{(\xi,d)}$.

Proof. We prove the second claim. The proof of the first one is similar. If $H^1(C_i, E_{|C_i}) \neq 0$, then there is a surjective homomorphism $\phi : E \to \mathcal{O}_{C_i}(-1-k)$, $k > 0$. By our assumption on E, $F := \ker \phi$ is a μ-stable sheaf with $\gamma(F) = \gamma(E) - (0, C_i, -k)$. Then we have

$$\begin{aligned}\dim M_H(\gamma(F)) =&((x - C_i)^2) + 2(\xi, x - C_i) + d - 2rk + q \\ <&2r + q - 2rk \leq q.\end{aligned}$$

This is impossible. Hence the claim holds. Q.E.D.

Corollary 4.16. *Let* E *be an element of* $M_H(r, \xi + x, \chi_0)^\mu$, $x \in B_{(\xi,d)}$. *For a subspace* $V \subset \operatorname{Hom}(E, \mathcal{O}_{C_i}(-1))$, $\phi : E \to V^\vee \otimes \mathcal{O}_{C_i}(-1)$ *is surjective and* $\ker \phi$ *is a* μ*-stable locally free sheaf with the Chern character* $\operatorname{ch}(F) = \operatorname{ch}(E) - (\dim V)(0, C_i, 0)$.

We set

$$\mathfrak{P}^{(n)}_{\mathcal{O}_{C_i}(-1)}(r, \xi + x, \chi_0) := \left\{ (E, U^\vee) \;\middle|\; \begin{aligned} E \in M_H(r, \xi + x, \chi_0)^\mu, \dim U = n \\ U^\vee \subset \operatorname{Hom}(E, \mathcal{O}_{C_i}(-1)) \end{aligned} \right\}$$

and define operators e_i, f_i, h_i. Then we have the following which is due to Nakajima [N2, sect. 5].

Proposition 4.17. *Let* $\mathfrak{g}$ *be a finite Lie algebra generated by* $\mathcal{O}_{C_i}(-1)$. *Then* $\mathfrak{g}$ *acts on* $\bigoplus_{x \in B_{(\xi,d)}} H_*(M_H(r, \xi + x, \chi_0)^\mu)$, *provided that the moduli spaces are non-empty.*

Remark 4.4. In order to compare the correspondence in Theorem 2.1, we need to set $\mathbb{F} := E[1]$ (cf. (1.10)).

Example 4.3. Let $\pi : X \to \mathbb{P}^1$ be an elliptic $K3$ surface with a section σ. Let f be a fiber of π. Then $H := \sigma + tf$, $t \gg 0$ is a nef and big divisor on X. Let C_i, $i = 1, 2, \ldots, n$ be (-2)-curves contracted by mH. Assume that $\gcd(r, (\xi, f)) = 1$. Then for $y \in \operatorname{NS}(X)$ with $(y, f) = 0$ and $k \in \mathbb{Z}$,

$$M_H(r, \xi + y, \chi)^\mu := \left\{ E \;\middle|\; \begin{aligned} &E \text{ is a torsion free sheaf with} \\ &\gamma(E) = (r, \xi + y, \chi) \text{ such that} \\ &E_{|\pi^{-1}(p)} \text{ is stable for a point } p \in \mathbb{P}^1 \end{aligned} \right\}.$$

In particular, $M_H(r, \xi + y, \chi)^\mu$ is projective and coincides with $M_{H'}(r, \xi + y, \chi)$ where H' is an ample divisor which is sufficiently close to H. Therefore $M_H(r, \xi + y, \chi)^\mu$ is not empty, provided the expected dimension is non-negative (cf. [Y2]). We set $\lambda := (\xi, f)$. Then since

$((\xi+\mu f)^2)=(\xi^2)+2\lambda\mu$, we can find $\chi_0\in\mathbb{Z}$ and $\mu\in\mathbb{Z}$ such that $d=((\xi+\mu f)^2)+2r\chi_0+(r^2+1)\chi(\mathcal{O}_X)$. Thus replacing ξ by $\xi+\mu f$, we can find a desired χ_0. Therefore all the requirements are satisfied and we have an action of finite Lie algebra. By Corollary 7.4, a similar result holds for a rational elliptic case.

Remark 4.5. Let $X\to C$ be an elliptic surface in Proposition 4.12. We use the same notations. Let $\mathcal{E}$ be the universal family on $Y\times X$. Then we have a Fourier–Mukai transform $\Phi_{X\to Y}^{\mathcal{E}^\vee}:\mathbf{D}(X)\to\mathbf{D}(Y)$. We set $G':=\Phi_{X\to Y}^{\mathcal{E}^\vee}(\mathcal{O}_\sigma)$. Let $\xi:=\sigma+D$ with $(f,D)=0$ be an effective divisor such that $(\xi^2)=(\sigma^2)$. Assume that

$$2(\xi,x)+(x^2)<2r \text{ for all } x\in\bigoplus_{i=1}^n\mathbb{Z}C_i.$$

For a G-twisted stable sheaf $E\in M_H^G(0,\xi+x,\chi)$, $E\otimes\pi^*(L)\in M_H^G(0,\xi+x,\chi)$, $L\in\operatorname{Pic}^0(C)$. Since $(E\otimes\pi^*(L))_{|\sigma}/(\text{torsion})=(E_{|\sigma}/(\text{torsion}))\otimes L$, we get $(\xi+x)^2+1+p_g=\dim M_H^G(0,\xi+x,\chi)\geq\dim\operatorname{Pic}^0(C)=g(C)$. Since $(\xi+x)^2+1+p_g=(\xi+x)^2+g(C)+\chi(\mathcal{O}_X)$ and $(\xi^2)=(\sigma^2)=-\chi(\mathcal{O}_X)$, we get $2(\xi,x)+(x^2)\geq 0$. Assume that $E\in M_H^G(0,\xi+x,\chi)$ contains the sheaf E_0 in Lemma 4.10. By the choice of H, E/E_0 is a G-twisted stable sheaf with $(\xi,c_1(E/E_0))+(c_1(E/E_0)^2)=(\xi,x+\sum_{i>0}a_iC_i)+((x+\sum_{i>0}a_iC_i)^2)-2r<0$, which is a contradiction. Therefore $E\in M_H^G(0,\xi+x,\chi)$ does not contain E_0. We also see that E does not contain $\mathcal{E}_{|\{x\}\times Y}$, $x\in X$. Then we have $\operatorname{Hom}(\mathcal{E}_{|\{x\}\times Y},E)=0$ for all $x\in X$. By the proof of [Y4, Thm. 3.13], $\Phi_{X\to Y}^{\mathcal{E}^\vee}$ induces an isomorphism $M_H^G(0,\xi+x,\chi)\cong M_H^{G'^\vee}(r,\xi',\chi')$, where $r\chi-d(\xi+x,\sigma)>0$, $(c_1(G'),f)=(\xi',f)$ and $((\xi+x)^2)=(\xi'^2)-2r\chi'-r(K_X,\xi')+r^2\chi(\mathcal{O}_X)$. Moreover $\Phi_{X\to Y}^{\mathcal{E}^\vee}(\mathcal{O}_{C_i}(-1))\cong\mathcal{O}_{C_i'}(k_i)$ for some k_i with $\chi_{G'^\vee}(\mathcal{O}_{C_i'}(k_i))=0$. The action of $\mathfrak{g}$ generated by $\Phi_{X\to Y}^{\mathcal{E}^\vee}(\mathcal{O}_{C_i}(-1))$ is similar to the action in this section. Indeed if $((C_i,C_j)_{i,j})$ is of type E_8, then there is a divisor $D'=\sum_{i=1}^8 b_iC_i'$ such that $r(D',C_i')=-(c_1(G'),C_i')=r(k_i+1)$. Then replacing $\mathcal{E}$ be $\mathcal{E}\otimes\mathcal{O}_Y(-D')$, we may assume that $k_i=-1$ for all $i>0$.

Remark 4.6. Let Y be a projective surface with rational double points as singularities and H' an ample Cartier divisor on Y. Assume that there is a morphism $\pi:X\to Y$ which gives the minimal resolution and $H=\pi^{-1}(H')$. For simplicity, we assume that there is a unique singular point $p\in Y$. Let $Z:=\pi^{-1}(p)$ be the fundamental cycle. For $E\in M_H(r,\xi+x,\chi)^\mu$, we have an exact sequence

$$0\to E'\to E\to F\to 0 \tag{4.3}$$

such that F is a successive extensions of $\mathcal{O}_{C_i}(-1)$ and $\operatorname{Hom}(E', \mathcal{O}_{C_i}(-1)) = 0$, $i = 1, 2, \dots, n$. Then we have $E'_{|C_i} \cong \mathcal{O}_{C_i}(1)^{\oplus a_i} \oplus \mathcal{O}_{C_i}^{\oplus b_i}$. For all C_i, there is an exact sequence

$$0 \to G \to \mathcal{O}_Z \to \mathcal{O}_{C_i} \to 0,$$

where G is a successive extensions of $\mathcal{O}_{C_j}(-1)$ (cf. Example 4.1). Hence we see that $H^1(Z, E'_{|Z}) = 0$. Then we see that $R^1\pi_*(E') = 0$. Since $R^1\pi_* F = 0$, we get that $\pi_*(E) \cong \pi_*(E')$ and $R^1\pi_*(E) \cong R^1\pi_*(E') = 0$. Therefore we have a morphism

$$\begin{array}{cccc} \pi_* : & M_H(r, \xi + x, \chi)^\mu & \to & M_{H'}(r, \xi + x, \chi)^\mu \\ & E & \mapsto & \pi_*(E), \end{array}$$

where $M_{H'}(r, \xi+x, \chi)^\mu$ is the moduli space of μ-stable sheaves on Y. By this morphism, we have a contraction of the Brill–Noether locus. We can also show that $R^1\pi_*(E'^\vee) = 0$ and $E'_{|Z}$ is generated by global sections. Thus $\pi_*(E) \cong \pi_*(E')$ is a reflexive sheaf and E' is a full sheaf. Hence the local structure of this contraction map is an example of the studies of Ishii [I1],[I2]. More generally, for each moduli space $M_H(r, \xi, \chi)^\mu$, let $M_H(r, \xi, \chi)^\#$ be the open subset consisting of E such that E is locally free, Condition 1 holds and $R^1\pi_*(E) = R^1\pi_*(E^\vee) = 0$. Then we see that $H^1(C_i, E_{|C_i}) = H^1(C_i, E^\vee_{|C_i}) = 0$ for all i and Corollary 4.16 holds. Since $R^j\pi_*(\mathcal{O}_{C_i}(-1)) = 0$ for all j and $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{C_i}(-1), \mathcal{O}_X) \cong \mathcal{O}_{C_i}(-1)$, $\ker\phi$ in Corollary 4.16 belongs to $M_H(r, \xi - (\dim V)C_i, \chi)^\#$. Therefore we also have similar claims for $M_H(r, \xi, \chi)^\#$.

§5. Equivariant sheaves

In this section, we give a remark for the moduli of equivariant sheaves. Let G be a finite group acting on X. Let E_0 be an irreducible G-sheaf of dimension 0, i.e. E_0 does not have a non-trivial G-subsheaf. Then $\operatorname{Hom}(E_0, E_0)^G = \mathbb{C}$.

Lemma 5.1. *Let E_0 be an irreducible G-sheaf of dimension 0. Let E be a torsion free (resp. purely 1-dimensional) G-sheaf.*

(1) *Then every non-trivial extension*

$$0 \to E \to F \to E_0 \to 0$$

defines a torsion free (resp. purely 1-dimensional) G-sheaf.

(2) *Let V be a subspace of* $\operatorname{Hom}(E, E_0)$. *Then $\phi : E \to V^\vee \otimes E_0$ is surjective. Moreover,* $\ker\phi$ *is a torsion free (resp. purely 1-dimensional) G-sheaf.*

Let H be a G-equivariant line bundle on X which is ample.

Definition 5.1. A G-sheaf E is μ-stable, if E is torsion free and

$$\frac{(c_1(F),H)}{\operatorname{rk} F} < \frac{(c_1(E),H)}{\operatorname{rk} E}$$

for all G-subsheaf F of E with $0 < \operatorname{rk} F < \operatorname{rk} E$.

For a G-sheaf E on X, $v(E)$ denotes the class of E in $K^G(X)$. For a $v \in K^G(X)$, $M_H(v)^\mu$ is the moduli of μ-stable G-sheaves E with $v(E) = v$. Assume that

$$\operatorname{Ext}^2(E,E)^G \to H^2(X,\mathcal{O}_X)^G$$

is an isomorphism for any $E \in M_H(v)^\mu$. We set

$$\langle v(E), v(F)\rangle := -G\text{-}\chi(E,F) = -\sum_i (-1)^i \dim \operatorname{Ext}^i(E,F)^G.$$

Let $E_1, E_2, \ldots, E_s$ be a configuration of irreducible G-sheaves of dimension 0 such that

$$E_i \otimes K_X \cong E_i,$$
$$\operatorname{Ext}^1(E_i,E_i)^G = 0.$$

Then $v(E_i)$ are (-2)-vectors. We set

$$\mathfrak{P}^{(n)}_{E_i}(v) := \{(E,U^\vee) | E \in M_H(v)^\mu, U^\vee \subset \operatorname{Hom}(E[1],E_i[1]), \dim U = n\}$$

and define operators e_i, f_i, h_i. Then we have an action of the Lie algebra $\mathfrak{g}$ generated by $v(E_1), v(E_2), \ldots, v(E_s)$ on $\bigoplus_v H^*(M_H(v)^\mu, \mathbb{C})$.

Remark 5.1. Assume that $X = \mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$ with an action of a Klein group $G \subset SL(\mathbb{C}^2)$. Let W be a G-vector space. We consider the moduli of framed G-sheaves (E,Φ), where E is a torsion free G-sheaf on $\mathbb{P}^2$ and $\Phi : E_{|\ell_\infty} \to \mathcal{O}_{\ell_\infty} \otimes W$ is a G-isomorphism. This is an example of Nakajima's quiver variety and we have an action of affine Lie algebra associated to G on the homology groups [N2]. In this case, we set $\langle v(E), v(F)\rangle := -G\text{-}\chi(E, F(-\ell_\infty))$ and we use the vanishing $\operatorname{Ext}^2(E, E \to (\mathcal{O}_{\ell_\infty} \otimes W \oplus E_i)) = 0$ to show the smoothness of $\mathfrak{P}^{(n)}_{E_i}$.

§6. Perverse coherent sheaves on a resolution of rational double points

6.1. Perverse coherent sheaves

In this section, we shall give examples of the action of affine Lie algebras on the moduli of stable perverse coherent sheaves. Let X be a smooth projective surface and $\pi : X \to Y$ a birational map such that $\mathbf{R}\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$. We first recall perverse coherent sheaves introduced by Bridgeland [B1].

Definition 6.1. [B1] Let $\mathrm{Per}(X/Y)$ be the subcategory of $\mathbf{D}(X)$ such that an object $E \in \mathbf{D}(X)$ belongs to $\mathrm{Per}(X/Y)$ if and only if

(i) $H^i(E) = 0$ for $i \neq -1, 0$,
(ii) $\pi_*(H^{-1}(E)) = 0$ and $R^1\pi_*(H^0(E)) = 0$,
(iii) $\mathrm{Hom}(H^0(E), c) = 0$ for all sheaf c on X with $\mathbf{R}\pi_*(c) = 0$.

An object $E \in \mathrm{Per}(X/Y)$ is called *perverse coherent sheaf.*

$\mathrm{Per}(X/Y)$ is an abelian category. For $E \in \mathrm{Per}(X/Y)$, we get $H^i(\pi_*(E)) = 0$, $i \neq 0$. Thus $\pi_*(E) \in \mathrm{Coh}(Y)$. The following is due to Bridgeland [B1] (cf. [N-Y, Lem. 1.2]).

Lemma 6.1. (1) *For a coherent sheaf F on Y, we have an exact sequence*

$$0 \to R^1\pi_*(L^{-1}\pi^*(F)) \to F \to \pi_*\pi^*(F) \to 0.$$

In particular, if F is torsion free, then $F \cong \pi_*\pi^*(F)$.

(2) *Let E be a coherent sheaf on X. For the natural map $\phi : \pi^*\pi_*(E) \to E$, we have (i) $\mathbf{R}\pi_*(\ker\phi) = 0$, (ii) $\pi_*(\mathrm{im}\,\phi) \to \pi_*(E)$ is isomorphic, (iii) $\pi_*(\mathrm{coker}\,\phi) = 0$ and (iv) $R^1\pi_*(E) \cong R^1\pi_*(\mathrm{coker}\,\phi)$.*

(3) *A coherent sheaf E belongs to $\mathrm{Per}(X/Y)$ if and only if $\phi : \pi^*\pi_*(E) \to E$ is surjective.*

(4) *For a coherent sheaf F on Y, $\mathrm{Ext}^1(\pi^*(F), c) = 0$ for all $c \in \mathrm{Coh}(X)$ with $\mathbf{R}\pi_*(c) = 0$.*

6.2. A family of perverse coherent sheaves

Let $\mathcal{Y} \to S$ be a flat family of surfaces and $\pi : \mathcal{X} \to \mathcal{Y}$ a family of projective birational maps such that $\mathcal{X} \to S$ is smooth and $\mathbf{R}\pi_*(\mathcal{O}_\mathcal{X}) = \mathcal{O}_\mathcal{Y}$.

Definition 6.2. Let $\mathcal{M}^p_{\mathcal{X}/S}(v)$ be the moduli stack of perverse coherent sheaves $E \in \mathrm{Per}(\mathcal{X}_s/\mathcal{Y}_s) \cap \mathrm{Coh}(\mathcal{X}_s)$, $s \in S$ with topological invariant $v(E) = v$ or $\gamma(E) = v$ in Section 4.2 such that $\pi_*(E)$ is torsion free or purely 1-dimensional.

By Lemma 6.1 and the base change theorem, $\mathcal{M}^p_{\mathcal{X}/S}(v)$ is an open substack of the moduli stack of coherent sheaves E on $\mathcal{X}_s, s \in S$. Let w be a numerical invariant of $\pi_*(E), E \in \mathcal{M}^p_{\mathcal{X}/S}(v)_s$ and $\mathcal{M}_{\mathcal{Y}/S}(w)$ be the moduli stack of torsion free sheaves or purely 1-dimensional sheaves F on $\mathcal{Y}_s, s \in S$.

Proposition 6.2. *We have a "proper" map $f : \mathcal{M}^p_{\mathcal{X}/S}(v) \to \mathcal{M}_{\mathcal{Y}/S}(w)$ by sending E to $\pi_*(E)$. More precisely, let T be a scheme of finite type over S and $\mathcal{F}$ a flat family of torsion free or purely 1-dimensional sheaves on $\mathcal{Y} \times_S T$. Then $\mathcal{M}^p_{\mathcal{X}/S}(v) \times_{\mathcal{M}_{\mathcal{Y}/S}(w)} T \to T$ is proper.*

Let $T' \to T$ be a morphism and $\mathcal{E}$ a flat family of coherent sheaves parametrized by T' such that $\mathcal{E}_t \in \operatorname{Per}(\mathcal{X}_t/\mathcal{Y}_t)$, $v(\mathcal{E}_t) = v$ and $\pi_*(\mathcal{E}) \cong \mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$. Since $\mathcal{E}$ is a quotient of $\pi^*(\mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'})$ in the category of coherent sheaves with a fixed topological invariant v, $\mathcal{M}^p_{\mathcal{X}/S}(v) \times_{\mathcal{M}_{\mathcal{Y}/S}(w)} T \to T$ is of finite type.

In order to prove the properness, we use the valuative criterion. Let R be a discrete valuation ring and K the quotient field of R. Let s be the closed point of $S = \operatorname{Spec}(R)$. Let $W \subset Y := \mathcal{Y}_s$ be the closed subset such that π_s is isomorphic over $Y \setminus W$.

Lemma 6.3. *Let $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ be a homomorphism of R-flat families of coherent sheaves $\mathcal{E}_i$, $i = 1, 2$ on $\mathcal{Y}$. Assume that $(\mathcal{E}_1)_s$ is torsion free or purely 1-dimensional, and ϕ is an isomorphism on $\mathcal{Y} \setminus W$. Then ϕ is injective and* $\operatorname{coker}\phi$ *is R-flat. Moreover if ϕ is isomorphic over K, then ϕ is an isomorphism.*

Proof. Since ϕ_s is isomorphic on $Y \setminus W$ and $(\mathcal{E}_1)_s$ is torsion free or purely 1-dimensional, ϕ_s is injective. Hence ϕ is injective and $\operatorname{coker}\phi$ is R-flat. If ϕ is isomorphic over K, then $(\operatorname{coker}\phi) \otimes_R K = 0$, which implies that $\operatorname{coker}\phi = 0$. Hence ϕ is an isomorphism. Q.E.D.

Corollary 6.4. *Let $\mathcal{F}$ be a R-flat family of torsion free or purely 1-dimensional sheaves on $\mathcal{Y}$. Let $\widehat{\pi^*(\mathcal{F})}$ be the R-torsion free quotient of $\pi^*(\mathcal{F})$. Then $\mathcal{F} \to \pi_*(\pi^*(\mathcal{F})) \to \pi_*(\widehat{\pi^*(\mathcal{F})})$ is injective and the cokernel is R-flat.*

By the following proposition, we have Proposition 6.2.

Proposition 6.5. (1) *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be R-flat families of coherent sheaves on $\mathcal{X}$ such that (i) $(\mathcal{E}_i)_s \in \operatorname{Per}(\mathcal{X}_s/\mathcal{Y}_s)$, (ii) $\pi_*(\mathcal{E}_1)_s$ is torsion free or purely 1-dimensional, and (iii) there are isomorphisms $\phi_K : \mathcal{E}_1 \otimes_R K \to \mathcal{E}_2 \otimes_R K$, $\psi : \pi_*(\mathcal{E}_1) \to \pi_*(\mathcal{E}_2)$ such that ϕ_K induces ψ over K. Then there is an isomorphism $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ extending ϕ_K and ψ.*

(2) *Let $\mathcal{E}_K$ be a coherent sheaf such that $\mathcal{E}_K \in \operatorname{Per}(\mathcal{X}_K/\mathcal{Y}_K)$, i.e., $R^1\pi_*(\mathcal{E}_K) = 0$ and $\pi^*\pi_*(\mathcal{E}_K) \to \mathcal{E}_K$ is surjective. Let $\mathcal{F}$ be a R-flat family of torsion free or purely 1-dimensional sheaves on $\mathcal{Y}$ with an isomorphism $\psi_K : \pi_*(\mathcal{E}_K) \to \mathcal{F} \otimes_R K$. Then there is a R-flat family $\mathcal{E}$ of perverse coherent sheaves which is an extension of $\mathcal{E}_K$ with an extension $\psi : \pi_*(\mathcal{E}) \to \mathcal{F}$ of ψ_K.*

Proof. (1) Let $\widehat{\pi^*(\pi_*(\mathcal{E}_i))}$ be the R-flat quotient of $\pi^*(\pi_*(\mathcal{E}_i))$ by the R-torsions and L_i the kernel of $\widehat{\pi^*(\pi_*(\mathcal{E}_i))} \to \mathcal{E}_i$. Then L_i are R-flat. Since $L_1 \otimes_R K \to \mathcal{E}_2 \otimes_R K$ is a 0-map, L_1 is contained in L_2. Hence we have a homomorphism $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ and we get a commutative diagram:

$$\begin{tikzcd} \widehat{\pi^*(\pi_*(\mathcal{E}_1))} \arrow[r] \arrow[d, "\pi^*(\psi)"'] & \mathcal{E}_1 \arrow[d, "\phi"] \\ \widehat{\pi^*(\pi_*(\mathcal{E}_2))} \arrow[r] & \mathcal{E}_2. \end{tikzcd}$$

Thus the claim holds.

(2) Let $\phi : \widehat{\pi^*(\mathcal{F})} \to \mathcal{E}_K$ be a homomorphism defined by the compositions $\phi : \widehat{\pi^*(\mathcal{F})} \to \widehat{\pi^*(\mathcal{F})} \otimes_R K \to \pi^*(\pi_*(\mathcal{E}_K)) \to \mathcal{E}_K$. We set $\mathcal{E} := \operatorname{im}(\phi)$. Then $\mathcal{E}$ is a R-flat family of coherent sheaves such that $R^1\pi_*(\mathcal{E}) = 0$ and $\mathcal{E} \otimes_R K \cong \mathcal{E}_K$. By Lemma 6.3, $\mathcal{F} \to \pi_*(\widehat{\pi^*(\mathcal{F})}) \to \pi_*(\mathcal{E})$ is an isomorphism. Q.E.D.

The following definition of the stability is slightly different from [N-Y, Lem. 2.9].

Definition 6.3. Let H be an ample Cartier divisor on Y. An object $E \in \operatorname{Per}(X/Y)$ is stable with respect to H if E is a sheaf and $\pi_*(E)$ is stable with respect to H. If $\pi_*(E)$ is μ-stable, we say that E is μ-stable.

Lemma 6.6. *Let $E \in \operatorname{Coh}(X)$ be a perverse coherent sheaf and F a coherent sheaf such that $\pi_*(F)$ is torsion free. Then* $\operatorname{Hom}(E, F) \to \operatorname{Hom}(\pi_*(E), \pi_*(F))$ *is injective. In particular, a stable perverse coherent sheaf is simple.*

Proof. Since $\pi^*(\pi_*(E)) \to E$ is surjective, we have an injective homomorphism $\operatorname{Hom}(E, F) \to \operatorname{Hom}(\pi^*(\pi_*(E)), F)$. Since

$$\operatorname{Hom}(\pi^*(\pi_*(E)), F) \cong \operatorname{Hom}(\pi_*(E), \pi_*(F)),$$

we get our claim. Q.E.D.

Theorem 6.7. [N-Y, Thm. 2.14] *There is a coarse moduli scheme $M_H^p(v)$ of stable perverse coherent sheaves E with the topological invariant v. $M_H^p(v)^\mu$ denotes the open subscheme of μ-stable perverse coherent sheaves. More generally, for a family of resolutions $\pi : \mathcal{X} \to \mathcal{Y} \to S$ and a relatively ample Cartier divisor $\mathcal{H}$ on $\mathcal{Y}$, we have a relative moduli space of stable perverse coherent sheaves $M_{\mathcal{X}/S,\mathcal{H}}^p(v)$, which is quasi-projective over S.*

Lemma 6.8. (1) *Let $v := (r, \xi, a) \in \mathbb{Z} \times \mathrm{NS}(X) \times \mathbb{Z}$ be a topological invariant. Assume that there is a line bundle $\mathcal{L}$ on $\mathcal{X}$ with $\xi = c_1(\mathcal{L}_s)$, $s \in S$. If* $\mathrm{tr} : \mathrm{Ext}^2(E, E) \to H^2(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s})$ *is injective for all $E \in M^p_{\mathcal{X}/S,\mathcal{H}}(v)_s$, then $M^p_{\mathcal{X}/S,\mathcal{H}}(v) \to S$ is smooth over $s \in S$.*

(2) *We set $X := \mathcal{X}_s$, $Y := \mathcal{Y}_s$ and assume that $X \to Y$ is a minimal resolution of rational double points. Let E be a stable perverse coherent sheaf on X. If*

$$H^0(Y, K_Y) \to \mathrm{Hom}(\pi_*(E), \pi_*(E) \otimes K_Y)$$

is surjective, then $\mathrm{tr} : \mathrm{Ext}^2(E, E) \to H^2(X, \mathcal{O}_X)$ *is injective.*

Proof. (1) is a consequence of a standard deformation theory.

(2) By the Serre duality, it is sufficient to prove that $\mathrm{Hom}(X, K_X) \to \mathrm{Hom}(E, E \otimes K_X)$ is surjective. Since $K_X = \pi^*(K_Y)$, the claim follows from Lemma 6.6. Q.E.D.

Proposition 6.9. *Let $\pi : X \to Y$ be a contraction of (-2)-curves by a linear system $|nH|$ on X. We set $v := (r, \xi, a)$. Assume that $-K_X$ is effective and* $\gcd(r, (\xi, H), a) = 1$. *Then $M^p_H(v)$ is smooth and projective over $\mathbb{C}$. If there is a polarized deformation $\phi : (\mathcal{X}, \mathcal{L}) \to S$ of $\mathcal{X}_{s_0} = X$ with a family of Mukai vectors v and a family of dvisors $\mathcal{H}$ such that $\mathcal{H}_{s_0} = H$ and $\mathcal{H}_s^{\perp}$ does not contain (-2) curves for a general $s \in S$. Then $M^p_H(v)$ is deformation equivalent to $M_H(v)$.*

Proof. Since $\mathbf{R}\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$, $H^1(X, \mathcal{O}_X(nH)) = 0$ for $n \gg 0$. Hence the base change theorem implies that $\phi_*(\mathcal{O}_{\mathcal{X}}(n\mathcal{H}))$ is a locally free sheaf on S and we get a flat family of contractions $\pi : \mathcal{X} \to \mathcal{Y}$ such that $\mathbf{R}\pi_*(\mathcal{O}_{\mathcal{X}}) = \mathcal{O}_{\mathcal{Y}}$. We set $\mathcal{H}_n := n\mathcal{H} + \mathcal{L}$. For a sufficiently large n, let $M_{\mathcal{X}/S,\mathcal{H}_n}(v) \to S$ be the relative moduli space of $(\mathcal{H}_n)_s$-stable sheaves on $\mathcal{X}_s$. Let S_0 be the open subscheme of S such that $\mathcal{H}_s$ is ample. Then $M_{\mathcal{X}/S,\mathcal{H}_n}(v)$ coincides with $M^p_{\mathcal{X}/S,\mathcal{H}}(v)$ over S_0. Hence we get our claim. Q.E.D.

Corollary 6.10. *Let X be a smooth projective surface with a contraction $\pi : X \to Y$ of (-2) curves, and let H a divisor which is the pull-back of an ample divisor on Y.*

(1) *Assume that X be a rational surface and $-K_X$ is effective. If* $\gcd(r, (\xi, H), a) = 1$, *then $M^p_H(r, \xi, a)$ is deformation equivalent to $M_H(r, \xi, a)$ and $H^*(M^p_H(r, \xi, a))$ is identified with $H^*(M_H(r, \xi, a))$ by an algebraic correspondence.*

(2) *Assume that X be a $K3$ surface with $\rho(Y) \geq 2$. If* $\gcd(r, (\xi, H))$ $= 1$, *then $M^p_H(r, \xi, a)$ is deformation equivalent to $M_H(r, \xi, a)$*

and $H^*(M_H^p(r,\xi,a))$ is identified with $H^*(M_H(r,\xi,a))$ by an algebraic correspondence.

Proof. We prove (2). Let N be a primitive sublattice of $\operatorname{Pic}(X)$ spanned by H and ξ. Replacing ξ by $\xi + rm\pi^*(\eta)$, $\eta \in \operatorname{NS}(Y)$, we may assume that $\dim_{\mathbb{Q}} \pi_*(N \otimes \mathbb{Q}) = 2$. Indeed $\gcd(r, (\xi, H)) = 1$ means that the stability does not change under the change $E \mapsto E(mD)$, if D is the pull-back of a Cartier divisor on Y. Then there is no (-2)-curve in $N \cap H^\perp$. Let R be the set of (-2)-vectors on $H^\perp \cap \operatorname{Pic}(X)$. Since R is a finite set, we can take an ample divisor L such that $L \notin \mathbb{Q}u + N \otimes \mathbb{Q}$ for all $u \in R$. We shall consider a deformation of (X, L, H, ξ). Then H deforms to an ample divisor, which implies that we can apply Proposition 6.9 to get the claim. Q.E.D.

Remark 6.1. If ξ is relatively ample, then we can take $L = \xi + rmH$. Then the same assertion holds if $\gcd(r, (\xi, H), a) = 1$ and $H^\perp \cap N$ does not contain (-2) vectors.

6.3. An action of the affine Lie algebra

From now on, we assume that $\pi : X \to Y$ is a minimal resolution of rational double points. For simplicity, we assume that Y has one singular point $p \in Y$. Let $C_1, C_2, \dots, C_n$ be the irreducible components of the exceptional divisor and Z the fundamental cycle on X.

Lemma 6.11. (1) *Let c be a coherent sheaf on X such that $\pi_*(c) = 0$. Then there is a filtration*

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = c \tag{6.1}$$

such that each F_k/F_{k-1} is a subsheaf of $\mathcal{O}_{C_i}(-1)$, $i > 0$. In particular, if $\operatorname{Hom}(c, \mathcal{O}_{C_i}(-1)) = 0$ *for all i, then $c = 0$.*

(2) *If $\mathbf{R}\pi_*(c) = 0$, then c is a semi-stable 1-dimensional sheaf and* $\operatorname{gr}(c) = \bigoplus_{i=1}^n \mathcal{O}_{C_i}(-1)^{\oplus r_i}$.

Proof. (1) Assume that $c \neq 0$. Since $\pi_*(c) = 0$, c is of pure dimension 1. Since $((C_i, C_j)_{i,j})$ is negative definite, $\chi(\mathcal{O}_{C_i}(-1), c) > 0$ for an i. Then there is a non-zero homomorphism $\phi : \mathcal{O}_{C_i}(-1) \to c$ or $\phi : c \to \mathcal{O}_{C_i}(-1)$. For the first case, ϕ is injective and $\pi_*(\operatorname{coker}\phi) = 0$. For the second case, $\pi_*(\ker\phi) = 0$ and $\operatorname{im}(\phi)$ is a subsheaf of $\mathcal{O}_{C_i}(-1)$. Applying the same procedure to $\operatorname{coker}\phi$ or $\ker\phi$, we get the claim.

(2) We first note that $\chi(c) = 0$. Let E be a subsheaf of c. Then $H^0(X, E) = 0$, which implies that $\chi(E) \leq 0$. Therefore c is a semi-stable 1-dimensional sheaf. Obviously $\mathcal{O}_{C_i}(-1)$ are stable. We take a filtration (6.1). Then $\chi(F_k/F_{k-1}) \leq 0$ and the equality holds if

$F_k/F_{k-1} \cong \mathcal{O}_{C_i}(-1)$. Hence $F_k/F_{k-1} \cong \mathcal{O}_{C_i}(-1)$ for all k. Therefore c is S-equivalent to $\oplus_i \mathcal{O}_{C_i}(-1)^{\oplus r_i}$. Q.E.D.

Corollary 6.12. *$E \in \mathrm{Coh}(X)$ belongs to $\mathrm{Per}(X/Y)$ if and only if $\mathrm{Hom}(E, \mathcal{O}_{C_i}(-1)) = 0$ for all i.*

Proof. Obviously $E \in \mathrm{Per}(X/Y) \cap \mathrm{Coh}(X)$ satisfies $\mathrm{Hom}(E, \mathcal{O}_{C_i}(-1)) = 0$ for all i. We prove the converse direction. We shall prove that the homomorphism $\phi : \pi^*(\pi_*(E)) \to E$ is surjective. By Lemma 6.1 (2), the cokernel of $\pi_*(E) \to \pi_*(\operatorname{im}\phi)$ satsifies $\pi_*(\operatorname{coker}\phi) = 0$. Since $\mathrm{Hom}(\operatorname{coker}\phi, \mathcal{O}_{C_i}(-1)) \subset \mathrm{Hom}(E, \mathcal{O}_{C_i}(-1)) = 0$, by Lemma 6.11 (1), we get $\operatorname{coker}\phi = 0$. Thus ϕ is surjective. Q.E.D.

Lemma 6.13. *Let E be a coherent sheaf belonging to $\mathrm{Per}(X/Y)$. If $\mathrm{Hom}(\mathcal{O}_{C_i}(-1), E) = 0$, $i = 1, 2, ..., n$ and $\mathrm{Ext}^1(\mathcal{O}_Z, E) = 0$, then E is locally free along Z and $\pi_*(E)$ is reflexive at p.*

Proof. Replacing X by an open neighborhood of Z, we may assume that E is locally free on $X \setminus Z$. Assume that E is not torsion free. Then for the torsion submodule T of E, there is a surjection $T \to \mathbb{C}_x$. We note that there is an exact sequence $0 \to c \to \mathcal{O}_Z \to \mathbb{C}_x \to 0$ such that $c \in \mathrm{Coh}(X)$ with $\mathbf{R}\pi_*(c) = 0$. Since $\mathrm{Hom}(c, T) = 0$, $\mathrm{Ext}^1(\mathbb{C}_x, T) \to \mathrm{Ext}^1(\mathcal{O}_Z, T)$ is injective. Since $\chi(\mathbb{C}_x, T) = 0$, $\mathrm{Ext}^1(\mathbb{C}_x, T) \neq 0$. Thus $\mathrm{Ext}^1(\mathcal{O}_Z, T) \neq 0$, which is a contradiction. Therefore E is torsion free. By the exact sequence

$$\mathrm{Hom}(\mathcal{O}_Z, E^{\vee\vee}) \to \mathrm{Hom}(\mathcal{O}_Z, E^{\vee\vee}/E) \to \mathrm{Ext}^1(\mathcal{O}_Z, E),$$

we get $\mathrm{Hom}(\mathcal{O}_Z, E^{\vee\vee}/E) = 0$. Since $\mathrm{Hom}(\mathbb{C}_x, E^{\vee\vee}/E) \neq 0$ for a point $x \in \mathrm{Supp}(E^{\vee\vee}/E)$, $E^{\vee\vee}/E = 0$. Thus E is locally free. Then we get $R^1\pi_*(E^{\vee}_{|Z}) = 0$, which implies that $R^1\pi_*(E^{\vee}) = 0$. Therefore $\pi_*(E)$ is a reflexive sheaf. Q.E.D.

Lemma 6.14. (1) (a) *Let E be a coherent sheaf on X such that $E \in \mathrm{Per}(X/Y)$ and $\pi_*(E)$ is torsion free. For a subspace $U \subset \mathrm{Hom}(\mathcal{O}_{C_i}(-1), E)$, the evaluation map $\phi : U \otimes \mathcal{O}_{C_i}(-1) \to E$ is injective in $\mathrm{Coh}(X)$, $\operatorname{coker}\phi \in \mathrm{Per}(X/Y)$ and $\pi_*(\operatorname{coker}\phi)$ is torsion free.*

(b) *Let F be a coherent sheaf on X such that $F \in \mathrm{Per}(X/Y)$ and $\pi_*(F)$ is torsion free. For a subspace V of $\mathrm{Hom}(F, \mathcal{O}_{C_i}(-1)[1])$, the associated extension in $\mathrm{Coh}(X)$*

$$0 \to V^{\vee} \otimes \mathcal{O}_{C_i}(-1) \to E \to F \to 0$$

defines $E \in \mathrm{Per}(X/Y)$ and $\pi_*(E)$ is torsion free.

(2) (a) *Let E be a coherent sheaf on X such that $E \in \mathrm{Per}(X/Y)$ and $\pi_*(E)$ is torsion free. Let $U \subset \mathrm{Hom}(\mathcal{O}_Z[-1], E)$ be a subspace. For the associated extension in $\mathrm{Coh}(X)$*

$$0 \to E \to F \to \mathcal{O}_Z \otimes U \to 0,$$

$F \in \mathrm{Per}(X/Y)$ and $\pi_*(F)$ is torsion free.

(b) *Let F be a coherent sheaf on X such that $F \in \mathrm{Per}(X/Y)$ and $\pi_*(F)$ is torsion free. Let $V \subset \mathrm{Hom}(F, \mathcal{O}_Z)$ be a subspace. Then $\phi : F \to \mathcal{O}_Z \otimes V^\vee$ is surjective in $\mathrm{Coh}(X)$, $E := \ker \phi \in \mathrm{Per}(X/Y)$ and $\pi_*(E)$ is torsion free.*

Proof. (1) (a) Since $\mathbf{R}\pi_*(\mathcal{O}_{C_i}(-1)) = 0$ and $\pi_*(E)$ is torsion free, $\pi_*(\ker \phi) = 0$ and $R^1\pi_*(\ker \phi) \cong \pi_*(\mathrm{im}\, \phi) = 0$. By Lemma 6.11, we see that $\ker \phi \cong \mathcal{O}_{C_i}(-1)^{\oplus r}$. Since ϕ induces an injective homomorphism $U \to \mathrm{Hom}(\mathcal{O}_{C_i}(-1), E)$, we have $r = 0$. Since $\pi_*(E) \cong \pi_*(\mathrm{coker}\, \phi)$, $R^1\pi_*(\mathrm{coker}\, \phi) = 0$ and $\pi^*\pi_*(E) \to E \to \mathrm{coker}\, \phi$ is surjective, $\mathrm{coker}\, \phi \in \mathrm{Per}(X/Y)$.

(b) We note that $\pi_*(E) \cong \pi_*(F)$ and $R^1\pi_*(E) = 0$. Hence we shall prove that $\mathrm{Hom}(E, \mathcal{O}_{C_j}(-1)) = 0$. If $j \neq i$, then obviously the claim holds. If $j = i$, then we have a non-zero map $V^\vee \otimes \mathcal{O}_{C_i}(-1) \to E \to \mathcal{O}_{C_i}(-1)$. By our choice of the extension class, this is impossible. Hence $E \in \mathrm{Per}(X/Y)$.

(2) (a) Obviously $\pi^*\pi_*(F) \to F$ is surjective and $R^1\pi_*(F) = 0$. If $\pi_*(F)$ has a torsion, then we have a non-trivial map $\mathcal{O}_Z = \pi^*(\mathbb{C}_p) \to F$. Then $\mathcal{O}_Z \to F \to \mathcal{O}_Z \otimes U$ is injective. By our choice of the extension class, this is impossible. Hence $\pi_*(F)$ is torsion free.

(b) Since $\mathrm{Hom}(F, \mathcal{O}_Z) \to \mathrm{Hom}(\pi^*\pi_*(F), \mathcal{O}_Z) = \mathrm{Hom}(\pi_*(F), \mathbb{C}_p)$ is injective, $\pi_*(F) \to \mathbb{C}_p \otimes V^\vee$ is surjective. Since $\pi^*(\pi_*(F)) \to \pi^*(\mathbb{C}_p) \otimes V^\vee$ is the composition of $\pi^*(\pi_*(F)) \to F$ and $F \to \mathcal{O}_Z \otimes V^\vee$, ϕ is surjective. Since $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_{C_j}(-1)) = 0$ for all j, $\mathrm{Hom}(E, \mathcal{O}_{C_j}(-1)) = 0$ for all j. Thus $E \in \mathrm{Per}(X/Y)$ and $\pi_*(E)$ is torsion free. Q.E.D.

We set $E_0 := \mathcal{O}_Z$, $E_i := \mathcal{O}_{C_1}(-1)[1]$, $i = 1, 2, ..., n$ and set

$$\mathfrak{P}_{E_i}^{(m)} := \{(E, U) | E \in M_H^p(v), U^\vee \subset \mathrm{Hom}(E, E_i), \dim U = m\}.$$

By Lemma 6.14, the Brill–Noether locus with respect to E_i, $i = 0, 1, ..., n$ behaves very well and we have the following.

Proposition 6.15. *The affine Lie algebra associated to E_i, $i = 0, 1, ..., n$ acts on $\bigoplus_v H_*(M_H^p(v))$.*

Remark 6.2. $f : M_H^p(v) \to M_H(w)$ gives the contraction map of the Brill–Noether locus with respect to E_i, $i = 1, 2, ..., n$.

Remark 6.3. Let X be an abelian surface or a K3 surface with a symplectic G-action. Assume that there is a fixed point. By the McKay correspondence [BKR], we have an equivalence $\Phi : \mathbf{D}^G(X) \cong \mathbf{D}(\widetilde{X/G})$, where $\widetilde{X/G} \to X/G$ is the minimal resolution of X/G. Moreover we can choose an equivalence so that Φ induces an equivalence $\mathrm{Coh}^G(X) \to \mathrm{Per}((\widetilde{X/G})/(X/G))$. By this equivalence, we have an isomorphism $M_H(v)^\mu \to M_H^p(w)^\mu$, where w is the Mukai vector corresponding to v via Φ. By this identification, the actions of the Lie algebras in Section 5 and Section 6 are the same.

§7. Appendix

7.1. Moduli of coherent systems

In this subsection, we shall explain how to construct the moduli space of coherent systems $\mathfrak{P}_{E_i}^{(n)}(v)$. We start with a definition of a flat family.

Definition 7.1. Let S be a scheme and $\mathcal{E}_\bullet : \cdots \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \cdots$ a bounded complex on $S \times X$.

(i) $\mathcal{E}_\bullet$ is a flat family of stable complexes, if $\mathcal{E}_i$ are coherent sheaves on $S \times X$ which are flat over S and $(\mathcal{E}_\bullet)_s$ are stable complexes for all $s \in S$.

(ii) $(\mathcal{E}_\bullet, \mathcal{U})$ is a family of coherent systems, if $\mathcal{E}_\bullet$ is a flat family of stable complexes and $\mathcal{U}$ is a locally free subsheaf of $\mathrm{Hom}_{p_S}(\mathcal{O}_S \boxtimes E_i, \mathcal{E}_\bullet)$ of rank n such that $\mathcal{U}_s \to \mathrm{Hom}(E_i, (\mathcal{E}_\bullet)_s)$ is injective for all $s \in S$. In this case, we have a resolution of E_i

$$W_\bullet : W_{-2} \to W_{-1} \to W_0$$

with a morphism $\mathcal{U} \boxtimes W_\bullet \to \mathcal{E}_\bullet$ as complexes which induces the inclusion $\mathcal{U} \to \mathrm{Hom}_{p_S}(\mathcal{O}_S \boxtimes E_i, \mathcal{E}_\bullet)$.

For a quasi-isomorphism $\mathcal{E}_\bullet \to \mathcal{E}'_\bullet$ of families of stable complexes over S, we take a resolution of E_i

$$W_\bullet : W_{-2} \to W_{-1} \to W_0$$

such that $\mathrm{Ext}^p(W_j, (\mathcal{E}_k)_s) = \mathrm{Ext}^p(W_j, (\mathcal{E}'_k)_s) = 0$, $p > 0$ for $j = 0, -1$, $k \in \mathbb{Z}$ and all $s \in S$. Then we see that $\mathrm{Ext}^p(W_{-2}, (\mathcal{E}_k)_s) = \mathrm{Ext}^p(W_{-2}, (\mathcal{E}'_k)_s) = 0$, $p > 0$ for $k \in \mathbb{Z}$ and all $s \in S$. By this choice of $W_\bullet$, we have an isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{K}(S \times X)}(\mathcal{O}_S \boxtimes W_\bullet, \mathcal{E}_\bullet[p]) \to \\ &\qquad \mathrm{Hom}_{\mathbf{K}(S \times X)}(\mathcal{O}_S \boxtimes W_\bullet, \mathcal{E}'_\bullet[p]) (\cong \mathrm{Ext}^p(\mathcal{O}_S \boxtimes E_i, \mathcal{E}'_\bullet)) \end{aligned}$$

where $\mathbf{K}(Z)$ is the homotopy category of complexes on Z. Hence for a family of coherent systems $(\mathcal{E}'_\bullet, \mathcal{U})$ and a quasi-isomorphism $\mathcal{E}_\bullet \to \mathcal{E}'_\bullet$ of flat families of stable complexes, there is a resolution of E_i and a family of coherent systems $(\mathcal{E}_\bullet, \mathcal{U})$ such that we have a homotopy commutative diagram:

$$\begin{array}{ccc} \mathcal{U} \boxtimes W_\bullet & \xrightarrow{\phi} & \mathcal{E}_\bullet \\ \| & & \downarrow \\ \mathcal{U} \boxtimes W_\bullet & \longrightarrow & \mathcal{E}'_\bullet. \end{array}$$

The choice of ϕ is unique, up to homotopy equivalence. In this case, we say that $(\mathcal{E}_\bullet, \mathcal{U})$ is equivalent to $(\mathcal{E}'_\bullet, \mathcal{U})$.

Let $q : Q_H(v) \to M_H(v)$ be a standard $PGL(N)$-covering of $M_H(v)$ which is an open subscheme of a suitable quot-scheme and satisfies the following properties:

(i) There is a flat family of stable complexes $\mathcal{V}_\bullet : \mathcal{V}_{-1} \to \mathcal{V}_0$ on $Q_H(v) \times X$, which is $GL(N)$-equivariant.

(ii) For a flat family of stable complexes $\mathcal{E}_\bullet$ parametrized by S, if we take a suitable open covering $S = \cup_\lambda S_\lambda$, then we have a morphisms $f_\lambda : S_\lambda \to Q_H(v)$ such that $\mathcal{E}_{\bullet|S_\lambda}$ is quasi-isomorphic to $f_\lambda^*(\mathcal{V}_\bullet)$. In particular $(q \circ f_\lambda)_{|S_\lambda \cap S_\mu} = (q \circ f_\mu)_{|S_\lambda \cap S_\mu}$ and we have a morphism $f : S \to M_H(v)$.

We take a locally free resolution of E_i

$$0 \to W_{-2} \to W_{-1} \to W_0 \to E_i \to 0$$

such that $\operatorname{Ext}^p(W_j, (\mathcal{V}_k)_t) = 0$, $p > 0$ for $j = 0, -1$, $k = -1, 0$ and all $t \in Q_H(v)$. Then $\operatorname{Ext}^p(W_{-2}, (\mathcal{V}_k)_t) = 0$, $p > 0$ for $k = -1, 0$ and all $t \in Q_H(v)$. We set

$$\mathcal{H}_n := \bigoplus_{-j+k=n} \operatorname{Hom}_{p_{Q_H(v)}}(\mathcal{O}_{Q_H(v)} \boxtimes W_j, \mathcal{V}_k).$$

$\mathcal{H}_n$, $n \in \mathbb{Z}$ are locally free sheaves on $Q_H(v)$. We take a complex

$$0 \to \mathcal{H}_{-1} \xrightarrow{\psi_{-1}} \mathcal{H}_0 \xrightarrow{\psi_0} \mathcal{H}_1 \xrightarrow{\psi_1} \cdots$$

associated to $\mathbf{R}\operatorname{Hom}_{p_{Q_H(v)}}(\mathcal{O}_{Q_H(v)} \boxtimes E_i, \mathcal{V}_\bullet)$. Since

$$\ker(\psi_{-1})_t \cong \operatorname{Hom}(E_i, \mathcal{E}_t[-1]) = 0$$

for all $t \in Q_H(v)$, ψ_{-1} is injective as a vector bundle homomorphism. Hence $\mathcal{H}'_0 := \operatorname{coker} \psi_{-1}$ is a locally free sheaf on $Q_H(v)$. For the morphism $f_\lambda : S_\lambda \to Q_H(v)$ and a locally free subsheaf $\mathcal{U} \subset \operatorname{Hom}_{p_S}(\mathcal{O}_S \boxtimes$

$E_i, \mathcal{E}_\bullet)$ such that $\mathcal{U}_s \to \mathrm{Hom}(E_i, (\mathcal{E}_\bullet)_s)$ is injective for all $s \in S$, we have an inclusion as a vector bundle homomorphism:

$$\mathcal{U}_{|S_\lambda} \hookrightarrow \mathrm{Hom}_{p_S}(\mathcal{O}_S \boxtimes E_i, \mathcal{E}_\bullet)_{|S_\lambda} = \ker(f_\lambda^*(\mathcal{H}_0') \to f_\lambda^*(\mathcal{H}_1)) \hookrightarrow f_\lambda^*(\mathcal{H}_0').$$

We take a Grassmann bundle $Gr(\mathcal{H}_0', n) \to Q_H(v)$ over $Q_H(v)$ parametrizing n-dimensional subspaces U of $(\mathcal{H}_0')_t$, $t \in Q_H(v)$. Then we have a lifting $\widetilde{f_\lambda} : S_\lambda \to Gr(\mathcal{H}_0', n)$ of f_λ and an equivalence between $(\mathcal{E}_\bullet, \mathcal{U}_{|S_\lambda})$ and $(\widetilde{f}_\lambda^*(\mathcal{V}_\bullet), \mathcal{U}_{|S_\lambda})$. Hence $\mathfrak{P}_{E_i}^{(n)}(v)$ is constructed as a closed subscheme of $Gr(\mathcal{H}_0', n)/PGL(N)$.

7.2. The existence of semi-stable sheaves on a $K3$ surface

Proposition 7.1. *Let X be a $K3$ surface and H an ample divisor on X. For $v = r + \xi + a\rho$, $r \in \mathbb{Z}_{>0}, \xi \in \mathrm{NS}(X), a \in \mathbb{Z}$ with $\langle v^2 \rangle \geq -2$, the moduli space of semi-stable sheaves $\overline{M}_H(v)$ is not empty.*

Proof. We may assume that v is primitive. In $H^*(X, \mathbb{Q})$, we can write v as

$$v = r + (dH + D) + a\rho, D \in H^\perp.$$

Since $ve^{nH} = r + (d + rn)H + D + (a + (dn + rn^2/2)(H^2))\rho$, $n \in \mathbb{Z}$, we see that

$$\langle ve^{nH}, ve^{nH} \rangle - (D^2) = \langle v, v \rangle - (D^2).$$

Hence replacing v by ve^{nH}, $n \gg 0$, we may assume that d is sufficiently larger than $\langle v^2 \rangle - (D^2)$. We shall consider the Fourier–Mukai transform

$$\begin{array}{rccc} \Phi_{X \to X}^{I_\Delta} : & \mathbf{D}(X) & \to & \mathbf{D}(X) \\ & E & \mapsto & \mathbf{R}p_{2*}(p_1^*(E) \otimes I_\Delta), \end{array}$$

where $p_1, p_2 : X \times X \to X$ are projections and I_Δ is the ideal sheaf of the diagonal $\Delta \subset X \times X$. By [Y5, Thm. 3.1], $\Phi_{X \to X}^{I_\Delta}$ induces an isomorphism $\overline{M}_H(r + \xi + a\rho) \cong \overline{M}_H(a - \xi + r\rho)$. Moreover [Y5, Cor. 2.14] says that every μ-semi-stable sheaf F with $v(F) = a - \xi + r\rho$ is semi-stable. For a sufficiently small $\epsilon \in \mathrm{NS}(X) \otimes \mathbb{Q}$, [Y3, Thm. 8.1] implies that there is a stable sheaf F with respect to $H + \epsilon$ with $v(F) = a - \xi + r\rho$. Then F is μ-semi-stable with respect to H, which implies that $\overline{M}_H(a - \xi + r\rho) \neq \emptyset$. Therefore $\overline{M}_H(v) \neq \emptyset$. Q.E.D.

7.3. The existence of stable sheaves on a rational elliptic surface

We shall find the conditions for the existence of stable sheaves on a rational elliptic surface $\pi : X \to \mathbb{P}^1$ with a section σ. We first note that a divisor C with $(C^2) = (C, K_X) = -1$ is effective. Indeed since

$(K_X - C, f) = -1$, $H^2(X, \mathcal{O}_X(C)) = 0$. By the Riemann–Roch theorem, $\dim H^0(X, \mathcal{O}_X(C)) \geq \chi(\mathcal{O}_X(C)) = 1$. The following is the result for the case of rank 0.

Proposition 7.2. *Let X be a rational elliptic surface with a section σ. Let D be a divisor with $(D^2) \geq 0$. Assume that $(0, D, \chi)$ is primitive. Then $M_H^G(0, D, \chi)$ is not empty for a general H and G if and only if $(D, C) \geq 0$ for all divisor C with $(C^2) = (C, K_X) = -1$.*

Proof. We use the notation in Subsection 4.3. Since $\overline{M}^G_{(\mathcal{X},\mathcal{H})/T}(0, D, \chi) \to T$ is smooth, it is sufficient to prove the claim for a nodal rational elliptic surface X. Let C be a divisor with $(C^2) = (C, K_X) = -1$. Since every fiber is irreducible, C must be a section of π. If $(D, C) < 0$, then $\chi(\mathcal{O}_C(k), E) = -(D, C) > 0$ for all sheaves E with $c_1(E) = D$. We set $n := \max\{k | \operatorname{Hom}(\mathcal{O}_C(k), E) \neq 0\}$. Then $\operatorname{Hom}(\mathcal{O}_C(n), E) \neq 0$ and $\operatorname{Hom}(E, \mathcal{O}_C(n))^\vee = \operatorname{Ext}^2(\mathcal{O}_C(n+1), E) \neq 0$. This means that E is not semi-stable, unless $E \cong \mathcal{O}_C(n)$.

Conversely, we assume that $(D, C) \geq 0$ for all sections C with $(C^2) = (C, K_X) = -1$. Then D is a nef divisor. If $(D, f) = 1$, then there is a section τ of π such that $D = \tau + nf$, $n > 0$. In this case, $M_H(0, \tau + nf, \chi) \cong \operatorname{Hilb}_X^n \neq \emptyset$ via the relative Fourier–Mukai transform. Since the non-emptyness does not depend on the choice of G [Y4], we get our claim. Hence we may assume that $(D, f) \geq 2$. We shall show that there is a reduced and irreducible curve $C \in |D|$. Then a line bundle E on C with $\chi(E) = \chi$ belongs to $M_H(0, D, \chi)$.

(1) If $(D^2) \geq 1$ or $(D, f) \geq 3$, then $D' := D - K_X$ is a nef divisor with $({D'}^2) \geq 5$. In this case, we shall prove that $D = D' + K_X$ is base point free by using Reider's result [R, Thm. 1]. If D is not base point free, then there is an effective divisor B such that (a) $(B, D') = 1$ and $(B^2) = 0$, or (b) $(B, D') = 0$ and $(B^2) = -1$. Since $0 \leq (D, B) \leq (D', B) \leq 1$, (i) $(f, B) = 0$ and $(D, B) \leq 1$ or (ii) $(f, B) = 1$ and $(D, B) = 0$. In the first case, $B = nf$. Since $(D, f) \geq 2$, this is impossible. In the second case, there is a section τ and $B = \tau + nf$. Then $(B^2) = 2n - 1 \neq 0$. Therefore $D = D' + K_X$ is base point free.

(2) If $(D^2) = 0$ and $(D, f) = 2$, then $D = 2\tau_1 + f$ or $D = \tau_1 + \tau_2$ with $(\tau_1, \tau_2) = 1$, where τ_1, τ_2 are sections of π. In the first case, $(D, \tau_1) = -1$, which is a contradiction. In the second case, D is connected and D is base point free.

Applying Bertini's theorem to both cases (1), (2), we have a reduced and irreducible curve $C \in |D|$.

Q.E.D.

Definition 7.2. We set

$$\mathcal{C} := \left\{ D \in \operatorname{Pic}(X) \;\middle|\; \begin{array}{l} (D, C) \geq 0 \text{ for all divisors } C \\ \text{with } (C^2) = (C, K_X) = -1 \end{array} \right\}.$$

Let $W := W(E_8^{(1)})$ be the Weyl group of the sublattice $f^\perp \cong E_8^{(1)}$ of $\operatorname{Pic}(X)$. W acts on $\operatorname{Pic}(X)$ and $\mathcal{C}$ is a W-invariant subset of $\operatorname{Pic}(X)$. Let $\mathcal{C}^+ \subset \mathcal{C}$ be the set of nef divisors. If X is nodal, then $\mathcal{C}^+ = \mathcal{C}$.

Theorem 7.3. *Let r and d be relatively prime integers with $r \geq 0$.*

(i) *For any $D \in \langle \sigma, f \rangle^\perp$, there is a stable vector bundle E_D such that $\operatorname{rk}(E_D) = r$, $c_1(E_D) \equiv d\sigma + D \mod \mathbb{Z}f$ and $\chi(E_D, E_D) = 1$. E_D is unique up to replacing it with $E_D(nf)$, $n \in \mathbb{Z}$. We set*

$$\mathcal{E}(r, d) := \{E_D | (D, \sigma) = (D, f) = 0\}.$$

(ii) *Let $F \in K(X)$ be a primitive class with $\operatorname{rk}(F) = lr$ and $(c_1(F), f) = ld$. Assume that $\chi(F, F) \leq 0$. We take an ample divisor H which is sufficiently close to f. Then F is represented by a stable sheaf if and only if $\chi(E_D, F) \leq 0$ for all $E_D \in \mathcal{E}(r, d)$. Moreover F is represented by a μ-stable vector bundle, if $lr > 1$.*

Proof. We may assume that $lr > 0$. By the deformation argument in the proof of Proposition 7.2, we may assume that X is nodal. We first prove (i). We note that $M_H(0, rf, -d) \cong X$. Let $\mathcal{E}$ be a universal family on $X \times X$. Since every fiber is irreducible, we have $\sigma - D = \tau - ((\sigma, \tau) + 1)f$, where τ is a section of π. Then $\mathcal{E}^\vee_{|X \times \tau}$ is a stable sheaf with the desired invariant. We next prove (ii). The proof of the necessary condition is similar to the proof of Proposition 7.2. We shall show that the condition is sufficient. Let $\Phi^{\mathcal{E}}_{X \to X} : \mathbf{D}(X) \to \mathbf{D}(X)$ be the relative Fourier–Mukai transform defined by the sheaf $\mathcal{E}$. Then $\Phi^{\mathcal{E}}_{X \to X}(E_D)[1] = \mathcal{O}_\tau$, where τ is a section of π such that $\tau - \sigma \equiv -D \mod \mathbb{Z}f$. Then $\operatorname{rk}(\Phi^{\mathcal{E}}_{X \to X}(F)[1]) = 0$ and $c_1(\Phi^{\mathcal{E}}_{X \to X}(F)[1]) \in \mathcal{C}$. Therefore $\Phi^{\mathcal{E}}_{X \to X}(F)[1]$ is represented by a line bundle L on a reduced and irreducible curve. Then the inverse $\Phi^{\mathcal{E}^\vee}_{X \to X}(L)[1]$ is a μ-stable sheaf. Q.E.D.

By the proof of the theorem, we also get the following.

Corollary 7.4. *If $\gcd(r, (\xi, f)) = 1$ and the expected dimension is non-negative, then $M_H(r, \xi, \chi)$ is not empty, where H is sufficiently close to f.*

Let X be a rational elliptic surface with a section σ such that there is a singular fiber $\pi^{-1}(o) = \sum_{i=0}^{8} a_i C_i$, $o \in \mathbb{P}^1$ of type $E_8^{(1)}$, where C_i are smooth (-2)-curves. We assume that $a_0 = 1$. Let C be a divisor with $(C^2) = (C, K_X) = -1$. Then $C = \sigma + \sum_{i=0}^{8} n_i C_i$, $n_i \geq 0$. Hence

$$\mathcal{C}^+ = \{D \in \operatorname{Pic}(X) \,|(D,\sigma) \geq 0, (D, C_i) \geq 0, 0 \leq i \leq 8\}.$$

Thus $D := r\sigma + nf + \xi$, $\xi \in \bigoplus_{i=1}^{8} \mathbb{Z}C_i$ is nef if and only if

$$\begin{cases} n \geq r, \\ (\xi, C_i) \geq 0, \ 1 \leq i \leq 8 \\ \sum_{i=1}^{8} a_i(\xi, C_i) \leq r. \end{cases}$$

Let W be the affine Weyl group of $E_8^{(1)}$. Then $M_H(0, D', \chi) \neq \emptyset$ if and only if $D' = w(D)$ with $D \in \mathcal{C}^+, w \in W$.

Acknowledgements. I would like to thank Hiraku Nakajima for valuable discussions on this subject for years. I would also like to thank the referee for many comments.

References

[B1] T. Bridgeland, Flops and derived categories, Invent. Math., **147** (2002), 613–632.

[BKR] T. Bridgeland, A. King and M. Reid, The McKay correspondence as an equivalence of derived categories, J. Amer. Math. Soc., **14** (2001), 535–554.

[GKV] V. Ginzburg, M. Kapranov and É. Vasserot, Langlands reciprocity for algebraic surfaces, Math. Res. Lett., **2** (1995), 147–160.

[H1] D. Huybrechts, Compact hyper-Kähler manifolds: basic results, Invent. Math., **135** (1999), 63–113.

[H2] D. Huybrechts, The Kähler cone of a compact hyperkähler manifold, Math. Ann., **326** (2003), 499–513.

[Iq] A. Iqbal, A note on E-strings, Adv. Theor. Math. Phys., **7** (2003), 1–23.

[I1] A. Ishii, On the moduli of reflexive sheaves on a surface with rational double points, Math. Ann., **294** (1992), 125–150.

[I2] A. Ishii, Versal deformation of reflexive modules over rational double points, Math. Ann., **317** (2000), 239–262.

[K-S] M. Kashiwara and P. Shapira, Sheaves on Manifolds, Grundlehren Math. Wiss., **292**, Springer-Verlag, Berlin, 1994.

[K-Y] T. Kawai and K. Yoshioka, String partition functions and infinite products, Adv. Theor. Math. Phys., **4** (2000), 397–485, arXiv:hep-th/0002169.

[Mr] E. Markman, Brill–Noether duality for moduli spaces of sheaves on K3 surfaces, J. Algebraic Geom., **10** (2001), 623–694.

[MNWV] J. A. Minahan, D. Nemeschansky, C. Vafa and N. P. Warner, E-strings and N=4 topological Yang–Mills theories, Nucl.Phys. B, **527** (1998), 581–623.

[Mu1] S. Mukai, Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface, Invent. Math., **77** (1984), 101–116.

[Mu2] S. Mukai, On the moduli space of bundles on $K3$ surfaces. I, In: Vector Bundles on Algebraic Varieties, Bombay, 1984, Tata Inst. Fund. Res., Bombay, 1987, pp. 341–413.

[N1] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras, Duke Math. J., **76** (1994), 365–416.

[N2] H. Nakajima, Gauge theory on resolution of simple singularities and simple Lie algebras, Inter. Math. Res. Notices, **2** (1994), 61–74.

[N2] H. Nakajima, Quiver varieties and Kac–Moody algebras, Duke Math. J., **91** (1998), 515–560.

[N3] H. Nakajima, Lectures on Hilbert Schemes of Points on Surfaces, Univ. Lecture Ser., **18**, Amer. Math. Soc., 1999.

[N4] H. Nakajima, Quiver varieties and finite dimensional representations of quantum affine algebras, J. Amer. Math. Soc., **14** (2001), 145–238.

[N5] H. Nakajima, Quiver varieties and McKay correspondence, In: Proceeding of Open Calabi–Yau manifolds—approaches from algebraic geometry and string theory, 2001 Dec., Hokkaido, 2002, pp. 1–49.

[N6] H. Nakajima, Convolution on homology groups of moduli spaces of sheaves on K3 surfaces, In: Vector Bundles and Representation Theory, Columbia, MO, 2002, Contemp. Math., **322**, Amer. Math. Soc., Providence, RI, 2003, pp. 75–87.

[N-Y] H. Nakajima and K. Yoshioka, Perverse coherent sheaves on blow-up. II. Wall-crossing and Betti numbers formula, arXiv:0806.0463.

[O-Y] N. Onishi and K. Yoshioka, Singularities on the 2-dimensional moduli spaces of stable sheaves on $K3$ surfaces, Internat. J. Math., **14** (2003), 837–864, arXiv:math.AG/0208241.

[R] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math. (2), **127** (1988), 309–316.

[YZ] S. T. Yau and E. Zaslow, BPS States, String Duality, and Nodal Curves on K3, Nucl. Phys. B, **471** (1996), 503–512.

[Y1] K. Yoshioka, Euler characteristics of $SU(2)$ instanton moduli spaces on rational elliptic surfaces, Commun. Math. Phys., **205** (1999), 501–517.

[Y2] K. Yoshioka, Some examples of Mukai's reflections on K3 surfaces, J. Reine Angew. Math., **515** (1999), 97–123.

[Y3] K. Yoshioka, Moduli spaces of stable sheaves on abelian surfaces, Math. Ann., **321** (2001), 817–884, arXiv:math.AG/0009001.

[Y4] K. Yoshioka, Twisted stability and Fourier–Mukai transform II, Manuscripta Math., **110** (2003), 433–465.

[Y5] K. Yoshioka, Stability and the Fourier–Mukai transform II, Compositio Math., **145** (2009), 112–142.

Department of Mathematics, Faculty of Science, Kobe University, Kobe, 657, Japan
E-mail address: `yoshioka@math.kobe-u.ac.jp`