Stability and arithmetic

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Dedicated to the memory of my father Jiahua Weng
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Abstract.

Stability plays a central role in arithmetic. In this article, we explain some basic ideas and present certain constructions for our studies. There are two aspects: namely, general Class Field Theories for Riemann surfaces using semi-stable parabolic bundles and for p-adic number fields using what we call semi-stable filtered $(\varphi, N; \omega)$-modules; and non-abelian zeta functions for function fields over finite fields using semi-stable bundles and for number fields using semi-stable lattices.

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Part B. High rank zeta functions and stability

Introduction

In the past a decade or so, importance of stability, which originally appeared and has played key roles in algebraic geometry, was gradually recognized by many people working in arithmetic. As typical examples, we now have

(i) Existence theorem and reciprocity law of a non-abelian class field theory for function fields over complex numbers, based on Seshadri's work of semi-stable parabolic bundles over Riemann surfaces;

(ii) High rank zeta functions for global fields, defined as natural integrations over moduli spaces of semi-stable bundles/lattices; and

(iii) Characterization of the so-called semi-stable representations for absolute Galois groups of p-adic number fields, in terms of weakly admissible filtered (φ, N)-modules, or better, semi-stable filtered (φ, N)-modules of slope zero.

Along with this line, as an integrated part of our Program on Geometric Arithmetic, in this article, we explain some basic ideas and present certain constructions using stability to study two non-abelian aspects of arithmetic, one at a micro level and the other on large scale.

I) Micro level

We, at this micro level, want to give a characterization for each individual Galois representation. For this, first we, according to the associated base field and coefficients, classify Galois representations into four types, namely, v-adic/adelic representations for local/global (number) fields. As such, then our aim becomes to expose some totally independent structures from which the original Galois representations can be reconstructed.

In general, arbitrary Galois representations are too complicated to have clearer structures, certain natural restrictions should be imposed. In this direction, we, as a natural continuation of existing theories of Galois representations, choose to make the following rather standard restrictions:

Fields\Coefs	v-Adic	Adelic
Local	Fin Monodromy & Nilpotency	Compatible System
Global	+ Finite Ramification	+ Admissible System

To be more precise, they are:

(i) v-adic Galois representations for

(i.a) *Local Field K_w*: Here Galois representations $\rho_{w,v} : G_{K_w} \to GL_n(F_v)$ involved are for the absolute Galois group G_{K_w} of a local w-adic number field K_w with coefficients in a fixed v-adic number field F_v. Motivated by (α) Grothendieck's Monodromy Theorem for v-adic Galois representations of w-adic number fields, where $v \nmid w$, i.e., v and w are with different residual characteristics; and

(β) Fontaine∥Berger's Monodromy Theorem for v-adic Galois representations of w-adic number fields, with $v\|w$, i.e., v and w are with the same residual characteristics,

we assume that
(pST) $\rho_{w,v}$ *is potentially semi-stable.*

Clearly, when $v \nmid w$, this is equivalent to the following
(pFM&U) $\rho_{w,v}$ *is potentially of finite monodromy and unipotent.*
In other words, we assume that there exists a finite Galois extension $L_{w'}/K_w$ such that for the induced Galois representation $\rho_{w',v} : G_{L_{w'}} \to GL_n(F_v)$, the image of the associated ramification group $I_{L_{w'}}$ is then nilpotent.

(i.b) *Global Field K*: Here Galois representations $\rho_{K,v} : G_K \to GL_n(F_v)$ involved are for the absolute Galois group G_K of a global number field K with coefficients in a fixed v-adic number field F_v. Motivated by etale cohomology theory of algebraic varieties, we assume that
(pST) *For all local completions K_w, the associated local v-adic representations $\rho_{w,v} : G_{K_w} \to GL_n(F_v)$ satisfies condition* **pST** *of (i.a); and*
(Unr) *For almost all w, the associated v-adic representations $\rho_{w,v} : G_{K_w} \to GL_n(F_v)$ are unramified.*

(ii) Adelic Galois representations for

(ii.a) *Local Field K_w*: Here Galois representations $\rho_{w,\mathbb{A}_F} : G_{K_w} \to GL_n(\mathbb{A}_F)$ involved are for the absolute Galois group G_{K_w} of a w-adic number field K_w with coefficients in the adelic space $\mathbb{A}_F$ associated to a number field F. Continuity of $\rho_{w,\mathbb{A}_F}$ proves to be too loose. Stronger algebraic condition should be imposed. Motivated by Grothendieck's etale cohomology theory of algebraic varieties, and Deligne's solution to the Weil conjecture when $v \nmid w$, together with Katz–Messing's modification when $v\|w$, we assume that
(Unr) *For almost all v (in coefficients), the associated v-adic representation $\rho_{w,v} : G_{K_w} \to GL_n(F_v)$ are unramified; and*

(**Inv**) *For all v, i.e., for v satisfying either $v\|w$ or $v \nmid w$, the associated characteristic polynomials of the Frobenius induced from $\rho_{w,v}$ are the same, particularly, independent of v.*

We call such a representation a *thick one*, as the invariants do not depend on the coefficients chosen.

Remark. The compatibility conditions stated here are standard. (See e.g. [Se2], [Hi], [Tay].) However, from our point of view, the **Inv** condition appears to be to practical—yes, it is very convenient and extremely useful to impose the independence for the associated characteristic polynomials of Frobenius; on the other hand, this independence should not be the cause but rather an ultimate goal. In other words, it would be much better if the Inv condition can be replaced by other principles, e.g., certain compatibility from class field theory. (See e.g., [Kh1,2,3].) We leave the details to the reader.

(**ii.b**) *Global Field K*: Here Galois representations $\rho_{K,\mathbb{A}_F} : G_K \to GL_n(\mathbb{A}_F)$ involved are for the absolute Galois group G_K of a global number field K with coefficients in the adelic space $\mathbb{A}_F$ associated to a number field F. As above, only continuity of $\rho_{w,\mathbb{A}_F}$ appears to be too weak to get a good theory. Much stronger algebraic conditions should be imposed. Certainly, there are two different directions to be considered, namely, the horizontal one consisting of places w of K, and the vertical one consisting of places v of coefficients field F. From ii.a), we assume that

(**Comp**) *For every fixed place w of K, the induced representation $\rho_{w,\mathbb{A}_F} : G_{K_w} \to \mathrm{GL}_n(\mathbb{A}_F)$ forms a compatible system.*

As such, the corresponding theory is a thick one. Hence, by **Inv**, we are able to select good representatives for $\rho_{w,\mathbb{A}_F}$, e.g., the induced $\rho_{w,v} : G_{K_w} \to \mathrm{GL}_n(F_v)$ where $v\|w$. In this language, we then further assume that the admissible conditions for the other direction v can be read from these selected $\rho_{w,v}$, $v\|w$. More precisely, we assume that

(**dR**) *All $\rho_{w,v}$, $v\|w$, are of de Rham type;*

(**Crys**) *For almost all w and v, $\rho_{v,w}$ are crystalline.*

For this reason, we may form what we call the *anleric ring*

$$\mathbb{B}_\mathbb{A} := \prod{}' \left(\mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{crys}}^+ \right),$$

where $\mathbb{B}_{\mathrm{dR}}$ denotes the ring of de Rham periods, and $\mathbb{B}_{\mathrm{crys}}^+$ the ring of crystalline periods, and $\prod'$ means the restricted product. As such, the final global condition we assume is the following:

(**Adm**) *$\{\rho_{w,v}\}_{v\|w}$ are $\mathbb{B}_\mathbb{A}$-admissible.*

Even this admissibility is not clearly stated due to 'the lack of space', which will be discussed in details elsewhere, one may sense it say via determinant formalism from abelian CFT, (see e.g., the reformulation by Serre for rank one case ([Se2]) and the conjecture of Fontaine–Mazur on geometric representations ([FM], see also [Tay]). For the obvious reason, we will call such a representation a *thin* one.

With the restrictions on Galois representations stated, let us next turn our attention to their characterizations. Here by a characterization, we mean a certain totally independent but intrinsic structure from which the original Galois representation can be reconstructed. There are two different approaches, analytic one and algebraic one.

• **Analytic One** Here the objects seeking are supposed to be equipped with analytic structures such as connections and residues (at least for v-adic representations). Good examples are the related works of Weil on flat bundles, of Seshadri on logarithmic unitary flat bundles, and of Dwork on p-adic differential equations;

• **Algebraic One** Here the structures involved are supposed to be purely alebraic. Good examples are Mumford's semi-stable bundles, Seshadri's parabolic bundles, Fontaine's various rings of periods, and semi-stable filtered $(\varphi, N; \omega)$-modules. We will leave the details to the main text. Instead, let me point out that for local theories, when $l \neq p$, we should equally have l-adic analogues $\mathbb{B}_{\text{total}}, \mathbb{B}_{\text{pFM\&N}}, \mathbb{B}_{\text{ur}}$ of Fontaine's p-adic ring of de Rham, semi-stable, crystalline periods, namely, $\mathbb{B}_{\text{dR}}, \mathbb{B}_{\text{st}}, \mathbb{B}_{\text{crys}}$, respectively. Practically, this is possible due to the following reasons.

• *Hodge–Tate Filtration*: Since every l-adic representation, $l \neq p$, is geometric. Hence, it can be realized in terms of etale cohomology over which by the comparison theorem there is a natural Hodge–Tate filtration structure;

• *Monodromy Operator*: This is a direct consequence of Grothendieck's Monodromy Theorem for l-adic Galois Representations;

• *Frobenius*, or equivalently, **Dieudonne Filtration**: This should be put into the context that Weil's conjecture works in both l-adic and p-adic settings mentioned above;

• *Ramifications*, or equivalently, ω-**structures**: This may be read from the so-called theory of breaks and conductors for l-adic Galois representations. For details see e.g., the main text and Chapter 1 of [Ka3].

To uniform the notation, denote the corresponding rings of periods in both l-adic theory and p-adic theory by $\mathbb{B}_{\text{dR}}, \mathbb{B}_{\text{st}}, \mathbb{B}_{\text{ur}}^+$. Accordingly, for

adelic representations of local fields, we then can formulate a huge *anleric ring* $\mathbb{B}_{\mathbb{A}} := \prod' \left(\mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{ur}}^{+} \right)$, *of adelic periods*, namely, the restricted product of $\mathbb{B}_{\mathrm{dR}}$ with respect to $\mathbb{B}_{\mathrm{ur}}^{+}$. In this language, the algebraic condition for thin adelic Galois representations of global fields along with the vertical direction may also be stated as:

(**Adm**) *It is $\mathbb{B}_{\mathbb{A}}$-admissible.*

II) Large scale

A characterization of each individual Galois representation in terms of pure algebraic structures may be called a Micro Reciprocity Law, MRL for short, as it exposes an intrinsic connection between Galois representations and certain algebraic aspects of the base fields. Assuming such a MRL, we then are in a position to understand the mathematics involved in a global way. There are also two different approaches, at least when the coefficients are local. Namely, the categorical theoretic one, based on the fact that Galois representations selected automatically form a Tannakian category, and the moduli theoretic one, based on the fact that the associated algebraic structures admit GIT stability interpretations. (In the case when the coefficients are global adelic spaces, the existing standard Tannakian category theory and GIT should be extended. Indeed, as pointed out by Hida, it is already an interesting problem to see whether our restricted adelic Galois representations form a Tannakian category: After all, the forgetful functor now is not to the category of finite vector spaces over local fields but to that of adelic spaces.)

• **Tannakian Categories** The main aim here is to offer a general Class Field Theory, CFT for short, for the associated base field. Roughly speaking, this goes as follows, at least when the coefficients are local fields. With the Micro Reciprocity Law, we then can get a clone Tannakian category, consisting of certain intrinsically defined pure algebraic objects associated to the base fields, for the Tannakian category consisting of selected Galois representations. As a direct consequence of the finite monodromy and nilpotence, using the so-called finitely generated sub-Tannakian categories and automorphism groups of the associated restrictions of the fiber functors, one then can establish an existence theorem and a global reciprocity law for all finite (non-abelian) extensions of the base fields so as to obtain a general CFT for them. As one may expect here, much refined results can be obtained. Indeed, via a certain truncation process, not only the associated Galois groups but the whole system of high ramification groups can be reproduced. For details, see Part C.

• **Moduli Spaces** From the MRL, Galois representations selected can be characterized by intrinsically defined algebraic structures associated to based fields. These algebraic structures are further expected to be able to put together to form well-controlled moduli spaces. Accordingly, we have certain geometric objects to work with. The importance of such geometric spaces can hardly be overestimated since, with such spaces, we can introduce intrinsic (non-abelian) invariants for the base fields. Good examples are high rank zeta functions and their associated abelian parts. For details, see Part B.

To achieve this, we clearly need to have a good control of objects selected. As usual, this is quite delicate: If the selection is too restrictive, then there might not be enough information involved; on the other hand, it should not be too loose, as otherwise, it is too complicated to see structures in a neat manner, even we know many things are definitely there. (The reader can sense this from our current studies of the Langlands Program.) It is for the purpose of overcoming such difficulties that we introduce the following

Key to the Success: **Stability**

This is supposed to be a condition which helps us to make *good selections* and hence to get nice portions among all possibilities. Particularly, for the algebraic objects selected, we then expect to establish a general MRL (using them) so that the Tannakian category formalism can be applied and a general CFT can be established; and to construct moduli spaces (for them) so that intrinsic invariants can be introduced naturally. This condition is *Stability*. In accordance with what said above, as a general principle of selection, the condition of stability then should be
(a) algebraic, (b) intrinsic, and (c) rigid,
so that, with it, we can
(i) have a nice characterization of Galois representations in terms of semi-stable algebraic structures;
(ii) form a Tannakian category for these semi-stable objects; and hence
(iii) construct natural moduli spaces.
Good examples are for (parabolic) bundles, filtered $(\varphi, N; \omega)$-modules, etc. For details, please see Parts A, B, and C in the main text.

This paper consists of three parts. In Part A, we indicate how a general non-abelian CFT for Riemann surfaces can be established using Tannakian category theory based on Seshadri's work on semi-stable parabolic bundles. This serves as a general guidance for our discussions

in later parts. In Part B, we, motivated by yet another CFT, the conformal field theory, for Riemann surfaces, discussed in Part A, make an intensive study on non-abelian invariants, namely, the high rank zetas for global fields defined using stability. Along with the course, we give a geometric characterization for rank two semi-stable lattices using generalized Siegel type distances between moduli points and cusps, an analytic characterization of stability using Arthur's truncation, and a definition of general non-abelain L-functions using Langlands' theory of Eisenstein series and spectral decompositions. In addition, we also briefly recall abelian zetas associated to (G, P), with G reductive groups and P their maximal parabolic groups, which may be viewed as abelian parts of our non-abelian zetas. These abelian parts, naturally related with constant terms of Eisenstein series are expected to help us to understand the hidden role played by symmetry in the Riemann Hypothesis. Finally, in Part C, we outline a program aiming at establishing a general CFT for p-adic number fields. Key points are the notion of semi-stable filtered $(\varphi, N; \omega)$-modules of slope zero and a conjectural Micro Reciprocity Law claiming that there is a natural one-to-one and onto correspondence between de Rham representations and semi-stable modules of slope zero. Key ingredients of Fontaine's theory of p-adic Galois representations are recalled as well.

Part A. Guidances from geometry

Chapter I. Micro Reciprocity Law in geometry

§1.   Narasimhan–Seshadri correspondence

1.1.   Uniformization

Let M be a compact Riemann surface of genus g and $M^o \hookrightarrow M$ a punctured Riemann surface with $M \backslash M^o := \{P_1, P_2, \ldots, P_N\}$. Assume that $2g - 2 + N > 0$ so that by uniformization theorem there exists a Fuchsian group of first type $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ and the associated universal covering map

$$(\pi^o : \mathfrak{H} \to \Gamma \backslash \mathfrak{H} \simeq M^o) \hookrightarrow (\pi : \mathfrak{H}^+ \to \Gamma \backslash \mathfrak{H}^+ \simeq M)$$

where $\mathfrak{H}$ denotes the usual upper half plane and $\mathfrak{H}^+$ denotes the extended upper half plane, namely, $\mathfrak{H}$ together with cusps associated to (M^o, M), or better, to Γ.

1.2.   (Narasimhan–)Seshadri correspondence

Let $\rho : \pi_1(M^o, *) \to \mathrm{GL}(V)$ be a *unitary representation* of the fundamental group $\pi_1(M^o, *)(\simeq \Gamma)$ of M^o. For simplicity, assume that it is irreducible. Then we know that ρ satisfies the *finite monodromy* property at all P_i's. This then implies that there exists a finite Galois covering

$$\pi' : M' \to M$$

of compact Riemann surfaces ramified possibly at P_i's such that ρ naturally induces a unitary representation

$$\rho' : \pi_1(M', *) \to \mathrm{GL}(V)$$

of the fundamental group of the *compact* Riemann surface M' on V. As such, by the uniformization theorem, we obtain a *unitary flat bundle* over M' equipped with a natural action of the Galois group $\mathrm{Gal}(\pi')$, namely, the four-tuple

$$\left(M', E_{\rho'} := \big(\pi_1(M', *), \rho'\big) \backslash \big(\mathfrak{H}^{(+)} \times V\big), \nabla_{\rho'}; \mathrm{Gal}(\pi') \right).$$

One checks that the $\mathrm{Gal}(\pi')$-invariants of the direct image of the differentials of M' with coefficients in $E_{\rho'}$ coincides with the logarithmic differentials on (M, Z) with coefficients in E_ρ, namely,

$$\left(\pi'_* \big(E_{\rho'} \otimes \Omega^1_{M'} \big) \right)^{\mathrm{Gal}(\pi')} = E_\rho \otimes \Omega^1_M(\log Z)$$

where $Z = P_1 + P_2 + \cdots + P_N$ denotes the reduced branch divisor on M. Consequently, we then obtain a *logarithmic unitary flat bundle* $\left(E_\rho, \nabla_\rho(\log Z)\right)$ on the compact Riemann surface M. Thus by using $\mathrm{Res}_{P_i} \nabla_\rho(\log Z)$, that is, by taking *residues* of logarithmic unitary connection $\nabla_\rho(\log Z)$ at P_i's, we then obtain Seshadri's *parabolic structures* on the fibers of E_ρ, which is nothing but the quotient bundle $\left(\pi_1(M, *), \rho\right) \backslash \left(\mathfrak{H}^{(+)} \times V\right)$, at punctures P_i's. As such, an important discovery of Seshadri is that the parabolic bundle obtained then is stable of degree zero. More strikingly, the converse is correct as well. Namely, any stable parabolic bundle of degree zero can be constructed in this manner.

§2. Micro Reciprocity Law

2.1. Weil's program

This result of Seshadri, obtained with the help of Metha ([MS]), is in fact motivated by an earlier fundamental work of Narasimhan–Seshadri ([NS]), which claims that there is a natural one-to-one and onto correspondence between irreducible unitary representations of fundamental group $\pi_1(M, *)$ of compact Riemnn surface M and stable bundles of degree zero on M. In this sense, Seshadri's result on parabolic bundles above is a generalization of Narasimhan–Seshadri's work from compact Riemann surfaces to punctured Riemann surfaces, in which vector bundles are replaced by parabolic bundles.

In (algebraic) geometry, Narasimhan–Seshadri's work then leads to a natural moduli space for irreducible unitary representations for fundamental groups of compact Riemann surfaces via Mumford's Geometric Invariant Theory, GIT for short. Indeed, by Narasimhan–Seshadri's result, it suffices to consider that for stable bundles of degree zero. While being stable and of degree zero for vector bundle are conditions in terms of intersection theory, it can be shown that this condition is equivalent to a certain GIT-stability. As such, via GIT quotient technique of Mumford ([M]), we can naturally realize the moduli space of stable bundles of degree zero on a compact Riemann surfaces as a quasi-projective variety. Moreover, following GIT, a natural compactification can be made by adding the so-called semi-stable points, which in terms of bundles means (Seshadri classes of) semi-stable vector bundles of degree zero. As Seshadri class corresponding naturally to equivalence class of unitary representations of fundamental group of the compact Riemann surface in question (modulo unipotency, or better after taking semi-simplification), this then gives also an algebraic construction for moduli spaces of these representations of fundamental groups.

However, moduli spaces of semi-stable bundles of degree zero over compact Riemann surfaces in general are singular. It was once a central problem to resolve these singularities in a natural manner. In terms of what was happened, there were in fact two different approaches, one of which due to Seshadri. It is this work of Seshadri that leads to the notion of parabolic bundles.

Before the notion of parabolic bundles, Seshadri also studied the so-called *π-bundles* ([S2]), a notion introduced by Grothendieck ([G]). In particular, Seshadri's main discovery may be stated as that there is a natural one-to-one and onto correspondence between the so-called π-bundles and bundles with parabolic structures (say, when π is a finite ramified covering). For more details, see e.g., Biswas related work on orbifold bundles and parabolic bundles ([Bis]).

Despite their huge successes in (algebraic) geometry, these fundamental works on stability have not made any serious impact in arithmetic (see however Nori's basic work ([Nor]) on fundamental groups via Tannakian category, even in which stability plays no role): until the time around the beginning of 90's of last century, the above works had been largely ignored by mathematicians working in arithmetic. This is in fact very much unfortunate and shows us how interesting mathematics is exposed as human being's activities. By contrast, as we now know, not just as a result these works play a central role in establishing a general non-abelian class field theory for Riemann surfaces, or the same, for function fields over complex numbers, but, all these works are generalizations of Weil's pioneer work claiming that the assignment $\rho \leftrightarrow E_\rho$ (resp. $\rho \leftrightarrow (E_\rho, \nabla_\rho)$) gives a canonical one-to-one and onto correspondence between irreducible representations of fundamental groups of compact Riemann surfaces and indecomposable degree zero bundles (resp. and indecomposable flat bundles) on the associated Riemann surfaces. And in history, it was

(i) aiming at establishing a general CFT for Riemann surfaces that motivated Weil to prove such a result in his master piece on generalization of abelian functions ([We1]); And
(ii) clearly with arithmetic applications in mind that Grothendieck gave a Bourbaki seminar explaining Weil's work in which the notion of π-bundles was introduced ([G]).

This unfortunate situation has been graduatelly changed. Say, at the end of 90's, there was a short note [W1]. This note is a rediscovery of

Weil's program, starting with a crucial observation that the above correspondences of Weil, Narasimhan–Seshadri and Seshadri can be viewed as a kind of reciprocity law; after all,

(a) the correspondences are relating fundamental groups (reading as analogue of Galois groups) with certain intrinsic algebraic structures (reading as non-abelian analogues and generalizations of ideal classes); and

(b) by using parabolic structures, ramification information can be taken care of completely.

Along with such a line, naturally, these works on stability then further leads to the part of our Program aiming at establishing a general CFT for various fields (using stability) [W1].

2.2. Micro Reciprocity Law

Seshadri's fundamental works may be summarized as the follows.

Theorem. *Let (M^0, M) be a punctured Riemann surface. Then we have*

(i) **Micro Reciprocity Law** ((Weil, Mumford, Narasimhan–Seshadri), Seshadri)

There exists a natural one-to-one and onto correspondence
$$\Big\{ \text{irreducible unitary representations of } \pi_1(M^o, *) \Big\}$$
$$\Updownarrow$$
$$\Big\{ \text{stable parabolic bundles of degree zero on } (M^o, M) \Big\};$$

(ii) **Ramifications versus Parabolic Structures** ((Grothendieck), Seshadri)

There exists a natural one-to-one and onto correspondence
$$\Big\{ \text{vector bundles } W/M' \text{ with compatible action } Gal(M'/M) \Big\}$$
$$\Updownarrow$$
$$\Big\{ \text{parabolic bundles } E_*/(M^o, M) \text{ with compatible parabolic weights} \Big\}$$
such that

(i) the correspondence induces a natural one on sub-objects of W and of $E_*(W)$; and

(ii) the degrees satisfy the relation

$$\deg(W) = \deg(M'/M) \cdot \text{par.deg}\big(E_*(W)\big).$$

Chapter II. CFTs in geometry

§3. Arithmetic CFT: class field theory

Building on the above detailed micro study of individual representation of fundamental groups of Riemann surfaces and hence individual semi-stable parabolic bundle, we can study them from a more global point of view. There are two approaches, one using category theory and the other using moduli theory.

As a starting point of the category approach, let us first consider the category consisting of semi-stable parabolic bundles of (parabolic) degree zero over (M^o, M). Note that, as building blocks of general semi-stable objects, stable ones are very rigid. That is to say, there is no non-trivial morphisms between two stable objects, a fact corresponding to Schur's Lemma in representation theory (for irreducible representations). Consequently, we conclude that the just formed category admits much finer structures: It is clearly abelian, has a tensor product structure and admits a natural functor $\mathbb{F}$ to the category of finite dimensional vector spaces (the fibers of base bundles to a fixed point of M^o). Thus, from the rigid properties mentioned above, which guarantees the faithfulness of the functor just mentioned, we see that the category is in fact Tannakian. Denote it by $\left(\mathbb{PV}^{ss;0}_{M^o,M}; \mathbb{F} \right)$. Then, from Tannakian category theory, we obtain the following main theorem of CFT for Riemann surfaces, or the same, for function fields over complex numbers;

Main Theorem of Arithmetic CFT ([W1])
- (**Existence**) *There exists a canonical one-to-one and onto correspondence*

$$\left\{ Finitely\ Generated\ SubTannakian\ Cats\ (\Sigma, \mathbb{F}|_\Sigma)\ of\ \left(\mathbb{PV}^{ss;0}_{M^o,M}; \mathbb{F} \right) \right\}$$

$$\Updownarrow \Pi$$

$$\left\{ Finite\ Galois\ Coverings\ M' \to (M^o, M) \right\}$$

which induces naturally an isomorphism
- (**Reciprocity Law**)

$$\mathrm{Aut}^\otimes \left(\Sigma, \mathbb{F}|_\Sigma \right) \simeq \mathrm{Gal}\left(\Pi(\Sigma, \mathbb{F}|_\Sigma) \right).$$

§4. Geometric CFT: conformal field theory

Here we give some most important aspects of the second global approach, namely the one using moduli spaces. As a starting point, for

a fixed compact Riemann surface M, denote by $\mathcal{M}_M(r,0)$ the moduli spaces of rank r semi-stable bundles of degree zero on M. (Recall that then we squeeze semi-stable bundles into their associated Seshadri classes, defined using graded pieces of the associated Jordan–Hölder filtrations.) Over such moduli spaces, we can construct many global invariants. Analytically we may expect that a still ill-defined Feymann integral would give us something interesting. We will not pursue this line further, instead, let us start with an algebraic construction.

Since each moduli point corresponds to a semi-stable vector bundle, it makes sense to talk about the associated cohomology groups. As such, then we may form the so-called Grothendieck–Mumford determinant line of cohomologies, i.e., the alternative tensor products of determinants of cohomologies. Consequently, if we move our moduli points over all moduli spaces, we can glue the above determinant lines to form the so-called Grothendieck–Mumford determinant line bundles λ_M on $\mathcal{M}_M(r,0)$. Note that the Picard group of $\mathcal{M}_M(r,0)$ is isomorphic to $\mathbb{Z}$, we see that a suitable multiple of λ_M is indeed very ample. (For all this, we in fact need to restrict ourselves only to the stable part. Let us assume it was the case now while leaving the details on how to fix it to the literatures, or better to the reader.) It then makes sense to talk about the $\mathbb{C}$-vector space $H^0\big(\mathcal{M}_M(r,0), \lambda_M^{\otimes n}\big)$ (for n sufficiently away from 0). This is a finite dimensional vector space naturally associated to M, whose dimension is given by the so-called Verlinde formula.

The most interesting and certain a very deep point is somehow we expect that the space itself $H^0\big(\mathcal{M}_M(r,0), \lambda_M^{\otimes n}\big)$, also called *conformal blocks*, does not really very much related with the complex structure on the compact Riemann surface M used. (For more details on Conformal Field Theory, intitated by Belavin–Polykov–Zamodolochikov, see e.g., [US].) More precisely, let us now move M in $\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}_g}$, the moduli space of compact Riemann surfaces of genus $g = g(M)$ and its stable compactification of Deligne–Mumford ([DM]). Denote by Δ_{bdy} the boundary of $\mathcal{M}_g$, which is a normal crossing divisor by Deligne–Mumford theory. Then the conformal blocks form a natural vector bundle $\Pi_*\big(\lambda_M^{\otimes n}\big)\big|_{\mathcal{M}_g}$ on $(\mathcal{M}_g \hookrightarrow)\overline{\mathcal{M}_g}$, with which, we may state the following:

Main Theorem in Geometric CFT: (Tsuchiya–Ueno–Yamada, see also [Hi]) *There exists a projectively flat logarithmic connection on the bundle* $\Pi_*\big(\lambda_M^{\otimes n}\big)\big|_{\mathcal{M}_g}$ *over* $(\mathcal{M}_g, \Delta_{\mathrm{bdy}})$.

Part B. High rank zeta functions and stability

Chapter III. High rank zeta functions

§5.   Function fields

5.1.   Definition and basic properties

Let C be a regular, geometrically irreducible projective curve of genus g defined over $\mathbb{F}_q$, the finite field with q elements, and $\mathcal{M}_{C,r}$ the moduli space of semi-stable bundles of rank r over C. These spaces are projective varieties. So following Weil, we may try to attach them with the standard Artin–Weil zeta functions. However, there is another more intrinsic way. Namely, instead of simply viewing these moduli spaces as algebraic varieties, we here want to fully use the moduli aspects of these spaces by viewing rational points of these varieties as rational bundles: This is possible at least for the stable part by a work of Harder–Narasimhan on Brauer groups ([HN]). Accordingly, for each rational moduli point, we can have a very natural weighted count. All this then leads to the following

Definition. (Weng) *The rank r zeta function for $C/\mathbb{F}_q$ is defined by*

$$\zeta_{C,\mathbb{F}_q;r}(s) := \sum_{V \in [V] \in \mathcal{M}_{C,r}} \frac{q^{h^0(C,V)} - 1}{\#\mathrm{Aut}(V)} \cdot \left(q^{-s}\right)^{\deg(V)}, \qquad \mathrm{Re}(s) > 1.$$

Here as usual, $[V]$ denotes the Seshadri class of (a rational) semi-stable bundle V, and $\mathrm{Aut}(V)$ denotes the automorphism group of V.

By semi-stable condition, the summation above is only taken over the part of moduli space whose points have non-negative degrees. Thus by the duality, Riemann–Roch and a Clifford type lemma for semi-stable bundles, we then can expose the following basic properties for our zeta functions of curves.

Zeta Facts(Weng) (0) $\zeta_{C,1,\mathbb{F}_q}(s)$ *is nothing but the classical Artin zeta function $\zeta_C(s)$ for curve C.*
(1) $\zeta_{C,r,\mathbb{F}_q}(s)$ *is well-defined for* $\mathrm{Re}(s) > 1$, *and admits a meromorphic continuation to the whole complex s-plane;*
(2) (**Rationality**) *Set $t := q^{-s}$ and introduce the non-abelian Z-function of C by*

$$\zeta_{C,r,\mathbf{F}_q}(s) =: Z_{C,r,\mathbf{F}_q}(t) := \sum_{V \in [V] \in \mathcal{M}_{C,r}(d), d \geq 0} \frac{q^{h^0(C,V)} - 1}{\#\mathrm{Aut}(V)} \cdot t^{d(V)}, \quad |t| < 1.$$

Then there exists a polynomial $P_{C,r,\mathbf{F}_q}(s) \in \mathbf{Q}[t]$ *such that*

$$Z_{C,r,\mathbf{F}_q}(t) = \frac{P_{C,r,\mathbf{F}_q}(t)}{(1-t^r)(1-q^r t^r)};$$

(3) (**Functional Equation**) *Set the rank* r *non-abelian* ξ-*function* $\xi_{C,r,\mathbf{F}_q}(s)$ *by*

$$\xi_{C,r,\mathbf{F}_q}(s) := \zeta_{C,r,\mathbf{F}_q}(s) \cdot (q^s)^{r(g-1)}.$$

Then

$$\xi_{C,r,\mathbf{F}_q}(s) = \xi_{C,r,\mathbf{F}_q}(1-s).$$

Remarks. (1) (**Count in Different Ways**) The above weighted count is designed for all rational semi-stable bundles, motivated by Harder–Narasimhan's interpretation on Siegel's work about Tamagawa numbers ([HN]). As such, even the moduli space is used, it does not really play a key role as all elements in a Seshadri class are counted. For this reason, modifications for the definition of high rank zetas can be given, say, count only one within a fixed Seshadri class, or count only what are called strongly semi-stable bundles, etc. . . .

(2) (**Stratifications and Cohomological Interpretations**) Deninger once asked whether there was a cohomological interpretation for our zeta functions. There is a high possibility for it: We expect that our earlier works on refined Brill–Noether loci would play a key role here, since refined Brill–Noether loci induce natural stratifications on moduli spaces. Thus, following Grothendieck's work on cohomological interpretation of Weil's zeta functions, what we have to do next is to expose a certain weighted fixed point formula.

5.2. Global high rank zetas via Euler products

Let $\mathcal{C}$ be a regular, reduced, irreducible projective curve of genus g defined over a number field F. Let S_{bad} be the collection of all infinite places and these finite places of F at which $\mathcal{C}$ does not have good reductions. As usual, a place v of F is called good if $v \notin S_{\mathrm{bad}}$. For any good place v of F, the v-reduction of $\mathcal{C}$, denoted as $\mathcal{C}_v$, gives a regular, reduced, irreducible projective curve defined over the residue field $F(v)$ of F at v. Denote the cardinal number of $F(v)$ by q_v. Then, we obtain the associated rank r non-abelian zeta function $\zeta_{\mathcal{C}_v,r,\mathbf{F}_{q_v}}(s)$. Moreover, from the rationality of $\zeta_{\mathcal{C}_v,r,\mathbf{F}_{q_v}}(s)$, there exists a degree $2rg$ polynomial $P_{\mathcal{C}_v,r,\mathbf{F}_{q_v}}(t) \in \mathbf{Q}[t]$ such that

$$Z_{\mathcal{C}_v,r,\mathbf{F}_{q_v}}(t) = \frac{P_{\mathcal{C}_v,r,\mathbf{F}_{q_v}}(t)}{(1-t^r)(1-q^r t^r)}.$$

Clearly, $P_{\mathcal{C}_v,r,\mathbf{F}_{q_v}}(0) \neq 0$. Set

$$\tilde{P}_{\mathcal{C}_v,r,F(v)}(t) := \frac{P_{\mathcal{C}_v,r,F(v)}(t)}{P_{\mathcal{C}_v,r,F(v)}(0)}.$$

Definition. (Weng) *The rank r non-abelian zeta function $\zeta_{\mathcal{C},r,F}(s)$ of $\mathcal{C}$ over F is defined as the following Euler product*

$$\zeta_{\mathcal{C},r,F}(s) := \prod_{v:\text{good}} \frac{1}{\tilde{P}_{\mathcal{C}_v,r,\mathbf{F}_{q_v}}(q_v^{-s})}, \qquad\qquad \mathrm{Re}(s) \gg 0.$$

Clearly, when $r = 1$, $\zeta_{\mathcal{C},r,F}(s)$ coincides with the classical Hasse–Weil zeta function for $\mathcal{C}$ over F.

Conjecture. *For a regular, reduced, geometrically irreducible projective curve $\mathcal{C}$ of genus g defined over a number field F, its associated rank r global non-abelian zeta function $\zeta_{\mathcal{C},r,F}(s)$ admits a meromorphic continuation to the whole complex s-plane.*

Recall that even when $r = 1$, i.e., for the classical Hasse–Weil zeta functions, this statement, as a part of a series of high profile conjectures is still open. On the other hand, we have the following

Proposition. ([W4]) *When $\mathrm{Re}(s) > 1 + g + (r^2 - r)(g - 1)$, $\zeta_{\mathcal{C},r,F}(s)$ converges.*

Like in the theory for abelian zeta functions, we want to use our non-abelian zeta functions to study non-abelian aspect of arithmetic of curves. For this purpose, completed zetas, or better, local factors for 'bad' places, should be introduced:
(i) For Γ-factors, motivated by the local rationality, we take these associated to $\zeta_F(rs) \cdot \zeta_F\big(r(s-1)\big)$, where $\zeta_F(s)$ denotes the standard Dedekind zeta function for F; and
(ii) for finite bad factors, first choose a semi-stable model for $\mathcal{C}$ so as to get a semi-stable reduction for curves at bad places. Then, either (a) use Seshadri's moduli spaces of semi-stable parabolic bundles as suggested in [W4]; or
(b) use moduli space of semi-stable bundles over nodal curvces, as pointed out by Seshadri.
For the time being, even we know that each produces local factors for singular fibers, usually polynomials with degree lower than $2rg$, but we do not know which one is right. To test them, we propose the following functional equation.

Working Hypothesis. *The completed zeta function $\xi_{C,r,F}(s)$ of C/F admits a unique meromorphic continuation to the whole complex s-plane and satisfies the functional equation*

$$\xi_{C,r,F}\big(s\big) = \varepsilon \cdot \xi_{C,r,F}\Big(1 + \frac{1}{r} - s\Big)$$

with $|\varepsilon| = 1$.

§6. Number fields

6.1. Stability of $\mathcal{O}_F$-lattices

Let F be a number field with $\mathcal{O}_F$ the ring of integer and Δ_F the discriminant. By definition, an $\mathcal{O}_F$-*lattice* Λ of rank r consists of a pair (P, ρ), where P is a rank r projective $\mathcal{O}_F$-module and ρ is a metric on the space $\big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\big)^r = \big(\mathbb{R}^r\big)^{r_1} \times \big(\mathbb{C}^r\big)^{r_2}$, where r_1 (resp. r_2) denotes the number of real embeddings (resp. complex embeddings) of F. Recall that, being projective, there exists a fractional idea $\mathfrak{a}$ of F such that $P \simeq \mathcal{O}_F^{r-1} \oplus \mathfrak{a}$. Particularly, the natural inclusion $\mathcal{O}_F^{r-1} \oplus \mathfrak{a} \hookrightarrow F^r$ induces a natural embedding of P into $\big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\big)^r$ via the compositions

$$P \simeq \mathcal{O}_F^{r-1} \oplus \mathfrak{a} \hookrightarrow F^r \hookrightarrow \big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\big)^r \simeq \big(\mathbb{R}^r\big)^{r_1} \times \big(\mathbb{C}^r\big)^{r_2}.$$

As such, then the image of P naturally offers us a lattice Λ in the metrized space $\Big(\big(\mathbb{R}^r\big)^{r_1} \times \big(\mathbb{C}^r\big)^{r_2}, \rho\Big)$.

By definition, an $\mathcal{O}_F$-lattice is called *semi-stable* if for all sub-$\mathcal{O}_F$-lattice Λ_1 of Λ, we have

$$\mathrm{Vol}(\Lambda_1)^{\mathrm{rank}(\Lambda)} \geq \mathrm{Vol}(\Lambda)^{\mathrm{rank}(\Lambda_1)},$$

where the volume $\mathrm{Vol}(\Lambda)$ of Λ is usually called the covolume of Λ, namely,

$$\mathrm{Vol}(\Lambda) := \mathrm{Vol}\Big(\big(\big(\mathbb{R}^r\big)^{r_1} \times \big(\mathbb{C}^r\big)^{r_2}, \rho\big)\big/\Lambda\Big).$$

Denote by $\mathcal{M}_{F,r}$ the moduli space of semi-stable $\mathcal{O}_F$ lattices of rank r, i.e., the space of isomorphism classes of semi-stable $\mathcal{O}_F$ lattices of rank r. Then there is a natural topological structure on $\mathcal{M}_{F,r}$. In fact there is a much finer structure on it; Denote by $\mathcal{M}_{F,r}[T]$ the volume T part of $\mathcal{M}_{F,r}$, i.e., the part consisting of isomorphisms classes of rank r semi-stable $\mathcal{O}_F$-lattices of volume T, then

(i) there is a natural decomposition

$$\mathcal{M}_{F,r} = \bigcup_{T \in \mathbb{R}_{>0}} \mathcal{M}_{F,r}[T];$$

Moreover,

(ii) for each fixed T, $\mathcal{M}_{F,r}[T]$ is compact; and

(iii) there are natural measures $d\mu$ on $\mathcal{M}_{F,r}$ such that

$$d\mu = d\mu\Big|_{\mathcal{M}_{F,r}[|\Delta_F|^{\frac{r}{2}}]} \times \frac{dT}{T}.$$

(The compactness of $\mathcal{M}_{F,r}[T]$ is the main reason why we use the stability condition in the study of non-abelian zetas in [W2,3].)

6.2.   Geo-arithmetical cohomology

Let Λ be an $\mathcal{O}_F$-lattice. Then define its *geo-arthmetical cohomology groups* by

$$H^0(F,\Lambda) := \Lambda, \qquad \text{and} \qquad H^1(F,\Lambda) := \left(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\right)^r\Big/\Lambda.$$

As such, unlike in algebraic geometry and/or in arithmetic geometry, cohomological groups H^i are not vector spaces, but locally compact topological groups.

   With this simple but genuine definition, then the basic properties such as the duality and the Riemann–Roch theorem can be realized as follows;

Pontrjagin Duality (Weng) *There is a natural topological isomorphism*

$$H^1(F,\Lambda) \simeq H^0(\widehat{F,\omega_F \otimes \Lambda^\vee})$$

where $\omega_F := (\mathfrak{d}_F, \rho_{\mathrm{st}})$ denotes the differential lattice of F, namely, the (rank one) projective module given by the standard differential module $\mathfrak{d}_F$ of $\mathcal{O}_F$, and the metric given by the standard metric ρ_{st} on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

   Moreover, since $H^{i=0,1}(F,\Lambda)$ are locally compact topological groups, we can apply Fourier analysis to introduce quantitive invariants for them ([F]), say, for h^0, or better for $exp(h^0)$, counting each element $\mathbf{x} \in H^0(F,\Lambda)$, (which is nothing but the lattice Λ itself,) with weight of the Gaussian distribution

$$e^{-\pi \sum_{\sigma:\mathbb{R}} \|\mathbf{x}\|_{\rho_\sigma} - 2\pi \sum_{\tau:\mathbb{C}} \|\mathbf{x}\|_{\rho_\tau}}.$$

(As such, this definition then coincides with the one previously introduced by van der Geer and Schoof, for which an arithmetic analogue of effectivity is used ([GS]).)

Geo-Arithmetical Riemann–Roch Theorem. (Weng) *For an $\mathcal{O}_F$-lattice Λ,*

$$h^0(F, \Lambda) - h^1(F, \Lambda) = \deg(V) - \frac{\mathrm{rank}(\Lambda)}{2} \cdot \log \left| \Delta_F \right|.$$

Our Riemann–Roch is a direct consequence of the Fourier inversion formula, reflecting the topological Pointrjagin duality above, and the standard Poission summation formula. So it has its roots in Tate's Thesis ([Ta1]), even our result is not really there.

In the above RR, $\deg(V)$ denotes what we call Arakelov degree of V. In fact, in Arakelov geometry, there is the following

Arakelov Riemann–Roch Theorem. (See e.g. [L1,2,3])

$$-\log\left(\mathrm{Vol}(\Lambda)\right) = \deg(V) - \frac{\mathrm{rank}(\Lambda)}{2} \cdot \log \left| \Delta_F \right|.$$

From this, it is simple to see that the above definition of ours for semi-satble $\mathcal{O}_F$ lattices is equivalent to the following definition in [St1]: an $\mathcal{O}_F$-lattice is semi-stable if for all sub-$\mathcal{O}_F$-lattice Λ_1 of Λ, we have

$$\frac{\deg(\Lambda_1)}{\mathrm{rank}(\Lambda_1)} \leq \frac{\mathrm{rank}(\Lambda)}{\mathrm{rank}(\Lambda)},$$

an arithmetic-geometric analogue of the slope stability condition of Mumford for vector bundles over compact Riemann surfaces: A vector bundle V over a compact Riemann surface M is semi-stable if for all subbundles V_1,

$$\frac{\deg(V_1)}{\mathrm{rank}(V_1)} \leq \frac{\deg(V)}{\mathrm{rank}(V)}.$$

6.3. High rank zetas

With the above preperation, we are ready to state the following

Definition. (Weng) *The rank r zeta function of F is defined by*

$$\xi_{F,r}(s) := \left(\left| \Delta_F \right|^s \right)^{\frac{r}{2}} \cdot \int_{\mathcal{M}_{F,r}} \left(e^{h^0(F,\Lambda)} - 1 \right) \cdot \left(e^{-s} \right)^{\deg(\Lambda)} d\mu(\Lambda), \ \mathrm{Re}(s) > 1.$$

Tautologically, from the duality and the geo-arithmetical Riemann–Roch, we obtain the following standard properties for the high rank zeta functions (see however [We2]):

Zeta Facts. (Weng) (0) (Iwasawa) $\xi_{F,1}(s) \doteq \xi_F(s)$, *the completed Dedekind zeta for F;*

(1) (**Meromorphic Extension**) *Non-abelian zeta function*

$$\xi_{F,r}(s) := \left(|\Delta_F|^{\frac{r}{2}}\right)^s \int_{\Lambda \in \mathcal{M}_{F,r}} \left(e^{h^0(F,\Lambda)} - 1\right)\left(e^{-s}\right)^{\deg(\Lambda)} \cdot d\mu$$

converges absolutely and uniformly when $\mathrm{Re}(s) \geq 1 + \delta$ *for any* $\delta > 0$. *Moreover,* $\xi_{F,r}(s)$ *admits a unique meromorphic continuation to the whole complex s-plane;*

(2) (**Functional Equation**) *The extended* $\xi_{F,r}(s)$ *satisfies the functional equation*

$$\xi_{F,r}(s) = \xi_{F,r}(1-s);$$

(3) (**Singularities**) *The extended* $\xi_{F,r}(s)$ *has two singularities, all simple poles, at* $s = 0\,1$, *with*

$$\mathrm{Res}_{s=0}\,\xi_{F,r}(s) \;=\; -\mathrm{Res}_{s=0}\,\xi_{F,r}(s) \;=\; \mathrm{Vol}\left(\mathcal{M}_{F,r}[|\Delta_F|^{\frac{r}{2}}]\right).$$

Chapter IV. Geometric characterization of stability

Here we give an example on how to characterize stability in geometric terms. More precisely, in this chapter, we will offer a characterization of semi-stable rank two $\mathcal{O}_F$-lattices in terms of a Siegel type distance to cusps. We will present the materials in a classical way in which many fundamental results of algebraic number theory will be used. The main results are listed in §§8, 9 and 10.

§7. Upper half space model

7.1. Upper half plane

As usual, denote by

$$\mathcal{H} := \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, y \in \mathbb{R}_+^*\},$$

the upper half plane. The group $SL(2, \mathbb{R})$ naturally acts on $\mathcal{H}$ via:

$$M\,z := \frac{az + b}{cz + d}, \qquad \forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad z \in \mathcal{H}.$$

The stablizer of $i = (0, 1) \in \mathcal{H}$ is equal to $SO(2) := \{A \in O(2) : \det A = 1\}$. Since this action is transitive, we can identify the quotient $SL(2, \mathbb{R})/SO(2)$ with $\mathcal{H}$ by the quotient map induced from $SL(2, \mathbb{R}) \to \mathcal{H}$, $g \mapsto g \cdot i$.

$\mathcal{H}$ admits the real line $\mathbb{R}$ as its boundary. Consequently, to compactify it, we add on it the real projective line $\mathbb{P}^1(\mathbb{R})$ with $\infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Naturally, the above action of $SL(2, \mathbb{R})$ also extends to $\mathbb{P}^1(\mathbb{R})$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

7.2. Upper half space

Similarly, 3-dimensional hyperbolic space is defined to be

$$\mathbb{H} := \mathbb{C} \times]0, \infty[= \left\{ (z, r) : z = x + iy \in \mathbb{C}, r \in \mathbb{R}_+^* \right\}$$
$$= \left\{ (x, y, r) : x, y \in \mathbb{R}, r \in \mathbb{R}_+^* \right\}.$$

We will think of $\mathbb{H}$ as a subset of Hamilton's quaternions with $1, i, j, k$ the standard $\mathbb{R}$-basis. Write points P in $\mathbb{H}$ as

$$P = (z, r) = (x, y, r) = z + rj \qquad \text{where } z = x + iy, \ j = (0, 0, 1).$$

The natural action of $SL(2,\mathbb{C})$ on $\mathbb{H}$ and on its boundary $\mathbb{P}^1(\mathbb{C})$ may be described as follows: We represent an element of $\mathbb{P}^1(\mathbb{C})$ by $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x, y \in \mathbb{C}$ with $(x, y) \neq (0, 0)$. Then the action of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ is defined to be

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

Moreover, if we represent points $P \in \mathbb{H}$ as quaternions whose fourth component equals zero, then the action of M on $\mathbb{H}$ is defined to be

$$P \mapsto MP := (aP + b)(cP + d)^{-1},$$

where the inverse on the right is taken in the skew field of quaternions.

Furthermore, with this action, the stablizer of $j = (0, 0, 1) \in \mathbb{H}$ in $SL(2, \mathbb{C})$ is equal to $SU(2) := \{A \in U(2) : \det A = 1\}$. Since the action of $SL(2, \mathbb{C})$ on $\mathbb{H}$ is transitive, we obtain also a natural identification $\mathbb{H} \simeq SL(2, \mathbb{C})/SU(2)$ via the quotient map induced from $SL(2, \mathbb{C}) \to \mathbb{H}$, $g \mapsto g \cdot j$.

7.3. Rank two $\mathcal{O}_F$-lattices: upper half space model

Identify $\mathcal{H}$ with $SL(2, \mathbb{R})/SO(2)$ and $\mathbb{H}$ with $SL(2, \mathbb{C})/SU(2)$. Denote by $\mathcal{M}_{F,2;\mathfrak{a}}$ the moduli space of semi-stable lattices of rank two whose associated projective models are isomorphic to $\mathcal{O}_F \oplus \mathfrak{a}$ for a certain ideal $\mathfrak{a}$, and denote its volume T part by $\mathcal{M}_{F,2;\mathfrak{a}}[T]$. Make the identification

$$\mathcal{M}_{F,2;\mathfrak{a}}\big[N(\mathfrak{a}) \cdot \Delta_F\big] \simeq \left(SL(\mathcal{O}_F \oplus \mathfrak{a})\big\backslash \Big(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}\Big)\right)_{\mathrm{ss}},$$

where as usual ss means the subset consisting of points corresponding to rank two semi-stable $\mathcal{O}_F$-lattices in the quotient space

$$SL(\mathcal{O}_F \oplus \mathfrak{a})\big\backslash \left(\Big(SL(2, \mathbb{R})/SO(2)\Big)^{r_1} \times \Big(SL(2, \mathbb{C})/SU(2)\Big)^{r_2}\right).$$

Hence clearly, if the metric on $\mathcal{O}_F \oplus \mathfrak{a}$ is given by $g = (g_\sigma)_{\sigma \in S_\infty}$ with $g_\sigma \in SL(2, F_\sigma)$, then the corresponding points on the right hand side is $g(\mathrm{Im}J)$ with $\mathrm{Im}J := (i^{(r_1)}, j^{(r_2)})$, i.e., the point given by $(g_\sigma \tau_\sigma)_{\sigma \in S_\infty}$ where $\tau_\sigma = i_\sigma := (0, 1)$ if σ is real and $\tau_\sigma = j_\sigma := (0, 0, 1)$ if σ is complex.

§8. Cusps

8.1. Definition

The working site now is shifted to the space $SL(\mathcal{O}_F \oplus \mathfrak{a}) \backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2} \right)$. Here the action of $SL(2, \mathcal{O}_F \oplus \mathfrak{a})$ is via the action of $SL(2, F)$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. More precisely, F^2 admits natural embeddings $F^2 \hookrightarrow \left(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \right)^2 \simeq \left(\mathbb{R}^2 \right)^{r_1} \times \left(\mathbb{C}^2 \right)^{r_2}$ so that $\mathcal{O}_F \oplus \mathfrak{a}$ naturally embeds into $\left(\mathbb{R}^2 \right)^{r_1} \times \left(\mathbb{C}^2 \right)^{r_2}$ as a rank two $\mathcal{O}_F$-lattice. As such, $SL(\mathcal{O}_F \oplus \mathfrak{a})$ acts on the image of $\mathcal{O}_F \oplus \mathfrak{a}$ in $\left(\mathbb{R}^2 \right)^{r_1} \times \left(\mathbb{C}^2 \right)^{r_2}$ as automorphisms. Our task here is to understand the cusps of this action of $SL(\mathcal{O}_F \oplus \mathfrak{a})$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. For this, we go as follows.

First, the space $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ admits a natural boundary $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, in which the field F is imbedded via Archmidean places of F: $F \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Consequently, $\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R})^{r_1} \times \mathbb{P}^1(\mathbb{C})^{r_2}$ with $\begin{bmatrix} 1 \\ 0 \end{bmatrix} := \infty \mapsto (\infty^{(r_1)}, \infty^{(r_2)})$. As usual, via fractional linear transformations, $SL(2, \mathbb{R})$ acts on $\mathbb{P}^1(\mathbb{R})$, and $SL(2, \mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$, hence so does $SL(2, F)$ on

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R})^{r_1} \times \mathbb{P}^1(\mathbb{C})^{r_2}.$$

Being a discrete subgroup of $SL(2, \mathbb{R})^{r_1} \times SL(2, \mathbb{C})^{r_2}$, for the action of $SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $\mathbb{P}^1(F)$, we call the corresponding orbits (of $SL(\mathcal{O}_F \oplus \mathfrak{a})$ on $\mathbb{P}^1(F)$) the *cusps*. Very often we also call their associated representatives cusps.

8.2. Cusp and ideal class correspondence

With this, we have the following fundamental result rooted back to Maaβ.

Cusp and Ideal Class Correspondence. (Maaβ) *There is a natural bijection Π between the ideal class group $CL(F)$ of F and the cusps $\mathcal{C}_\Gamma$ of $\Gamma = SL(\mathcal{O}_F \oplus \mathfrak{a})$ acting on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ given by*

$$\mathcal{C}_\Gamma \to CL(F), \qquad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \mapsto \left[\mathcal{O}_F\, \alpha + \mathfrak{a}\, \beta \right].$$

Easily, one checks that the inverse map Π^{-1} is given as follows: For a fractional ideal $\mathfrak{b}$, by Chinese Reminder Theorem, choose $\alpha_\mathfrak{b}, \beta_\mathfrak{b} \in F$ such that $\mathcal{O}_F \cdot \alpha_\mathfrak{b} + \mathfrak{a} \cdot \beta_\mathfrak{b} = \mathfrak{b}$; Define $\Pi^{-1}([\mathfrak{b}])$ simply by the class of

the point $\begin{bmatrix} \alpha_\mathfrak{b} \\ \beta_\mathfrak{b} \end{bmatrix}$ in $SL(2, \mathcal{O}_F \oplus a) \backslash \mathbb{P}^1(F)$. Recall also that there always

exists $M_{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}} := \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \in SL(2, F)$ such that $M_{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}} \cdot \infty = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

8.3. Stablizer groups of cusps

Recall that under the Cusp-Ideal Class Correspondence, there are exactly h inequivalence cusps η_i, $i = 1, 2, \ldots, h$, where $h := \#\mathrm{CL}(F)$. Moreover, if we write the cusp $\eta := \eta_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$ for suitable α_i, $\beta_i \in F$, then the associated ideal class is exactly the one for the fractional ideal $\mathcal{O}_F \alpha_i + a \beta_i =: \mathfrak{b}_i$. Denote the stablizer group of η in $SL(\mathcal{O}_F \oplus a)$ by Γ_η.

Lemma. ([W-2,5]) *The associated 'lattice' for the cusp η is given by* $a\mathfrak{b}^{-2}$. *Namely,*

$$A^{-1} \Gamma_\eta A = \left\{ \begin{pmatrix} u & z \\ 0 & u^{-1} \end{pmatrix} : u \in U_F, z \in a\mathfrak{b}^{-2} \right\},$$

where U_F denotes the group of units of F.

Set $\Gamma'_\eta := \left\{ A \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} A^{-1} : z \in a\mathfrak{b}^{-2} \right\}$, Then

$$\Gamma_\eta = \Gamma'_\eta \times \left\{ A \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} A^{-1} : u \in U_F \right\}.$$

Note that also componentwisely, $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} z = \frac{uz}{u^{-1}} = u^2 z$. So, in practice, what we really get is the following decomposition

$$\Gamma_\eta = \Gamma'_\eta \times U_F^2$$

with

$$U_F^2 \simeq \left\{ A \cdot \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \cdot A^{-1} : u \in U_F \right\} \simeq \left\{ A \begin{pmatrix} 1 & 0 \\ 0 & u^2 \end{pmatrix} A^{-1} : u \in U_F \right\}.$$

8.4. Fundamental domain for Γ_∞ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$

We are now ready to construct a fundamental domain for the action of $\Gamma_\eta \subset SL(\mathcal{O}_F \oplus a)$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. This is based on a construction of a fundamental domain for the action of Γ_∞ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. More precisely, with an element $A = \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \in SL(2, F)$ (always exists!), we have

i) $A \cdot \infty = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$; and

ii) The isotropy group of η in $A^{-1}SL(\mathcal{O}_F \oplus \mathfrak{a})A$ is generated by translations $\tau \mapsto \tau + z$ with $z \in \mathfrak{a}\mathfrak{b}^{-2}$ and by dilations $\tau \mapsto u\tau$ where u runs through the group U_F^2.

(Here, we use A, α, β, $\mathfrak{b}$ as running symbols for A_i, α_i, β_i, $\mathfrak{b}_i := \mathcal{O}_F\alpha_i + \mathfrak{a}\beta_i$.)

Consider then the map

$$\mathrm{ImJ}: \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} \quad \to \quad \mathbb{R}_{>0}^{r_1+r_2},$$
$$(z_1, \cdots, z_{r_1}; P_1, \cdots, P_{r_2}) \quad \mapsto \quad (\Im(z_1), \cdots, \Im(z_{r_1}); J(P_1), \cdots, J(P_{r_2})),$$

where if $z = x + iy \in \mathcal{H}$, resp. $P = z + rj \in \mathbb{H}$, we set $\Im(z) = y$, resp. $J(P) = r$. It induces a map

$$\left(A^{-1} \cdot \Gamma_\eta \cdot A\right) \backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}\right) \to U_F^2 \backslash \mathbb{R}_{>0}^{r_1+r_2},$$

which exhibits $\left(A^{-1} \cdot \Gamma_\eta \cdot A\right) \backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}\right)$ as a torus bundle over $U_F^2 \backslash \mathbb{R}_{>0}^{r_1+r_2}$ with fiber the $n = r_1 + 2r_2$ dimensional torus $\left(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\right) \big/ \mathfrak{a}\mathfrak{b}^{-2}$. Having factored out the action of the translations, we only have to construct a fundamental domain for the action of U_F^2 on $\mathbb{R}_{>0}^{r_1+r_2}$. For this, we look first at the action of U_F^2 on the norm-one hypersurface $\mathbf{S} := \left\{y \in \mathbb{R}_{>0}^{r_1+r_2} : N(y) =: \prod_i y_i = 1\right\}$. By taking logarithms, it is transformed bijectively into a trace-zero hyperplane which is isomorphic to the space $\mathbb{R}^{r_1+r_2-1}$

$$\mathbf{S} \overset{\log}{\to} \mathbb{R}^{r_1+r_2-1} := \left\{(a_1, \cdots a_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2} : \sum a_i = 0\right\},$$

$$y \mapsto \quad \left(\log y_1, \cdots, \log y_{r_1+r_2}\right),$$

where the action of U_F^2 on $\mathbf{S}$ is carried out over an action on $\mathbb{R}^{r_1+r_2-1}$ by the translations $a_i \mapsto a_i + \log \varepsilon^{(i)}$. By Dirichlet's Unit Theorem ([L1], [Ne]), the logarithm transforms U_F^2 into a lattice in $\mathbb{R}^{r_1+r_2-1}$. Accordingly, the exponential map transforms a fundamental domain, e.g., a fundamental parallelopiped, for this action back into a fundamental domain $\mathbf{S}_{U_F^2}$ for the action of U_F^2 on $\mathbf{S}$. The cone over $\mathbf{S}_{U_F^2}$, that is, $\mathbb{R}_{>0} \cdot \mathbf{S}_{U_F^2} \subset \mathbb{R}_{>0}^{r_1+r_2}$, is then a fundamental domain for the action of U_F^2 on $\mathbb{R}_{>0}^{r_1+r_2}$. Denote by $\mathcal{T}$ a fundamental domain for the action of the translations by elements of $\mathfrak{a}\mathfrak{b}^{-2}$ on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, and set

$$\mathrm{ReZ}\left(z_1, \cdots, z_{r_1}; P_1, \cdots, P_{r_2}\right) := \left(\Re(z_1), \cdots, \Re(z_{r_1}); Z(P_1), \cdots, Z(P_{r_2})\right)$$

with $\Re(z) := x$, resp. $Z(P) := z$ if $z = x + iy \in \mathcal{H}$, resp. $P = z + rj \in \mathbb{H}$, then what we have just said proves the following

Proposition. ([W-2,5]) *A fundamental domain for the action of $A^{-1}\Gamma_\eta A$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ is given by*

$$\mathbf{E} := \left\{ \tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mathrm{ReZ}\,(\tau) \in \mathcal{T},\ \mathrm{ImJ}\,(\tau) \in \mathbb{R}_{>0} \cdot \mathbf{S}_{U_F^2} \right\}.$$

For later use, we also set $\mathcal{F}_\eta := A_\eta^{-1} \cdot \mathbf{E}$.

§9. Fundamental domain

9.1. Siegel type distance

Guided by Siegel's discussion on totally real fields [Sie] and the discussion above, we are now ready to construct fundamental domains for $SL(\mathcal{O}_F \oplus \mathfrak{a}) \big\backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2} \right)$.

As the first step, we generalize Siegel's 'distance to cusps'. For this, recall that for a cusp $\eta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{P}^1(F)$, by the Cusp-Ideal Class Correspondence, we obtain a natural ideal class associated to the fractional ideal $\mathfrak{b} := \mathcal{O}_F \cdot \alpha + \mathfrak{a} \cdot \beta$. Moreover, by assuming that α, β are all contained in $\mathcal{O}_F$, as we may, we know that the corresponding stablizer group Γ_η is given by

$$A^{-1} \cdot \Gamma_\eta \cdot A = \left\{ \gamma = \begin{pmatrix} u & z \\ 0 & u^{-1} \end{pmatrix} \in \Gamma : u \in U_F, z \in \mathfrak{a}\mathfrak{b}^{-2} \right\},$$

where $A \in SL(2, F)$ satisfying $A\infty = \eta$ which may be further chosen in the form $A = \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \in SL(2, F)$ so that $\mathcal{O}_F \beta^* + \mathfrak{a}^{-1}\alpha^* = \mathfrak{b}^{-1}$.

Now for $\tau = (z_1, \ldots, z_{r_1}; P_1, \cdots, P_{r_2}) \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$, set

$$N(\tau) := N\left(\mathrm{ImJ}(\tau) \right) = \prod_{i=1}^{r_1} \Im(z_i) \cdot \prod_{j=1}^{r_2} J(P_j)^2 = \left(y_1 \cdot \ldots \cdot y_{r_1} \right) \cdot \left(v_1 \cdot \ldots \cdot v_{r_2} \right)^2.$$

Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, F)$,

$$N\left(\mathrm{ImJ}(\gamma \cdot \tau) \right) = \frac{N(\mathrm{ImJ}(\tau))}{\|N(c\tau + d)\|^2}. \tag{$*$}$$

(Note that here only the second row of γ appears.) Moreover, following [W-2,5], define the *reciprocal distance $\mu(\eta, \tau)$ from the point $\tau \in \mathcal{H}^{r_1} \times$*

$\mathbb{H}^{r_2}$ *to the cusp* $\eta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ *in* $\mathbb{P}^1(F)$ *by*

$$\mu(\eta, \tau) := N\left(\mathfrak{a}^{-1} \cdot (\mathcal{O}_F \alpha + \mathfrak{a}\beta)^2\right)$$

$$\times \frac{\Im(z_1) \cdots \Im(z_{r_1}) \cdot J(P_1)^2 \cdots J(P_{r_2})^2}{\prod_{i=1}^{r_1} |(-\beta^{(i)} z_i + \alpha^{(i)})|^2 \prod_{j=1}^{r_2} \|(-\beta^{(j)} P_j + \alpha^{(j)})\|^2}$$

$$= \frac{1}{N(\mathfrak{a}\mathfrak{b}^{-2})} \cdot \frac{N(\mathrm{Im}J(\tau))}{\|N(-\beta\tau + \alpha)\|^2}.$$

Lamma 1. *([W-2,5]) (i)* μ *is well-defined;*
(ii) μ *is invariant under the action of* $SL(\mathcal{O}_F \oplus \mathfrak{a})$. *That is to say,*

$$\mu(\gamma\eta, \gamma\tau) = \mu(\eta, \tau), \qquad \forall \gamma \in SL(\mathcal{O}_F \oplus \mathfrak{a}).$$

(iii) There exists a positive constant C *depending only on* F *and* $\mathfrak{a}$ *such that if* $\mu(\eta, \tau) > C$ *and* $\mu(\eta', \tau) > C$ *for* $\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ *and* $\eta, \eta' \in \mathbb{P}^1(F)$, *then* $\eta = \eta'$.
(iv) There exists a positive real number $T := T(F)$ *depending only on* F *such that for* $\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$, *there exists a cusp* η *such that* $\mu(\eta, \tau) > T$.

Now for the cusp $\eta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{P}^1(F)$, define the 'sphere of influence' of η by

$$F_\eta := \left\{ \tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mu(\eta, \tau) \geq \mu(\eta', \tau), \forall \eta' \in \mathbb{P}^1(F) \right\}.$$

Lemma 2. *([W-2,5]) The action of* $SL(\mathcal{O}_F \oplus \mathfrak{a})$ *in the interior* F_η^0 *of* F_η *reduces to that of the isotropy group* Γ_η *of* η, *i.e., if* τ *and* $\gamma\tau$ *both belong to* F_η^0, *then* $\gamma\tau = \tau$.

Consequently, we arrive at the following way to decompose the orbit space $SL(\mathcal{O}_F \oplus \mathfrak{a}) \backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2} \right)$ into h pieces glued in some way along pants of their boundary.

Proposition. *([W-2,5]) Let* $i_\eta : \Gamma_\eta \backslash F_\eta \hookrightarrow SL(\mathcal{O}_F \oplus \mathfrak{a}) \backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2} \right)$
denote the natural map. Then

$$SL(\mathcal{O}_F \oplus \mathfrak{a}) \backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2} \right) = \bigcup_\eta i_\eta \left(\Gamma_\eta \backslash F_\eta \right),$$

where the union is taken over a set of h *cusps representing the ideal classes of* F. *Each piece corresponds to an ideal class of* F.

Note that the action of Γ_η on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ is free. Consequently, all fixed points of $SL(\mathcal{O}_F \oplus \mathfrak{a})$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ lie on the boundaries of F_η.

9.2. Fundamental domains

We can give a more precise description of the fundamental domain, based on our understanding of that for stablizer groups of cusps. To state it, denote by $\eta_1, \ldots, \eta_h$ inequivalent cusps for the action of $SL(\mathcal{O}_F \oplus \mathfrak{a})$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. Choose $A_{\eta_i} \in SL(2, F)$ such that $A_{\eta_i} \infty = \eta_i$, $i = 1, 2, \ldots, h$. Write $\mathbf{S}$ for the norm-one hypersurface $\mathbf{S} := \Big\{ y \in \mathbb{R}_{>0}^{r_1+r_2} : N(y) = 1 \Big\}$, and $\mathbf{S}_{U_F^2}$ for the fundamental domain of the action of U_F^2 on $\mathbf{S}$. Denote by $\mathcal{T}$ a fundamental domain for the action of the translations by elements of $\mathfrak{a}\mathfrak{b}^{-2}$ on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, and

$$\mathbf{E} := \Big\{ \tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mathrm{Re}Z\,(\tau) \in \mathcal{T}, \ \mathrm{Im}J\,(\tau) \in \mathbb{R}_{>0} \cdot \mathbf{S}_{U_F^2} \Big\}$$

a fundamental domain for the action of $A_\eta^{-1}\Gamma_\eta A_\eta$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. The intersections of $\mathbf{E}$ with $i_\eta(F_\eta)$ are connected. Consequently, we have the following

Proposition.$'$ (Siegel, Weng) (1) $D_\eta := A_\eta^{-1}\mathbf{E} \cap F_\eta$ *is a fundamental domain for the action of* Γ_η *on* F_η;
(2) *There exist* $\alpha_1, \cdots, \alpha_h \in SL(\mathcal{O}_F \oplus \mathfrak{a})$ *such that* $\cup_{i=1}^h \alpha(D_{\eta_i})$ *is connected and hence a fundamental domain for* $SL(\mathcal{O}_F \oplus \mathfrak{a})$.

That is to say, a fundamental domain may be given as $S_Y \cup \mathcal{F}_1(Y_1) \cup \cdots \cup \mathcal{F}_h(Y_h)$ with S_Y bounded, $\mathcal{F}_i(Y_i) = A_i \cdot \widetilde{\mathcal{F}}_i(Y_i)$ and

$$\widetilde{\mathcal{F}}_i(Y_i) := \Big\{ \tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mathrm{Re}Z(\tau) \in \Sigma, \ \mathrm{Im}J(\tau) \in \mathbb{R}_{>T} \cdot \mathbf{S}_{U_F^2} \Big\}.$$

Moreover, all $\mathcal{F}_i(Y_i)$'s are disjoint from each other when Y_i are sufficiently large.

§10. Stability in rank two

10.1. Stability and distances to cusps

Define now the *distance of* τ *to the cusp* η by

$$d(\eta, \tau) := \frac{1}{\mu(\eta, \tau)} \geq 1.$$

Then, with the use of a crucial result of Tsukasa Hayashi [Ha], we are ready to state the following fundamental result, which exposes a beautiful intrinsic relation between stability and the distance to cusps.

Theorem. (Weng) *The lattice Λ is semi-stable if and only if the distances of corresponding point $\tau_\Lambda \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ to all cusps are all bigger or equal to 1.*

10.2. Moduli space of rank two semi-stable $\mathcal{O}_F$-lattices

For a rank two $\mathcal{O}_F$-lattice Λ, denote by $\tau_\Lambda \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ the corresponding module point. Then, by the previous subsection, Λ is semi-stable if and only if for all cusps η, $d(\eta, \tau_\Lambda) := \frac{1}{\mu(\eta, \tau_\Lambda)}$ are bigger than or equal to 1. This then leads to the consideration of the following truncation of the fundamental domain $\mathcal{D}$ of $SL(\mathcal{O}_F \oplus \mathfrak{a}) \backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2} \right)$: For $T \geq 1$, denote by

$$\mathcal{D}_T := \left\{ \tau \in \mathcal{D} : d(\eta, \tau) \geq T^{-1}, \ \forall \text{cusp } \eta \right\}.$$

The space $\mathcal{D}_T$ may be precisely described in terms of $\mathcal{D}$ and certain neighborhood of cusps. To explain this, we first recall the following

Lemma. ([W-2,5]) *For a cusp η, denote by*

$$X_\eta(T) := \left\{ \tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : d(\eta, \tau) < T^{-1} \right\}.$$

Then for $T \geq 1$,

$$X_{\eta_1}(T) \cap X_{\eta_2}(T) \neq \emptyset \qquad \Leftrightarrow \qquad \eta_1 = \eta_2.$$

With this, we are ready to state the following

Theorem. (Weng) *There is a natural identification between*
(a) moduli space of rank two semi-stable $\mathcal{O}_F$-lattices of volume $N(\mathfrak{a}) \cdot |\Delta_F|$ with underlying projective module $\mathcal{O}_F \oplus \mathfrak{a}$; and
(b) truncated compact domain $\mathcal{D}_1$ consisting of points in the fundamental domain $\mathcal{D}$ whose distances to all cusps are bigger than 1.

In other words, the truncated compact domain $\mathcal{D}_1$ is obtained from the fundamental domain $\mathcal{D}$ of $SL(\mathcal{O}_F \oplus \mathfrak{a}) \backslash \left(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2} \right)$ by deleting the disjoint open neighborhoods $\cup_{i=1}^h \mathcal{F}_i(1)$ associated to inequivalent cusps $\eta_1, \eta_2, \ldots, \eta_h$, where $\mathcal{F}_i(T)$ denotes the neighborhood of η_i consisting of $\tau \in \mathcal{D}$ whose distance to η_i is strictly less than T^{-1}.

Chapter V. Algebraic characterization of stability

§11. Canonical filtrations

11.1. Canonical filtrations

Following Lafforgue [Laf], we call a proto-abelian category $\mathcal{A}$ together with two additive morphisms

$$\mathrm{rk} : \mathcal{A} \to \mathbb{N}, \qquad \deg : \mathcal{A} \to \mathbb{R}$$

a *category with slope structure*. In particular, for non-zero $A \in \mathcal{A}$,

(1) define the *slope* of A by $\mu(A) := \frac{\deg(A)}{\mathrm{rk}A}$;

(2) If $0 = A_0 \subset A_1 \subset \cdots \subset A_l = A$ is a filtration of A in $\mathcal{A}$ with $\mathrm{rk}(A_0) < \mathrm{rk}(A_1) < \cdots < \mathrm{rk}(A_l)$, define the *associated polygon* to be the function $[0, \mathrm{rk}A] \to \mathbb{R}$ such that
(i) its values at 0 and $\mathrm{rk}(A)$ are 0;
(ii) it is affine on the intervals $[\mathrm{rk}(A_{i-1}), \mathrm{rk}(A_i)]$ with slope $\mu(A_i/A_{i-1}) - \mu(A)$;

(3) If $\mathfrak{a}$ is a collection of subobjects of A in $\mathcal{A}$, then $\mathfrak{a}$ is said to be *nice* if
(i) $\mathfrak{a}$ is stable under intersection and finite summation;
(ii) $\mathfrak{a}$ is Noetherian, i.e., every increasing chain of elements in $\mathfrak{a}$ has a maximal element in $\mathfrak{a}$;
(iii) if $A_1 \in \mathfrak{a}$ then $A_1 \neq 0$ if and only if $\mathrm{rk}(A_1) \neq 0$; and
(iv) for $A_1, A_2 \in \mathfrak{a}$ with $\mathrm{rk}(A_1) = \mathrm{rk}(A_2)$. Then $A_1 \subset A_2$ is proper implies that $\deg(A_1) < \deg(A_2)$;

(4) For any nice $\mathfrak{a}$, set

$$\mu^+(A) := \sup \Big\{ \mu(A_1) : A_1 \in \mathfrak{a}, \mathrm{rk}(A_1) \geq 1 \Big\},$$

$$\mu^-(A) := \inf \Big\{ \mu(A/A_1) : A_1 \in \mathfrak{a}, \mathrm{rk}(A_1) < \mathrm{rk}(A) \Big\}.$$

Then we say $(A, \mathfrak{a})$ is *semi-stable* if $\mu^+(A) = \mu(A) = \mu^-(A)$. Moreover if $\mathrm{rk}(A) = 0$, set also $\mu^+(A) = -\infty$ and $\mu^-(A) = +\infty$.

Proposition 1. ([Laf]) *Let $\mathcal{A}$ be a category with slope structure, A an object in $\mathcal{A}$ and $\mathfrak{a}$ a nice family of subobjects of A in $\mathcal{A}$. Then*
(1) **(Canonical Filtration)** *A admits a unique filtration $0 = \overline{A}_0 \subset \overline{A}_1 \subset \cdots \subset \overline{A}_l = A$ with elements in $\mathfrak{a}$ such that*
(i) *$\overline{A}_i, 0 \leq i \leq k$ are maximal in $\mathfrak{a}$;*

(ii) $\overline{A}_i/\overline{A}_{i-1}$ *are semi-stable; and*

(iii) $\mu(\overline{A}_1/\overline{A}_0) > \mu(\overline{A}_2/\overline{A}_1) > \cdots > \mu(\overline{A}_k/\overline{A}_{k-1})$;

(2) (**Boundness**) *All polygons of filtrations of A with elements in $\mathfrak{a}$ are bounded from above by $\overline{p}$, where $\overline{p} := \overline{p}^A$ is the associated polygon for the canonical filtration in (1);*

(3) *For any $A_1 \in \mathfrak{a}, \mathrm{rk}(A_1) \geq 1$ implies $\mu(A_1) \leq \mu(A) + \frac{\overline{p}(\mathrm{rk}(A_1))}{\mathrm{rk}(A_1)}$;*

(4) *The polygon $\overline{p}$ is convex with maximal slope $\mu^+(A) - \mu(A)$ and minimal slope $\mu^-(A) - \mu(A)$;*

(5) *If $(A', \mathfrak{a}')$ is another pair, and $u : A \to A'$ is a homomorphism such that $\mathrm{Ker}(u) \in \mathfrak{a}$ and $\mathrm{Im}(u) \in \mathfrak{a}'$. Then $\mu^-(A) \geq \mu^+(A')$ implies that $u = 0$.*

This results from a Harder–Narasimhan type filtration consideration. A detailed proof may be found at pp. 87–88 in [Laf]. (There are some interesting approaches related to the topics here in literatures. For examples, [An2], [Ch].)

11.2. Examples of lattices

As an example, we have the following

Proposition 2. ([W2,3]) *Let F be a number field. Then*

(1) *proto-abelian category of hermitian vector sheaves on $\mathrm{Spec}\,\mathcal{O}_F$ together with the natural rank and the Arakelov degree is a category with slopes;*

(2) *For any hermitian vector sheaf (E, ρ), $\mathfrak{a}$ consisting of pairs (E_1, ρ_1) with E_1 sub vector sheaves of E and ρ_1 the restrictions of ρ, forms a nice family.*

Indeed, (1) is obvious, while (2) is a direct consequence of the following standard facts:

(i) For a fixed (E, ρ), $\left\{ \deg(E_1, \rho_1) : (E_1, \rho_1) \in \mathfrak{a} \right\}$ is discrete subset of $\mathbb{R}$; and

(ii) for any two sublattices Λ_1, Λ_2 of Λ,

$$\mathrm{Vol}\Big(\Lambda_1/(\Lambda_1 \cap \Lambda_2)\Big) \geq \mathrm{Vol}\Big((\Lambda_1 + \Lambda_2)/\Lambda_2\Big).$$

Consequently, there exists canonical filtrations of Harder–Narasimhan type for hermitian vector sheaves over $\mathrm{Spec}\,\mathcal{O}_F$. Recall that hermitian vector sheaves over $\mathrm{Spec}\,\mathcal{O}_F$ are $\mathcal{O}_F$-lattices in $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^{r=\mathrm{rk}(E)}$ in the language of Arakelov theory: Say, corresponding $\mathcal{O}_F$-lattices are induced from their H^0 via the natural embedding $F^r \hookrightarrow (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^r$ where r_1 (resp. r_2) denotes the real (resp. complex) embeddings of F.

§12. Algebraic characterization

12.1. A GIT principle

In Geometric Invariant Theory ([M], [Kem], [RR]), a fundamental principle, the Micro-Global Principle, claims that if a point is not GIT stable then there exists a parabolic subgroup which destroys the corresponding stability.

In the setting of $\mathcal{O}_F$-lattices, even we do not have a proper definition of GIT stability for lattices, in terms of intersection stability, an analogue of the Micro-Global Principle does hold.

12.2. Micro-global relation for Geo-Ari truncations

Let $\Lambda = \Lambda^g$ be a rank r lattice associated to $g \in \mathrm{GL}_r(\mathbb{A})$ and P a parabolic subgroup. Denote the sublattices filtration associated to P by

$$0 = \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{|P|} = \Lambda.$$

Assume that P corresponds to the partition $I = (d_1, d_2, \cdots, d_{n=:|P|})$. Consequently, we have

$$\mathrm{rk}(\Lambda_i) = r_i := d_1 + d_2 + \cdots + d_i, \qquad \text{for } i = 1, 2, \cdots, |P|.$$

Let $p, q : [0, r] \to \mathbb{R}$ be two polygons such that $p(0) = q(0) = p(r) = q(r) = 0$. Then following Lafforgue, we say q is *bigger than p with respect to P* and denote it by $q >_P p$, if $q(r_i) - p(r_i) > 0$ for all $i = 1, \cdots, |P| - 1$. Introduce also the characteristic function $\mathbf{1}(\overline{p}^* \leq p)$ by

$$\mathbf{1}(\overline{p}^g \leq p) = \begin{cases} 1, & \text{if } \overline{p}^g \leq p; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that for a parabolic subgroup P, p_P^g denotes the polygon induced by P for (the lattice corresponding to) the element $g \in G(\mathbb{A})$.

Fundamental Relation. (Lafforgue, Weng) *Let $p : [0, r] \to \mathbb{R}$ be a fixed convex polygon such that $p(0) = p(r) = 0$. Then we have*

$$\mathbf{1}(\overline{p}^g \leq p) = \sum_{P:\, stand\ para} (-1)^{|P|-1} \sum_{\delta \in P(F) \backslash G(F)} \mathbf{1}(p_P^{\delta g} >_P p), \quad \forall g \in G(\mathbb{A}).$$

Chapter VI. Analytic characterization of stability

§13. Arthur's analytic truncation

13.1. Parabolic subgroups

Let F be a number field with $\mathbb{A} = \mathbb{A}_F$ the ring of adeles. Let G be a connected reductive group defined over F. Recall that a subgroup P of G is called *parabolic* if G/P is a complete algebraic variety. Fix a minimal F-parabolic subgroup P_0 of G with its unipotent radical $N_0 = N_{P_0}$ and fix a F-Levi subgroup $M_0 = M_{P_0}$ of P_0 so as to have a Levi decomposition $P_0 = M_0 N_0$. An F-parabolic subgroup P is called *standard* if it contains P_0. For such a parabolic subgroup P, there exists a unique Levi subgroup $M = M_P$ containing M_0 which we call the *standard Levi subgroup* of P. Let $N = N_P$ be the unipotent radical. Let us agree to use the term parabolic subgroups and Levi subgroups to denote standard F-parabolic subgroups and standard Levi subgroups repectively, unless otherwise is stated.

Let P be a parabolic subgroup of G. Write T_P for the maximal split torus in the center of M_P and T'_P for the maximal quotient split torus of M_P. Set $\tilde{\mathfrak{a}}_P := X_*(T_P) \otimes \mathbb{R}$ and denote its real dimension by $d(P)$, where $X_*(T)$ is the lattice of 1-parameter subgroups in the torus T. Then it is known that $\tilde{\mathfrak{a}}_P = X_*(T'_P) \otimes \mathbb{R}$ as well. The two descriptions of $\tilde{\mathfrak{a}}_P$ show that if $Q \subset P$ is a parabolic subgroup, then there is a canonical injection $\tilde{\mathfrak{a}}_P \hookrightarrow \tilde{\mathfrak{a}}_Q$ and a natural surjection $\tilde{\mathfrak{a}}_Q \twoheadrightarrow \tilde{\mathfrak{a}}_P$. We thus obtain a canonical decomposition $\tilde{\mathfrak{a}}_Q = \tilde{\mathfrak{a}}_Q^P \oplus \tilde{\mathfrak{a}}_P$ for a certain subspace $\tilde{\mathfrak{a}}_Q^P$ of $\tilde{\mathfrak{a}}_Q$. In particular, $\tilde{\mathfrak{a}}_G$ is a summand of $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_P$ for all P. Set $\mathfrak{a}_P := \tilde{\mathfrak{a}}_P / \tilde{\mathfrak{a}}_G$ and $\mathfrak{a}_Q^P := \tilde{\mathfrak{a}}_Q^P / \tilde{\mathfrak{a}}_G$. Then we have

$$\mathfrak{a}_Q = \mathfrak{a}_Q^P \oplus \mathfrak{a}_P$$

and $\mathfrak{a}_P$ is canonically identified as a subspace of $\mathfrak{a}_Q$. Set $\mathfrak{a}_0 := \mathfrak{a}_{P_0}$ and $\mathfrak{a}_0^P = \mathfrak{a}_{P_0}^P$ then we also have $\mathfrak{a}_0 = \mathfrak{a}_0^P \oplus \mathfrak{a}_P$ for all P.

13.2. Logarithmic map

For a real vector space V, write V^* its dual space over $\mathbb{R}$. Then dually we have the spaces $\mathfrak{a}_0^*, \mathfrak{a}_P^*, \left(\mathfrak{a}_0^P\right)^*$ and hence the decompositions

$$\mathfrak{a}_0^* = \left(\mathfrak{a}_0^Q\right)^* \oplus \left(\mathfrak{a}_Q^P\right)^* \oplus \mathfrak{a}_P^*.$$

In particular, $\mathfrak{a}_P^* = X(M_P) \otimes \mathbb{R}$ with $X(M_P) := \mathrm{Hom}_F\left(M_P, GL(1)\right)$ i.e., collection of characters on M_P. It is known that $\mathfrak{a}_P^* = X(A_P) \otimes \mathbb{R}$

where A_P denotes the split component of the center of M_P. Clearly, if $Q \subset P$, then $M_Q \subset M_P$ while $A_P \subset A_Q$. Thus via restriction, the above two expressions of $\mathfrak{a}_P^*$ also naturally induce an injection $\mathfrak{a}_P^* \hookrightarrow \mathfrak{a}_Q^*$ and a sujection $\mathfrak{a}_Q^* \twoheadrightarrow \mathfrak{a}_P^*$, compactible with the decomposition $\mathfrak{a}_Q^* = \left(\mathfrak{a}_Q^P\right)^* \oplus \mathfrak{a}_P^*$.

Every $\chi = \sum s_i \chi_i$ in $\mathfrak{a}_{P,\mathbb{C}}^* := \mathfrak{a}_P^* \otimes \mathbb{C}$ determines a morphism $P(\mathbb{A}) \to \mathbb{C}^*$ by $p \mapsto p^\chi := \prod |\chi_i(p)|^{s_i}$. Consequently, we have a natural logarithmic map $H_P : P(\mathbb{A}) \to \mathfrak{a}_P$ defined by

$$\langle H_P(p), \chi \rangle = p^\chi, \qquad \forall \chi \in \mathfrak{a}_P^*.$$

The kernel of H_P is denoted by $P(\mathbb{A})^1$ and we set $M_P(\mathbb{A})^1 := P(\mathbb{A})^1 \cap M_P(\mathbb{A})$.

Let also A_+ be the set of $a \in A_P(\mathbb{A})$ such that
(1) $a_v = 1$ for all finite places v of F; and
(2) $\chi(a_\sigma)$ is a positive number independent of infinite places σ of F for all $\chi \in X(M_P)$.
Then $M(\mathbb{A}) = A_+ \cdot M(\mathbb{A})^1$.

13.3.   Roots, coroots, weights and coweights

We now introduce standard bases for above spaces and their duals. Let Δ_0 and $\widehat{\Delta}_0$ be the subsets of simple roots and simple weights in $\mathfrak{a}_0^*$ respectively. (Recall that elements of $\widehat{\Delta}_0$ are non-negative linear combinations of elements in Δ_0.) Write $\Delta_0^\vee$ (resp. $\widehat{\Delta}_0^\vee$) for the basis of $\mathfrak{a}_0$ dual to $\widehat{\Delta}_0$ (resp. Δ_0). Being the dual of the collection of simple weights (resp. of simple roots), $\Delta_0^\vee$ (resp. $\widehat{\Delta}_0^\vee$) is the set of coroots (resp. coweights).

For every P, let $\Delta_P \subset \mathfrak{a}_P^*$ be the set of non-trivial *restrictions* of elements of Δ_0 to $\mathfrak{a}_P$. Denote the dual basis of Δ_P by $\widehat{\Delta}_P^\vee$. For each $\alpha \in \Delta_P$, let $\alpha^\vee$ be the projection of $\beta^\vee$ to $\mathfrak{a}_P$, where β is the root in Δ_0 whose restriction to $\mathfrak{a}_P$ is α. Set $\Delta_P^\vee := \left\{ \alpha^\vee : \alpha \in \Delta_P \right\}$, and define the dual basis of $\Delta_P^\vee$ by $\widehat{\Delta}_P$.

More generally, if $Q \subset P$, write Δ_Q^P to denote the *subset* $\alpha \in \Delta_Q$ appearing in the action of T_Q in the unipotent radical of $Q \cap M_P$. (Indeed, $M_P \cap Q$ is a parabolic subgroup of M_P with nilpotent radical $N_Q^P := N_Q \cap M_P$. Thus Δ_Q^P is simply the set of roots of the parabolic subgroup $(M_P \cap Q, A_Q)$. And one checks that the map $P \mapsto \Delta_Q^P$ gives a natural bijection between parabolic subgroups P containin Q and subsets of Δ_Q.) Then $\mathfrak{a}_P$ is the subspace of $\mathfrak{a}_Q$ annihilated by Δ_Q^P. Denote

by $(\widehat{\Delta}^{\vee})^P_Q$ the dual of Δ^P_Q. Let $(\Delta^P_Q)^{\vee} := \left\{ \alpha^{\vee} : \alpha \in \Delta^P_Q \right\}$ and denote by $\widehat{\Delta}^P_Q$ the dual of $(\Delta^P_Q)^{\vee}$.

13.4. Positive cone and positive chamber

Let $Q \subset P$ be two parabolic subgroups of G. We extend the linear functionals in Δ^P_Q and $\widehat{\Delta}^P_Q$ to elements of the dual space $\mathfrak{a}^*_0$ by means of the canonical projection from $\mathfrak{a}_0$ to $\mathfrak{a}^P_Q$ given by the decomposition $\mathfrak{a}_0 = \mathfrak{a}^Q_0 \oplus \mathfrak{a}^P_Q \oplus \mathfrak{a}_P$. Let τ^P_Q be the characteristic function of the *positive chamber*

$$\left\{ H \in \mathfrak{a}_0 : \langle \alpha, H \rangle > 0 \text{ for all } \alpha \in \Delta^P_Q \right\}$$
$$= \mathfrak{a}^Q_0 \oplus \left\{ H \in \mathfrak{a}^P_Q : \langle \alpha, H \rangle > 0 \text{ for all } \alpha \in \Delta^P_Q \right\} \oplus \mathfrak{a}_P$$

and let $\widehat{\tau}^P_Q$ be the characteristic function of the *positive cone*

$$\left\{ H \in \mathfrak{a}_0 : \langle \varpi, H \rangle > 0 \text{ for all } \varpi \in \widehat{\Delta}^P_Q \right\}$$
$$= \mathfrak{a}^Q_0 \oplus \left\{ H \in \mathfrak{a}^P_Q : \langle \varpi, H \rangle > 0 \text{ for all } \varpi \in \widehat{\Delta}^P_Q \right\} \oplus \mathfrak{a}_P.$$

Note that elements in $\widehat{\Delta}^P_Q$ are non-negative linear combinations of elements in Δ^P_Q, we have

$$\widehat{\tau}^P_Q \geq \tau^P_Q.$$

13.5. Partial truncation and first estimations

Denote τ^G_P and $\widehat{\tau}^G_P$ simply by τ_P and $\widehat{\tau}_P$.

Basic Estimation. (Arthur) *Suppose that we are given a parabolic subgroup P, and a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_P$. Then there are constants c and N such that for all $x \in G(\mathbb{A})^1$ and $X \in \mathfrak{a}_P$,*

$$\sum_{\delta \in P(F) \backslash G(F)} \widehat{\tau}_P \left(H(\delta x) - X \right) \leq c \left(\|x\| e^{\|X\|} \right)^N.$$

Moreover, *the sum is finite.*

As a direct consequence, we have the following
Corollary. ([Ar2,3]) *Suppose that $T \in \mathfrak{a}_0$ and $N \geq 0$. Then there exist constants c' and N' such that for any function ϕ on $P(F) \backslash G(\mathbb{A})^1$, and $x, y \in G(\mathbb{A})^1$,*

$$\sum_{\delta \in P(F) \backslash G(F)} \left| \phi(\delta x) \right| \cdot \widehat{\tau}_P \left(H(\delta x) - H(y) - X \right)$$

is bounded by

$$c'\|x\|^{N'} \cdot \|y\|^{N'} \cdot \sup_{u \in G(\mathbb{A})^1} \left(|\phi(u)| \cdot \|u\|^{-N} \right).$$

§14.   Reduction theory

14.1.   Langlands' combinatorial lemma

If $P_1 \subset P_2$, following Arthur [Ar2], set

$$\sigma_1^2(H) := \sigma_{P_1}^{P_2} := \sum_{P_3 : P_2 \supset P_2} (-1)^{\dim(A_3/A_2)} \tau_1^3(H) \cdot \hat{\tau}_3(H),$$

for $H \in \mathfrak{a}_0$. Then we have

Lemma 1. ([Ar2]) *If $P_1 \subset P_2$, σ_1^2 is a characteristic function of the subset of $H \in \mathfrak{a}_1$ such that*
(i) $\alpha(H) > 0$ for all $\alpha \in \Delta_1^2$;
(ii) $\sigma(H) \leq 0$ for all $\sigma \in \Delta_1 \backslash \Delta_1^2$; and
(iii) $\varpi(H) > 0$ for all $\varpi \in \hat{\Delta}_2$.

As a spacial case, with $P_1 = P_2$, we get the following important consequence:

Langlands' Combinatorial Lemma. *If $Q \subset P$ are parabolic subgroups, then for all $H \in \mathfrak{a}_0$,*

$$\sum_{R : Q \subset R \subset P} (-1)^{\dim(A_R/A_P)} \tau_Q^R(H) \hat{\tau}_R^P(H) = \delta_{QP};$$

$$\sum_{R : Q \subset R \subset P} (-1)^{\dim(A_Q/A_R)} \hat{\tau}_Q^R(H) \tau_R^P(H) = \delta_{QP}.$$

Suppose now that $Q \subset P$ are parabolic subgroups. Fix a vector $\Lambda \in \mathfrak{a}_0^*$. Let

$$\varepsilon_Q^P(\Lambda) := (-1)^{\#\{\alpha \in \Delta_Q^P : \Lambda(\alpha^\vee) \leq 0\}},$$

and let

$$\phi_Q^P(\Lambda, H), \qquad H \in \mathfrak{a}_0,$$

be the characteristic function of the set

$$\left\{ H \in \mathfrak{a}_0 : \begin{array}{ll} \varpi(H) > 0, & \text{if } \Lambda(\alpha^\vee) \leq 0 \\ \varpi(H) \leq 0, & \text{if } \Lambda(\alpha^\vee) > 0 \end{array}, \forall \alpha \in \Delta_Q^P \right\}.$$

Lemma 2.([Ar2,3]) With the same notation as above,

$$\sum_{R : Q \subset R \subset P} \varepsilon_Q^R(\Lambda) \cdot \phi_Q^R(\Lambda, H) \cdot \tau_R^P(H) = \begin{cases} 0, & \text{if } \Lambda(\alpha^\vee) \leq 0, \ \exists \alpha \in \Delta_Q^P \\ 1, & \text{otherwise} \end{cases}.$$

14.2. Langlands–Arthur's partition: reduction theory

Our aim here is to derive Langlands–Arthur's partition of $G(F)\backslash G(\mathbb{A})$ into disjoint subsets, one for each (standard) parabolic subgroup.

To start with, suppose that ω is a compact subset of $N_0(\mathbb{A})M_0(\mathbb{A})^1$ and that $T_0 \in -\mathfrak{a}_0^+$. For any parabolic subgroup P_1, introduce the associated *Siegel set* $\mathfrak{s}^{P_1}(T_0, \omega)$ as the collection of

$$pak, \qquad p \in \omega, \ a \in A_0(\mathbb{R})^0, \ k \in K,$$

where $\alpha\Big(H_0(a) - T_0\Big)$ is positive for each $\alpha \in \Delta_0^1$. Then from classical reduction theory, we conclude that *for sufficiently big ω and sufficiently small T_0, $G(\mathbb{A}) = P_1(F) \cdot \mathfrak{s}^{P_1}(T_0, \omega)$.*

Suppose now that P_1 is given. Let $\mathfrak{s}^{P_1}(T_0, T, \omega)$ be the set of x in $\mathfrak{s}^{P_1}(T_0, \omega)$ such that $\varpi\Big(H_0(x) - T\Big) \leq 0$ for each $\varpi \in \hat{\Delta}_0^1$. Let $F^{P_1}(x, T) := F^1(x, T)$ be the characteristic function of the set of $x \in G(\mathbb{A})$ such that δx belongs to $\mathfrak{s}^{P_1}(T_0, T, \omega)$ for some $\delta \in P_1(F)$.

As such, $F^1(x, T)$ is left $A_1(\mathbb{R})^0 N_1(\mathbb{A})M_1(F)$-invariant, and can be regarded as the characteristic function of the projection of $\mathfrak{s}^{P_1}(T_0, T, \omega)$ onto $A_1(\mathbb{R})^0 N_1(\mathbb{A})M_1(F)\backslash G(\mathbb{A})$, a compact subset of the quotient space $A_1(\mathbb{R})^0 N_1(\mathbb{A})M_1(F)\backslash G(\mathbb{A})$.

For example, $F(x, T) := F^G(x, T)$ admits the following more direct description which will play a key role in our study of Arthur's periods:

If $P_1 \subset P_2$ are (standard) parabolic subgroups, we write $A_1^\infty := A_{P_1}^\infty$ for $A_{P_1}(\mathbb{A})^0$, the identity component of $A_{P_1}(\mathbb{R})$, and

$$A_{1,2}^\infty := A_{P_1, P_2}^\infty := A_{P_1} \cap M_{P_2}(\mathbb{A})^1.$$

Then the logarthmic map H_{P_1} maps $A_{1,2}^\infty$ isomorphically onto $\mathfrak{a}_1^2$, the orthogonal complement of $\mathfrak{a}_2$ in $\mathfrak{a}_1$. If T_0 and T are points in $\mathfrak{a}_0$, set $A_{1,2}^\infty(T_0, T)$ to be the set

$$\Big\{a \in A_{1,2}^\infty : \alpha\Big(H_1(a) - T\Big) > 0, \ \alpha \in \Delta_1^2; \ \varpi\Big(H_1(a) - T\Big) < 0, \ \varpi \in \hat{\Delta}_1^2\Big\},$$

where $\Delta_1^2 := \Delta_{P_1 \cap M_2}$ and $\hat{\Delta}_1^2 := \hat{\Delta}_{P_1 \cap M_2}$. In particular, for T_0 such that $-T_0$ is suitably regular, $F(x, T)$ *is the characteristic function of the compact subset of $G(F)\backslash G(\mathbb{A})^1$ obtained by projecting*

$$N_0(\mathbb{A}) \cdot M_0(\mathbb{A})^1 \cdot A_{P_0, G}^\infty(T_0, T) \cdot K$$

onto $G(F)\backslash G(\mathbb{A})^1$.

All in all, we arrive at the following

Arthur's Partition. (Arthur) *Fix P and let T be any suitably point in $T_0 \in \mathfrak{a}_0^+$. Then*

$$\sum_{P_1: P_0 \subset P_1 \subset P} \sum_{\delta \in P_1(F) \backslash G(F)} F^1(\delta x) \cdot \tau_1^P\Big(H_0(\delta x) - T\Big) = 1, \quad \forall x \in G(\mathbb{A}).$$

§15. Arthur's analytic truncation

15.1. Definition

Following Arthur, we make the following

Definition. (Arthur) *Fix a suitably regular point $T \in \mathfrak{a}_0^+$. If ϕ is a continuous function on $G(F) \backslash G(\mathbb{A})^1$, define Arthur's analytic truncation $\Big(\Lambda^T \phi\Big)(x)$ to be the function*

$$\Big(\Lambda^T \phi\Big)(x) := \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(F) \backslash G(F)} \phi_P(\delta x) \cdot \hat{\tau}_P\Big(H(\delta x) - T\Big),$$

where

$$\phi_P(x) := \int_{N(F) \backslash N(\mathbb{A})} \phi(nx) \, dn$$

denotes the constant term of ϕ along P, and the sum is over all (standard) parabolic subgroups.

The main purpose for introducing analytic truncation is to give a natural way to construct integrable functions: even from the example of GL_2, we know that automorphic forms are generally not integrable over the total fundamental domain $G(F) \backslash G(\mathbb{A})^1$ mainly due to the fact that in the Fourier expansions of such functions, constant terms are only of moderate growth (hence not integrable). Thus in order to naturally obtain integrable functions, we should truncate the original function along the cuspidal regions by removing constant terms. Simply put, Arthur's analytic truncation is a well-designed divice in which constant terms are tackled in such a way that different levels of parabolic subgroups are suitably counted at the corresponding cuspidal region so that the whole truncation will not be overdone while there will be no parabolic subgroups left untackled.

Note that all parabolic subgroups of G can be obtained from standard parabolic subgroups by taking conjugations with elements from $P(F) \backslash G(F)$. So we have:

(a) $\Big(\Lambda^T \phi\Big)(x) = \sum_P (-1)^{\dim(A/Z)} \phi_P(x) \cdot \hat{\tau}_P\Big(H(x) - T\Big)$, *where the sum is over all, both standard and non-standard, parabolic subgroups;*

(b) *If ϕ is a cusp form, then $\Lambda^T \phi = \phi$;*

This is because by definition, all constant terms along proper P : $P \neq G$ are zero. Moreover, as a direct consequence of the Basic Estimation for partial truncation, we have

(c) *If ϕ is of moderate growth in the sense that there exist some constants C, N such that $\left|\phi(x)\right| \leq c\|x\|^N$ for all $x \in G(\mathbb{A})$, then so is $\Lambda^T \phi$.*

15.2. Basic properties

Recall that an element $T \in \mathfrak{a}_0^+$ is called *sufficiently regular*, if for any $\alpha \in \Delta_0$, $\alpha(T) \gg 0$. Fundamental properties of Arthur's analytic truncation may be summarized as follows:

Proposition. (Arthur) *For sufficiently regular T in $\mathfrak{a}_0$,*
(1) *Let $\phi : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ be a locally L^1 function. Then*

$$\Lambda^T \Lambda^T \phi(g) = \Lambda^T \phi(g)$$

for almost all g. If ϕ is also locally bounded, then the above is true for all g;
(2) *Let ϕ_1, ϕ_2 be two locally L^1 functions on $G(F)\backslash G(\mathbb{A})$. Suppose that ϕ_1 is of moderate growth and ϕ_2 is rapidly decreasing. Then*

$$\int_{Z_{G(\mathbb{A})} G(F)\backslash G(\mathbb{A})} \overline{\Lambda^T \phi_1(g)} \cdot \phi_2(g)\, dg = \int_{Z_{G(\mathbb{A})} G(F)\backslash G(\mathbb{A})} \overline{\phi_1(g)} \cdot \Lambda^T \phi_2(g)\, dg;$$

(3) *Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$, and r, r' are two positive real numbers. Then there exists a finite subset $\left\{X_i \: : \: i = 1, 2, \cdots, N\right\} \subset \mathfrak{u}$, the universal enveloping algebra of the Lie algebra associated to $G(\mathbb{A}_\infty)$, such that the following is satisfied: Let ϕ be a smooth function on $G(F)\backslash G(\mathbb{A})$, right invariant under K_f and let $a \in A_{G(\mathbb{A})}$, $g \in G(\mathbb{A})^1 \cap S$. Then*

$$\left|\Lambda^T \phi(ag)\right| \leq \|g\|^{-r} \sum_{i=1}^N \sup\left\{ |\delta(X_i)\phi(ag')| \, \|g'\|^{-r'} : g' \in G(\mathbb{A})^1 \right\},$$

where S is a Siegel domain with respect to $G(F)\backslash G(\mathbb{A})$.

15.3. Truncation $\Lambda^T \mathbf{1}$

To go further, let us give a much more detailed study of Authur's analytic truncation for the constant function $\mathbf{1}$. Fix a sufficiently regular

$T \in \mathfrak{a}_0$. Introduce the truncated subset $\Sigma(T) := \left(Z_{G(\mathbb{A})} G(F) \backslash G(\mathbb{A}) \right)_T$ of the space $G(F) \backslash G(\mathbb{A})^1$ by

$$\Sigma(T) := \left(Z_{G(\mathbb{A})} G(F) \backslash G(\mathbb{A}) \right)_T := \left\{ g \in Z_{G(\mathbb{A})} G(F) \backslash G(\mathbb{A}) : \Lambda^T \mathbf{1}(g) = 1 \right\}.$$

We claim that $\Sigma(T)$ or the same $\left(Z_{G(\mathbb{A})} G(F) \backslash G(\mathbb{A}) \right)_T$, is compact. In fact, much stronger result is correct. Namely, we have the following

Lemma. (Arthur) *For sufficiently regular $T \in \mathfrak{a}_0^+$, $\Lambda^T \mathbf{1}(x) = F(x, T)$. That is to say, $\Lambda^1 \mathbf{1}$ is the characteristic function of the compact subset $\Sigma(T)$ of $G(F) \backslash G(\mathbb{A})^1$ obtained by projecting $N_0(\mathbb{A}) \cdot M_0(\mathbb{A})^1 \cdot A_{P_0, G}^\infty(T_0, T) \cdot K$ onto $G(F) \backslash G(\mathbb{A})^1$.*

§16.   Analytic characterization of stability

16.1.   A micro bridge

For simplicity, we in this subsection work only with the field of rationals $\mathbb{Q}$ and use mixed languages of adeles and lattices. Also, without loss of generality, we assume that $\mathbb{Z}$-lattices are of volume one. Accordingly, set $G = SL_r$.

For a rank r lattice Λ of volume one, denote the sublattices filtration associated to a parabolic subgroup P by

$$0 = \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{|P|} = \Lambda.$$

Assume that P corresponds to the partition $I = (d_1, d_2, \cdots, d_{|P|})$. A polygon $p : [0, r] \to \mathbb{R}$ is called *normalized* if $p(0) = p(r) = 0$. For a (normalized) polygon $p : [0, r] \to \mathbb{R}$, define the associated (real) character $T = T(p)$ of M_0 by the condition that

$$\alpha_i(T) = \Big[p(i) - p(i-1) \Big] - \Big[p(i+1) - p(i) \Big]$$

for all $i = 1, 2, \cdots, r - 1$. Then one checks that $T(p)$ coincides with

$$\Big(p(1), p(2) - p(1), \cdots, p(i) - p(i-1), \cdots, p(r-1) - p(r-2), -p(r-1) \Big).$$

Now take $g = g(\Lambda) \in G(\mathbb{A})$. Denote its lattice by Λ^g, and its induced filtration from P by

$$0 = \Lambda_0^{g,P} \subset \Lambda_1^{g,P} \subset \cdots \subset \Lambda_{|P|}^{g,P} = \Lambda^g.$$

Consequently, the polygon $p_P^g = p_P^{\Lambda^g} : [0, r] \to \mathbb{R}$ is characterized by

(1) $p_P^g(0) = p_P^g(r) = 0$;

(2) p_P^g is affine on $[r_i, r_{i+1}]$, $i = 1, 2, \cdots, |P| - 1$; and

(3) $p_P^g(r_i) = \deg(\Lambda_i^{g,P}) - r_i \cdot \dfrac{\deg(\Lambda^g)}{r}$, $i = 1, 2, \cdots, |P| - 1$.

Note that the volume of Λ is assumed to be one, therefore (3) is equivalent to

(3)' $p_P^g(r_i) = \deg(\Lambda_i^{g,P})$, $i = 1, 2, \cdots, |P| - 1$.

The advantage of partially using adelic language is that the values of p_P^g may be written down precisely. Indeed, using Langlands decompositon $g = n \cdot m \cdot a(g) \cdot k$ with $n \in N_P(\mathbb{A}), m \in M_P(\mathbb{A})^1, a \in A_+$ and $k \in K := \prod_p SL(\mathcal{O}_{\mathbb{Q}_p}) \times SO(r)$. Write

$$a = a(g) = \mathrm{diag}\Big(a_1 I_{d_1}, a_2 I_{d_2}, \cdots, a_{|P|} I_{d_{|P|}} \Big)$$

where $r = d_1 + d_2 + \cdots + d_{|P|}$ is the partition corresponding to P. Then it is a standard fact that

$$\deg\Big(\Lambda_i^{g,P}\Big) = -\log\Big(\prod_{j=1}^i a_j^{d_j} \Big) = -\sum_{j=1}^i d_j \log a_j, \qquad i = 1, \cdots, |P|.$$

Set now $\mathbf{1}(p_P^* >_P p)$ to be the characteristic function of the subset of g's such that $p_P^g >_P p$. Then by a certain calculation, we obtain the following

Micro Bridge. (Lafforgue, Weng) *For a fixed convex normalized polygon $p : [0, r] \to \mathbb{R}$, and $g \in SL_r(\mathbb{A})$, with respect to any parabolic subgroup P, we have*

$$\hat{\tau}_P\Big(-H_0(g) - T(p) \Big) = \mathbf{1}\Big(p_P^g >_P p\Big).$$

16.2. Analytic truncations and stability

With the micro bridge above, now we are ready to state the following analytic characterization of stability.

Global Bridge. (Lafforgue, Weng) *For a fixed normalized convex polygon $p : [0, r] \to \mathbb{R}$, let $T(p) \in \mathfrak{a}_0$ be the associated vector defined by*

$$\Big(p(1), p(2) - p(1), \cdots, p(i) - p(i-1), \cdots, p(r-1) - p(r-2), -p(r-1)\Big).$$

If $T(p)$ is sufficiently positive, then

$$\mathbf{1}(\overline{p}^g \leq p) = \Big(\Lambda^{T(p)} \mathbf{1}\Big)(g).$$

Chapter VII. Non-abelian L-functions

§17.   High rank zetas and Eisenstein series

17.1.   Epstein zeta functions and high rank zetas

Recall that the rank r non-abelian zeta function $\xi_{\mathbb{Q},r}(s)$ of $\mathbb{Q}$ is given by

$$\xi_{\mathbb{Q},r}(s) = \int_{\mathcal{M}_{\mathbb{Q},r}} \left(e^{h^0(\mathbb{Q},\Lambda)} - 1 \right) \cdot \left(e^{-s} \right)^{\deg(\Lambda)} d\mu(\Lambda), \qquad \mathrm{Re}(s) > 1,$$

with $e^{h^0(\mathbb{Q},\Lambda)} := \sum_{x \in \Lambda} \exp\left(-\pi |x|^2 \right)$ and $\deg(\Lambda) = -\log \mathrm{Vol}\left(\mathbb{R}^r / \Lambda \right)$.

Decompose according to their volumes to get $\mathcal{M}_{\mathbb{Q},r} = \cup_{T>0} \mathcal{M}_{\mathbb{Q},r}[T]$. Using the natural morphism $\mathcal{M}_{\mathbb{Q},r}[T] \to \mathcal{M}_{\mathbb{Q},r}[1]$, $\Lambda \mapsto T^{\frac{1}{r}} \cdot \Lambda$, we obtain

$$\xi_{\mathbb{Q},r}(s) = \int_{\cup_{T>0} \mathcal{M}_{\mathbb{Q},r}[T]} \left(e^{h^0(\mathbb{Q},\Lambda)} - 1 \right) \cdot \left(e^{-s} \right)^{\deg(\Lambda)} d\mu(\Lambda)$$

$$= \int_0^\infty T^s \frac{dT}{T} \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \left(e^{h^0(\mathbb{Q}, T^{\frac{1}{r}} \cdot \Lambda)} - 1 \right) \cdot d\mu(\Lambda).$$

But

$$h^0(\mathbb{Q}, T^{\frac{1}{r}} \cdot \Lambda) = \log \left(\sum_{x \in \Lambda} \exp\left(-\pi |x|^2 \cdot T^{\frac{2}{r}} \right) \right)$$

and

$$\int_0^\infty e^{-AT^B} T^s \frac{dT}{T} = \frac{1}{B} \cdot A^{-\frac{s}{B}} \cdot \Gamma\left(\frac{s}{B}\right), \qquad B \neq 0,$$

we have $\xi_{\mathbb{Q},r}(s) = \frac{r}{2} \cdot \pi^{-\frac{r}{2} s} \Gamma(\frac{r}{2} s) \cdot \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \left(\sum_{x \in \Lambda \setminus \{0\}} |x|^{-rs} \right) \cdot d\mu_1(\Lambda)$.

Accordingly, introduce the completed Epstein zeta function for Λ by

$$\hat{E}(\Lambda; s) := \pi^{-s} \Gamma(s) \cdot \sum_{x \in \Lambda \setminus \{0\}} |x|^{-2s}.$$

Proposition. (Weng) (Eisenstein Series and High Rank Zetas)

$$\xi_{\mathbb{Q},r}(s) = \frac{r}{2} \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \hat{E}(\Lambda, \frac{r}{2} s) \, d\mu_1(\Lambda).$$

17.2.   Rankin–Selberg method: an example with SL_2

Consider the action of $\mathrm{SL}(2, \mathbb{Z})$ on the upper half plane $\mathcal{H}$. Then a standard 'fundamental domain' is given by $D = \{z = x+iy \in \mathcal{H} : |x| \leq$

$\frac{1}{2}, y > 0, x^2 + y^2 \geq 1\}$. Recall also the completed standard Eisenstein series

$$\hat{E}(z; s) := \pi^{-s}\Gamma(s) \cdot \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mz + n|^{2s}}.$$

Naturally, we are led to the integral $\int_D \hat{E}(z, s)\frac{dx\,dy}{y^2}$. However, this integration diverges. Indeed, near the only cusp $y = \infty$, $\hat{E}(z, s)$ has the Fourier expansion

$$\hat{E}(z; s) = \sum_{n=-\infty}^{\infty} a_n(y, s)e^{2\pi inx}$$

with

$$a_n(y, s) = \begin{cases} \xi(2s)y^s + \xi(2 - 2s)y^{1-s}, & \text{if } n = 0; \\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y), & \text{if } n \neq 0, \end{cases}$$

where $\xi(s)$ is the completed Riemann zeta function, $\sigma_s(n) := \sum_{d|n} d^s$, and $K_s(y) := \frac{1}{2}\int_0^\infty e^{-y(t+\frac{1}{t})/2}t^s\frac{dt}{t}$ is the K-Bessel function. Moreover,

$$|K_s(y)| \leq e^{-y/2}K_{\text{Re}(s)}(2), \text{ if } y > 4, \qquad \text{and} \qquad K_s = K_{-s}.$$

So $a_{n\neq0}(y, s)$ decay exponentially, and the problematic term comes from $a_0(y, s)$, which is of slow growth.

Therefore, to make the original integration meaningful, we need to cut-off the slow growth part. Recall from the discussions in previous three chapters, we have two different ways to do so: one is geometric and hence rather direct and simple; while the other is analytic, and hence rather technical and traditional, dated back to Rankin–Selberg.

(a) Geometric Truncation

Draw a horizontal line $y = T \geq 1$ and set

$$D_T = \{z = x + iy \in D : y \leq T\}, \qquad D^T = \{z = x + iy \in D : y \geq T\}.$$

Then $D = D_T \cup D^T$. Introduce a well-defined integration

$$I_T^{\text{Geo}}(s) := \int_{D_T} \hat{E}(z, s)\frac{dx\,dy}{y^2}.$$

(b) Analytic Truncation

Define a truncated Eisenstein series $\hat{E}_T(z; s)$ by

$$\hat{E}_T(z; s) := \begin{cases} \hat{E}(z; s), & \text{if } y \leq T; \\ \hat{E}(z; s) - a_0(y; s), & \text{if } y > T. \end{cases}$$

Introduce a well-defined integration

$$I_T^{\text{Ana}}(s) := \int_D \hat{E}_T(z; s) \frac{dx\,dy}{y^2}.$$

With this, from the Rankin–Selberg method, one checks that we have the following:

Proposition. ([W2,3,5]) (**Analytic Truncation=Geometric Truncation in Rank Two**)

$$I_T^{\text{Geo}}(s) = \xi(2s)\frac{T^{s-1}}{s-1} - \xi(2s-1)\frac{T^{-s}}{s} = I_T^{\text{Ana}}(s).$$

Each of the above two integrations has its own merit: for the geometric one, we keep the Eisenstein series unchanged, while for the analytic one, we keep the original fundamental domain of $\mathcal{H}$ under $\text{SL}(2,\mathbb{Z})$ as it is.

Note that the nice point about the fundamental domain is that it admits a modular interpretation. Thus it would be very idealistic if we could at the same time keep the Eisenstein series unchanged, while offer some integration domains which appear naturally in certain moduli problems. Guided by this, in the follows, we will introduce non-abelian L-functions using integrations of Eisenstein series over generalized moduli spaces.

(c) **Arithmetic Truncation**

Now we explain why above discussion and Rankin–Selberg method have anything to do with our non-abelian zeta functions. For this, we introduce yet another truncation, the algebraic, or better arithmetic, one.

So back to the moduli space of rank 2 lattices of volume 1 over $\mathbb{Q}$. Then classical reduction theory gives a natural map from this moduli space to the fundamental domain D of $\text{SL}(2,\mathbb{Z})$ on $\mathcal{H}$: For any lattice Λ, fix $\mathbf{x}_1 \in \Lambda$ such that its length gives the first Minkowski minimum λ_1 of Λ ([Mi]). Then via rotation, we may assume that $\mathbf{x}_1 = (\lambda_1, 0)$. Further, from the reduction theory $\frac{1}{\lambda_1}\Lambda$ may be viewed as the lattice of the volume $\lambda_1^{-2} = y_0$ which is generated by $(1,0)$ and $\omega = x_0 + iy_0 \in D$. That is to say, the points in D_T are in one-to-one corresponding to the rank two lattices of volume one whose first Minkowski minimum $\lambda_1^{-2} \leq T$, i.e, $\lambda_1 \geq T^{-\frac{1}{2}}$. Set $\mathcal{M}_{\mathbb{Q},2}^{\leq \frac{1}{2}\log T}[1]$ be the moduli space of rank 2 lattices Λ of volume 1 over $\mathbb{Q}$ whose sublattices Λ_1 of rank 1 have degrees $\leq \frac{1}{2}\log T$. As a direct consequence, we have the following

Proposition. (Geometric Truncation = Algebraic Truncation) *There is a natural one-to-one, onto morphism*

$$\mathcal{M}_{\mathbb{Q},2}^{\leq \frac{1}{2}\log T}[1] \simeq D_T.$$

In particular,

$$\mathcal{M}_{\mathbb{Q},2}^{\leq 0}[1] = \mathcal{M}_{\mathbb{Q},2}[1] \simeq D_1.$$

Consequently, we have the following

Example in Rank 2. $\xi_{\mathbb{Q},2}(s) = \frac{\xi(2s)}{s-1} - \frac{\xi(2s-1)}{s}.$

§18. Non-abelian *L*-functions: definitions

18.1. Automorphic forms and Eisenstein series

To faciliate our ensuing discussion, we make the following preparations. Here, as usual, instead of parabolic subgroups P, we adopt their Levi subgroups M as running symbols. For details, see e.g., [MW] and [W-1].

Fix a connected reduction group G defined over F, denote by Z_G its center. Fix a minimal parabolic subgroup P_0 of G. Then $P_0 = M_0 N_0$, where as usual we fix once and for all the Levi M_0 and the unipotent radical N_0. Recall that a parabolic subgroup P is G is called standard if $P \supset P_0$. For such groups, write $P = MN$ with $M_0 \subset M$ the standard Levi and N the unipotent radical. Denote by $\mathrm{Rat}(M)$ the group of rational characters of M, i.e, the morphism $M \to \mathbb{G}_m$ where $\mathbb{G}_m$ denotes the multiplicative group. Set $\mathfrak{a}_M^* := \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{C}$, $\mathfrak{a}_M := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{C})$, and $\mathrm{Re}\mathfrak{a}_M^* := \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R}$, $\mathrm{Re}\mathfrak{a}_M := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{R})$. For any $\chi \in \mathrm{Rat}(M)$, we obtain a (real) character $|\chi| : M(\mathbb{A}) \to \mathbb{R}^*$ defined by $m = (m_v) \mapsto m^{|\chi|} := \prod_{v \in S} |m_v|_v^{\chi_v}$ with $|\cdot|_v$ the v-absolute values. Set then $M(\mathbb{A})^1 := \cap_{\chi \in \mathrm{Rat}(M)} \mathrm{Ker}|\chi|$, which is a normal subgroup of $M(\mathbb{A})$. Set X_M to be the group of complex characters which are trivial on $M(\mathbb{A})^1$. Denote by $H_M := \log_M : M(\mathbb{A}) \to \mathfrak{a}_M$ the map such that $\forall \chi \in \mathrm{Rat}(M) \subset \mathfrak{a}_M^*, \langle \chi, \log_M(m) \rangle := \log(m^{|\chi|})$. Clearly, $M(\mathbb{A})^1 = \mathrm{Ker}(\log_M)$; $\log_M(M(\mathbb{A})/M(\mathbb{A})^1) \simeq \mathrm{Re}\mathfrak{a}_M$. Hence in particular there is a natural isomorphism $\kappa : \mathfrak{a}_M^* \simeq X_M$. Set $\mathrm{Re}X_M := \kappa(\mathrm{Re}\mathfrak{a}_M^*)$, $\mathrm{Im}X_M := \kappa(i \cdot \mathrm{Re}\mathfrak{a}_M^*)$. Moreover define our working space X_M^G to be the subgroup of X_M consisting of complex characters of $M(\mathbb{A})/M(\mathbb{A})^1$ which are trivial on $Z_{G(\mathbb{A})}$.

Fix a maximal compact subgroup K such that for all standard parabolic subgroups $P = MN$ as above, $P(\mathbb{A}) \cap K = M(\mathbb{A}) \cap K \cdot U(\mathbb{A}) \cap K$.

Hence we get the Langlands decomposition $G(\mathbb{A}) = M(\mathbb{A}) \cdot N(\mathbb{A}) \cdot K$. Denote by $m_P : G(\mathbb{A}) \to M(\mathbb{A})/M(\mathbb{A})^1$ the map $g = m \cdot n \cdot k \mapsto M(\mathbb{A})^1 \cdot m$ where $g \in G(\mathbb{A}), m \in M(\mathbb{A}), n \in N(\mathbb{A})$ and $k \in K$.

Fix Haar measures on $M_0(\mathbb{A}), N_0(\mathbb{A}), K$ respectively such that the induced measure on $N_0(F)$ is the counting measure and the volumes of $N(F)\backslash N_0(\mathbb{A})$ and K are all 1.

Such measures then also induce Haar measures via $\log_M$ to $\mathfrak{a}_{M_0}, \mathfrak{a}_{M_0}^*$, etc. Furthermore, if we denote by ρ_0 the half of the sum of the positive roots of the maximal split torus T_0 of the central Z_{M_0} of M_0, then $f \mapsto \int_{M_0(\mathbb{A}) \cdot N_0(\mathbb{A}) \cdot K} f(mnk) \, dk \, dn \, m^{-2\rho_0} dm$ defined for continuous functions with compact supports on $G(\mathbb{A})$ defines a Haar measure dg on $G(\mathbb{A})$. This in turn gives measures on $M(\mathbb{A}), N(\mathbb{A})$ and hence on $\mathfrak{a}_M, \mathfrak{a}_M^*$, $P(\mathbb{A})$, etc, for all parabolic subgroups P. In particular, the following compactibility condition

$$\int_{M_0(\mathbb{A}) \cdot N_0(\mathbb{A}) \cdot K} f(mnk) \, dk \, dn \, m^{-2\rho_0} dm$$

$$= \int_{M(\mathbb{A}) \cdot N(\mathbb{A}) \cdot K} f(mnk) \, dk \, dn \, m^{-2\rho_P} dm$$

holds for all continuous functions f with compact supports on $G(\mathbb{A})$, where ρ_P denotes the half of the sum of the positive roots of the maximal split torus T_P of the central Z_M of M. For later use, denote also by Δ_P the set of positive roots determined by (P, T_P) and $\Delta_0 = \Delta_{P_0}$.

Fix an isomorphism $T_0 \simeq \mathbb{G}_m^R$. Embed $\mathbb{R}_+^*$ by the map $t \mapsto (1; t)$. Then we obtain a natural injection $(\mathbb{R}_+^*)^R \hookrightarrow T_0(\mathbb{A})$ which splits. Denote by $A_{M_0(\mathbb{A})}$ the unique connected subgroup of $T_0(\mathbb{A})$ which projects onto $(\mathbb{R}_+^*)^R$. More generally, for a standard parabolic subgroup $P = MN$, set $A_{M(\mathbb{A})} := A_{M_0(\mathbb{A})} \cap Z_{M(\mathbb{A})}$ where as used above Z_* denotes the center of the group $*$. Clearly, $M(\mathbb{A}) = A_{M(\mathbb{A})} \cdot M(\mathbb{A})^1$. For later use, set also $A_{M(\mathbb{A})}^G := \{a \in A_{M(\mathbb{A})} : \log_G a = 0\}$. Then $A_{M(\mathbb{A})} = A_{G(\mathbb{A})} \oplus A_{M(\mathbb{A})}^G$.

Note that $K, M(F)\backslash M(\mathbb{A})^1$ and $N(F)\backslash N(\mathbb{A})$ are all compact, thus with the Langlands decomposition $G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K$ in mind, the reduction theory for $G(F)\backslash G(\mathbb{A})$ or more generally $P(F)\backslash G(\mathbb{A})$ is reduced to that for $A_{M(\mathbb{A})}$ since $Z_G(F) \cap Z_{G(\mathbb{A})}\backslash Z_{G(\mathbb{A})} \cap G(\mathbb{A})^1$ is compact as well. As such for $t_0 \in M_0(\mathbb{A})$ set $A_{M_0(\mathbb{A})}(t_0) := \{a \in A_{M_0(\mathbb{A})} : a^\alpha > t_0^\alpha \forall \alpha \in \Delta_0\}$. Then, for a fixed compact subset $\omega \subset P_0(\mathbb{A})$, we have the corresponding Siegel set $S(\omega; t_0) := \{p \cdot a \cdot k : p \in \omega, a \in A_{M_0(\mathbb{A})}(t_0), k \in K\}$. In particular, for big enough ω and small enough t_0, i.e, t_0^α is very close to 0 for all $\alpha \in \Delta_0$, the classical reduction theory may be restated as $G(\mathbb{A}) = G(F) \cdot S(\omega; t_0)$. More generally set

$A^P_{M_0(\mathbb{A})}(t_0) := \{a \in A_{M_0(\mathbb{A})} : a^\alpha > t_0^\alpha \forall \alpha \in \Delta_0^P\}$, and $S^P(\omega; t_0) := \{p \cdot a \cdot k : p \in \omega, a \in A^P_{M_0(\mathbb{A})}(t_0), k \in K\}$. Then similarly as above for big enough ω and small enough t_0, $G(\mathbb{A}) = P(F) \cdot S^P(\omega; t_0)$. (Here Δ_0^P denotes the set of positive roots for $(P_0 \cap M, T_0)$.)

Fix an embedding $i_G : G \hookrightarrow SL_n$ sending g to (g_{ij}). Introducing a hight function on $G(\mathbb{A})$ by setting $\|g\| := \prod_{v \in S} \sup\{|g_{ij}|_v : \forall i, j\}$. It is well-known that up to $O(1)$, hight functions are unique. This implies that the following growth conditions do not depend on the height function we choose.

A function $f : G(\mathbb{A}) \to \mathbb{C}$ is said to have *moderate growth* if there exist $c, r \in \mathbb{R}$ such that $|f(g)| \leq c \cdot \|g\|^r$ for all $g \in G(\mathbb{A})$. Similarly, for a standard parabolic subgroup $P = MN$, a function $f : N(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is said to have moderate growth if there exist $c, r \in \mathbb{R}, \lambda \in \mathrm{Re}X_{M_0}$ such that for any $a \in A_{M(\mathbb{A})}, k \in K, m \in M(\mathbb{A})^1 \cap S^P(\omega; t_0)$, $|f(amk)| \leq c \cdot \|a\|^r \cdot m_{P_0}(m)^\lambda$.

Also a function $f : G(\mathbb{A}) \to \mathbb{C}$ is said to be *smooth* if for any $g = g_f \cdot g_\infty \in G(\mathbb{A}_f) \times G(\mathbb{A}_\infty)$, there exist open neighborhoods V_* of g_* in $G(\mathbb{A})$ and a C^∞-function $f' : V_\infty \to \mathbb{C}$ such that $f(g'_f \cdot g'_\infty) = f'(g'_\infty)$ for all $g'_f \in V_f$ and $g'_\infty \in V_\infty$.

By contrast, a function $f : S(\omega; t_0) \to \mathbb{C}$ is said to be *rapidly decreasing* if there exists $r > 0$ and for all $\lambda \in \mathrm{Re}X_{M_0}$ there exists $c > 0$ such that for $a \in A_{M(\mathbb{A})}, g \in G(\mathbb{A})^1 \cap S(\omega; t_0)$, $|\phi(ag)| \leq c \cdot \|a\| \cdot m_{P_0}(g)^\lambda$. And a function $f : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is said to be rapidly decreasing if $f|_{S(\omega; t_0)}$ is so.

By definition, a function $\phi : N(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is called *automorphic* if
(i) ϕ has moderate growth;
(ii) ϕ is smooth;
(iii) ϕ is K-finite, i.e, the $\mathbb{C}$-span of all $\phi(k_1 \cdot * \cdot k_2)$ parametrized by $(k_1, k_2) \in K \times K$ is finite dimensional; and
(iv) ϕ is $\mathfrak{z}$-finite, i.e, the $\mathbb{C}$-span of all $\delta(X)\phi$ parametrized by all $X \in \mathfrak{z}$ is finite dimensional. Here $\mathfrak{z}$ denotes the center of the universal enveloping algebra $\mathfrak{u} := \mathfrak{U}(\mathrm{Lie}G(\mathbb{A}_\infty))$ of the Lie algebra of $G(\mathbb{A}_\infty)$ and $\delta(X)$ denotes the derivative of ϕ along X.

For automorphic function ϕ, set $\phi_k : M(F)\backslash M(\mathbb{A}) \to \mathbb{C}$ by $m \mapsto m^{-\rho_P}\phi(mk)$ for all $k \in K$. Then one checks that ϕ_k is an automorphic form in the usual sense. Set $A(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ be the space of automorphic forms on $N(\mathbb{A})M(F)\backslash G(\mathbb{A})$.

For a measurable locally L^1-function $f : N(F)\backslash G(\mathbb{A}) \to \mathbb{C}$, define its *constant term* along with the standard parabolic subgroup $P = NM$ to

be the function $f_P : N(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{C}$ given by $g \to \int_{N(F)\backslash G(\mathbb{A})} f(ng)dn$.
By definition, an automorphic form $\phi \in A(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is called
cuspidal if for any standard parabolci subgroup P' properly contained in
P, $\phi_{P'} \equiv 0$. Denote by $A_0(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ the space of cusp forms
on $N(\mathbb{A})M(F)\backslash G(\mathbb{A})$. Obviously, all cusp forms are rapidly decreasing.
Hence, there is a natural pairing

$$\langle \cdot, \cdot \rangle : A_0(N(\mathbb{A})M(F)\backslash G(\mathbb{A})) \times A(N(\mathbb{A})M(F)\backslash G(\mathbb{A})) \to \mathbb{C}$$

defined by

$$\langle \psi, \phi \rangle := \int_{Z_{M(\mathbb{A})}N(\mathbb{A})M(F)\backslash G(\mathbb{A})} \psi(g)\bar{\phi}(g)\, dg.$$

Moreover, for a (complex) character $\xi : Z_{M(\mathbb{A})} \to \mathbb{C}^*$, set

$$A(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi := \Big\{ \phi \in A(N(\mathbb{A})M(F)\backslash G(\mathbb{A})) :$$

$$\phi(zg) = z^{\rho_P} \cdot \xi(z) \cdot \phi(g), \forall z \in Z_{M(\mathbb{A})}, g \in G(\mathbb{A}) \Big\},$$

and $A_0(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi$ its subspace consisting of cusp forms.
    Set now

$$A_{(0)}(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_Z := \sum_{\xi \in \mathrm{Hom}(Z_{M(\mathbb{A})}, \mathbb{C}^*)} A_{(0)}(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi.$$

Then the natural morphism

$$\mathbb{C}[\mathrm{Re}\mathfrak{a}_M] \otimes A_{(0)}(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_Z \to \qquad A_{(0)}(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))$$

$$(Q, \phi) \mapsto \quad \big(g \mapsto Q(\log_M(m_P(g)))\big) \cdot \phi(g)$$

is an isomorphism.
    Let $\Pi_0(M(\mathbb{A}))_\xi$ be isomorphism classes of irreducible representa-
tions of $M(\mathbb{A})$ occuring in the space $A_0(M(F)\backslash M(\mathbb{A}))_\xi$, and

$$\Pi_0(M(\mathbb{A}) := \cup_{\xi \in \mathrm{Hom}(Z_{M(\mathbb{A})}, \mathbb{C}^*)} \Pi_0(M(\mathbb{A}))_\xi.$$

(In fact, we should use $M(\mathbb{A}_f) \times (M(\mathbb{A}) \cap K, \mathrm{Lie}(M(\mathbb{A}_\infty)) \otimes_{\mathbb{R}} \mathbb{C}))$ instead
of $M(\mathbb{A})$.) For any $\pi \in \Pi_0(M(\mathbb{A}))_\xi$, set $A_0(M(F)\backslash M(\mathbb{A})_\pi$ to be the
isotypic component of type π of $A_0(M(F)\backslash M(\mathbb{A}))_\xi$, i.e, the set of cusp
forms of $M(\mathbb{A})$ generating a semi-simple isotypic $M(\mathbb{A}_f) \times (M(\mathbb{A}) \cap$
$K, \mathrm{Lie}(M(\mathbb{A}_\infty)) \otimes_{\mathbb{R}} \mathbb{C}))$-module of type π. Set

$$A_0(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi := \Big\{ \phi \in A_0(N(\mathbb{A})M(F)\backslash G(\mathbb{A})) :$$

$$\phi_k \in A_0(M(F)\backslash M(\mathbb{A}))_\pi, \forall k \in K \Big\}.$$

It is quite clear that

$$A_0(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi = \oplus_{\pi\in\Pi_0(M(\mathbb{A}))_\xi} A_0(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi.$$

More generally, let $V \subset A(M(F)\backslash M(\mathbb{A}))$ be an irreducible $M(\mathbb{A}_f) \times (M(\mathbb{A})\cap K, \mathrm{Lie}(M(\mathbb{A}_\infty)) \otimes_\mathbb{R} \mathbb{C}))$-module with π_0 the induced representation of $M(\mathbb{A}_f) \times (M(\mathbb{A})\cap K, \mathrm{Lie}(M(\mathbb{A}_\infty)) \otimes_\mathbb{R} \mathbb{C}))$. Then we call π_0 an automorphic representation of $M(\mathbb{A})$. Denote by $A(M(F)\backslash M(\mathbb{A}))_{\pi_0}$ the isotypic subquotient module of type π_0 of $A(M(F)\backslash M(\mathbb{A}))$. One checks that

$$V \otimes \mathrm{Hom}_{M(\mathbb{A}_f)\times(M(\mathbb{A})\cap K, \mathrm{Lie}(M(\mathbb{A}_\infty))\otimes_\mathbb{R}\mathbb{C}))}(V, A(M(F)\backslash M(\mathbb{A})))$$
$$\simeq A(M(F)\backslash M(\mathbb{A}))_{\pi_0}.$$

Set

$$A(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_{\pi_0} := \Big\{ \phi \in A(N(\mathbb{A})M(F)\backslash G(\mathbb{A})) :$$
$$\phi_k \in A(M(F)\backslash M(\mathbb{A}))_{\pi_0}, \forall k \in K \Big\}.$$

Moreover if $A(M(F)\backslash M(\mathbb{A}))_{\pi_0} \subset A_0(M(F)\backslash M(\mathbb{A}))$, we call π_0 cuspidal.

Automorphic representations π and π_0 of $M(\mathbb{A})$ are said to be equivalent if $\pi \simeq \pi_0 \otimes \lambda$ for some $\lambda \in X_M^G$. This, in practice, means that $A(M(F)\backslash M(\mathbb{A}))_\pi = \lambda \cdot A(M(F)\backslash M(\mathbb{A}))_{\pi_0}$. Consequently,

$$A(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi = (\lambda \circ m_P) \cdot A(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_{\pi_0}.$$

Denote by $\mathfrak{P} := [\pi_0]$ the equivalence class of π_0. Then $\mathfrak{P}$ is an X_M^G-principal homogeneous space, hence admits a natural complex structure. Usually we call $(M, \mathfrak{P})$ a cuspidal datum of G if π_0 is cuspidal. Also for $\pi \in \mathfrak{P}$ set $\mathrm{Re}\pi := \mathrm{Re}\chi_\pi = |\chi_\pi| \in \mathrm{Re}X_M$, where χ_π is the central character of π, and $\mathrm{Im}\pi := \pi \otimes (-\mathrm{Re}\pi)$.

For $\phi \in A(N(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi$ with π an irreducible automorphic representation of $M(\mathbb{A})$, define the associated *Eisenstein series* $E(\phi, \pi)$: $G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ by

$$E(\phi, \pi)(g) := \sum_{\delta \in P(F)\backslash G(F)} \phi(\delta g).$$

Then there is an open cone $\mathcal{C} \subset \mathrm{Re}X_M^G$ such that if $\mathrm{Re}\pi \in \mathcal{C}$, $E(\lambda \cdot \phi, \pi \otimes \lambda)(g)$ converges uniformly for g in a compact subset of $G(\mathbb{A})$ and λ in an open neighborhood of 0 in X_M^G. For example, if $\mathfrak{P} = [\pi]$ is cuspidal, we may even take $\mathcal{C}$ to be the cone $\{\lambda \in \mathrm{Re}X_M^G : \langle \lambda - \rho_P, \alpha^\vee \rangle > 0, \forall \alpha \in \Delta_P^G\}$. As a direct consequence, then $E(\phi, \pi) \in A(G(F)\backslash G(\mathbb{A}))$ is an automorphic form.

18.2.   Non-abelian L-functions

Being automorphic forms, Eisenstein series are of moderate growth. Consequently, they are not integrable over $G(F)\backslash G(\mathbb{A})^1$ in general. On the other hand, Eisenstein series are also smooth and hence integrable over compact subsets of $G(F)\backslash G(\mathbb{A})^1$. So it is very natural for us to search for compact domains which are intrinsically defined.

As such, let us now return to the group $G = GL_r$. Then, we obtain compact moduli spaces

$$\mathcal{M}_{F,r}^{\leq p}[\Delta_F^{\frac{r}{2}}] := \left\{ g \in GL_r(F)\backslash GL_r(\mathbb{A}) : \deg g = 0, \bar{p}^g \leq p \right\}$$

for a fixed convex polygon $p : [0,r] \to \mathbb{R}$. For example, $\mathcal{M}_{\mathbb{Q},r}^{\leq 0}[1] = \mathcal{M}_{\mathbb{Q},r}[1]$, (the adelic inverse image of) the moduli space of rank r semi-stable $\mathbb{Z}$-lattices of volume 1.

More generally, for the standard parabolic subgroup P of GL_r, we introduce the moduli spaces

$$\mathcal{M}_{F,r}^{P;\leq p}[\Delta_F^{\frac{r}{2}}] := \left\{ g \in P(F)\backslash GL_r(\mathbb{A}) : \deg g = 0, \bar{p}_P^g \leq p, \bar{p}_P^g \geq -p \right\}.$$

One checks that these moduli spaces $\mathcal{M}_{F,r}^{P;\leq p}[\Delta_F^{\frac{r}{2}}]$ are all compact.

As usual, we fix the minimal parabolic subgroup P_0 corresponding to the partition $(1, \cdots, 1)$ with M_0 consisting of diagonal matrices. Then $P = P_I = N_I M_I$ corresponds to a certain partition $I = (r_1, \cdots, r_{|P|})$ of r with M_I the standard Levi and N_I the unipotent radical.

Now for a fixed irreducible automorphic representation π of $M_I(\mathbb{A})$, choose

$$\phi \in A(N_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))_\pi \cap L^2(N_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))$$
$$:= A^2(N_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))_\pi,$$

with $L^2(N_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))$ the space of L^2 functions on the space $Z_{G(\mathbb{A})}N_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A})$. Denote the associated Eisenstein series by $E(\phi, \pi) \in A(G(F)\backslash G(\mathbb{A}))$.

Definition. (Weng) *The rank r non-abelian L-function $L_{F,r}^{\leq p}(\phi, \pi)$ associated to the L^2-automorphic form $\phi \in A^2(N_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))_\pi$ for the number field F is defined by the following integration*

$$L_{F,r}^{\leq p}(\phi, \pi) := \int_{\mathcal{M}_{F,r}^{\leq p}[\Delta_F^{\frac{r}{2}}]} E(\phi, \pi)(g)\, dg, \qquad \operatorname{Re} \pi \in \mathcal{C}.$$

More generally, for any standard parabolic subgroup $P_J = N_J M_J \supset P_I$ (so that the partition J is a refinement of I), we obtain a relative

Eisenstein series

$$E_I^J(\phi, \pi)(g) := \sum_{\delta \in P_I(F) \backslash P_J(F)} \phi(\delta g), \qquad \forall g \in P_J(F) \backslash G(\mathbb{A}).$$

There is an open cone $\mathcal{C}_I^J$ in $\mathrm{Re} X_{M_I}^{P_J}$ s.t. if $\mathrm{Re}\pi \in \mathcal{C}_I^J$, then $E_I^J(\phi, \pi) \in A(P_J(F) \backslash G(\mathbb{A}))$, where $X_{M_I}^{P_J}$ is defined similarly as X_M^G with G replaced by P_J. As such, we are able to define the associated non-abelian L-function by

$$L_{F,r}^{P_J; \leq p}(\phi, \pi) := \int_{\mathcal{M}_{F,r}^{P_J; \leq p}[\Delta_F^{\frac{r}{2}}]} E_I^J(\phi, \pi)(g) \, dg, \qquad \mathrm{Re}\pi \in \mathcal{C}_I^J.$$

Remark. Here when defining non-abelian L-functions we assume that ϕ comes from a single irreducible automorphic representations. But this restriction is rather artifical and can be removed easily: such a restriction only serves the purpose of giving the constructions and results in a very neat form.

§19. Basic properties of non-abelian L-functions

19.1. Meromorphic extension and functional equations

With the same notation as above, set $\mathfrak{P} = [\pi]$. For $w \in W$ the Weyl group of $G = GL_r$, fix once and for all representative $w \in G(F)$ of w. Set $M' := wMw^{-1}$ and denote the associated parabolic subgroup by $P' = N'M'$. W acts naturally on the automorphic representations, from which we obtain an equivalence classes $w\mathfrak{P}$ of automorphic representations of $M'(\mathbb{A})$. As usual, define the associated *intertwining operator* $M(w, \pi)$ by

$$(M(w, \pi)\phi)(g) := \int_{N'(F) \cap wN(F)w^{-1} \backslash N'(\mathbb{A})} \phi(w^{-1}n'g) dn', \qquad \forall g \in G(\mathbb{A}).$$

One checks that if $\langle \mathrm{Re}\pi, \alpha^\vee \rangle \gg 0, \forall \alpha \in \Delta_P^G$,
(i) for a fixed ϕ, $M(w, \pi)\phi$ depends only on the double coset $M'(F)wM(F)$. So $M(w, \pi)\phi$ is well-defined for $w \in W$;
(ii) the above integral converges absolutely and uniformly for g varying in a compact subset of $G(\mathbb{A})$;
(iii) $M(w, \pi)\phi \in A(N'(\mathbb{A})M'(F) \backslash G(\mathbb{A}))_{w\pi}$; and if ϕ is L^2, which from now on we always assume, so is $M(w, \pi)\phi$.

Basic Facts of Non-Abelian L-Functions. (Langlands, Weng)
● (**Meromorphic Continuation**) $L_{F,r}^{\leq p}(\phi, \pi)$ *for* $\mathrm{Re}\pi \in \mathcal{C}$ *is well-defined and admits a unique meromorphic continuation to the whole space* $\mathfrak{P}$;

• (**Functional Equation**) *As meromorphic functions on* $\mathfrak{P}$,

$$L_{F,r}^{\leq p}(\phi, \pi) = L_{F,r}^{\leq p}(M(w, \pi)\phi, w\pi), \qquad \forall w \in W.$$

This is a direct consequence of the fundamental results of Langlands on Eisenstein series and spectrum decompositions and explains why only L^2-automorphic forms are used in the definition of non-abelian Ls. (See e.g, [Ar1], [La1], [MW] and/or [W2]).

19.2. Holomorphicity and singularities

Let $\pi \in \mathfrak{P}$ and $\alpha \in \Delta_M^G$. Define the function $h : \mathfrak{P} \to \mathbb{C}$ by $\pi \otimes \lambda \mapsto \langle \lambda, \alpha^\vee \rangle, \forall \lambda \in X_M^G \simeq \mathfrak{a}_M^G$. Here as usual, $\alpha^\vee$ denotes the coroot associated to α. Set $H := \{\pi' \in \mathfrak{P} : h(\pi') = 0\}$ and call it a root hyperplane. Clearly the function h is determined by H, hence we also denote h by h_H. Note also that root hyperplanes depend on the base point π we choose.

Let D be a set of root hyperplanes. Then
(i) the singularities of a meromorphic function f on $\mathfrak{P}$ is said to be supported by D if for all $\pi \in \mathfrak{P}$, there exist $n_\pi : D \to \mathbb{Z}_{\geq 0}$ zero almost everywhere such that $\pi' \mapsto \left(\Pi_{H \in D} h_H(\pi')^{n_\pi(H)}\right) \cdot f(\pi')$ is holomorphic at π';
(ii) the singularities of f are said to be without multiplicity at π if $n_\pi \in \{0, 1\}$;
(iii) D is said to be locally finite, if for any compact subset $C \subset \mathfrak{P}$, $\{H \in D : H \cap C \neq \emptyset\}$ is finite.

Basic Facts of Non-Abelian L-Functions. (Langlands, Weng)
• (**Holomorphicity**) (i) *When* $\mathrm{Re}\pi \in \mathcal{C}$, $L_{F,r}^{\leq p}(\phi, \pi)$ *is holomorphic;*
(ii) $L_{F,r}^{\leq p}(\phi, \pi)$ *is holomorphic at* π *where* $\mathrm{Re}\pi = 0$;
• (**Singularities**) *Assume further that* ϕ *is a cusp form. Then*
(i) *There is a locally finite set of root hyperplanes D such that the singularities of* $L_{F,r}^{\leq p}(\phi, \pi)$ *are supported by D;*
(ii) *Singularities of* $L_{F,r}^{\leq p}(\phi, \pi)$ *are without multiplicities at* π *if* $\langle \mathrm{Re}\pi, \alpha^\vee \rangle \geq 0, \forall \alpha \in \Delta_M^G$;
(iii) *There are only finitely many of singular hyperplanes of* $L_{F,r}^{\leq p}(\phi, \pi)$ *which intersect* $\{\pi \in \mathfrak{P} : \langle \mathrm{Re}\pi, \alpha^\vee \rangle \geq 0, \forall \alpha \in \Delta_M\}$.
As above, this is a direct consequence of the fundamental results of Langlands on Eisenstein series and spectrum decompositions. (See e.g, [Ar1], [La1], [MW] and/or [W2]).

Chapter VIII. Symmetries and the Riemann Hypothesis

§20. Abelian parts of high rank zetas

20.1. Analytic studies of high rank zetas

Associated to a number field F is the genuine high rank zeta function $\xi_{F,r}(s)$ for every fixed $r \in \mathbb{Z}_{>0}$. Being natural generalizations of (completed) Dedekind zeta functions, these functions satisfy canonical properties for zetas as well. Namely, they admit meromorphic continuations to the whole complex s-plane, satisfy the functional equation $\xi_{F,r}(1-s) = \xi_{F,r}(s)$ and have only two singularities, all simple poles, at $s = 0, 1$. Moreover, we expect that the Riemann Hypothesis holds for all zetas $\xi_{F,r}(s)$, namely, all zeros of $\xi_{F,r}(s)$ lie on the central line $\mathrm{Re}(s) = \frac{1}{2}$.

Recall that $\xi_{F,r}(s)$ is defined by

$$\xi_{F,r}(s) := \left(|\Delta_F|\right)^{\frac{rs}{2}} \int_{\mathcal{M}_{F,r}} \left(e^{h^0(F,\Lambda)} - 1\right) \cdot \left(e^{-s}\right)^{\deg(\Lambda)} d\mu(\Lambda), \ \mathrm{Re}(s) > 1$$

where Δ_F denotes the discriminant of F, $\mathcal{M}_{F,r}$ the moduli space of semi-stable $\mathcal{O}_F$-lattices of rank r (here $\mathcal{O}_F$ denotes the ring of integers), $h^0(F,\Lambda)$ and $\deg(\Lambda)$ denote the 0-th geo-arithmetic cohomology and the Arakelov degree of the lattice Λ, respectively, and $d\mu(\Lambda)$ a certain Tamagawa type measure on $\mathcal{M}_{F,r}$. Defined using high rank lattices, these zetas then are expected to be naturally related with non-abelian aspects of number fields.

On the other hand, algebraic groups associated to $\mathcal{O}_F$-lattices are general linear group GL and special linear group SL. A natural question then is whether principal lattices associated to other reductive groups G and their associated zeta functions can be introduced and studied.

While arithmetic approach using stability seems to be complicated, analytic one using analytic truncation is ready to be exposed. To start with, let us go back to high rank zetas. For simplicity, take F to be the field $\mathbb{Q}$ of rationals. Then, via a Mellin transform, high rank zeta $\xi_{\mathbb{Q},r}(s)$ can be written as

$$\xi_{\mathbb{Q},r}(s) = \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \widehat{E}(\Lambda, s) \, d\mu(\Lambda), \quad \mathrm{Re}(s) > 1,$$

where $\mathcal{M}_{\mathbb{Q},r}[1]$ denotes the moduli space of $\mathbb{Z}$-lattices of rank r and volume 1 and $\widehat{E}(\Lambda, s)$ the completed Epstein zeta functions associated to Λ. Recall that the moduli space $\mathcal{M}_{\mathbb{Q},r}[1]$ may be viewed as a compact subset in $SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R}) / SO(r)$ and Epstein zeta functions may be

written as the relative Eisenstein series $E^{SL(r)/P_{r-1,1}}(\mathbf{1}; s; g)$ associated to the constant function $\mathbf{1}$ on the maximal parabolic subgroup $P_{r-1,1}$ corresponding to the partition $r = (r-1) + 1$ of $SL(r)$, we have

$$\frac{2}{r} \cdot \xi_{\mathbb{Q},r}\left(\frac{2}{r} \cdot s\right) = \int_{\mathcal{M}_{\mathbb{Q},r}[1] \subset SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R})/SO(r)} \widehat{E}(\Lambda, s) \, d\mu(g)$$

$$= \int_{SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R})/SO(r)} \mathbf{1}_{\mathcal{M}_{\mathbb{Q},r}[1]}(g) \cdot \widehat{E}(\mathbf{1}; s; g) \, d\mu(g)$$

where $\mathbf{1}_{\mathcal{M}_{\mathbb{Q},r}[1]}(g)$ denotes the characteristic function of the compact subset $\mathcal{M}_{\mathbb{Q},r}[1]$.

Recall also that, in parallel, to remedy the divergence of integration

$$\int_{SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R})/SO(r)} \widehat{E}(\mathbf{1}; s; g) \, d\mu(g),$$

in theories of automorphic forms and trace formula, Rankin, Selberg and Arthur introduced an analytic truncation for smooth functions $\phi(g)$ over $SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R})/SO(r)$. Simply put, Arthur's analytic truncation is a device to get rapidly decreasing functions from slowly increasing functions by cutting off slow growth parts near all types of cusps uniformly. Being truncations near cusps, a rather large, or better, sufficiently regular, new parameter T must be introduced. In particular, when applying to Eisenstein series $\widehat{E}(\mathbf{1}; s; g)$ and to $\mathbf{1}$ on $SL(r,\mathbb{R})$, we get the truncated function $\Lambda^T \widehat{E}(\mathbf{1}; s; g)$ and $(\Lambda^T \mathbf{1})(g)$, respectively. Consequently, by using basic properties on Arthur's truncation, we obtain the following well-defined integrations

$$\int_{SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R})/SO(r)} \Lambda^T \widehat{E}(\mathbf{1}; s; g) \, d\mu(g)$$

$$= \int_{SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R})/SO(r)} (\Lambda^T \mathbf{1})(g) \cdot \widehat{E}(\mathbf{1}; s; g) \, d\mu(g)$$

$$= \int_{\mathfrak{F}(T) \subset SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R})/SO(r)} \widehat{E}(\mathbf{1}; s; g) \, d\mu(g)$$

where $\mathfrak{F}(T)$ is the compact subset in (a fundamental domain for the quotient space) $SL(r,\mathbb{Z}) \backslash SL(r,\mathbb{R})/SO(r)$ whose characteristic function is given by $(\Lambda^T \mathbf{1})(g)$.

20.2. Advanced Rankin–Selberg and Zagier methods

As such, we find an analytic way to understand our high rank zetas, provided that the above analytic discussion for sufficiently positive parameter T can be further strengthened so as to work for smaller T, in

particular, for $T = 0$, as well. In general, it is very difficult. Fortunately, as recalled in the previous two chapters, in the case of SL, this can be achieved based on an intrinsic geo-arithmetic result, called the Micro-Global Bridge, an analogue of the following basic principle in Geometric Invariant Theory for unstability: A point is not stable, then there is a parabolic subgroup which destroys the stability. Consequently, we have

$$\frac{2}{r} \cdot \xi_{\mathbb{Q},r}\left(\frac{2}{r} \cdot s\right) = \left(\int_{G(\mathbb{Z}) \backslash G(\mathbb{R})/K} \Lambda^T \widehat{E}(\mathbf{1}; s; g) \, d\mu(g) \right)\Big|_{T=0}.$$

This then leads to evaluation of the special Eisenstein periods

$$\int_{G(\mathbb{Z}) \backslash G(\mathbb{R})/K} \Lambda^T \widehat{E}(\mathbf{1}; s; g) \, d\mu(g),$$

and more generally the evaluation of *Eisenstein periods*

$$\int_{G(\mathbb{Z}) \backslash G(\mathbb{R})/K} \Lambda^T E(\phi; \lambda; g) \, d\mu(g),$$

where K a certain maximal compact subgroup of a reductive group G, ϕ is a P-level automorphic forms with P parabolic, and $E(\phi; \lambda; g)$ the relative Eisenstein series from P to G associated to a P-level L^2 form ϕ.

Unfortunately, in general, it is quite difficult to find a close formula for Eisenstein periods. But, when ϕ is cuspidal, then the corresponding Eisenstein period can be calculated, thanks to the work of [JLR] and [W4], an advanced version of Rankin–Selberg & Zagier method.

20.3. Discovery of maximal parabolics: SL, Sp and G_2

Back to high rank zeta functions, the bad news is that this powerful calculation cannot be applied directly, since in the specific Eisenstein series, i.e., the classical Epstein zeta, used, the function $\mathbf{1}$, corresponding to ϕ in general picture, on the maximal parabolic $P_{r-1,1}$ is only L^2, far from being cuspidal. To overcome this technical difficulty, we, partially also motivated by our earlier work on the so-called abelian part of high rank zeta functions ([W2,4]) and Venkov's trace formula for $SL(3)$ ([Ve]), introduce Eisenstein series $E^{G/B}(\mathbf{1}; \lambda; g)$ associated to the constant function $\mathbf{1}$ on $B = P_{1,1,\ldots,1}$, the Borel, into our study, since
1) being over the Borel, the constant function $\mathbf{1}$ is cuspidal. So the associated Eisenstein period $\omega_{\mathbb{Q}}^{G;T}(\lambda)$ can be evaluated following [JLR]/[W4]; and
2) $E(\mathbf{1}; s; g)$ used in high rank zetas can be realized as residues of $E^{G/B}(\mathbf{1}; \lambda; g)$ along with suitable singular hyper-planes, a result essentially due to Siegel and Langlands, but carried out by Diehl ([D]).

In particular, for 1), we now know that

$$\omega_{\mathbb{Q}}^{G;T}(\lambda) = \sum_{w \in W} \left(\frac{e^{\langle w\lambda - \rho, T\rangle}}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee \rangle} \cdot \prod_{\alpha > 0, w\alpha < 0} \frac{\xi_{\mathbb{Q}}(\langle \lambda, \alpha^\vee \rangle)}{\xi_{\mathbb{Q}}(\langle \lambda, \alpha^\vee \rangle + 1)} \right).$$

Here W denotes the associated Weyl group, Δ_0 the collection of simple roots, $\rho := \frac{1}{2} \sum_{\alpha > 0} \alpha$, and $\alpha^\vee$ the co-root associated to α.

With all this, it is clear that to get genuine zetas associated to reductive groups G, it may be more economical to use the period $\omega_{\mathbb{Q}}^G(\lambda)$ defined by

$$\omega_F^G(\lambda) := \sum_{w \in W} \left(\frac{1}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee \rangle} \cdot \prod_{\alpha > 0, w\alpha < 0} \frac{\xi_F(\langle \lambda, \alpha^\vee \rangle)}{\xi_F(\langle \lambda, \alpha^\vee \rangle + 1)} \right)$$

which make sense for all reductive groups G defined over F. Here as usual, $\xi_F(s)$ denotes the completed Dedekind zeta function of F.

Back to the field of rationals. The period $\omega_{\mathbb{Q}}^G(\lambda)$ of G over $\mathbb{Q}$ is of rank(G) variables. To get a single variable zeta out from it, we need to take residues along with rank$(G) - 1$ (linearly independent) singular hyper-planes. So proper choices for singular spaces should be made. This is done for SL and Sp in [W7], thanks to Diehl's paper ([D]). (In fact, Diehl dealt with Sp only. But due to the fact that positive definite matrices are naturally associated to $\mathbb{Z}$-lattices and Siegel upper spaces, SL can be also treated successfully with a bit extra care.) Simply put, for each $G = SL(r)$ (or $= Sp(2n)$), within the framework of classical Eisenstein series, there exists *only one* choice of rank$(G) - 1$ singular hyper-planes $H_1 = 0, H_2 = 0, \ldots, H_{\text{rank}(G)-1} = 0$. Moreover, after taking residues along with them, that is,

$$\text{Res}_{H_1=0, H_2=0, \ldots, H_{\text{rank}(G)-1}=0} \, \omega_{\mathbb{Q}}^G(\lambda),$$

with suitable normalizations, we can get a new zeta $\xi_{G;\mathbb{Q}}(s)$ for G.

At this point, the role played in new zetas $\xi_{G;\mathbb{Q}}(s)$ by maximal parabolic subgroups has not yet emerged. It is only after the study done for G_2 that we understand such a key role. Nevertheless, what we do observe from these discussions on SL and Sp is the follows: all singular hyper-planes are taken from only a single term appeared in the period $\omega_{\mathbb{Q}}^G(\lambda)$. More precisely, the term corresponding to $w = \text{Id}$, the Weyl element Identity. In other words, singular hyper-planes are taken from the denominator of the expression

$$\frac{1}{\prod_{\alpha \in \Delta_0} \langle \lambda - \rho, \alpha^\vee \rangle}.$$

(Totally, there are rank(G) factors, among which we have carefully chosen rank(G) -1 for $G = SL$, Sp.) In particular, for the exceptional G_2, being a rank two group and hence an obvious choice for our next test, this reads as

$$\frac{1}{\langle \lambda - \rho, \alpha_{\text{short}}^\vee \rangle \cdot \langle \lambda - \rho, \alpha_{\text{long}}^\vee \rangle}$$

where α_{short}, α_{long} denote the short and long roots of G_2 respectively. So two possibilities,

a) $\operatorname{Res}_{\langle \lambda - \rho, \alpha_{\text{short}}^\vee \rangle = 0} \omega_{\mathbb{Q}}^{G_2}(\lambda)$, leading to $\xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(s)$ after suitable normalization; and

b) $\operatorname{Res}_{\langle \lambda - \rho, \alpha_{\text{long}}^\vee \rangle = 0} \omega_{\mathbb{Q}}^{G_2}(\lambda)$, leading to $\xi_{\mathbb{Q}}^{G_2/P_{\text{short}}}(s)$ after suitable normalization.

With this, by the fact that there exists a natural one-to-one and onto correspondence between collection of conjugation classes of maximal parabolic groups and simple roots, we are able to detect in [W7] the crucial role played by maximal parabolic subgroups and hence are able to offer the proper definition for the genuine zetas associated to pairs of reductive groups and their maximal parabolic subgroups.

§21. Abelian zetas for (G, P)

21.1. Definition

Motivated by the above discussion, we can introduce a genuine abelian zeta function for pairs (G, P) defined over number fields, consisting of reductive groups G and their maximal reductive groups. As the details is explained in [W7] collected in this volume, we here only sketch key features of such zetas.

Thus let G be a reductive group and P a maximal parabolic subgroup of G both defined over $\mathbb{Q}$. Denote by Δ_0 the collection of simple roots. For any root α denotes by $\alpha^\vee$ the corresponding co-root and $\rho := \frac{1}{2} \sum_{\alpha > 0} \alpha$. Denote by W the associated Weyl group. The for any λ in a suitable positive chamber of the root space, define the abelian zeta function associated to (G, P) over $\mathbb{Q}$ by

$$\xi_{\mathbb{Q}}^{G/P}(s) := \operatorname{Norm}\left[\operatorname{Res}_{\langle \lambda - \rho, \alpha^\vee \rangle = 0, \alpha \in \Delta_0 \setminus \{\alpha_P\}} \omega_{\mathbb{Q}}^G(\lambda) \right)\Big]$$

where as above,

$$\omega_{\mathbb{Q}}^G(\lambda) := \sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee \rangle} \cdot \prod_{\alpha > 0, w\alpha < 0} \frac{\xi_{\mathbb{Q}}(\langle \lambda, \alpha^\vee \rangle)}{\xi_{\mathbb{Q}}(\langle \lambda, \alpha^\vee \rangle + 1)},$$

α_P denotes the unique simple root corresponding to the maximal parabolic subgroup P, $s := \langle \lambda - \rho, \alpha_P^\vee \rangle$, and Norm means a certain normalization, the details of which may be found in [W7].

21.2. Conjectural FE and the RH

As such, then easily, $\xi_{\mathbb{Q}}^{G/P}(s)$ is a well-defined meromorphic function on the whole complex s-plane. And strikingly, the structures of all this zetas can be summarized by the following

Main Conjecture. (i) (**Functional Equation**)

$$\xi_{\mathbb{Q}}^{G/P}(1 - s) = \xi_{\mathbb{Q}}^{G/P}(s);$$

(ii) (**The Riemann Hypothesis**)

$$\xi_{\mathbb{Q}}^{G/P}(s) = 0 \text{ implies that } \mathrm{Re}(s) = \tfrac{1}{2}.$$

Remarks. (i) Funational equation is checked in [W7] for 10 examples listed in the appendix there, namely for the groups $SL(2, 3, 4, 5), Sp(4)$ and G_2; More generally, in April 2008, Henry Kim in a joint effort with the author obtained a proof of the functional equation for $\xi_{\mathbb{Q}}^{SL(n)/P_{n-1,1}}(s)$ ([KW2]); Independently, in June, 2009, Yasushi Komori ([Ko]) found an elegant proof of the functional equation for all zetas $\xi_{\mathbb{Q}}^{G/P}(s)$:

Functional Equation. *For zeta functions* $\xi_{\mathbb{Q}}^{G/P}(s)$, *we have*

$$\xi_{\mathbb{Q}}^{G/P}(1 - s) = \xi_{\mathbb{Q}}^{G/P}(s).$$

(ii) Based on symmetries, the Riemann Hypothesis for the above 10 examples is solved partially by J. Lagarias–M. Suzuki, Suzuki, and fully by H. Ki. Ki's method is expected to have more applications. For details, please go to ([LS], [Su1,2], [SW], [Ki1,2]).

§22. Abelian parts of high rank zetas

In a certain sense, $\xi_{\mathbb{Q}}^{SL(r)/P_{r-1,1}}(s)$ may be viewed as an abelian part of the high rank zeta $\xi_{\mathbb{Q},r}(s)$, since it is naturally related to the so-called constant terms of the Eisenstein series $E^{SL/B}(\mathbf{1}; \lambda; g)$. Formally, starting from Eisenstein series $E^{G/B}(\mathbf{1}; \lambda; g)$, we can get high rank zeta functions by first taking the residues along suitable singular hyperplanes, then taking integration over moduli spaces of semi-stable lattices. That is to say, $\xi_{\mathbb{Q},r}(s)$ corresponds to (Res $\to \int$)-ordered construction. In this sense, the zeta function $\xi_{SL(r),\mathbb{Q}}(s)$ corresponds to ($\int \to$ Res)-ordered construction.

Since there is no needs to take residues, for $SL(2)$, we have $\xi_{\mathbb{Q},2}(s) = \xi_{\mathbb{Q}}^{SL(2)/P_{1,1}}(s)$. However, in general, there is a discrepancy between $\xi_{\mathbb{Q},r}(s)$ and $\xi_{\mathbb{Q}}^{SL(r)/P_{r-1,1}}(s)$, because of the obstruction for the exchanging of $\int$ and Res.

Remarks. (i) Non-abelain zetas were essentially introduced around 2000. Contrary to the publishing order, the zetas for number fields was first introduced, and it was for the purpose to get some concrete feelings that we started our examples with function fields;
(ii) There are a few flaws in our works on the zeta associated to SL(3) in the final chapter of [W2]. More precisely, what we have done there is the abelian zeta $\xi_{\mathbb{Q}}^{SL(3)/P_{2,1}}(s)$, instead of the non-abelian rank 3 zeta $\xi_{\mathbb{Q},3}(s)$; Moreover, there are sign mistakes in the formula for $\xi_{\mathbb{Q}}^{SL(3)/P_{2,1}}(s)$. The right one should be

(1)
$$
\begin{aligned}
\xi_{\mathbb{Q}}^{SL(3)/P_{2,1}}(s) = {} & \xi_{\mathbb{Q}}(2) \cdot \frac{1}{3s-3} \cdot \xi_{\mathbb{Q}}(3s) \\
& - \xi_{\mathbb{Q}}(2) \cdot \frac{1}{3s} \cdot \xi_{\mathbb{Q}}(3s-2) \\
& - \frac{1}{3} \cdot \frac{1}{3s-3} \cdot \xi_{\mathbb{Q}}(3s-1) \\
& + \frac{1}{3} \cdot \frac{1}{3s} \cdot \xi_{\mathbb{Q}}(3s-1) \\
& + \frac{1}{2} \cdot \frac{1}{3s-1} \cdot \xi_{\mathbb{Q}}(3s-2) \\
& - \frac{1}{2} \cdot \frac{1}{3s-2} \cdot \xi_{\mathbb{Q}}(3s).
\end{aligned}
$$

(iii) Combinatorial techniques used by Arthur for reduction theory and analytic truncations are discussed in details in our preprint (arXiv:math/ 0505016). But we remind the reader that τ, the characteristic function in §13.4, does not work well for analytic truncations.

Part C. General CFT and stability

In this part, we will propose a general CFT for p-adic number fields using stability of what we call filtered $(\varphi, N; \omega)$-modules, built on Fontaine's theory of p-adic Galois representations. The key points are

1) (Fontaine‖Berger) p-adic monodromy theorem for p-adic representations which claims that a de Rham representation is a potentially semi-stable representation;
2) (Fontaine‖Fontaine, Colmez–Fontaine) characterization of semi-stable representations in terms of weakly admissible filtered (φ, N)-modules;
3) a notion of ω-structures measuring (higher) ramifications of de Rham representations;
4) a conjectural Micro Reciprocity Law, characterizing de Rham representations in terms of semi-stable filtered $(\varphi, N; \omega)$-modules of slope zero.

Chapter IX. l-adic representations for p-adic fields

§23.   Finite monodromy and nilpotency

23.1.   Absolute Galois group and its pro-l structures

Let K be a p-adic number field, i.e., a finite extension of $\mathbb{Q}_p$. Denote by k its residue field. Fix an algebraic closure $\overline{K}$ of K. Let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K with I_K its inertial subgroup and P_K its wild ramification group. Then from the theory of local fields, we have the following structural exact sequences

$$1 \to I_K \to G_K \to G_k \to 1 \quad \text{and} \quad 1 \to P_K \to I_K \to \prod_{l(\neq p)} \mathbb{Z}_l(1) \to 1.$$

With its application to l-adic representation in mind, let us fix a prime $l \neq p$. To avoid the pro-l part systematically, define $P_{K,l}$ to be the inverse image of $\prod_{l'(\neq p,l)} \mathbb{Z}_{l'}(1)$. Accordingly, we have an induced exact sequence

$$1 \to P_K \to P_{K,l} \to \prod_{l' \neq p,l} \mathbb{Z}_{l'}(1) \to 1.$$

By contrast, the pro-l part can be read from the exact sequence

$$1 \to \mathbb{Z}_l(1) \to G_{K,l} \to G_k \to 1,$$

where the group $G_{K,l}$ is defined via the exact sequence

$$1 \to P_{K,l} \to G_K \to G_{K,l} \to 1.$$

Consequently, $g \in G_k$ acts naturally on $\gamma \in P_{K,l}$ via

$$\gamma \mapsto g\gamma g^{-1}.$$

We are ready to state one of the most intrinsic relations for Galois groups of local fields:

Tame Relation. (Iwasawa) *For any* $\gamma \in \mathbb{Z}_l(1)$ *and* $\mathrm{Fr}_k \in G_k$ *the absolute arithmetic Frobenius, a topological generator, we have*

$$\mathrm{Fr}_k \cdot \gamma \cdot \mathrm{Fr}_k^{-1} = \gamma^q$$

where $q := \#k$.

23.2. Finite monodromy

We say that a representation $\rho : G_K \to \mathrm{Aut}_{\mathbb{Q}_l}(V)$ is *a l-adic representation of* G_K if $V/\mathbb{Q}_l$ is finite dimensional and ρ is continuous. The following is the basic result on the structure of l-adic Galois representations:

Finite Monodromy. (Grothendieck) *If* $\rho : G_K \to \mathrm{Aut}_{\mathbb{Q}_l}(V)$ *is a l-adic representation, then* $\rho(P_{K,l})$ *is finite.*

Sketch of a proof. Since it is a profinite group, the Galois group G_K is compact. Consequently, there exists a maximal G_K-stable $\mathbb{Z}_l$-lattice Λ in V such that ρ admits an integral form

$$\rho_{\mathbb{Z}_l} : G_K \to \mathrm{Aut}_{\mathbb{Z}_l}(\Lambda).$$

As such, for any $n \in \mathbb{N}$, define a subgroup N_n of $\mathrm{Aut}_{\mathbb{Z}_l}(\Lambda)$ to be the kernel of mod l^n map

$$1 \to N_n \to \mathrm{Aut}_{\mathbb{Z}_l}(\Lambda) \to \mathrm{Aut}_{\mathbb{Z}_l}(\Lambda/l^n\Lambda) \to 1.$$

Clearly, N_1/N_n is a finite group of order equal to a power of l and hence $N_1 = \lim_{\leftarrow n} N_n$ is a pro-l group.

On the other hand, by definition, $P_{K,l}$ is a projective limit of finite groups whose orders are prime to l, thus $\rho_{\mathbb{Z}_l}(P_{K,l}) \cap N_1 = \{1\}$. Consequently, $\rho(P_{K,l}) = \rho_{\mathbb{Z}_l}(P_{K,l})$ is naturally embedded in $\mathrm{Aut}_{\mathbb{Z}_l}(\Lambda/l\Lambda)$ which is a finite group. Q.E.D.

23.3.   Unipotency

Based on finite monodromy property, we further have the following

Monodromy Theorem. (Grothendieck) *Let $\rho : G_K \to \mathrm{Aut}_{\mathbb{Q}_l}(V)$ be a l-adic representation. Then there exists a finite Galois extension L/K such that for the induced representation $\rho|_{G_L} : G_L(\subset G_K) \to \mathrm{Aut}_{\mathbb{Q}_l}(V)$, the inertial subgroup $I_L(\subset G_L)$ acts unipotently.*

Sketch of a proof. This is a direct consequence of the Tame relation. Indeed, by the finite monodromy result in the previous subsection, replacing K by a finite Galois extension, we may assume that $P_{K,l}$ acts on V trivially. Consequently, since $G_K/P_{K,l} = G_{K,l}$, the representation ρ factors through $G_{K,l}$:

$$\rho : G_K \twoheadrightarrow G_{K,l} \xrightarrow{\bar{\rho}} \mathrm{Aut}_{\mathbb{Q}_l}(V).$$

Recall now that we have the following structural exact sequence

$$1 \to \mathbb{Z}_l(1) \to G_{K,l} \to G_k \to 1$$

and the tame relation, recalled above, implies that for any $t \in \mathbb{Z}_l(1), n \in \mathbb{N}$,

$$\mathrm{Fr}_k^n \cdot t \cdot \mathrm{Fr}_k^{-n} = t^{nq},$$

with Fr_k the absolute Frobenius of k and $q = \#k$. Consequently, if λ is an eigenvalues of $\bar{\rho}(t) = \rho(t)$, then so is λ^n. This implies that all such λ's are roots of unity. Namely, all elements of $\mathbb{Z}_l(1) \subset G_{K,l}$ act unipotently. But $\mathbb{Z}_l(1)$ is rank one, so if we choose t_0 as a topological generator, then the topological closure $\overline{\langle t_0 \rangle}$ of the subgroup generated by t_0 acts unipotently on V. Since $\overline{\langle t_0 \rangle}$ is clearly an open subgroup of $\mathbb{Z}_l(1)$, so the whole $\mathbb{Z}_l(1)$ acts on V unipotently. With this, to complete the proof, it suffices to note that the induced action of inertia subgroup I_K factors through $\mathbb{Z}_l(1)$. From the exact sequences

$$0 \to P_K \to P_{K,l} \to \prod_{l' \neq p,l} \mathbb{Z}_{l'}(1) \to 0 \text{ and } 0 \to P_{K,l} \to G_K \to G_{K,l} \to 0,$$

we conclude that the induced action on I_K factors through $\mathbb{Z}_l(1)$ via the natural projection map

$$I_K \twoheadrightarrow I_K/P_K \simeq \mathbb{Z}_l(1) \times \prod_{l' \neq p,l} \mathbb{Z}_{l'}(1) \twoheadrightarrow \mathbb{Z}_l(1),$$

and hence is unipotent.   Q.E.D.

Example. If $V/\mathbb{Q}_l$ is one dimensional, from the Monodromy Theorem above, there exists a finite Galois extension L/K such that the induced action of I_L on V is unipotent. That means that the image of I_L is a finite group. As such, replacing L with a further extension, we may assume that I_L acts trivially on V. Particularly, this works for the Tate module $\mathbb{Z}_l(1)$.

Definition. Let $\rho : G_K \to \mathrm{Aut}(V)$ be a l-adic representation. Then ρ is called
1.a) *unramified* if I_K acts on V trivially;
1.b) *potentially unramified* if there exists a finite Galois extension $L/K \subset \overline{K}/K$ such that the induced action of I_L on V is trivial;

2.a) *semi-stable* if I_K acts on V unipotently;
2.b) *potentially semi-stable* if there exists a finite Galois extension $L/K \subset \overline{K}/K$ such that the induced action of I_L on V is unipotent.

In terms of this language, then Grothendieck's Monodromy Theorem claims that all l-adic Galois representation of a p-adic number field, $l \neq p$, is potentially semi-stable.

Chapter X. Primary theory of p-adic representations

In this chapter, we expose some elementary structures of p-adic Galois representations following [FO].

§24.   Preliminary structures of absolute Galois groups

24.1.   Galois theory: a p-adic consideration

Let K be a p-adic number field with k its residue field. Fix an algebraic closure $\overline{K}$. $\overline{K}$ is not complete with respect to the natural extension of the p-adic valuation of K. Denote the corresponding completion of $\overline{K}$ by $\mathbb{C} = \mathbb{C}_p$.

Denote by $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. Then, from p-adic theory point of view, G_K can be naturally decomposed into two parts, namely arithmetic one corresponding to the cyclotomic extensions by p^n-th roots of unity, and the geometric one, corresponding to the so-called field of norms.

More precisely, let $K_n := K(\mu_{p^n})$ where μ_{p^n} denotes the collection of p^n-th roots of unity in $\overline{K}$ and set $K_\infty := \cup_n K_n$. Denote the corresponding Galois groups by $H_K := \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K := \mathrm{Gal}(K_\infty/K)$. Clearly, $G_K/H_K \simeq \Gamma_K$.

24.2.   Arithmetic structure: cyclotomic character

Denote by $K_0 := \mathrm{Fr}\, W(k)$ the fractional field of the ring of Witt vectors with coefficients in k. Then it is known that K_0 is the maximal unramified extension of $\mathbb{Q}_p$ contained in K and Γ_{K_0} is canonically isomorphic to $\mathbb{Z}_p^*$ via the cyclotomic character $\chi_{\mathrm{cyc},p} = \chi_{\mathrm{cyc}}$. Clearly, Γ_K may be viewed as an open subgroup of Γ_{K_0} via χ_{cyc}.

The natural exponential map gives a $\mathbb{Z}_p$-module structure on $\mathbb{Z}_p^*$. One can easily checks that it is of rank one and its torsion part is given by

$$(\mathbb{Z}_p^*)_{\mathrm{tor}} = \begin{cases} \mathbb{F}_p^*, & p \neq 2 \\ \mathbb{Z}/2\mathbb{Z}, & p = 2. \end{cases}$$

Consequently, if we denote by Δ_K the torsion subgroup of Γ_K, then $K_\infty^{\Delta_K} = (K_{0,\infty})^{\Delta_{K_0}} \cdot K/K$ is a $\mathbb{Z}_p$-extension with the same residue field k of K.

For later use, denote by k' the residue field of K_∞. From the discussion above, we see that it may happen that k' is different from k.

24.3. Geometric structure: fields of norms

24.3.1. *Definition* With Γ_K understood, let us turn our attention to H_K part. This then leads to the theory of fields of norms due to Wintenberger. Roughly speaking, this theory says that the arithmetically defined Galois group $H_K := \mathrm{Gal}(\overline{K}/K_\infty)$ of fields of characteristic zero admits a natural geometric interpretation in terms of Galois group of localizations of function fields over finite fields, due to the fact that the natural norm map $N_{K_n/K_{n-1}}$ is quite related with the p-th power map.

More precisely, motivated by a work of Tate, for fields K_n, consider norm maps $N_{K_n/K_{n-1}}$. Clearly, $\left\{(K_n, N_{K_n/K_{n-1}})\right\}_{n\in\mathbb{N}}$ forms a projective system. Let $\mathcal{N}_K := \varprojlim_n K_n$ be the corresponding limit. That is,

(i) as a set,

$$\mathcal{N}_K = \left\{(x^{(0)}, x^{(1)}, \ldots, x^{(n)}, \ldots) : x^{(n)} \in K_n,\ N_{K_n/K_{n-1}}\left(x^{(n)}\right) = x^{(n-1)}\right\};$$

(ii) for the ring structure, the addition and multiplication on $\mathcal{N}_K$ are given by

$$\left(x + y\right)^{(n)} := \lim_{m\to\infty} N_{K_{n+m}/K_n}\left(x^{(n+m)} + y^{(n+m)}\right)$$
$$\left(x \cdot y\right)^{(n)} := x^{(n)} \cdot y^{(n)}$$

for $x = \left(x^{(n)}\right)$, $y = \left(x^{(n)}\right) \in \mathcal{N}_K$.

Much more holds:

Theorem. (Wintenberger) $\mathcal{N}_K$ *is a field, the so-called field of norms of* K_∞/K, *such that its separable closure* $\mathcal{N}_K^s$ *is given by*

$$\bigcup_{L/K:\text{finite Galois}} \mathcal{N}_L,$$

and $G_{\mathcal{N}_K} := \mathrm{Gal}(\mathcal{N}_K^s/\mathcal{N}_K)$ *is isomorphic to* H_K.

In particular,

(i) for every finite Galois extension L/K *in* $\overline{K}/K$, $\mathcal{N}_L/\mathcal{N}_K$ *is a finite Galois extension with*

$$\mathrm{Gal}\left(\mathcal{N}_L/\mathcal{N}_K\right) \simeq \mathrm{Gal}\left(L_\infty/K_\infty\right);$$

(ii) for every finite Galois extension $\mathcal{N}_*/\mathcal{N}_K$, *there exists a finite Galois extension* L/K *such that* $\mathcal{N}_L = \mathcal{N}_*$.

24.3.2. *Geometric interpretation* To give a geometric interpretation of $\mathcal{N}_K$, let us start with $\mathcal{N}_{K_0}$. If we set $E_{K_0} := k\big((\pi_{K_0})\big)$, for a certain indeterminant π_{K_0} over k, then

$$\mathcal{N}_{K_0} \simeq E_{K_0} = k\big((\pi_{K_0})\big).$$

And more generally, for a certain indeterminant π_K over k',

$$\mathcal{N}_K \simeq E_K = k'((\pi_K)).$$

To be more precise, this is realized via the following consideration. First, by ramification theory, we see that the norm map $N_{K_n/K_{n-1}}$ is not far away from being the p-th power map. Accordingly, it is natural to introduce the ring

$$\widetilde{\mathbb{E}^+} := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}} := \left\{ (x^{(0)}, x^{(1)}, \dots) : x^{(n)} \in \mathcal{O}_{\mathbb{C}}, \ \left(x^{(n+1)}\right)^p = x^{(n)} \right\}$$

where $\mathcal{O}_{\mathbb{C}}$ denotes the ring of integers of $\mathbb{C}$. Define the ring structure on $\widetilde{\mathbb{E}^+}$ by

$$\big(x + y\big)^{(n)} := \lim_{m \to \infty} \left(x^{(n+m)} + y^{(n+m)}\right)^{p^m} \quad \& \quad \big(x \cdot y\big)^{(n)} := x^{(n)} \cdot y^{(n)}$$

for $x = \left(x^{(n)}\right), \ y = \left(x^{(n)}\right) \in \widetilde{\mathbb{E}^+}$.

One can easily check that $\widetilde{\mathbb{E}^+}$ is *perfect*. It is also of characteristic p. Indeed, there is a bijection

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}} \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}}.$$

This implies that

$$\widetilde{\mathbb{E}^+} \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}},$$

since $\mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}} \simeq \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$.

Moreover, if we set $\varepsilon = \left(\varepsilon^{(n)}\right) \in \widetilde{\mathbb{E}^+}$ with $\varepsilon^{(0)} = 1, \varepsilon^{(1)} \neq 1$ defined by primitive p^n-th roots of unity, and set

$$\widetilde{\mathbb{E}} = \widetilde{\mathbb{E}^+}\big[(\varepsilon - 1)^{-1}\big].$$

Then this is the completion of the algebraic (yet non-separable) closure of $\mathbb{F}_p\big((\varepsilon - 1)\big)$.

By definition, there is a natural action of H_K on $\widetilde{\mathbb{E}}$. With the interpretation of $\widetilde{\mathbb{E}^+} \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ in terms of $\mathcal{O}_{\overline{K}}$ (not the one

from the definition in terms of the completion $\mathcal{O}_{\mathbb{C}}$), this action can be read clearly as follows:

We have a natural injective morphism

$$
\begin{aligned}
\mathcal{N}_K &\quad \to \quad \widetilde{\mathbb{E}} \\
\left(x^{(n)}\right) &\quad \mapsto \quad \left(y^{(n)} \quad := \lim_{m\to\infty}(x^{(n+m)})^{p^m}\right).
\end{aligned}
$$

Moreover, one checks that

(i) $\mathcal{N}_{K_0} \simeq k\big((\pi)\big)$ with $\pi = \varepsilon - 1$;

(ii) $\mathbb{E}_K = \left(\widetilde{\mathbb{E}}\right)^{H_K}$ coincides with the image of $\mathcal{N}_K$;

(iii) $H_{L/K} := H_K/H_L = \mathrm{Gal}\big(L_\infty/K_\infty\big) \simeq \mathrm{Gal}\big(\mathcal{N}_L/\mathcal{N}_K\big) \simeq \mathrm{Gal}\big(\mathbb{E}_L/\mathbb{E}_K\big).$

§25. Galois representations: characteristic p-theory

In this section we concentrate on Galois representations of fields of characteristic p, motivated by the geometric interpretation of H_K.

25.1. $\mathbb{F}_p$-representations

Assume that E is a field of characteristic $p > 0$. Fix a separable closure E^s and let $G_E := \mathrm{Gal}(E^s/E)$ be the corresponding absolute Galois group. Denote by $\sigma : \lambda \mapsto \lambda^p$ the absolute Frobenius of E. Let V be a mod p representation of G_E of dimension d, i.e., a $\mathbb{F}_p$-vector space V of dimension d equipped with a linear and continuous action of G_E.

Since G_E acts naturally on E^s, it makes sense to talk about the E^s-representation $E^s \otimes_{\mathbb{F}_p} V$ equipped with G_E. The advantage of taking this extension of scalars is that, by Hilbert Theorem 90, one checks that if we set $\mathbb{D}(V) := \left(E^s \otimes_{\mathbb{F}_p} V\right)^{G_E}$, then

(i) $\mathbb{D}(V)$ is a E-vector space of dimension d; and

(ii) the natural map

$$
\alpha_V : E^s \otimes_E \mathbb{D}(V) \to E^s \otimes_{\mathbb{F}_p} V
$$

is an isomorphim of G_E-modules. Here, as usual, on the left hand side, the action concentrates on the coefficients E^s, while on the right, it is given by the diagonal action.

Moreover, since the absolute Frobenius σ commutes with the action of G_E, via the natural definition $\varphi : \lambda \otimes v \mapsto \sigma(\lambda) \otimes v$, we obtain a Frobenius on $E^s \otimes_{\mathbb{F}_p} V$ such that if $x \in \mathbb{D}(V)$ then so is $\varphi(x)$. Consequently, we obtain a natural Frobenius $\varphi : \mathbb{D}(V) \to \mathbb{D}(V)$.

25.2. Etale φ-modules

Motivated by the above discussion, we call a finite dimensional E-vector space M equipped with a σ-semi-linear map $\varphi : M \to M$ a φ-module over E.

We call a φ-module *etale* if $M = E \cdot \varphi(M)$.

Proposition. (See e.g.,[FO]) *If V is a $\mathbb{F}_p$-representation of G_E of dimension d, then $\mathbb{D}(V) := \left(E^s \otimes V \right)^{G_E}$ is an etale φ-module of dimension d over E. Moreover, as G_E-modules, we have an isomorphism*

$$\alpha_V : E^s \otimes_E \mathbb{D}(V) \to E^s \otimes_{\mathbb{F}_p} V.$$

25.3. Characteristic p representation and etale φ-module

Denote by $\mathbf{Rep}_{\mathbb{F}_p}(G_E)$ the category of all mod p representations of G_E and $\mathcal{M}_\varphi^{\mathrm{et}}(E)$ the category of etale φ-modules over E with morphisms being E-linear maps which commute with φ. Then from the paragraph above we have a natural functor

$$\mathbb{D}_E : \mathbf{Rep}_{\mathbb{F}_p}(G_E) \to \mathcal{M}_\varphi^{\mathrm{et}}(E).$$

Proposition. (Fontaine) *The natural functor*

$$\begin{aligned} \mathbb{D}_E : \quad \mathbf{Rep}_{\mathbb{F}_p}(G_E) \quad &\to \quad \mathcal{M}_\varphi^{\mathrm{et}}(E) \\ V \quad &\mapsto \quad \mathbb{D}_E(V) \quad := \left(E^s \otimes_{\mathbb{F}_p} V \right)^{G_E} \end{aligned}$$

gives an equivalence of categories and its quasi-inverse is given by

$$\begin{aligned} \mathbb{V}_E : \quad \mathcal{M}_\varphi^{\mathrm{et}}(E) \quad &\to \quad \mathbf{Rep}_{\mathbb{F}_p}(G_E) \\ M \quad &\mapsto \quad \mathbb{V}_E(M) \quad := \left(E^s \otimes_E M \right)^{\varphi=1}. \end{aligned}$$

§26. Lifting to characteristic zero

As our final aim is to study p-adic representations of Galois groups of local fields, it is natural to see how the discussions above on $\mathbb{F}_p$-representations, a characteristic p-theory, can be lifted to p-adic representations, a characteristic zero theory. We present the ralated materials following [FO] (and [B2]).

26.1. Witt vectors and Teichmüller lift

Let us start with a preparation on the coefficients, particularly, the theory of Witt vectors.

So let R be a perfect ring of characteristic p. We want to construct a ring $W(R)$, the so-called *ring of Witt vectors with coefficients in R*, such that p is not nilpotent and $W(R)$ is separated and complete for the topology defined by $p^n W(R)$. It is well known that *such a ring $W(R)$ does exists, unique up to isomorphism, and has R as its residual ring.* Consequently, if $\sigma : R \to S$ is a morphism, then σ lifts to a morphism $W(\sigma) =: \sigma : W(R) \to W(S)$. Particularly, all Witt ring admits a lift of Frobenius σ!

Examples:
(i) $W(\mathbb{F}_p) = \mathbb{Z}_p$;
(ii) If k is a finite field, then $W(k)$ is the ring of integers of the unique unramified extension of $\mathbb{Q}_p$ whose residue field is k. Consequently,
(iii) $W(\overline{\mathbb{F}_p}) = \mathcal{O}_{\widehat{\mathbb{Q}_p^{un}}}$ is the ring of integers of the p-adic completion of the maximal unramified extension $\mathbb{Q}_p^{un}$ of $\mathbb{Q}_p$.

For $x = x_0 \in R$, since R is perfect, it makes sense to talk about $x^{p^{-n}}$ in R for all n. (This is in fact the key condition for a field to be perfect.) Up to $W(R)$, choose then an element $\widetilde{x_n} \in W(R)$ such that its residue class coincides with $x^{p^{-n}}$. Then the sequence $\{\widetilde{x_n}\}_{n \geq 0}$ converges in $W(R)$, say, to an element $[x]$. This $[x]$ is known to depend only on x, not on the choices of $\widetilde{x_n}$. As such, we obtain a multiplicative map, the so-called *Teichmüller lift*:

$$
\begin{array}{rccc}
[\cdot]: & R & \to & W(R) \\
& x & \mapsto & [x].
\end{array}
$$

Clearly,
(i) the Teichmüller lift is a special section to the natural reduction map;
(ii) every element $x \in W(R)$ can be written uniquely as $x = \sum_{n=0}^{\infty} p^n [x_n]$ wth $x_n \in R$. Moreover,
(iii) there exist universal homogeneous polynomials
$S_n, P_n \in \mathbb{Z}[X_i^{p^{-n}}, Y_i^{p^{-n}} : i = 0, 1, \ldots, n]$ of degree 1 (where $\deg X_i := 1 =: \deg Y_i$) such that for all $x, y \in W(R)$, we have

$$
\begin{aligned}
x + y &= \sum_{n=0}^{\infty} p^n \Big[S_n(x_0, y_0, \ldots, x_n, y_n) \Big], \\
xy &= \sum_{n=0}^{\infty} p^n \Big[P_n(x_0, y_0, \ldots, x_n, y_n) \Big].
\end{aligned}
\qquad (*)
$$

For instance,

$$S_0(X_0, Y_0) := X_0 + Y_0;$$

$$S_1(X_0, Y_0, X_1, Y_1) := X_1 + Y_1 + p^{-1}\Big((X_0^{1/p} + Y_0^{1/p})^p - X_0 - Y_0\Big).$$

Indeed, with the help of the polynomials S and P, we can construct $W(R)$ by setting
(a) as a set, $W(R) := \prod_{n=0}^{\infty} R$, and
(b) for the ring structure, set the addition and the multiplication according to the above relations $(*)$.

Furthermore, the concept of Witt ring can be extended to the case when R is *not* perfect. In this later case, we call the result ring a *Cohen ring* $C(R)$. Cohen rings are not really unique, but still they are of characteristic zero with residual ring $C(R)/pC(R) = R$. For example, $C\Big(\mathbb{F}_p[[X]]\Big) = \mathbb{Z}_p[[X]]$.

26.2.   *p*-adic representations of fields of characteristic 0

26.2.1. *Lift of base fields* Let $\mathbb{E}_K \subset \widetilde{\mathbb{E}}$ be the field isomorphic to the field of norms $\mathcal{N}_K$ introduced before. It is of characteristic p and may not be perfect. Denote its associated Cohen ring $\mathcal{C}(\mathbb{E}_K)$ by $\mathcal{O}_{\mathcal{E}_K}$ and write $\mathcal{E}_K$ the associated fraction field which is of characteristic 0. Denote by $\varphi : \mathcal{E}_K \to \mathcal{E}_K$ a lift of the Frobenius $\sigma : \mathbb{E}_K \to \mathbb{E}_K$. Consequently,

$$\mathcal{O}_{\mathcal{E}_K} = \lim_{\leftarrow n} \mathcal{O}_{\mathcal{E}_K}/p^n \mathcal{O}_{\mathcal{E}_K}, \quad \mathcal{O}_{\mathcal{E}_K}/p\mathcal{O}_{\mathcal{E}_K} = \mathbb{E}_K \quad \text{and} \quad \mathcal{E}_K = \mathcal{O}_{\mathcal{E}_K}[\frac{1}{p}].$$

Let $\mathcal{F}$ be a finite extension of $\mathcal{E}_K$ and $\mathcal{O}_{\mathcal{F}}$ be the ring of integers. We say that $\mathcal{F}/\mathcal{E}_K$ is *unramified* if
(i) p is a generator of the maximal ideal of $\mathcal{O}_{\mathcal{F}}$; and
(ii) $F = \mathcal{O}_{\mathcal{F}}/p\mathcal{O}_{\mathcal{F}}$ is a separable extension of $\mathbb{E}_K$.

For any finite separable extension F of $\mathbb{E}_K$, the inclusion $\mathbb{E}_K \hookrightarrow F$ induces a local homomorphism $\mathcal{C}(\mathbb{E}_K) \to \mathcal{C}(F)$ through which we may identify $\mathcal{C}(\mathbb{E}_K)$ and a subring of $\mathcal{C}(F)$ and $\mathrm{Fr}\,\mathcal{C}(F)$ as a field extension of $\mathrm{Fr}\,\mathcal{C}(\mathbb{E}_K)$, which in particular is unramified. Much more is correct: By the field of norms, all finite unramified extensions of $\mathcal{E}_K$ are obtained in this way. If we let $\mathcal{E}^{\mathrm{ur}} := \lim_{\to F \in S} \mathcal{E}_F$ and let $\widehat{\mathcal{E}^{\mathrm{ur}}}$ be the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$ with $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ its ring of integers, then $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is a local ring and

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} = \lim_{\leftarrow} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p^n \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}.$$

Clearly, all are equipped with Frobenious φ which commute with the natural action of H_K. Moreover, one checks directly the following holds:

(i) $\left(\widehat{\mathcal{E}^{\mathrm{ur}}}\right)^{H_K} = \mathcal{E}_K$, $\left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\right)^{H_K} = \mathcal{O}_{\mathcal{E}_K}$;

(ii) $\left(\widehat{\mathcal{E}^{\mathrm{ur}}}\right)^{\varphi=1} = \mathbb{Q}_p$, $\left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\right)^{\varphi=1} = \mathbb{Z}_p$.

26.2.2. *p-adic representations* For simplicity, write $\mathcal{E}$ for $\mathcal{E}_K$. We say that a φ-*module* M over $\mathcal{E}$ is a finite dimensional $\mathcal{E}$-vector space equipped with a σ-semi-linear morphism $\varphi : M \to M$; and a φ-module is called *etale* if $M = \mathcal{E} \cdot \varphi(M)$. One can easily check that *for a p-adic representation V of H_K*,

$$\mathbb{D}(V) := \left(\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V\right)^{H_K}$$

is an etale φ-module over $\mathcal{E}$ such that the natural map

$$\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{E}} \mathbb{D}(V) \to \widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V$$

is a H_K-equivariant isomorphism.

26.3. *p*-adic representations and etale (φ, Γ)-modules

Let V be a $\mathbb{Q}_p$-representation of G_K, set

$$\mathbb{D}(V) := \left(\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V\right)^{H_K},$$

then $\mathbb{D}(V)$ admits natural Γ_K-actions. We say that D is a (φ, Γ)-*module* over $\mathcal{O}_{\mathcal{E}}$ (resp. over $\mathcal{E}$) if it a φ-module over $\mathcal{O}_{\mathcal{E}}$ (resp. over $\mathcal{E}$) together with a σ-semi-linear action of Γ_K commuting with φ. Moreover, D is called *etale* if it is an etale φ-module and the action of Γ_K is continuous.

Denote by $\mathbf{Rep}_{\mathbb{Q}_p}(G_K)$ the category of p-adic representations of G_K and $\mathcal{M}^{\mathrm{et}}_{\varphi,\Gamma}(\mathcal{E})$ the category of etale (φ, Γ)-modules over $\mathcal{E}$. Then we have the following

Corollary. (Fontaine) *The natural functor*

$$\mathbb{D} : \quad \mathbf{Rep}_{\mathbb{Q}_p}(G_K) \quad \to \quad \mathcal{M}^{\mathrm{et}}_{\varphi,\Gamma}(\mathcal{E})$$
$$V \quad \mapsto \quad \mathbb{D}(V) \quad := \left(\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V\right)^{H_K}$$

gives an equivalence of categories and its quasi-inverse is given by

$$\mathbb{V} : \quad \mathcal{M}^{\mathrm{et}}_{\varphi,\Gamma}(\mathcal{E}) \quad \to \quad \mathbf{Rep}_{\mathbb{Q}_p}(G_K)$$
$$M \quad \mapsto \quad \mathbb{V}(M) \quad := \left(\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{E}} M\right)^{\varphi=1}.$$

Chapter XI. p-adic Hodge and properties of periods

To expose basic structures of p-adic Galois representations, we shift our attentions to the so-called p-adic Hodge theory, based on the following reason: etale cohomology not only offers natural examples of Galois representations, but provides all the fine structures which play key roles in the theory of p-adic Galois representations.

§27. Hodge theory over $\mathbb{C}$

Let X be a projective smooth variety over a field E of characteristic zero. Then we have the associated complex of sheaf of differential forms

$$\Omega_{X/E}^* : \mathcal{O}_{X/E} \to \Omega_{X/E}^1 \to \Omega_{X/E}^2 \to \cdots .$$

By definition, the de Rham cohomology groups $H_{\mathrm{dR}}^m(X/E)$ are the hyper-cohomology groups $\mathbb{H}^m(\Omega_{X/E}^*)$ for all m.

On the other hand, for any embedding $E \hookrightarrow \mathbb{C}$, since $X(\mathbb{C})$ is a compact complex manifold, the singular cohomology $H^m(X(\mathbb{C}), \mathbb{Q})$, being the dual of $H_m(X(\mathbb{C}))$, is a finite dimensional $\mathbb{Q}$-vector space. The comparison theorem in the classical Hodge theory then says that there exists a canonical isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} H^m(X(\mathbb{C}), \mathbb{Q}) \simeq \mathbb{C} \otimes_E H_{\mathrm{dR}}^m(X/E).$$

Thus without loss of generality, we may assume that E is simply $\mathbb{C}$.

For a complex smooth projective variety X, denote by $A^n(X)$, resp. by $A^{p,q}(X)$, the space of C^∞ n-forms, resp. C^∞ (p,q)-forms. Clearly, $A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X)$. With respect to the total differential operator $d : A^n(X) \to A^{n+1}(X)$, we have the cohomology groups

$$H^{p,q}(X) := \left\{ \phi \in A^{p,q}(X) : d\phi = 0 \right\} \Big/ dA^{n-1}(X) \cap A^{p,q}(X).$$

Then the Hodge decomposition theorem in the classical Hodge theory claims that there exists a canonical isomorphism

$$H_{\mathrm{dR}}^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

Furthermore, there is a decreasing filtration on $A^n(X)$ defined by

$$\mathrm{Fil}^p A^n(X) := A^{n,0}(X) \oplus A^{n-1,1}(X) \oplus \cdots \oplus A^{p,n-p}(X)$$

and the induced decreasing filtration of $H_{\mathrm{dR}}^n(X)$ defined by

$$\mathrm{Fil}^p H_{\mathrm{dR}}^n(X) := H^{n,0}(X) \oplus H^{n-1,1}(X) \oplus \cdots \oplus H^{p,n-p}(X).$$

Clearly,

$$\mathrm{Fil}^p H_{\mathrm{dR}}^n(X) = \Big\{ \phi \in \mathrm{Fil}^p A^n(X) : d\phi = 0 \Big\} \Big/ dA^{n-1}(X) \cap \mathrm{Fil}^p A^n(X),$$

$$H^{p,q}(X) = \overline{H^{q,p}(X)},$$

$$H^{p,q}(X) = \mathrm{Fil}^p H_{\mathrm{dR}}^n(X) \cap \overline{\mathrm{Fil}^q H_{\mathrm{dR}}^n(X)}.$$

§28. Admissible Galois representations

Before we go to the essentials of p-adic Hodge theory, let us make a further preparation.

Let G be a topological group and B a topological commutative ring equipped with a continuous G action. Then by a B-*representation* V of G, we mean a free B-module V of finite rank d together with a semi-linear and continuous action of G. Such a representation is said to be *trivial* if there exists a basis of V consisting of only elements of V^G, the invariants of V with respect to the action of G.

Assume that $E := B^G$ is a field and let F be a closed subfield of E. Then B is called (F, G)-*regular* if

(1) B is a domain;

(2) $B^G = \mathrm{Fr}\, B^G$, where the action of G on B extends naturally on its fraction field;

(3) all elements

$$\Big\{ b \in B - \{0\} : \ \forall g \in G, \ \exists \lambda(g) \in F \text{ s.t. } g(b) = \lambda(g) \cdot b \Big\}$$

are invertible in B.

Let V be a F-representation of G. Set then $\mathbb{D}_B(V) := (B \otimes_F V)^G$. Accordingly, we have a natural B-linear and G-equivariant morphism

$$\begin{array}{rccc} \alpha_V : & B \otimes_E \mathbb{D}_B(V) & \to & B \otimes_F V \\ & \lambda \otimes x & \mapsto & \lambda x. \end{array}$$

We say that V is B-*admissible* if $B \otimes_F V$ is a trivial B-representation of G.

Lemma. (See e.g., [FO]) *Assume B is (F, G)-regular and let V be a F-representation of G. Then*
(1) The map α_V is injective and

$$\dim_E \mathbb{D}_B(V) \leq \dim_F V;$$

(2) The following things are equivalent:
  (i) V is B-admissible;
  (ii) $\dim_E \mathbb{D}_B(V) = \dim_F V$;
  (iii) α_V *is an isomorphism.*

§29. Basic properties of various periods

With the above discussion and the p-adic Hodge structures (to be stated below) in mind, we then can summarize the essential properties of various p-adic periods rings. Our treatment follows [Tsu2].

29.1. Hodge–Tate periods

Define the *ring of Hodge–Tate periods* to be the graded ring

$$\mathbb{B}_{\mathrm{HT}} := \bigoplus_{i \in \mathbb{Z}} \mathbb{B}_{\mathrm{HT}}^i$$

where,
(i)$_{\mathrm{HT}}$ the i-th piece is given by $\mathbb{B}_{\mathrm{HT}}^i := \mathbb{C}(i)$; and
(ii)$_{\mathrm{HT}}$ the ring structure is given by the natural multiplication

$$\mathbb{C}(i) \otimes_{\mathbb{C}} \mathbb{C}(j) \to \mathbb{C}(i+j).$$

29.2. de Rham periods

Fix a p-adic number field K. Denote by $\mathbb{B}_{\mathrm{dR}}$ the ring of de Rham periods.
Basic Properties of $\mathbb{B}_{\mathrm{dR}}$:
(i)$_{\mathrm{dR}}$ $\mathbb{B}_{\mathrm{dR}}$ is a complete discrete valuation field with $\mathbb{C}_p$ its residue field;
(ii)$_{\mathrm{dR}}$ $\mathbb{B}_{\mathrm{dR}}$ admits a natural decreasing filtration

$$\mathrm{Fil}_{\mathrm{HT}}^i \mathbb{B}_{\mathrm{dR}} := \left\{ x \in \mathbb{B}_{\mathrm{dR}} : v(x) \geq i \right\}$$

(reflecting the structure of Hodge filtration). Here we have normalized the valuation so that $v(\mathbb{B}_{\mathrm{dR}}^*) = \mathbb{Z}$;
(iii)$_{\mathrm{dR}}$ $\mathbb{B}_{\mathrm{dR}}$ admits a natural G_K action which not only preserves the above filtration, but is compatible with the natural induced projection $\mathrm{Fil}^0 \mathbb{B}_{\mathrm{dR}} \to \mathbb{C}$;
(iv)$_{\mathrm{dR}}$ $\mathbb{B}_{\mathrm{dR}}$ satisfies the following additional fine structures/properties:

(1)$_{\mathrm{dR}}$ There is a natural G_K-equivariant embedding
$$P_0 := K_0^{\mathrm{ur}} \otimes_{K_0} \overline{K} \hookrightarrow \mathcal{O}_{\mathbb{B}_{\mathrm{dR}}} =: \mathbb{B}_{\mathrm{dR}}^+$$
such that its composition with the residue map $\mathbb{B}_{\mathrm{dR}}^+ \twoheadrightarrow \mathbb{C}$ coincides with the natural embedding $K_0^{\mathrm{ur}} \otimes_{K_0} \overline{K} \hookrightarrow \mathbb{C}$;

$(2)_{\mathrm{dR}}$ There is a natural G_K-equivariant injection $\mathbb{Q}_p(i) \hookrightarrow \mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}}$ such that one (and hence all) $a \in \mathbb{Q}_p(1), a \neq 0$, maps into a prime element of $\mathbb{B}_{\mathrm{dR}}$. In particular,

$(2.1)_{\mathrm{dR}}$ there are natural G_K-equivariant injections $\mathbb{Q}_p(i) \hookrightarrow \mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}}$;

$(2.2)_{\mathrm{dR}}$ there are natural G_K-equivariant isomorphisms
$$\mathbb{C}(i) \simeq \mathrm{Gr}^i_{\mathrm{HT}}\mathbb{B}_{\mathrm{dR}} := \mathrm{Fil}^i_{\mathrm{HT}}\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{i+1}_{\mathrm{HT}}\mathbb{B}_{\mathrm{dR}};$$

$(3)_{\mathrm{dR}}$ $\mathbb{B}_{\mathrm{dR}}^{G_K} = K$.

It appears that $\mathbb{B}_{\mathrm{dR}}$ depends on K. For this, we have

$(\mathrm{v})_{\mathrm{dR}}$ If L/K is a finite Galois extension contained in $\overline{K}/K$, then

$$\Big(\mathbb{B}_{\mathrm{dR}}(L), G_L\Big) \simeq \Big(\mathbb{B}_{\mathrm{dR}}(K), G_L(\subset G_K)\Big).$$

That is to say, $\mathbb{B}_{\mathrm{dR}}(L)$ together with its Galois action G_L coincides with $\mathbb{B}_{\mathrm{dR}}(K)$ associated to K together with the induced action of G_L as the restriction from G_K to its subgroup G_L.

29.3. Crystalline periods

Denote by $\mathbb{B}_{\mathrm{crys}}$ the ring of crystalline periods.

Basic Properties of $\mathbb{B}_{\mathrm{crys}}$:

$(\mathrm{i})_{\mathrm{crys}}$ $\mathbb{B}_{\mathrm{crys}}$ is a G_K-stable subring of $\mathbb{B}_{\mathrm{dR}}$ such that the induced decreasing filtration $\mathrm{Fil}^i\mathbb{B}_{\mathrm{crys}} := \mathbb{B}_{\mathrm{crys}} \cap \mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}}$ has the same graded pieces $\mathbb{C}(i)$;

$(\mathrm{ii})_{\mathrm{crys}}$ $\mathbb{B}_{\mathrm{crys}}$ satisfies the following additional structures/properties:

$(1)_{\mathrm{crys}}$ There is a natural σ-semi $(P_0\text{-})$linear action of G_K and a G_K-equivariant injective morphism $\varphi : \mathbb{B}_{\mathrm{crys}} \to \mathbb{B}_{\mathrm{crys}}$, the so-called Frobenius, such that the following holds

$(1.1)_{\mathrm{crys}}$ For $t \in \mathbb{Q}_p(1) \subset \mathbb{B}_{\mathrm{crys}}$, $\varphi(t) = pt$;

$(1.2)_{\mathrm{crys}}$ $\mathrm{Fil}^0\mathbb{B}_{\mathrm{crys}} \cap \mathbb{B}_{\mathrm{crys}}^{\varphi=1} = \mathbb{Q}_p$;

$(1.3)_{\mathrm{crys}}$ $\forall x \in \mathbb{Q}_p(i)$, $\varphi(x) = p^i x$ and $\mathrm{Fil}^i\mathbb{B}_{\mathrm{crys}} \cap \mathbb{B}_{\mathrm{crys}}^{\varphi=p^i} = \mathbb{Q}_p(i)$;

$(2)_{\mathrm{crys}}$ The natural map $K \otimes_{K_0} \mathbb{B}_{\mathrm{crys}} \to \mathbb{B}_{\mathrm{dR}}$ is injective;

$(3)_{\mathrm{crys}}$ $\mathbb{B}_{\mathrm{crys}}^{G_K} = K_0$;

$(4)_{\mathrm{crys}}$ All one dimensional G_K-stable $\mathbb{Q}_p$-vector subspaces of $\mathbb{B}_{\mathrm{crys}}$ are contained in $P_0 \cdot \mathbb{Q}_p(i), i \in \mathbb{Z}$.

Similarly, as for $\mathbb{B}_{\mathrm{dR}}$, we have

$(\mathrm{iii})_{\mathrm{crys}}$ If L/K is a finite Galois extension contained in $\overline{K}/K$, then

$$\Big(\mathbb{B}_{\mathrm{crys}}(L), G_L\Big) \simeq \Big(\mathbb{B}_{\mathrm{crys}}(K), G_L(\subset G_K)\Big).$$

29.4. Semi-stable periods

Denote by $\mathbb{B}_{\mathrm{st}}$ the ring of semi-stable periods.

Basic Properties of $\mathbb{B}_{\mathrm{st}}$:

(i)$_{st}$ $\mathbb{B}_{st}$ may be understood as a G_K-stable subring of $\mathbb{B}_{dR}$. However, different from $\mathbb{B}_{crys}$, such an embedding of $\mathbb{B}_{st}$ in $\mathbb{B}_{crys}$ depends on the choices of prime element π of K.

(ii)$_{st}$ $\mathbb{B}_{st}$ satisfies the following additional structures/properties:

(1)$_{st}$ Corresponding to a systematic choice of p^n-th root of π in $\overline{K}$: $s = (s_n)_{n \in \mathbb{N}}, s_0 = \pi, s_{n+1}^p = s_n$, there is a natural element $u_s \in \mathbb{B}_{st}$ such that

$\quad$ (1.1)$_{st}$ $\mathbb{B}_{st} = \mathbb{B}_{crys}[u_s]$;

$\quad$ (1.2)$_{st}$ $\forall g \in G_K$, $g(u_s) = u_{g(s)}$, where $g(s) = \big(g(s_n)\big)_{n \in \mathbb{N}}$;

$\quad$ (1.3)$_{st}$ If $s' = (s_n')$ is another choice, then $u_{s'} = u_s + t$, where
$\quad\quad (s_n' s_n^{-1})_{n \in \mathbb{N}} =: t \in \mathbb{Q}_p(1) \subset \mathbb{B}_{crys}$;

(2)$_{st}$ $\mathbb{B}_{st}$ admits a natural G_K-equivariant Frobenius $\varphi(u_s) = p \cdot u_s$ extending the Frobenius φ on $\mathbb{B}_{crys}$;

(3)$_{st}$ $\mathbb{B}_{st}$ admits a natural monodromy operator $N : \mathbb{B}_{st} \to \mathbb{B}_{st}$ satisfying

$\quad$ (3.0)$_{st}$ N is a $\mathbb{B}_{crys}$-derivation and $N(u_s) = 1$;

$\quad$ (3.1)$_{st}$ N is G_K-equivariant;

$\quad$ (3.2)$_{st}$ $N\varphi = p\varphi N$;

$\quad$ (3.3)$_{st}$ $\mathbb{B}_{st}^{N=0} = \mathbb{B}_{crys}$; and

$\quad$ (3.4)$_{st}$ $\mathrm{Fil}^0 \mathbb{B}_{dR} \cap \mathbb{B}_{st}^{N=0,\varphi=1} = \mathbb{Q}_p$;

(4)$_{st}$ The natural map $K \otimes_{K_0} \mathbb{B}_{st} \to \mathbb{B}_{dR}$ is injective; and

(5)$_{st}$ $\mathbb{B}_{st}^{G_K} = K_0$;

(6)$_{crys}$ All one dimensional G_K-stable $\mathbb{Q}_p$-vector subspaces of $\mathbb{B}_{st}$ are contained in $P_0 \cdot \mathbb{Q}_p(i), i \in \mathbb{Z}$.

$\quad$ Similarly,

(iii)$_{crys}$ If L/K is a finite Galois extension contained in $\overline{K}/K$, then

$$\Big(\mathbb{B}_{st}(L), G_L, e(L/K)^{-1} N\Big) \simeq \Big(\mathbb{B}_{st}(K), G_L(\subset G_K), N\Big).$$

Here $e(L/K)$ denotes the ramification index of the extension L/K.

§30. Hodge–Tate, de Rham, semi-stable and crystalline reps

30.1. Definition

Let V be a p-adic representation of G_K, and let

$$\mathbb{D}_{\bullet}(V) := \Big(\mathbb{B}_{\bullet} \otimes_{\mathbb{Q}_p} V\Big)^{G_K}$$

where $\bullet$ is the running symbol for HT, dR, st, crys, and G_K acts on $\mathbb{B}_{\bullet} \otimes_{\mathbb{Q}_p} V$ via diagonal action of G_K. Clearly, from the natural structure of the ring of periods, there is an induced structures on $\mathbb{D}_{\bullet}(V)$. In particular, since

$$\mathbb{C}^{G_K} = \mathbb{B}_{HT}^{G_K} = \mathbb{B}_{dR}^{G_K} = K, \quad \text{and} \quad \mathbb{B}_{st}^{G_K} = \mathbb{B}_{crys}^{G_K} = K_0,$$

(i) $\mathbb{D}_{\mathrm{HT}}(V), \mathbb{D}_{\mathrm{dR}}(V)$ are K-vector spaces; and

(ii) $\mathbb{D}_{\mathrm{st}}(V), \mathbb{D}_{\mathrm{crys}}(V)$ are K_0-vector spaces.

One checks easily that $\mathbb{B}_\bullet$ is $\left(\mathbb{B}_\bullet^{G_K}, G_K \right)$-regular. Accordingly, following Fontaine, we call a p-adic Galois representation V of G_K a $\bullet$-*representation*, where $\bullet$=Hodge–Tate, de Rham, semi-stable, crystalline, if V is $\mathbb{B}_\bullet$-admissible, that is to say, if

$$\dim_{\mathbb{B}_\bullet^{G_K}} \mathbb{D}_\bullet(V) = \dim_{\mathbb{Q}_p}(V).$$

30.2. Basic structures of $\mathbb{D}_\bullet(V)$

Induced from Fontaine's rings of various periods, there are natural structures on the space $\mathbb{D}_\bullet(V)$ associated to a p-adic Galois representation V of G_K.

• **Hodge–Tate**: The graded structure on $\mathbb{B}_{\mathrm{HT}}$ induces a natural graded structure on K-vector space $\mathbb{D}_{\mathrm{HT}}(V)$. More precisely,

$$\mathbb{D}_{\mathrm{HT}}(V) = \bigoplus_{i \in \mathbb{Z}} \mathbb{D}_{\mathrm{HT}}^i(V) \quad \text{where} \quad \mathbb{D}_{\mathrm{HT}}^i(V) := \left(\mathbb{C}(i) \otimes_{\mathbb{Q}_p} V \right)^{G_K}.$$

• **de Rham**: The decreasing filtration structure on $\mathbb{B}_{\mathrm{dR}}$ induces a natural decreasing filtration of K-vector subspaces on $\mathbb{D}_{\mathrm{dR}}(V)$. More precisely,

$$\mathrm{Fil}_{\mathrm{HT}} \mathbb{D}_{\mathrm{dR}}^i(V) := \left(\mathrm{Fil}_{\mathrm{HT}}^i \mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \right)^{G_K}.$$

This filtration is *exhaustive* and *separated*, that is, we have

$$\bigcup_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{HT}}^i \mathbb{D}_{\mathrm{dR}}(V) = \mathbb{D}_{\mathrm{dR}}(V) \quad \text{and} \quad \bigcap_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{HT}}^i \mathbb{D}_{\mathrm{dR}}(V) = 0.$$

Moreover, by $(2.2)_{\mathrm{dR}}$, we have the following natural injection of K-vector spaces

$$\mathrm{Gr}_{\mathrm{HT}} \mathbb{D}_{\mathrm{dR}}(V) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{HT}}^i \mathbb{D}_{\mathrm{dR}}(V) / \mathrm{Fil}_{\mathrm{HT}}^{i+1} \mathbb{D}_{\mathrm{dR}}(V) \hookrightarrow \mathbb{D}_{\mathrm{HT}}(V). \quad (*)$$

• **Semi-Stable**: By $(4)_{\mathrm{st}}$, we have a non-canonical embedding of $K \otimes_{K_0} \mathbb{B}_{\mathrm{st}} \hookrightarrow \mathbb{B}_{\mathrm{dR}}$, and hence a natural inclusion

$$K \otimes_{K_0} \mathbb{D}_{\mathrm{st}}(V) \hookrightarrow \mathbb{D}_{\mathrm{dR}}(V). \quad (**)$$

Consequently, there is a natural decreasing filtration by K-vector subspaces on $K \otimes_{K_0} \mathbb{D}_{\mathrm{st}}(V)$. Moreover, from the Frobenius structure φ and monodromy operator N on $\mathbb{B}_{\mathrm{st}}$, we get a natural Frobeinus structure

$\varphi : \mathbb{D}_{st}(V) \to \mathbb{D}_{st}(V)$ and a monodromy operator $N : \mathbb{D}_{st}(V) \to \mathbb{D}_{st}(V)$ which are all K_0-linear and satisfy the relation

$$N\varphi = p \cdot \varphi N.$$

• **Crystalline:** By $(2)_{crys}$, we have a canonical embedding $K \otimes_{K_0} \mathbb{B}_{crys} \hookrightarrow \mathbb{B}_{dR}$, and hence a natural inclusion

$$K \otimes_{K_0} \mathbb{D}_{crys}(V) \hookrightarrow \mathbb{D}_{dR}(V). \qquad (*_3)$$

Consequently, there is a natural decreasing filtration by K-vector subspaces on $K \otimes_{K_0} \mathbb{D}_{crys}(V)$. Moreover, from the Frobenius structure φ on $\mathbb{B}_{crys}$, we get a natural Frobeinus $\varphi : \mathbb{D}_{crys}(V) \to \mathbb{D}_{crys}(V)$ which is K_0-linear.

Finally, by $(3.3)_{st}$, we have $\mathbb{B}_{st}^{N=0} = \mathbb{B}_{crys}$ and hence

$$\mathbb{D}_{st}(V)^{N=0} = \mathbb{D}_{crys}(V). \qquad (*_4)$$

30.3. Relations among various p-adic representations

Let V be a p-adic representation of G_K. Then from Lemma in §28, $(*, **, *_3, *_4)$, and the fact that the natural $\mathbb{C}$-linear morphism

$$\bigoplus_{i \in \mathbb{Z}} \mathbb{C}(-i) \otimes_K \left(\mathbb{C}(i) \otimes_K V \right)^{G_K} \to \mathbb{C} \otimes_{\mathbb{Q}_p} V$$

is an injection, we obtain the following inequalities:

$$\begin{aligned}
\dim_{K_0} \mathbb{D}_{crys}(V) &\leq \dim_{K_0} \mathbb{D}_{st}(V) \\
&\leq \dim_K \mathbb{D}_{dR}(V) \leq \dim_K \mathbb{D}_{HT}(V) \\
&\leq \dim_{\mathbb{Q}_p} V.
\end{aligned}$$

Consequently,
(i) $\mathbb{D}_\bullet(V)$ are all finite dimensional $\mathbb{B}_\bullet^{G_K}$-vector spaces;
(ii) φ, whenever makes sense, is an isomorphism; and most importantly,
(iii) there are simple implications that

$$\text{crystalline} \;\Rightarrow\; \text{semistable} \;\Rightarrow\; \text{de Rham} \Rightarrow \text{Hodge Tate}.$$

Proposition. (Fontaine) *Let V be a p-adic representation of G_K. Use $\bullet$ as the runing symbol for HT, dR, st, crys. Then*
(1) the natural $\mathbb{B}_\bullet$-linear map

$$\mathbb{B}_\bullet \otimes_{\mathbb{B}_\bullet^{G_K}} \mathbb{D}_\bullet(V) \hookrightarrow \mathbb{B}_\bullet \otimes_{\mathbb{Q}_p} V$$

is a G_K-equivariant morphism which preserves the grads, where
(i) G_K acts on the left hand side via the action on $\mathbb{B}_\bullet$ and on the right hand side via the diagonal one; and
(ii) the graded structures are given on the left hand side by

$$\sum_{i=i_0+i_1} \mathrm{Fil}^{i_0}\mathbb{B}_\bullet \otimes_K \mathrm{Fil}^{i_0}\mathbb{D}_\bullet^{i_1}(V)$$

and on the right hand by $\mathrm{Fil}^i\mathbb{B}_\bullet \otimes_{\mathbb{Q}_p} V$;
(2) If V is $\bullet$-admissible, then the $\mathbb{B}_\bullet$-linear map

$$\mathbb{B}_\bullet \otimes_{\mathbb{B}_\bullet^{G_K}} \mathbb{D}_\bullet(V) \to \mathbb{B}_\bullet \otimes_{\mathbb{Q}_p} V$$

is an isomorphism; Moreover,
(3) (i) If V is of Hodge–Tate, then by considering the degree zero parts, we get a natural isomorphism, the so-called Hodge–Tate decomposition,

$$\oplus_{i \in \mathbb{Z}} \mathbb{C}(-i) \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{HT}}^i(V) \simeq \mathbb{C} \otimes_{\mathbb{Q}_P} V;$$

(ii) If V is semi-stable, then the natural $\mathbb{B}_{\mathrm{st}}$-linear map

$$\mathbb{B}_{\mathrm{st}} \otimes_{K_0} \mathbb{D}_{\mathrm{st}}(V) \simeq \mathbb{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$$

commutes with φ and N;
(iii) If V is crystalline, then the natural $\mathbb{B}_{\mathrm{crys}}$-linear map

$$\mathbb{B}_{\mathrm{crys}} \otimes_{K_0} \mathbb{D}_{\mathrm{crys}}(V) \simeq \mathbb{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V$$

commutes with φ.

30.4. Examples

(1) *Tate Twist*: $\mathbb{Q}_p(i)$ given by cyclotomic characters $\chi_{\mathrm{cyclo}}^i, i \in \mathbb{Z}$. All are crystalline. Indeed, $D = \mathbb{D}_{\mathrm{crys}}(\mathbb{Q}_p(i)) = K_0 \cdot e$ with $e = t^{-i} \otimes t^i$ and

$$\varphi(e) = p^{-i}e, \quad \mathrm{Fil}_{\mathrm{HT}}^{-i}D = D, \quad \mathrm{Fil}_{\mathrm{HT}}^{-i+1}D = 0.$$

(2) *Unramified Representations*: A unramified p-adic Galois representation, i.e., where the inertia group I_K acts trivially, is crystalline. Moreover, a crystalline representation is unramified if and only if its

associated Hodge–Tate filtration satisfies $\mathrm{Fil}^0_{\mathrm{HT}}\mathbb{D}_{\mathrm{dR}}(V) = \mathbb{D}_{\mathrm{dR}}(V)$ and $\mathrm{Fil}^1_{\mathrm{HT}}\mathbb{D}_{\mathrm{dR}}(V) = 0$.

(3) *Semi-Stable Representations*: All Tate modules $T_p(E)$ for Tate curves E are semi-stable representations.

(4) *de Rham and Hodge–Tate Representations*:
(i) Extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ is de Rham, but
(ii) Non-trivial extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ is Hodge–Tate but not de Rham.

(5) *One Dimensional Galois Representations*: In this case, there are following equivalences
(i) Hodge–Tate $\Leftrightarrow$ de Rham
$\Leftrightarrow$ There is an open subgroup I_L of I_K and an integer i such that the induced action of I_L on $V(-i)$ is trivial;
(ii) Semi-stable $\Leftrightarrow$ Crystalline
$\Leftrightarrow$ the induced action of I_K on $V(-i)$ is trivial.

(6) *Not Even Hodge–Tate*: V is a two dimensional $\mathbb{Q}_p$-vector space equipped with an action of G_K given by $\begin{pmatrix} 1 & \log_p \chi(g) \\ 0 & 1 \end{pmatrix}$. By §45, the Sen operator $\Theta_V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, it is not of Hodge–Tate.

§31. *p*-adic Hodge theory

Fix a p-adic number field K with absolute Galois group G_K.

Therem. (*p*-adic **Hodge Theory**) *Let X be a n-dimensional proper regular variety defined over K. Denote by $\left(V := H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p), \rho\right)$ the induced representation of G_K, where $H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ denotes the m-th p-adic etale cohomology group of X. Then the following conjectures hold:*

• **Hodge–Tate** *(i) The Galois representation $\left(H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p), \rho_K\right)$ is of Hodge–Tate type; and*
(ii) There is a natural graded preserved isomorphism

$$\mathbb{D}_{\mathrm{HT}}\left(H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)\right) \simeq \oplus_{i\in\mathbb{Z}} H^{m-i}(X, \Omega^i_{X/K}),$$

and hence the following G_K-equivariant Hodge–Tate decomposition

$$\mathbb{C} \otimes_{\mathbb{Q}_p} H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p) \simeq \oplus_{i=0}^m \mathbb{C}(-i) \otimes_K H^{m-i}(X, \Omega^i_{X/K});$$

• **de Rham** *(i) The Galois representation $\left(H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p), \rho_K\right)$ is of de Rham type. Moreover,*

(ii) $\mathbb{D}_{\mathrm{dR}}(V)$ *together with its associated Hodge filtration is isomorphic to the de Rham cohomology* $H^m_{\mathrm{dR}}(X_K/K)$ *equipped with the Hodge filtration;*

• **Semi-Stable** *(i) If X has a semi-stable reduction (Y, D), then the Galois representation* $\left(H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p), \rho_K\right)$ *is semi-stable. Moreover,*

(ii) The associated filtered (φ, N)-module $\mathbb{D}_{\mathrm{st}}(V)$ is canonically isomorphic to the following filtered (φ, N)-module on the log crystalline cohomology $H^m_{\mathrm{log}}((Y, D)/K_0)$: Choose a semi-stable model $\mathfrak{X} \to \mathcal{O}_K$ of X/K so that we obtain a log geometric structure (Y, D) on the special fiber. Then induced from the log crystalline cohomology of the special fiber, there is a natural weakly admissible filtered (φ, N)-module structure on $\left(H^m_{\mathrm{log}}((Y, D)/K_0), H^m_{\mathrm{dR}}(X_K/K)\right);$

• **Crystalline** *(i) If X has a good reduction, the Galois representation* $\left(H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p), \rho_K\right)$ *is crystalline. Moreover,*

(ii) The filtered φ-module $\mathbb{D}_{\mathrm{crys}}(V)$ is canonically isomorphic to the following filtered φ-module on the crystalline cohomology $H^m_{\mathrm{crys}}(Y/K_0)$: Choose a proper regular model $\mathfrak{X} \to \mathcal{O}_K$ of X/K. Then induced from crystalline cohomology of the special fiber, there is a natural weakly admissible filtered φ-module structure on $\left(H^m_{\mathrm{crys}}(Y/K_0), H^m_{\mathrm{dR}}(X_K/K)\right).$

The Hodge–Tate conjecture, due to mainly Tate, is a p-adic analogus of the standard Hodge theory for projective complex manifolds. This conjectures was solved by Tate for abelian varieties with good reduction, by Raynaud for all abelian varieties, by Bloch–Kato ([BK1]) for varieties with good reduction and finally by Faltings ([Fa1]) in general.

The de Rham conjecture and the crystalline conjecture are due to Fontaine ([Fon3]) and are solved by Fontaine–Messing ([FMe]) when $K = K_0$, $\dim X \leq p - 1$ and X has good reduction, and by Faltings ([Fa2]) in gereral. The above filtered φ-module structure on the de Rham cohomology is due to Berthelot–Ogus ([Ber1,2], [BO1,2]), and the independence issue for the filtered φ-structure on the de Rham cohomology on the model used is established by Gillet–Messing ([GM]).

The semi-stable conjecture is due to Fontaine and U. Jannsen ([Fon6]), solved by Fontaine for abelian varieties ([Fon6]), by Kato when $\dim X \leq (p-1)/2$ ([K]), by Tsuji ([Tsu1]), Niziol ([N1,2]) and Faltings ([Fa4]) independently in general. The above filtered (φ, N)-structure on the de Rham cohomology is due to Hyodo–Kato ([HK]) and the independence of the model chosen can be established via de Jong's alternation theory ([dJ]).

Chapter XII. Fontaine's rings of periods

In this chapter, for completeness, we explain the essentials of various rings of periods following Fontaine (e.g. [FO]).

§32. The Ring of de Rham periods $\mathbb{B}_{dR}$

To have a reasonable theory of p-adic Galois representations, the standard p-adic cyclotomic character should be involved in a natural way. Accordingly, to construct the ring of good period $\mathbb{B}_\bullet$, we need to find an element $t \in \mathbb{B}_\bullet$ which is a period for the cyclotomic character. That is to say, there should be an element $t \in \mathbb{B}_\bullet$ such that

$$g(t) = \chi_{\mathrm{cycl}}(g) \cdot t \qquad \text{for all } g \in G_K.$$

As a starting point, one may naively try $\mathbb{C}$. However it does not work since

$$\left\{ x \in \mathbb{C} : g(x) = \chi_{\mathrm{cycl}}(g) \cdot x, \forall g \in G_K \right\} = \{0\}.$$

Thus we need to enlarge it. This then leads to the Tate module $\mathbb{Z}(1)$ and hence the ring of Hodge–Tate periods

$$\mathbb{B}_{\mathrm{HT}} := \oplus_{i \in \mathbb{Z}} \mathbb{C}(i),$$

which in a certain sense is the simplest ring of periods.

With the simplest one found, it is then very natural for us to seek a sort of 'universal' one. With the theory of field of norms, we are led to the Cohen ring $\widetilde{\mathbb{A}}^+ := W(\widetilde{\mathbb{E}}^+)$ associated to $\widetilde{\mathbb{E}}^+$, or better, to its fractional field $\widetilde{\mathbb{B}}^+ := \widetilde{\mathbb{A}}^+[\frac{1}{p}]$. While this basically works, an essential modification should be made.

To be more precise, let $\varepsilon = (\varepsilon^{(n)}) \in \widetilde{\mathbb{E}}^+$ with $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$. Assume that

$$t := \log[\varepsilon] = -\sum_{n=1}^{\infty} \frac{(1 - [\varepsilon])^n}{n}$$

makes sense. That is to say, assume that the infinite power series above converges. Then, formally, we have for all $g \in G_{K_0}$,

$$g(t) = g\left(\log[\varepsilon] \right) = \log\left([g(\varepsilon^{(0)}, \varepsilon^{(1)}, \dots)] \right)$$
$$= \log\left([\varepsilon^{\chi_{\mathrm{cycl}}(g)}] \right) = \chi_{\mathrm{cycl}}(g) \cdot t.$$

In other words, whenever it makes sense, $t = \log[\varepsilon]$ is a cyclotomic period. Thus, we need to create a ring within which the above series defining $\log[\varepsilon]$ converges.

For the infinite series defining $\log[\varepsilon]$ to converge, it suffices to make $1 - [\varepsilon]$ small. However, in $\widetilde{\mathbb{E}}^+$, we have

$$v_{\mathbb{E}}(\varepsilon - 1) = \lim_{n \to \infty} v_p(\varepsilon^{(n)} - 1)^{p^n} = \frac{p}{p-1}.$$

In other words, within $\widetilde{\mathbb{E}}^+$, $\varepsilon - 1$ is not really very small. To overcome this difficulty, following Fontaine, we go as follows:

From the natural isomorphism $\widetilde{\mathbb{E}} \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}}$, we obtain an induced homomorphism

$$\theta : \quad \begin{array}{ccc} \widetilde{\mathbb{E}} & \to & \mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}} \\ (x^{(n)}) & \mapsto & x^{(0)}. \end{array}$$

Lift this construction to the characteristic zero world. Since

$$\widetilde{\mathbb{B}}^+ := \widetilde{\mathbb{A}}^+\left[\frac{1}{p}\right] := \Big\{ \sum_{k \gg -\infty} p^k[x_k] : x_k \in \widetilde{\mathbb{E}}^+ \Big\}$$

where $[x] \in \widetilde{\mathbb{A}}^+$ denotes the Teichmüller lift of $x \in \widetilde{\mathbb{E}}^+$, we obtain a natural morphism, a lift of θ,

$$\theta : \quad \begin{array}{ccc} \widetilde{\mathbb{B}}^+ & \to & \mathbb{C} \\ \sum p^k[x_k] & \mapsto & \sum p^k x_k^{(0)}. \end{array}$$

(Here we have used the isomorphism

$$\widetilde{\mathbb{E}}^+ \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}} \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}},$$

namely, a shift from $\mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}}$ a characteristic p one to $\mathcal{O}_{\mathbb{C}}$ a characteristic zero world, so that elements x take the forms $x = (x^{(n)})$ with $x^{(n)} \in \mathcal{O}_{\mathbb{C}}$.)

Recall that $\varepsilon = (\varepsilon^{(n)}) \in \widetilde{\mathbb{E}}^+$ with $\varepsilon^{(0)} = 1$, $\varepsilon^{(0)} \neq 1$. Set

$$\varepsilon_1 := \varepsilon^p = (\varepsilon^{(1)}, \varepsilon^{(2)}, \dots) \in \widetilde{\mathbb{E}}^+ \qquad \text{and} \qquad \omega := \frac{[\varepsilon] - 1}{[\varepsilon_1] - 1}.$$

Then $\theta(\omega) = 1 + \varepsilon^{(1)} + \cdots + (\varepsilon^{(1)})^{p-1} = 0$. In other words, $\langle \omega \rangle \subset \mathrm{Ker}(\theta)$.

Lemma. (Fontaine) $\mathrm{Ker}(\theta) = \langle \omega \rangle$.

Proof. Obviously, $\mathrm{Ker}(\theta)$ is an ideal of $\widetilde{\mathbb{E}}^+$ whose elements satisfying $v_{\mathbb{E}}(x) \geq 1$. Note that $\omega \in \mathrm{Ker}(\theta)$ with its modulo p reduction $\overline{\omega}$ satisfies

$v_{\mathbb{E}}(\overline{\omega}) = 1$. Thus the natural injection map $\langle \omega \rangle \to \mathrm{Ker}(\theta)$ is surjective modulo p. Since both sides are complete for the p-adic topology, this has to be an isomorphism.   Q.E.D.

Note that in particular $\theta([\varepsilon] - 1) = 0$ i.e., $[\varepsilon] - 1 \in \mathrm{Ker}(\theta) = \langle \omega \rangle$. Thus in order to make $[\varepsilon] - 1$ small, it suffices to introduce the $\mathrm{Ker}(\theta)$-adic, or the same ω-adic, topology. Accordingly, let

$$\mathbb{B}_{\mathrm{dR}}^{+} := \varprojlim_{n} \widetilde{\mathbb{B}}^{+}/(\mathrm{Ker}\theta)^{n},$$

namely, define $\mathbb{B}_{\mathrm{dR}}^{+}$ to be the ring obtained by completing $\widetilde{\mathbb{B}}^{+}$ with respect to the $\mathrm{Ker}(\theta)$-adic topology.

Clearly, $t = \log([\varepsilon]) \in \mathbb{B}_{\mathrm{dR}}^{+}$. Indeed, we have the follows.

Lemma. (Fontaine) *(1)* $\mathbb{B}_{\mathrm{dR}} := \mathbb{B}_{\mathrm{dR}}^{+}[\frac{1}{t}]$ *is a field;*
(2) There is a natural filtration $\mathrm{Fil}_{HT}^{i}\mathbb{B}_{\mathrm{dR}} = t^{i} \cdot \mathbb{B}_{\mathrm{dR}}^{+}$ *such that*

$$\mathrm{Gr}_{\mathrm{HT}}\mathbb{B}_{\mathrm{dR}} \simeq \oplus_{i \in \mathbb{Z}}\mathbb{C}(i);$$

(3) There is a natural G_{K} action on $\mathbb{B}_{\mathrm{dR}}$ with $\mathbb{B}_{\mathrm{dR}}^{G_{K}} = K$.

§33.   The Ring of crystalline periods $\mathbb{B}_{\mathrm{crys}}$

The point here is to create a subring $\mathbb{B}_{\mathrm{crys}}$ of $\mathbb{B}_{\mathrm{dR}}$ which contains the cyclotomic period t and is equipped with a natural Frobenius structure. Its construction is essentially based on the following two relations:
(1) $\varphi(t) = \log([\varepsilon^{p}]) = \log([\varepsilon]^{p}) = p\log([\varepsilon]) = p \cdot t$, and
(2) $\varphi\big(\mathrm{Ker}(\theta) + p \cdot W(\widetilde{\mathbb{E}^{+}})\big) \subset \mathrm{Ker}(\theta) + p \cdot W(\widetilde{\mathbb{E}^{+}})$.

Indeed, in order to have $t \in \mathbb{B}_{\mathrm{crys}}$, we need to analyze the terms $\frac{([\varepsilon]-1)^{n}}{n}$ appeared in the defining series of $t = \log[\varepsilon]$. Note that

$$\frac{([\varepsilon] - 1)^{n}}{n} = (n - 1)!([\varepsilon_{1}] - 1)^{n}\frac{\omega^{n}}{n!}.$$

Since, (i) p-adically, $(n - 1)! \to 0$, and (ii) both $[\varepsilon_{1}] - 1$ and $\varphi([\varepsilon_{1}] - 1)$ are in $W(\widetilde{\mathbb{E}^{+}})$, we need to understand how all $\varphi\left(\frac{\omega^{n}}{n!}\right)$ behave.

For this, recall that on $W(\widetilde{\mathbb{E}^{+}})$, we have a Frobenius map

$$\varphi : (a_{0}, a_{1}, \dots, a_{n}, \dots) \mapsto (a_{0}^{p}, a_{1}^{p}, \dots, a_{n}^{p}, \dots).$$

So, for all $b \in W(\widetilde{\mathbb{E}^{+}})$, $\varphi(b) \equiv b^{p} \bmod p$. In particular,

$$\varphi(\omega) = \omega^{p} + p\eta = p\Big(\eta + (p - 1)!\frac{\omega^{p}}{p!}\Big)$$

for a certain $\eta \in W(\widetilde{\mathbb{E}}^+)$. Consequently,

$$\varphi\left(\frac{\omega^m}{m!}\right) = \frac{p^m}{m!} \cdot \left(\eta + (p-1)!\frac{\omega^p}{p!}\right)^m$$

which are contained in $W(\widetilde{\mathbb{E}}^+)\left[\frac{\omega^p}{p!}\right]$.

All this then leads to the following constructions:

(1) Starting from $\widetilde{\mathbb{A}^+} = W(\widetilde{\mathbb{E}}^+)$, we introduce the ring $\mathbb{A}_{\mathrm{crys}}^0$ by adding all elements $\frac{a^m}{m!}$ for $a \in \mathrm{Ker}(\theta)$, the so-called *divided power envelope of* $\widetilde{\mathbb{A}^+} = W(\widetilde{\mathbb{E}}^+)$ *with respect to* $\mathrm{Ker}(\theta)$;

(2) To make $(n-1)!$ small, we need to use p-adic topology and hence to obtain the ring

$$\mathbb{A}_{\mathrm{crys}} := \varprojlim_n \mathbb{A}_{\mathrm{crys}}^0 / p^n \mathbb{A}_{\mathrm{crys}}^0 = \Big\{ \sum_{n=0}^{\infty} a_n \frac{\omega^n}{n!} : a_n \to 0 \ p\text{--adically in } W(\widetilde{\mathbb{E}}^+) \Big\};$$

(3) By inverting p, we get

$$\mathbb{B}_{\mathrm{crys}}^+ := \mathbb{A}_{\mathrm{crys}}\Big[\frac{1}{p}\Big] = \Big\{ \sum_{n=0}^{\infty} a_n \frac{\omega^n}{n!} : a_n \to 0 \ p\text{--adically in } W(\widetilde{\mathbb{E}}^+)\Big[\frac{1}{p}\Big] \Big\}.$$

Clearly, $\mathbb{B}_{\mathrm{crys}}^+$ contains t and is naturally contained in $\mathbb{B}_{\mathrm{dR}}^+$; (Indeed, we have

$$\mathbb{B}_{\mathrm{crys}}^+ = \Big\{ \sum_{n=0}^{\infty} a_n \frac{\omega^n}{n!} \in \mathbb{B}_{\mathrm{dR}} : a_n \to 0 \ \text{ in } \ W(\widetilde{\mathbb{E}}^+)\Big[\frac{1}{p}\Big] \Big\}.)$$

(4) Finally, define the ring of crystalline periods by $\mathbb{B}_{\mathrm{crys}} := \mathbb{B}_{\mathrm{crys}}^+[\frac{1}{t}]$ with the extension of Frobenius via $\varphi(\frac{1}{t}) := \frac{1}{pt}$.

Remark. The domain $\mathbb{B}_{\mathrm{crys}}$ is not a field. For example, $\omega - p$ is in $\mathbb{B}_{\mathrm{crys}} \backslash \mathbb{B}_{\mathrm{crys}}^*$.

§34. The Ring of semi-stable periods $\mathbb{B}_{\mathrm{st}}$

Since for semi-stable periods, $\mathbb{B}_{\mathrm{st}}^{N=0} = \mathbb{B}_{\mathrm{crys}}$, a natural way to construct $\mathbb{B}_{\mathrm{st}}$ is to enlarge $\mathbb{B}_{\mathrm{crys}}$. For this purpose, motivated by analysis, we may simply try to find a transcendental element T over $\mathbb{B}_{\mathrm{crys}}$, or better, over its fraction field $\mathbb{C}_{\mathrm{crys}} := \mathrm{Fr}\,\mathbb{B}_{\mathrm{crys}}$, such that
(1) $\varphi(T) = pT$;
(2) $N(T) = 1$, which implies $N\big(\sum a_n T^n\big) = \sum n a_n T^{n-1}$ for all $a_n \in \mathbb{C}_{\mathrm{crys}}$; and

(3) There is a natural action of G_K on T which commutes with the operators φ and N. That is to say, for all $g \in G_K$,

$$g\big(\varphi(T)\big) = \varphi\big(g(T)\big) \qquad \text{and} \qquad g\big(N(T)\big) = N\big(g(T)\big).$$

This, by (1) and (2), shows that, for all $g \in G_K$,

$$\varphi\big(g(T)\big) = p \cdot g(T) \qquad \text{and} \qquad N\big(g(T)\big) = 1.$$

Consequently, if such a T exists, $g(T)$ should satisfy an additive relation

$$g(T) = T + \eta(g)$$

for a certain $\eta(g) \in \mathbb{B}_{\mathrm{crys}}$ such that $\varphi(\eta(g)) = p \cdot \eta(g)$. A good choice of $\eta(g)$ is $\chi_{\mathrm{cycl}}(g) \cdot t$. This then leads to finding an element $T \in \mathbb{B}_{\mathrm{dR}}$ such that
(i) T is transcendental over $\mathbb{B}_{\mathrm{crys}}$;
(ii) $\varphi(T) = pT$; and
(iii) $g(T) = T + \chi_{\mathrm{cycl}}(g) \cdot t$ for all $g \in G_K$.
   From our experience, a natural way to obtain transcendental element is via logarithmic map. Thus, by applying the exponential map, we must find an element $\varpi \in \widetilde{\mathbb{E}}^+$ satisfying the multiplicative relation

$$g(\varpi) = \varpi \cdot \varepsilon^{\chi_{\mathrm{cycl}}(g)}.$$

But this is relatively easy since the element $\varpi := (\varpi^{(n)}) \in \widetilde{\mathbb{E}}^+$ with $\varpi^{(0)} = p$ does the job. Indeed, $\theta\big(\frac{[\varpi]}{p} - 1\big) = \frac{p}{p} - 1 = 0$. Thus

$$\log[\varpi] := \log\left(\frac{[\varpi]}{p}\right) = \sum_{i=0}^{\infty} (-1)^{n+1} \frac{\left(\frac{[\varpi]}{p} - 1\right)^n}{n} = -\sum_{n=0}^{\infty} \frac{\omega^n}{np^n},$$

which is clearly convergent in $\mathbb{B}_{\mathrm{dR}}$. As a by-product, this also offers us a (non-canonical) embedding

$$\mathbb{B}_{\mathrm{st}} := \mathbb{B}_{\mathrm{crys}}\big[\log[\varpi]\big] \hookrightarrow \mathbb{B}_{\mathrm{dR}}.$$

Chapter XIII. Micro Reciprocity Laws and general CFT

§35. Filtered (φ, N)-modules and semi-stable reps

35.1. Definition

Let $\rho : G_K \to \mathrm{GL}(V)$ be a p-adic Galois representation. Following Fontaine, define the associated spaces of periods by

$$\mathbb{D}_{\mathrm{HT}}(V) := \left(\mathbb{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}, \qquad \mathbb{D}_{\mathrm{dR}}(V) := \left(\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\right)^{G_K},$$

$$\mathbb{D}_{\mathrm{st}}(V) := \left(\mathbb{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}, \qquad \mathbb{D}_{\mathrm{crys}}(V) := \left(\mathbb{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}.$$

Then by the properties of corresponding rings of periods $\mathbb{B}_\bullet$, we know that $\mathbb{D}_{\mathrm{HT}}(V)$ (resp. $\mathbb{D}_{\mathrm{dR}}(V)$, resp. $\mathbb{D}_{\mathrm{st}}(V)$, resp. $\mathbb{D}_{\mathrm{crys}}(V)$) is finite dimensional K (resp. K, resp. K_0, resp. K_0)-vector space. Moreover, it is known that

(1) the following inequalities hold:

$$\dim_{K_0}\mathbb{D}_{\mathrm{crys}}(V) \leq \dim_{K_0}\mathbb{D}_{\mathrm{st}}(V)$$
$$\leq \dim_K\mathbb{D}_{\mathrm{dR}}(V) \leq \dim_K\mathbb{D}_{\mathrm{HT}}(V)$$
$$\leq \dim_{\mathbb{Q}_p}(V);$$

(2) Refined structures on the rings of periods $\mathbb{B}_\bullet$, where $\bullet =$HT, dR, st, crys, naturally induce additional structures on $\mathbb{D}_\bullet(V)$ as well. More precisely,

• **Hodge–Tate** On $D := \mathbb{D}_{\mathrm{HT}}(V)$, there is a natural filtration structure, the *Hodge–Tate filtration*, given by

$$\mathrm{Fil}^i_{\mathrm{HT}}\mathbb{D}_{\mathrm{HT}}(V) := \left(\mathrm{Fil}^i_{\mathrm{HT}}\mathbb{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}.$$

This is a decreasing filtration by K-vector subspaces on $\mathbb{D}_{\mathrm{HT}}(V)$. Define then the associated graded piece by

$$\mathrm{Gr}^i_{\mathrm{HT}}(D) := \mathrm{Fil}^i_{\mathrm{HT}}(D)/\mathrm{Fil}^{i+1}_{\mathrm{HT}}(D),$$

and its associated *Hodge–Tate slope* by

$$\mu_{\mathrm{HT}}(D) := \frac{1}{\dim_K D} \cdot \sum_{i \in \mathbb{Z}} i \cdot \dim_K \mathrm{Gr}^i_{\mathrm{HT}}(D).$$

• **de Rham** On $D := \mathbb{D}_{\mathrm{dR}}(V)$, there is a natural filtration structure given by

$$\mathrm{Fil}^i_{\mathrm{HT}}\mathbb{D}_{\mathrm{dR}}(V) := \left(\mathrm{Fil}^i_{\mathrm{HT}}\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}.$$

This is a decreasing filtration by K-vector subspaces on $\mathbb{D}_{dR}(V)$. Define then the associated graded piece by

$$\mathrm{Gr}^i_{dR}(D) := \mathrm{Fil}^i_{HT}(D)/\mathrm{Fil}^{i+1}_{HT}(D),$$

and its associated slope by

$$\mu_{dR}(D) := \frac{1}{\dim_K D} \cdot \sum_{i \in \mathbb{Z}} i \cdot \dim_K \mathrm{Gr}^i_{dR}(D).$$

Since

$$\mathrm{Gr}^i_{dR}\mathbb{B}_{dR} \simeq \mathbb{C}(i) \simeq \mathrm{Gr}^i_{HT}\mathbb{B}_{HT},$$

quite often, we will call the above filtration and its associated slope the *Hodge–Tate filtration* and the *Hodge–Tate slope* respectively;

• **Crystalline** Being naturally embedded in $\mathbb{B}_{dR}$, $K \otimes_{K_0} \mathbb{B}_{crys}$ admits a natural filtration. Consequently, this induces a natural Hodge–Tate filtration on $D := K \otimes_{K_0} \mathbb{D}_{crys}(V)$ by K-vector subspaces.

Denote by $D_0 := \mathbb{D}_{crys}(V)$. Since $\mathbb{B}_{crys}$ admits a natural Frobenius φ, we obtain a natural φ-module structure on D_0 induced from $\varphi \otimes \mathrm{Id}_V$ on $\mathbb{B}_{crys} \otimes_{\mathbb{Q}_p} V$. Thus working over $\overline{K}_0^{\mathrm{ur}}$, or better, via the natural residue map, working over $\overline{k}$, the associated algebraic closure of the residue field k of K, according to Dieudonne, there is a natural decomposition

$$\overline{D}_0 = \oplus_{l \in \mathbb{Q}} \overline{D_{0,l}}.$$

Here $\overline{D_{0,l=\frac{s}{r}}}$ denotes the p^s eigen-space of φ^r with $l = \frac{s}{r}$ the reduced expression of $l \in \mathbb{Q}$ in terms of quotient of integers r, s, i.e., $r, s \in \mathbb{Z}$, $(r, s) = 1$ and $r > 0$. Introduce then the associated $\mathbb{Q}$-indexed filtration by K_0 vector subspaces, called the *Dieudonne filtration* associated to the φ-module D_0, by

$$\mathrm{Fil}^{l_0}_{\mathrm{Dieu}} D_0 := \oplus_{l \geq l_0} \overline{D_{0,l}}.$$

Accordingly, define the associated graded K_0-vector space by

$$\mathrm{Gr}^{l_0}_{\mathrm{Dieu}}(D) := \mathrm{Fil}^{l_0}_{\mathrm{Dieu}}(D_0)/ \cup_{l < l_0} \mathrm{Fil}^l_{\mathrm{Dieu}}(D_0),$$

and the *Dieudonne slope* by

$$\mu_{\mathrm{Dieu}}(D_0) := \frac{1}{\dim_{K_0} D_0} \cdot \sum_{l \in \mathbb{Q}} l \cdot \dim_{K_0} \mathrm{Gr}^l_{\mathrm{Dieu}}(D_0).$$

• **Semi-Stablility** Unlike $\mathbb{B}_{crys}$, there is no natural embedding of $K \otimes_{K_0} \mathbb{B}_{st}$ in $\mathbb{B}_{dR}$. But still we can embed $K \otimes_{K_0} \mathbb{B}_{st}$ in $\mathbb{B}_{dR}$. Fix such an embedding. Then we obtain a filtration by K-vector subspaces on $D :=$

$K \otimes_{K_0} \mathbb{D}_{st}(V)$. One can easily check that this filtration does not depend on the choice used above, thus it is well-justified to call such a filtration the Hodge–Tate filtration on D.

Similarly, the Frobenius structure on $\mathbb{B}_{st}$ induces a natrual φ-module structure on the finite dimensional K_0-vector space $D_0 := \mathbb{D}_{st}(V)$, or better, on $\overline{D_0/K_0^{ur}}$. Accordingly, we can introduce the Dieudonne filtration on D_0 and hence its associated Dieudonne slope $\mu_{Dieu}(V)$.

Moreover, the natural monodromy operator N on $\mathbb{B}_{st}$ introduces a nilpotent monodromy operator N on D_0 via $N \otimes Id_V$ on $\mathbb{B}_{st} \otimes_{\mathbb{Q}_p} V$. Motivated by this, we say that $\mathbb{D} = (D_0, D)$ is a *filtered (φ, N)-module* if it consists of a finite dimensional K_0-vector space D_0 and a finite dimensional K-vector space D, equipped with a exhaustive and separated filtration by K-vector subspaces on D, a φ-module structure on D_0, and a monodromy operator N satisfying the following compatibility conditions:
(i) $D \simeq K \otimes_{K_0} D_0$;
(ii) $N \circ \varphi = p\varphi \circ N$.
    Set
$$\mu_{HT}(\mathbb{D}) := \mu_{HT}(D), \qquad \mu_{Dieu}(\mathbb{D}) := \mu_{Dieu}(D_0).$$

It is known that $\mu_{Dieu}(\mathbb{D})$ is equal to the Newton slope $\mu_N(\mathbb{D})$ of $\mathbb{D}$. Here, $\mu_N(\mathbb{D})$ is defined as follows:
(a) If D_0 is of dimension 1 over K_0, say, $D_0 = K_0 \cdot d$. Then, we can see that we have $N = 0$ and there exists a non-zero $\lambda \in K_0$ such that $\varphi(d) = \lambda \cdot d$. Consequently, we have

$$\mu_N(\mathbb{D}) = v_{K_0}(\lambda);$$

(b) In general, we have

$$\mu_N(\mathbb{D}) = \mu_N(\det \mathbb{D}),$$

where $\det \mathbb{D}$ denotes the determinant of $\mathbb{D}$ obtained by taking the maximal exterior products of D_0 and D.

Tautologically, we have also the notion of *saturated filtered (φ, N)-submodules*.

35.2. Weak admissibility and semi-stablility

Clearly, if V is a p-adic semi-stable representation of G_K, then $\mathbb{D}(V) := \left(\mathbb{D}_{st}(V), \mathbb{D}_{dR}(V) \right)$ admits a natural filtered (φ, N)-module structure, since in this case

$$\mathbb{D}_{dR}(V) = K \otimes_{K_0} \mathbb{D}_{st}(V).$$

Hence it makes sense to talk about the corresponding Hodge–Tate slopes and Newton slopes. Along with this line, an important discovery of Fontaine is the following basic:

Theorem. (Fontaine) *Let* $\rho : G_K \to \mathrm{GL}(V)$ *be a semi-stable p-adic representation of* G_K *and set* $\mathbb{D} := (D_0, D)$ *with*

$$D_0 := \mathbb{D}_{\mathrm{st}}(V) \quad and \quad D := \mathbb{D}_{\mathrm{dR}}(V).$$

Then
(i) $\mu_{\mathrm{HT}}(\mathbb{D}) = \mu_{\mathrm{N}}(\mathbb{D})$; *and*
(ii) $\mu_{\mathrm{HT}}(\mathbb{D}') \leq \mu_{\mathrm{N}}(\mathbb{D}')$ *for any saturated filtered* (φ, N)*-submodule* $\mathbb{D}' = (D_0', D')$ *of* $\mathbb{D} = (D_0, D)$.

If a filtered (φ, N)-module (D_0, D) satisfies the above two conditions (i) and (ii), following Fontaine, we call it a *weakly admissible filtered* (φ, N)*-module.* So the above result then simply says that for a semi-stable representation V, its associated periods $\mathbb{D} := \left(\mathbb{D}_{\mathrm{st}}(V), \mathbb{D}_{\mathrm{dR}}(V) \right)$ is weakly admissible. More surprisingly, the converse holds correctly. That is to say, we also have the following

Theorem. (Fontaine‖Colmez–Fontaine) *If* (D_0, D) *is a weakly admissible filtered* (φ, N)*-module. Then there exists a semi-stable representation* V *of* G_K *such that*

$$D = \mathbb{D}_{\mathrm{dR}}(V) \quad and \quad D_0 = \mathbb{D}_{\mathrm{st}}(V).$$

Remark. (A‖B), for contributors, means that the assertion is on one hand conjectured by A and on the other proved by B.

§36. Monodromy theorem for p-adic Galois representations

We have already explained two of fundamental results on p-adic Galois representations, namely, Theorems in §31 and §35. Here we introduce another one, the so-called Monodromy Theorem for p-adic Galois Representations.

To explain this, let us recall that a p-adic Galois representation $\rho : G_K \to \mathrm{GL}(V)$ is called *potentially semi-stable*, if there exists a finite Galois extension L/K such that the induced Galois representation $\rho|_{G_L} : G_L(\hookrightarrow G_K) \to \mathrm{GL}(V)$ is a semi-stable representation. One can easily check that every potentially semi-stable representation is de Rham. As a p-adic analogue of the Monodromy Theorem for l-adic Galois Representations, we have the following fundamental result:

Monodromy Theorem for p-adic Galois Reps. (Fontaine‖Berger)
All de Rham representations are potentially semi-stable representations.

Started with Sen's theory for $\mathbb{B}_{dR}$ of Fontaine, bridged by over-convergence of p-adic representations due to (Cherbonnier‖Cherbonnier–Colmez), Berger's proof is based on the so-called p-adic monodromy theorem (for p-adic differentials equations) of (Crew, Tsuzuki‖Crew, Tsuzuki, Andre, Kedelaya, Menkhout). For more details, please refer to the final chapter.

§37. Semi-stability of filtered $(\varphi, N; \omega)$-modules

37.1. Weak admissibility = stability and of slope zero

With the geometric picture in mind, particularly the works of Weil, Grothendieck, Mumford, Narasimhan–Seshadri and Seshadri, we then notice that weakly admissible condition for filtered (φ, N)-module $\mathbb{D} = (D_0, D)$ is an arithmetic analogue of the condition on semi-stable bundles of slope zero. Indeed, if we set $\mu_{\text{total}}(\mathbb{D}) := \mu_{\text{HT}}(D) - \mu_{\text{Dieu}}(D_0)$, then the first condition of weak admissibility, namely,
(i) $\mu_{\text{HT}}(D) = \mu_{\text{Dieu}}(D_0)$
is *equivalent* to the slope zero condition
(i)′ $\mu_{\text{total}}(\mathbb{D}) = 0$;
and the second condition
(ii) $\mu_{\text{HT}}(D') \leq \mu_{\text{Dieu}}(D_0')$ for any saturated filtered (φ, N)-submodule (D_0', D') of (D_0, D),
is *equivalent* to the semi-stability condition
(ii)′ $\mu_{\text{total}}(\mathbb{D}') \leq \mu_{\text{total}}(\mathbb{D}) = 0$ for all saturated filtered (φ, N)-submodule $\mathbb{D}'$ of $\mathbb{D}$.
Put in this way, the above correspondence between semi-stable Galois representations and weakly admissible filtered (φ, N)-modules may be understood as an arithmetic analogue of the Narasimhan–Seshadri correspondence between irreducible unitary representations and stable bundles of degree zero over compact Riemann surfaces.

Accordingly, in order to establish a general class field theory for p-adic number fields, we need to introduce some new structures to tackle ramifications. Recall that in algebraic geometry, as explained in Part A, there are two parallel theories for this purpose, namely, the π-bundle one on the covering space using Galois groups; and the parabolic bundle one on the base space using parabolic structures. Hence, in our current arithmetic setting, we would like to develop corresponding theories.

The π-bundle analogue is easy, based on Monodromy theorem for p-adic Galois Representations. In fact, we have the following orbifold version:

Theorem. (Fontaine‖Fontaine, Colmez–Fontaine, Berger) *There exists a natural one-to-one and onto correspondence*

$$\Big\{ de\ Rham\ Galois\ representations\ of\ G_K \Big\}$$

$$\Updownarrow$$

$$\Big\{ semi\text{-}stable\ filtered\ (\varphi, N; G_{L/K})\ of\ slope\ zero{:}\ \exists\ L/K\ finite\ Galois \Big\}.$$

37.2. Ramifications

In geometry, parabolic structures take care of ramifications. Recall that if $M^0 \hookrightarrow M$ is a punctured Riemann surface, then around the punctures $P_i \in M \backslash M^0, i = 1, 2, \ldots, N$, the associated monodromy groups generated by parabolic elements S_i are isomorphic to $\mathbb{Z}$, an abelian group. Thus for a unitary representation $\rho : \pi_1(M^0; *) \to \mathrm{GL}(V)$, the images of $\rho(S_i)$ are given by diagonal matrices with diagonal entries $\exp(2\pi\sqrt{-1}\alpha_{i;k})$, that is to say, they are determined by unitary characters $\exp(2\pi\sqrt{-1}\alpha)$, $\alpha \in \mathbb{Q}$. As such, to see the corresponding ramifications, one usually choose a certain cyclic covering with ramifications around P_i's such that the orbifold semi-stable bundles can be characterized by semi-stable parabolic bundles on (M^0, M).

However, in arithmetic side, the picture is much more complicated since there is no simple way to make each step abelian. By contrast, the good news is that there is a well-established theory in number theory to measure ramifications, namely, the theory of high ramification groups.

Let then $G_K^{(r)}$ be the upper-indexed high ramification groups of G_K, parametrized by non-negative reals $r \in \mathbb{R}_{\geq 0}$. (See e.g., [Se3].) Denote then by $V^{(r)} := V^{G_K^{(r)}}$ the invariant subspace of V under $G_K^{(r)}$, and $K^{(r)} := \overline{K}^{G_K^{(r)}}$. For a p-adic Galois representation V, define the associated r-th graded piece by

$$\mathrm{Gr}^{(r)}V := \bigcap_{s \geq r} V^{(s)} \Big/ \bigcup_{s < r} V^{(s)},$$

and its *Swan conductor* by

$$\mathrm{Sw}(\rho) := \sum_{r \in \mathbb{R}_{\geq 0}} r \cdot \dim_{\mathbb{Q}_p} \mathrm{Gr}^{(r)}V.$$

Proposition. *Let $\rho : G_K \to \mathrm{GL}(V)$ be a de Rham representation.*
*(i) (**Hasse–Arf Lemma**) All jumps of $\mathrm{Gr}^{(r)}V$ are rational;*

(ii) (**Artin, Fontaine**) *There exists a Swan representation* $\rho_{\mathrm{Sw}} : G_K \to$ $\mathrm{GL}(V_{\mathrm{Sw}})$ *such that*

$$\langle \rho_{\mathrm{Sw}}, \rho \rangle = \mathrm{Sw}(\rho).$$

In particular, $\mathrm{Sw}(\rho) \in \mathbb{Z}_{\geq 0}$.

37.3. ω-structures

Recall that in geometry ([MY]), parabolic structures, taking care of ramifications, can also be characterized via an $\mathbb{R}$-index filtration

$$E_t := \left(p_* \left(W \otimes \mathcal{O}_Y \left(- [\#\Gamma \cdot t] D \right) \right) \right)^{\Gamma},$$

and its associated parabolic degree is measured by

$$\sum_i \alpha_i \cdot \dim_{\mathbb{C}} \mathrm{Gr}^i V.$$

Moreover, it is known that the filtration E_t is
(i) left continuous;
(ii) has jumps only at $t = \alpha_i - \alpha_{i-1} \in \mathbb{Q}$; and
(iii) with parabolic degree in $\mathbb{Z}_{\geq 0}$.

Even we have not yet checked with geometers whether their ramification filtration constructions are motivated by the arithmetic one related to the filtration of upper indexed high ramification groups, the similarities between both constructions are quite apparent. Indeed, it is well-known that, for the filtrations on Galois groups G_K and on representations V induced from that of high ramification groups $G_K^{(r)}$,
(i) by definition, $G_K^{(r)}$ and hence $V^{(r)}$ are left continuous;
(ii) from the Hasse–Arf Lemma, all jumps of $G_K^{(r)}$ and hence of $V^{(r)}$ are rational; and
(iii) according to essentially a result of Artin, the Artin/Swan conductors are non-negative integers.

Motivated by this, for a finite dimensional K-vector space D, a ω-*filtration* $\mathrm{Fil}_\omega^r D$ is defined to be a $\mathbb{R}_{\geq 0}$-indexed *increasing* and *exhausive* filtration by finite dimensional K-vector subspaces on D satisfying the following properties:
(i) (**Continuity**) it is left continuous;
(ii) (**Hasse–Arf's Rationality**) it has all jumps at rationals;
Define then the associated r-th graded piece by

$$\mathrm{Gr}_\omega^{(r)} D := \bigcap_{s \geq r} \mathrm{Fil}_\omega^{(s)} D \Big/ \bigcup_{s < r} \mathrm{Fil}_\omega^{(s)} D,$$

and its ω-*slope* by

$$\mu_\omega(D) := \frac{1}{\dim_K D} \cdot \sum_{r \in \mathbb{R}_{\geq 0}} r \cdot \dim_K \mathrm{Gr}_\omega^{(r)} D.$$

(iii) (**Artin's Integrality**) The ω-degree

$$\deg_\omega(D) := \sum_{r \in \mathbb{R}_{\geq 0}} r \cdot \dim_K \mathrm{Gr}_\omega^{(r)} D = \dim_K D \cdot \mu_\omega(D)$$

is a non-negative integer.

37.4. Semi-stability of filtered $(\varphi, N; \omega)$-modules

By the monodromy theorem of p-adic Galois representations, for a de Rham representation V of G_K, there exists a finite Galois extension L/K such that V, as a representation of G_L, is semi-stable. As such, then, over the extension field L, the weakly admissible filtered (φ, N)-structure on $\left(\mathbb{D}_{\mathrm{st},L}(V), \mathbb{D}_{\mathrm{dR},L}(V) \right)$ is equipped with a compatible Galois action of $G_{L/K}$. By contrast, motivated by the non-abelian class field theory for Riemann surfaces, we expect that the ω-structures would play a similar role in our approach to a general CFT in arithmetic as that of parabolic structures in geometry. Accordingly, we make the following

Definition. *(i) A filtered $(\varphi, N; \omega)$-module* $\mathbf{D} := \left(D_0, D; \mathrm{Fil}_\omega^r D \right)$ *is a filtered (φ, N)-module (D_0, D) equipped with a compatible ω-structure on D;*
(ii) Tautologically, we have the notion of a saturated filtered $(\varphi, N; \omega)$-submodule $\mathbf{D}' := \left(D_0', D'; \mathrm{Fil}_\omega^r D' \right)$ *of* $\mathbf{D} = \left(D_0, D; \mathrm{Fil}_\omega^r D \right)$;
(iii) Define the total slope of a filtered $(\varphi, N; \omega)$-module $\mathbf{D} := \left(D_0, D; \mathrm{Fil}_\omega^r D \right)$ *by*

$$\mu_{\mathrm{total}}(\mathbf{D}) := \mu_{\mathrm{HT}}(D) - \mu_{\mathrm{Dieu}}(D_0) - \mu_\omega(D);$$

(iv) A filtered $(\varphi, N; \omega)$-module $\mathbf{D} = \left(D_0, D; \mathrm{Fil}_\omega^r D \right)$ *is called semi-stable and of slope zero if*
*(a) (**Slope 0**) it is of total slope zero, i.e.,*

$$\mu_{\mathrm{total}}(\mathbf{D}) = 0;$$

*(b) (**Semi-Stability**) For every saturated filtered $(\varphi, N; \omega)$-module* $\mathbf{D}'$ *of* $\mathbf{D}$, *we have*

$$\mu_{\mathrm{total}}(\mathbf{D}') \leq \mu_{\mathrm{total}}(\mathbf{D}).$$

§38. General CFT for p-adic number fields

38.1. Micro Reciprocity Law

With all these preparations, we are now ready to make the following:

Conjectural Micro Reciprocity Law. *There exists a canonical one-to-one correspondence*

$$\left\{ \text{de Rham representations of } G_K \right\}$$

$$\Updownarrow$$

$$\left\{ \text{semi-stable filtered } (\varphi, N; \omega)\text{-modules of slope zero over } K \right\}.$$

38.2. General CFT for p-adic number fields

Denote the category of semi-stable filtered $(\varphi, N; \omega)$-modules of slope zero over K by $\mathrm{FM}_K^{\mathrm{ss};0}(\varphi, N; \omega)$. Assuming the MRL, i.e., the micro reciprocity law, then we can easily show that, with respect to natural structures, $\mathrm{FM}_K^{\mathrm{ss};0}(\varphi, N; \omega)$ becomes a Tannakian category. Denote by $\mathbb{F}$ the natural fiber functor to the category of finite K-vector spaces. Then, from the standard Tannakian category theory, we obtain the following

General CFT for p-adic Number Fields

• **Existence Theorem** *There exists a canonical one-to-one correspondence*

$$\left\{ \text{Finitely Generated Sub-Tannakian Categories } \left(\Sigma, \mathbb{F}|_\Sigma\right) \right\}$$

$$\Updownarrow \Pi$$

$$\left\{ \text{Finite Galois Extensions L/K} \right\};$$

Moreover,

• **Reciprocity Law** *The canonical correspondence above induces a natural isomorphism*

$$\mathrm{Aut}^\otimes\left(\Sigma, \mathbb{F}|_\Sigma\right) \simeq \mathrm{Gal}\left(\Pi\left(\Sigma, \mathbb{F}|_\Sigma\right)\right).$$

In fact much refined result holds: By using ω-filtration, for all $r \in \mathbb{R}_{\geq 0}$, we may form a sub-Tannakian category $(\Sigma^{(r)}, \mathbb{F}|_{\Sigma^{(r)}})$ of $(\Sigma, \mathbb{F}|_\Sigma)$, consisting of objects admitting trivial $\mathrm{Fil}_\omega^{r'}$ for all $r' \geq r$.

• **Refined Reciprocity Law** *The natural correspondence Π induces, for all $r \in \mathbb{R}_{\geq 0}$, a canonical isomorphism*

$$\mathrm{Aut}^\otimes\left(\Sigma^{(r)}, \mathbb{F}|_{\Sigma^{(r)}}\right) \simeq \mathrm{Gal}\left(\Pi\left(\Sigma, \mathbb{F}|_\Sigma\right)\right) \Big/ \mathrm{Gal}^{(r)}\left(\Pi\left(\Sigma, \mathbb{F}|_\Sigma\right)\right).$$

Chapter XIV. GIT stability, moduli and invariants

§39.   Moduli spaces

Let $\mathbf{D} := \left(D_0, D; \mathrm{Fil}_\omega^r(D)\right)$ be a filtered $(\varphi, N; \omega)$-module of rank d over K. Then D_0 is a d-dimensional K_0-vector space equipped with a (φ, N)-module structure, which induces a K_0-vector subspace filtration of D_0, namely, the $\mathbb{Q}$-indexed Dieudonne filtration $\left\{\mathrm{Fil}_{\mathrm{Dieu}}^l(D_0)\right\}_{l\in\mathbb{Q}}$, $D = K \otimes_{K_0} D_0$, and there are two K-vector subspace filtrations of D, namely, the decreasing Hodge–Tate filtration $\left\{\mathrm{Fil}_{\mathrm{HT}}^i(D)\right\}_{i\in\mathbb{Z}}$, and the increasing ω-filtration $\left\{\mathrm{Fil}_\omega^r(D)\right\}_{r\in\mathbb{R}_{\geq 0}}$ which is compatible with φ and N.

Let $P(\kappa_{\mathrm{Dieu}})$ and $P(\kappa_{\mathrm{HT}})$ be the corresponding parabolic subgroups of $\mathrm{GL}(D_0)$ and of $\mathrm{GL}(D)$. Define the character $L_{\kappa_{\mathrm{HT}}}$ of $P(\kappa_{\mathrm{HT}})$ by

$$L_{\kappa_{\mathrm{HT}}} := \bigotimes_{i\in\mathbb{Z}} \left(\det \mathrm{Gr}_{\mathrm{HT}}^i(D)\right)^{\otimes -i}.$$

Similarly, define the (rational) character $L_{\kappa_{\mathrm{Dieu}}}$ of $P(\kappa_{\mathrm{Dieu}})$ by

$$L_{\kappa_{\mathrm{Dieu}}} := \bigotimes_{l\in\mathbb{Q}} \left(\det \mathrm{Gr}_{\mathrm{Dieu}}^l(D_0)\right)^{\otimes -l}.$$

(Unlike $L_{\kappa_{\mathrm{HT}}}$, which is an element of the group $X^*(P_{\kappa_{\mathrm{HT}}})$ of characters of $P_{\kappa_{\mathrm{HT}}}$, being rationally indexed, $L_{\kappa_{\mathrm{Dieu}}}$ is in general not an element of $X^*(P_{\kappa_{\mathrm{Dieu}}})$, but a rational character, i.e., it belongs to $X^*(P_{\kappa_{\mathrm{Dieu}}}) \otimes \mathbb{Q}$.)

Moreover, since all jumps of an ω-structure are rationals, it makes sense to define the associated parabolic subgroup $P(\kappa_\omega)$ and a (rational) character L_{κ_ω} of $P(\kappa_\omega)$ by

$$L_{\kappa_\omega} := \bigotimes_{r\in\mathbb{R}_{\geq 0}} \left(\det \mathrm{Gr}_\omega^r(D)\right)^{\otimes -r}.$$

As usual, identify $L_{\kappa_{\mathrm{HT}}}$ with an element of $\mathrm{Pic}^{\mathrm{GL}(D)}\left(\mathrm{Flag}(\kappa_{\mathrm{HT}})\right)$, where $\mathrm{Flag}(\kappa_{\mathrm{HT}})$ denotes the partial flag variety consisting of all filtrations of D with the same graded piece dimensions $\dim_K \mathrm{Gr}_{\mathrm{HT}}^k(D)$. (We have identified $\mathrm{Flag}(\kappa_{\mathrm{HT}})$ with $\mathrm{GL}(D)/P_{\kappa_{\mathrm{HT}}}$.) Similarly, we get an element L_{κ_ω} of $\mathrm{Pic}^{\mathrm{GL}(D)}\left(\mathrm{Flag}(\kappa_\omega)\right) \otimes \mathbb{Q}$, with $\mathrm{Flag}(\kappa_\omega)$ the partial flag variety consisting of all filtrations of D with the same $\dim_K \mathrm{Gr}_\omega^r(D)$. Thus, it makes sense to talk about the rational line bundle $\left(L_{\kappa_{\mathrm{HT}}} \boxtimes L_{\kappa_\omega}\right) \otimes L_{\kappa_{\mathrm{Dieu}}}$

on the product variety $\mathrm{Flag}(\kappa_{\mathrm{HT}}) \times \mathrm{Flag}(\kappa_\omega)$. Moreover, define $J = J_K$ be an algebraic group whose $\mathbb{Q}_p$-rational points consist of automorphisms of the filtered $(\varphi, N; \omega)$-module $\mathbf{D}$ over K. We infer the following Proposition essentially from the works of Langton, Mehta–Seshadri, Rapoport–Zink, and particularly, Totaro.

Proposition. ([Lan], [MS], [To]) *Assume k is algebraically closed. Then* $\left(D_0, D; \mathrm{Fil}_\omega^r(D) \right)$ *is semi-stable of slope zero if and only if the corresponding point*

$$\left(\mathrm{Fil}_{\mathrm{HT}}^i(D), \mathrm{Fil}_\omega^r(D) \right) \in \mathrm{Flag}(\kappa_{\mathrm{HT}}) \times \mathrm{Flag}(\kappa_\omega)$$

is semi-stable with respect to all one-parameter subgroups $\mathbb{G}_m \to J$ defined over $\mathbb{Q}_p$ and the rational J-line bundle

$$\left(L_{\kappa_{\mathrm{HT}}} \boxtimes L_{\kappa_\omega} \right) \otimes L_{\kappa_{\mathrm{Dieu}}}$$

on $\mathrm{Flag}(\kappa_{\mathrm{HT}}) \times \mathrm{Flag}(\kappa_\omega)$.

As a direct consequence, following Mumford's Geometric Invariant Theory ([M]), we then obtain the moduli space $\mathfrak{M}_{K;d,0}^{\varphi,N;\omega}$ of rank d semi-stable filtered $(\varphi, N; \omega)$-modules of slope zero over K. In particular, when there is no ω-structure involved, we denote the corresponding moduli space simply by $\mathfrak{M}_{K;d,0}^{\varphi,N}$.

Remark. The notion of semi-stable filtered $(\varphi, N; \omega)$-modules of slope s and the associated moduli space $\mathfrak{M}_{K;r,s}^{\varphi,N;\omega}$ for arbitrary s can also be introduced similarly. We leave the details to the reader.

§40. Polarizations and Galois cohomology

With moduli spaces of semi-stable filtered $(\varphi, N; \omega)$-modules built, next we want to introduce various invariants (using these spaces). Recall that in (algebraic) geometry for semi-stable vector bundles, this process is divided into two: First we construct natural polarizations via the so-called Mumford–Grothendieck determinant line bundles of cohomologies; then we study the cohomologies of these polarizations.

Moduli spaces of semi-stable filtered $(\varphi, N; \omega)$-modules, being projective, admit natural geometrized polarizations as well. However, such geometric polarizations, in general, are quite hard to be used arithmetically, due to the fact that it is difficult to reinterpret them in terms of arithmetic structures involved. To overcome this difficulty, we here want to use Galois cohomologies of p-adic representations, motivated by

the $(\mathfrak{g}, K)$-modules interpretations of cohomology of (certain types of) vector bundles over homogeneous spaces.

On the other hand, as said, such polarizations, or better, determinant line bundles, if exist, should be understood as arithmetic analogues of Grothendieck–Mumford determinant line bundles constructed using cohomologies of vector bundles. Accordingly, if we were seeking a perfect theory, we should first develop an analogue of sheaf cohomology for filtered $(\varphi, N; \omega)$-modules. We will discuss this elsewhere, but merely point out here the follows:

(i) a good cohomology theory in the simplest abelian case of $r = 1$ is already very interesting since it would naturally lead to a true arithmetic analogue of the theory of Picard varieties, an understanding of which is expected to play a key role in our intersectional approach to the Riemann Hypothesis proposed in our Program paper [W2];

(ii) the yet to be developed cohomology theory would help us to build up p-adic L-functions algebrically. This algebraically defined L-function for filtered $(\varphi, N; \omega)$-modules then should be compared to p-adic L-functions for Galois representations defined using Galois cohomology ([PR]). We expect that these two different types of L's correspond to each other in a canonical way and further can be globalized within the framework of the thin theory of adelic Galois representations proposed in the introduction.

§41. Iwasawa cohomology and dual exponetial map

In this section, we recall some basic facts about Iwasawa cohomology needed in defining p-adic L-functions following [Col1, Col3].

41.1. Galois cohomology

Let M be a $\mathbb{Z}_p$-representation of G_K. As usual, for any $n \in \mathbb{N}$, denote by $C_c^n(G_K, M)$ the collections of continuous maps $G_K^n \to M$, called n-cochains of G_K with coefficients in M. (Thus $C_c^0(G_K, M)$ is simply M.) Define the boundary map $d_n : C_c^n(G_K, M) \to C_c^{n+1}(G_K, M)$ by

$$(d_0 a)(g) := g(a) - a;$$
$$(d_1 f)(g_1 g_2) := g_1(f(g_2)) - f(g_1 g_2) + f(g_1);$$
$$\cdots \cdots$$

$$(d_n f)(g_1, g_2, \ldots, g_{n+1}) := g_1(f(g_2, g_3, \ldots, g_{n+1}))$$
$$+ \sum_{i=1}^{n} (-1)^i f(g_1, g_2, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n, g_{n+1})$$
$$+ (-1)^{n+1} f(g_1, g_2, \ldots, g_n).$$

One can easily check that $\left(C_c^*(G_K, M), d_*\right)$ forms a complex of abelian groups. Set $Z_c^n(G_K, M) := \operatorname{Ker} d_n$ be the collections of n-th cocycles, and $B^n(G_K, M) := \operatorname{Im} d_{n-1}$ the collections of n-th coboundaries. Then, the n-th *Galois cohomology* of M is defined by

$$H_c^n(G_K, M) := H^n(C_c^*(G_K, M), d_*) := Z_c^n(G_K, M)/B_c^n(G_K, M).$$

For examples, $H^0(G_K, M) = M^{G_K}$,

$$Z_c^1(G_K, M) = \left\{ f : G_K \to M : f \text{ continuous, } f(g_1 g_2) = g_1 f(g_2) + f(g_1) \right\}$$

and $B_c^1(G_K, M) := \left\{ f_m : g \mapsto gm - m : \exists\, m \in M \right\}$.

As usual, for a p-adic representation V of G_K, choose a maximal G_K-stable $\mathbb{Z}_p$-lattice M, and set

$$H^n(G_K, V) := H^n(G_K, M) \otimes \mathbb{Q}_p.$$

Proposition. (See e.g., [Hi]) *Let V be a p-adic representation of G_K. Then*
(i) $H^{n \geq 3}(G_K, V) = \{0\}$;
(ii) $H^2(G_K, V) = H^0(G_K, V^\vee(1))^\vee$ *and* $H^1(G_K, V) = H^1(G_K, V^\vee(1))^\vee$;
(iii) $\sum_{n=0}^2 (-1)^n \dim_{\mathbb{Q}_p} H^n(G_K, V) = -[K : \mathbb{Q}_p] \cdot \dim_{\mathbb{Q}_p} V$.

41.2. (φ, Γ)-modules and Galois cohomology

We already knew that the category of etale (φ, Γ)-modules is equivalent to that of p-adic Galois representations. Thus, in principle, it is possible to compute Galois cohomologies in terms of (φ, Γ_K)-modules.

Let K be a p-adic number field. As usual, denote by $K_n := K(\mu_{p^n})$, $n \geq 1$ and $K_\infty = \cup_{n \geq 1} K_n$ with μ_{p^n} the p^n-th roots of unity. Set $\Gamma_n := \operatorname{Gal}(K_\infty/K_n)$. For simplicity, in the sequel, assume that Γ_n is free and hence of rank 1 over $\mathbb{Z}_p$.

Let V be a p-adic representation of G_K. For a fixed generator $\gamma \in \Gamma_K$, introduce the complex $C_{\varphi, \gamma}(K, V)$ via:

$$0 \to \mathbb{D}(V) \xrightarrow{(\varphi-1, \gamma-1)} \mathbb{D}(V) \oplus \mathbb{D}(V) \xrightarrow{(\gamma-1)\mathrm{pr}_1 - (\varphi-1)\mathrm{pr}_2} \mathbb{D}(V) \to 0.$$

Lemma. (Herr) *Let V be a p-adic representation of G_K. Then the cohomology of the complex $C_{\varphi, \gamma}(K, V)$ is naturally isomorphic to the Galois cohomology of V.*

41.3. Iwasawa cohomology $H^i_{\mathrm{Iw}}(K, V)$

Choose a system of generators γ_n of Γ_n such that $\gamma_n = \gamma_1^{p^{n-1}}$. Then, $\mathbb{Z}_p[[\Gamma_K]]$, the so-called the Iwasawa algebra, may be realized as the topological ring $\mathbb{Z}_p[[T]]$ with the (p, T)-adic topology $(T \leftrightarrow \gamma - 1)$, and

$$\mathbb{Z}_p\big[[\Gamma_K]\big]/(\gamma_n - 1) \simeq \mathbb{Z}_p\Big[\mathrm{Gal}(K_n/K)\Big].$$

Moreover, via the quotient map $G_K \to \Gamma_K$, we obtain a natural G_K action on $\mathbb{Z}_p[[\Gamma_K]]$ and hence a G_K-action on $\mathbb{Z}_p[\mathrm{Gal}(K_n/K)]$.

Recall that for a $\mathbb{Z}_p[G_K]$-module M, using Shapiro's lemma, see e.g., [Hi], we have canonical isomorphisms

$$H^i\big(G_{K_n}, M\big) \simeq H^i\Big(G_K, \mathbb{Z}_p[\mathrm{Gal}(K_n/K)] \otimes M\Big),$$

which then make the corestriction maps $H^i(G_{K_{n+1}}, M) \to H^i(G_{K_n}, M)$ a projective system. Consequently, associated to a $\mathbb{Z}_p$-representation M of G_K, we obtain the well-defined *Iwasawa cohomology groups*

$$H^i_{\mathrm{Iw}}(K, M) := \varprojlim H^i(G_{K_n}, M).$$

Moreover, for a p-adic representation V of G_K, define its associated Iwasawa cohomology by

$$H^i_{\mathrm{Iw}}(K, V) := H^i_{\mathrm{Iw}}(K, \Lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where Λ is a (maximal) G_K-stable $\mathbb{Z}_p$-lattice of V.

41.4. Two descriptions of $H^i_{\mathrm{Iw}}(K, V)$

There are various ways to describe Iwasawa cohomologies. For example, we have the following:

Proposition. $H^i_{\mathrm{Iw}}(K, V) = H^i\Big(G_K, \mathbb{Z}_p[[\Gamma_K]] \otimes V\Big).$

Consequently, Iwasawa cohomologies admit natural $Z_p[[\Gamma_K]]$-module structures. Quite often we also call $H^i_{\mathrm{Iw}}(K, V)$ Iwasawa modules associated to V. Moreover, recall that there is a natural bijection

$$\begin{array}{ccc} \mathbb{Z}_p[[\Gamma_K]] \otimes V & \simeq & \mathcal{D}_0(\Gamma_K, V) \\ \gamma \otimes v & \mapsto & \delta_\gamma \otimes v, \end{array}$$

where $\mathcal{D}_0(\Gamma_K, V)$ denotes the set of p-adic measures from Γ_K to V, and δ_γ denotes the Dirac measure at γ. Therefore, we can interpret elements

of $H^1_{\mathrm{Iw}}(K, V)$ in terms of p-adic measures. In particular, if $\eta : \Gamma_K \to \mathbb{Q}_p^*$ is a continuous character, then, for any $n \geq 1$, we obtain a natural map

$$
\begin{array}{ccc}
H^1_{\mathrm{Iw}}(K, V) & \to & H^1(G_K, V \otimes \eta) \\
\mu & \mapsto & \int_{\Gamma_{K_n}} \eta\,\mu.
\end{array}
$$

We can also interpret Iwasawa modules in terms of (φ, Γ)-modules. Denote by ψ a left inverse of the Frobenius φ. If V is a $\mathbb{Z}_p$-representation of G_K, then there exists a unique operator $\psi : \mathbb{D}(V) \to \mathbb{D}(V)$ such that $\psi(\varphi(a)x) = a\psi(x)$ and $\psi(a\varphi(x)) = \psi(a)x$ for $a \in A_K, x \in \mathbb{D}(V)$ and ψ commutes with the action of Γ_K. Similarly, if D is an etale (φ, Γ)-module over A_K or B_K, there exists a unique operator $\psi : D \to D$ as above. In particular, for any $x \in D$, x can be written as $x = \sum_{i=0}^{p^n-1}[\varepsilon]^i\varphi^n(x_i)$ where $x_i := \psi^n([\varepsilon]^{-i}x)$.

Lamma. (See e.g., [Col3]) *(1) If D is an etale φ-module over A_K (resp. over B_K), then*

    (i) $D^{\psi=1}$ is compact (resp. locall compact);

    (ii) $D/(\psi - 1)$ is finitely generated over $\mathbb{Z}_p$ (resp. over $\mathbb{Q}_p$).

(2) Let V be a p-adic representation of G_K. Let $C_{\psi,\gamma}$ be the complex

$$
0 \to \mathbb{D}(V) \xrightarrow{(\psi-1,\gamma-1)} \mathbb{D}(V) \oplus \mathbb{D}(V) \xrightarrow{(\gamma-1)\mathrm{pr}_1 - (\psi-1)\mathrm{pr}_2} \mathbb{D}(V) \to 0.
$$

Then we have a commutative diagram of between complexes $C_{\varphi,\gamma}$ and $C_{\psi,\gamma}$:

$$
\begin{array}{ccccccccc}
0 \to & \mathbb{D}(V) & \xrightarrow{(\varphi-1,\gamma-1)} & \mathbb{D}(V) \oplus \mathbb{D}(V) & \xrightarrow{(\gamma-1)\mathrm{pr}_1 - (\varphi-1)\mathrm{pr}_2} & \mathbb{D}(V) & \to 0 \\
& \mathrm{Id} \downarrow & & -\psi \oplus \mathrm{Id} \downarrow & & \downarrow -\psi & \\
0 \to & \mathbb{D}(V) & \xrightarrow{(\psi-1,\gamma-1)} & \mathbb{D}(V) \oplus \mathbb{D}(V) & \xrightarrow{(\gamma-1)\mathrm{pr}_1 - (\psi-1)\mathrm{pr}_2} & \mathbb{D}(V) & \to 0,
\end{array}
$$

which induces an isomorphism on cohomologies.

Corollary. (See e.g., [Col3]) *If V is a $\mathbb{Z}_p/\mathbb{Q}_p$-representation of G_K, then $C_{\psi,\gamma}(K, V)$ computes the Galois cohomology of V. More precisely,*

(i) $H^0(G_K, V) = \mathbb{D}(V)^{\psi=1,\gamma=1}$;

(ii) $H^2(G_K, V) = \mathbb{D}(V)/(\psi - 1, \gamma - 1)$; and

(iii) there exists a short exact sequence

$$
0 \to \mathbb{D}(V)/(\gamma - 1) \to H^1(G_K, V) \to \left(\mathbb{D}(V)/(\psi - 1)\right)^{\gamma=1} \to 0.
$$

Consequently, $H^i_{\mathrm{Iw}}(K, V) = 0$ if $i \neq 1, 2$, and there are canonical isomorphisms

$$
\mathrm{Exp}^* : H^1_{\mathrm{Iw}}(K, V) = \mathbb{D}(V)^{\psi=1}, \quad \text{and} \quad H^2_{\mathrm{Iw}}(K, V) = \mathbb{D}(V)/(\psi - 1).
$$

41.5. Dual exponential maps

From now on, assume that V is de Rham. Then we have the following natural isomorphisms

$$\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \simeq \mathbb{B}_{\mathrm{dR}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V), \qquad \mathbb{D}_{\mathrm{dR}}(V) = H^0(G_K, \mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$$

and

$$H^1(G_K, \mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = H^1(G_K, \mathbb{B}_{\mathrm{dR}}) \otimes_K \mathbb{D}_{\mathrm{dR}}(V).$$

Recall also that
(i) for all $k \neq 0$, $H^i(G_K, \mathbb{C}_p(k)) = 0$ for all i;
(ii) for all $i \geq 2$, $H^i(G_K, \mathbb{C}_p) = 0$, $H^0(G_K, \mathbb{C}_p) = K$, and
(iii) $H^1(G_K, \mathbb{C}_p)$ is a one-dimensional K-vector space generated by $\log \chi \in H^1(G_K, \mathbb{Q}_p)$.
Consequently, the cup product $x \mapsto x \cup \log \chi$ gives isomorphisms

$$H^0(G_K, \mathbb{C}_p) \simeq H^1(G_K, \mathbb{C}_p) \quad \text{and} \quad \mathbb{D}_{\mathrm{dR}}(V) \simeq H^1(G_K, \mathbb{B}_{\mathrm{dR}} \otimes V).$$

All this then leads to the so-called *Bloch–Kato dual exponential map* ([BK2]) for a de Rham representation V of G_K, i.e., the composition

$$\exp^* : H^1(G_K, V) \to H^1(G_K, B_{\mathrm{dR}} \otimes V) \simeq \mathbb{D}_{\mathrm{dR}}(V).$$

Consequently, for any $\mu \in H^1_{\mathrm{Iw}}(K, V)$, for any $k \in \mathbb{Z}$, we obtain a natural element

$$\exp^* \left(\int_{\Gamma_{K_n}} \chi^k \mu \right) \in t^{-k} K_n \otimes_K \mathbb{D}_{\mathrm{dR}}(V),$$

which is zero when $k \gg 0$.

Moreover, from the overconvergent theory ([CC]), there exists $n(V)$ such that, for all $n \geq n(V)$, the natural map φ^{-n} sends $\mathbb{D}(V)^{\psi=1}$ into

$$\varphi^{-n} \left(\mathbb{D}(V)^{\psi=1} \right) \subset K_n((t)) \otimes_K \mathbb{D}_{\mathrm{dR}}(V).$$

Exp versus exp. *Let V be a de Rham representation of G_K, and $\mu \in H^1_{\mathrm{Iw}}(K, V)$. Then, for all $n \geq n(V)$,*

$$p^{-n} \cdot \varphi^{-n} \left(\mathrm{Exp}^*(\mu) \right) = \sum_{k \in \mathbb{Z}} \exp^* \left(\int_{\Gamma_{K_n}} \chi^k \mu \right).$$

That is to say, when V is de Rham, the isomorphism

$$\mathrm{Exp}^* : H^1_{\mathrm{Iw}}(K, V) \simeq \mathbb{D}(V)^{\psi=1}$$

and the Bloch–Kato dual exponential map admit much more refined arithmetic structures. This is particularly so when the representation is

semi-stable. In fact, following Perrin–Riou ([PR]), it is known that they are related to theory of p-adic L-functions. We leave the details to the literatures. Instead, to end this discussion of polarizations, let us simply point out that the associated determinants, or better, exterior products, are very important invariants and hence should be investigated from a more board point of view.

Chapter XV. Two approaches to conjectural MRL

§42.   Algebraic and geometric methods

There are two different approaches to establish the conjectural Micro Reciprocity Law. Namely, algebraic one and geometric one.

Let us start with algebraic approach. Here, we want to establish a correspondence between filtered $(\varphi, N; G)$-modules M and filtered $(\varphi, N; \omega)$-modules D. Obviously, this is an arithmetic analogue of Seshadri's correspondence between π-bundles and parabolic bundles over Riemann surfaces. Therefore, we expect further that our correspondence satisfies the following two compatibility conditions:

(i) it induces a natural correspondence between saturated subobjects M' and D' of M and D; and

(ii) it scales the slopes by a constant multiple of $\#G$. Namely,

$$\mu_{\text{total}}(M') = \#G \cdot \mu_{\text{total}}(D').$$

Assume the existence of such a correspondence. Then, as a direct consequence of the compatibility conditions, semi-stable filtered $(\varphi, N; G)$-modules M of slope zero correspond naturally to semi-stable filtered $(\varphi, N; \omega)$-modules D of slope zero. Indeed, if M is a semi-stable filtered $(\varphi, N; G)$-module of slope zero, then using the correspondence, we obtain a filtered $(\varphi, N; \omega)$-module D. Clearly, by (ii), we conclude that the slope of D is zero. Furthermore, D is semi-stable as well: Let D' be a saturated submodule of D. Then, via the induced correspondence (i) for saturated submodules, there exists a saturated submodule M' of M such that the slope of M' is a positive multiple of the slope of D'. On the other hand, since M is semi-stable, the slope of M' is at most zero. Consequently, the slope of D' is at most zero too. So D is semi-stable of slope zero. We are done. Conversely, if D is a semi-stable filtered $(\varphi, N; \omega)$-module of slope zero, then the corresponding filtered $(\varphi, N; G)$-module M can be similarly proved to be semi-stable of slope zero.

In this way, via the MRL with limited ramifications and the Monodromy Theorem for p-adic Galois Representations, we are able to establish the conjectural MRL.

With algebraic approach roughly discussed, let us say also a few wrods on the geometric approach here. Simply put, the main point we want to establish there is a direct correspondence between p-adic representations with finite monodromy around marks of fundamental

groups of curves defined over finite fields of characteristic p and what we call semi-stable rigid parabolic F-bundles in what should be called logarithmic rigid analytic geometry.

§43. MRL with limited ramifications

Before we give more details on our algebraic approach, for completion, let us in this section recall some of the key ingredients in establishing the natural connection between semi-stable Galois representations and weakly admissible filtered (φ, N)-modules.

43.1. Logarithmic map

We start with a description of refined structures of $\mathbb{B}_{\mathrm{crys}}$.

Set $\mathcal{O}_{\mathbb{C}}^{**} := \left\{ x \in \mathcal{O}_{\mathbb{C}} : \|x - 1\| < 1 \right\}$ be a subgroup of units of $\mathcal{O}_{\mathbb{C}}$. Clearly,
(i) if $x \in \mathcal{O}_{\mathbb{C}}^{**}$, then $x^{p^r} \to 1$ as $r \to +\infty$; and
(ii) for all $r \in \mathbb{Z}_{\geq 0}$, the map $x \mapsto x^{p^r}$ induces a surjective morphism from $\mathcal{O}_{\mathbb{C}}^{**}$ into itself with kernel $\mu_{p^r}(\mathbb{C})$. Consequently, any element in $\mathcal{O}_{\mathbb{C}}^{**}$ has exactly p^r numbers of p^r-th roots in $\mathcal{O}_{\mathbb{C}}^{**}$.

Let
$$U^* := \left\{ (x^{(n)}) \in \widetilde{\mathbb{E}^+} : x^{(0)} \in \mathcal{O}_{\mathbb{C}}^{**} \right\},$$
$$U_1^* := \left\{ (x^{(n)}) \in \widetilde{\mathbb{E}^+} : x^{(0)} \in 1 + 2p\mathcal{O}_{\mathbb{C}} \right\}.$$

From above, one can easily check that
(iii) the multiplicative group U_1^*, resp. U^*, admits a natural $\mathbb{Z}_p$-module structure, resp. $\mathbb{Q}_p$-vector space structure, such that

$$U^* \simeq \mathbb{Q}_P \otimes_{\mathbb{Z}_p} U_1^*;$$

(iv) if $x \in U_1^*$, then $[x] - 1 \in \mathrm{Ker}\,\theta + p \cdot W(\widetilde{\mathbb{E}^+})$.
    Consequently, the series

$$\log[x] := -\sum_{n=1}^{\infty} (-1)^n \frac{([x] - 1)^n}{n}$$

converges in $\mathbb{A}_{\mathrm{crys}}$. Hence, we get a logarithmic map $\log[\ ] : U_1^* \to \mathbb{A}_{\mathrm{crys}}$ which can also be extended to a logarithmic map $\log[\ ] : U^* \to \mathbb{B}_{\mathrm{crys}}^+$. Denote its image by U. Clearly, $\varphi([x]) = (x^p)$ and $\varphi(\log[x]) = p \cdot \log[x]$ for all $x \in U^*$.

43.2. Basic structures of $\mathbb{B}_{\mathrm{crys}}^{\varphi=1}$

As usual, let

$$\mathbb{B}_{\mathrm{crys}}^{\varphi=1} := \Big\{ x \in \mathbb{B}_{\mathrm{crys}};\, \varphi(x) = x \Big\}.$$

Also fix an element $v \in U(-1) - \mathbb{Q}_p$. Then, we have the following

Theorem. ([CF]) *(i.a)* $\mathrm{Fil}^0 \mathbb{B}_{\mathrm{crys}}^{\varphi=1} = \mathbb{Q}_p$;

(i.b) $\mathrm{Fil}^i \mathbb{B}_{\mathrm{crys}}^{\varphi=1} = 0$ *for all* $i > 0$;

(i.c) $\mathrm{Fil}^{-1} \mathbb{B}_{\mathrm{crys}}^{\varphi=1} = U(-1)$;

(i.d) All elements $b \in \mathrm{Fil}^{-i} \mathbb{B}_{\mathrm{crys}}^{\varphi=1}$, $i \geq 1$, *can be written in the form*

$$b = b_0 + b_1 v + \cdots + b_{r-1} v^{r-1}$$

where $b_0, b_1, \ldots, b_{r-1} \in U(-1)$;

(ii.a) For all $r \geq 1$, *there is an exact sequence*

$$0 \to \mathbb{Q}_p \to \mathrm{Fil}^{-r} \mathbb{B}_{\mathrm{crys}}^{\varphi=1} \to \Big(\mathrm{Fil}^{-r} \mathbb{B}_{\mathrm{dR}} / \mathbb{B}_{\mathrm{dR}}^{+} \Big) \to 0;$$

(ii.b) There is an exact sequence

$$0 \to \mathbb{Q}_p \to \mathbb{B}_{\mathrm{crys}}^{\varphi=1} \to \mathbb{B}_{\mathrm{dR}} / \mathbb{B}_{\mathrm{dR}}^{+} \to 0.$$

43.3. Rank one structures

Let V be a p-adic Galois representation, then we can form a filtered (φ, N)-module via

$$\mathbb{D}_{\mathrm{st}}(V) := \Big(\mathbb{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \Big)^{G_K}, \quad \mathbb{D}_{\mathrm{dR}}(V) := \Big(\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \Big)^{G_K}.$$

Following Fontaine, if V is semi-stable, then $\big(\mathbb{D}_{\mathrm{st}}(V), \mathbb{D}_{\mathrm{dR}}(V) \big)$ is weakably admissible. Conversely, for a filtered (φ, N)-module $\mathbf{D} = (D_0, D)$, we can introduce a Galois representation via the functor

$$\mathbb{V}_{\mathrm{st}}(\mathbf{D}) := \Big\{ v \in \mathbb{B}_{\mathrm{st}} \otimes D : \varphi v = v,\ Nv = 0\ \&\ 1 \otimes v \in \mathrm{Fil}_{\mathrm{HT}}^0 \Big(\mathbb{B}_{\mathrm{dR}} \otimes_K D \Big) \Big\}.$$

Moreover, following Colmez–Fontaine, if (D_0, D) is weakly admissible, then $\mathbb{V}_{\mathrm{st}}(\mathbf{D})$ is semi-stable.

While for general ranks, the proof of this equivalence between semi-stable representations and weakly admissible filtered (φ, N)-module is a bit twisted, the rank one case is rather transparent, thanks to the structural result above on $\mathbb{B}_{\mathrm{crys}}^{\varphi=1}$. As the statement, together with its proof, is a good place to understand the essentials involved, we decide to include full details.

Proposition. ([CF]) *Let* $\mathbf{D} = (D_0, D)$ *be a filtered* (φ, N)*-module of dimension 1 over* K.
(i) *If* $t_H(D) < t_N(D_0)$, $\mathbb{V}_{\mathrm{st}}(\mathbf{D}) = \{0\}$;
(ii) *If* $t_H(D) = t_N(D_0)$, $\dim_{\mathbb{Q}_p} \mathbb{V}_{\mathrm{st}}(\mathbf{D}) = 1$. *If* $\mathbb{V}_{\mathrm{st}}(\mathbf{D})$ *is generated by* $\alpha \cdot \mathbf{x}$, α *is an invertible element of* $\mathbb{B}_{\mathrm{st}}$;
(iii) *If* $t_H(D) > t_N(D_0)$, $\mathbb{V}_{\mathrm{st}}(\mathbf{D})$ *is infinite dimensional over* $\mathbb{Q}_p$.

Proof. The core is really the structural result on $\mathbf{B}_{\mathrm{crys}}^{\varphi=1}$ stated in the previous subsection. (In fact, only (i) and (ii) will be used.)
Step One: *Twisted by* $\mathbb{Q}(-m)$ *to make Hodge–Tate weight zero.* Since $\dim_{K_0} D_0 = 1$ and N is nilpotent, we have $D_0 = K_0 \mathbf{x}$ with $N\mathbf{x} = 0$. Let $\varphi(\mathbf{x}) = a \cdot \mathbf{x} = p^m a_0 \cdot \mathbf{x}$ with $m = v_p(a) = t_N(D)$ and $a_0 \in K_0$ satisfying $v_p(a_0) = 0$. Then there exists an element $\alpha_0 \in W(\overline{k})$ satisfying $\varphi(\alpha_0) = a_0 \alpha_0$. Set accordingly $\alpha = \alpha_0^{-1} \cdot t^{-m}$. Clearly, α is an invertible element in $\mathbb{B}_{\mathrm{crys}}$.
Step Two: *Deduced to Crystalline Periods.* If $\beta \mathbf{x} \in \mathbb{V}_{\mathrm{st}}(D)$ with $\beta \neq 0$, then

a) $0 = N(\beta \mathbf{x}) = N(\beta)\mathbf{x} + \beta \cdot N(\mathbf{x}) = N(\beta)\mathbf{x}$. Hence $N(\beta) = 0$;
b) $\beta \in \mathrm{Fil}^{-t_H(D)} \mathbb{B}_{\mathrm{st}}$ by definition; And
c) $\beta \mathbf{x} = \varphi(\beta)\varphi(\mathbf{x}) = \varphi(\beta) \cdot a\mathbf{x}$. So $\varphi(\beta) = a^{-1} \cdot \beta$.

Therefore,

$$\mathbb{V}_{\mathrm{st}}(D) = \left\{ \beta\mathbf{x} \,\middle|\, \beta \in \mathrm{Fil}^{-t_H(D)} \mathbb{B}_{\mathrm{st}}, N(\beta) = 0, \varphi(\beta) = a^{-1}\beta \right\}.$$

Set then $\beta = y\alpha \in \mathbb{B}_{\mathrm{st}}$, (since $\alpha \in \mathbb{B}_{\mathrm{crys}}$ is invertible, this is possible,) and we have

$$\varphi(\beta) = \varphi(y)\varphi(\alpha) = \varphi(y) \cdot \varphi(\alpha_0^{-1}) \cdot \varphi(t^{-m}) = \varphi(y) \cdot \varphi(\alpha_0)^{-1} p^{-m} t^{-m}$$

since $\varphi(t^{-1}) = (pt)^{-1}$. On the other hand,

$$\varphi(\beta) = a^{-1}\beta = a^{-1}y\alpha = a^{-1}y \cdot \alpha_0^{-1} t^{-m}$$
$$= y \cdot p^{-m} a_0^{-1} \alpha_0^{-1} \cdot t^{-m} = y \cdot \varphi(\alpha_0)^{-1} \cdot p^{-m} t^{-m}.$$

Consequently, $\varphi(y) = y$. Therefore,

$$\mathbb{V}_{\mathrm{st}}(D) = \left\{ y \cdot \alpha\mathbf{x} \,\middle|\, y \in \mathrm{Fil}^{t_N(D)-t_H(D)} \mathbb{B}_{\mathrm{st}}, N(y) = 0, \varphi(y) = 1 \right\}$$
$$= \left\{ y \cdot \alpha\mathbf{x} \,\middle|\, y \in \mathrm{Fil}^{t_N(D)-t_H(D)} \mathbb{B}_{\mathrm{crys}}, \varphi(y) = 1 \right\}$$
$$= \left\{ y \cdot \alpha\mathbf{x} \,\middle|\, y \in \mathrm{Fil}^{t_N(D)-t_H(D)} \mathbb{B}_{\mathrm{st}}^{\varphi=1} \right\}.$$

This then completes the proof of the Proposition. Q.E.D.

§44.   Filtration of invariant lattices

Now let come back to our algebraic approach to the conjectural MRL.

Let then $\mathbf{D}_L := (D_0, D)$ be a filtered $(\varphi, N; G_{L/K})$-module. So D_0 is defined over L_0 and D is over L. By the compactness of the Galois groups, there exists a lattice version of (D_0, D) which we denote by (Λ_0, Λ). In particular, Λ_0 is an $\mathcal{O}_{L_0}$-lattice with a group action G_{L_0/K_0}. Consider then the finite covering map

$$\pi_0 : \operatorname{Spec} \mathcal{O}_{L_0} \to \operatorname{Spec} \mathcal{O}_{K_0}.$$

We identify Λ_0 with its associated coherent sheaf on $\operatorname{Spec} \mathcal{O}_{L_0}$. Set

$$\Lambda_{0,K} := \left((\pi_0)_* \Lambda_0 \right)^{\operatorname{Gal}(L_0/K_0)}.$$

Clearly, there is a natural (φ, N)-structure on $\Lambda_{0,K}$.

Moreover, for the natural covering map

$$\pi : \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K,$$

view Λ as a coherent sheaf on $\operatorname{Spec} \mathcal{O}_L$ and form the coherent sheaf $\mathcal{O}_L\left(- [\deg(\pi) \cdot t]\mathfrak{m}_L \right)$, where $t \in \mathbb{R}_{\geq 0}$ and $\mathfrak{m}_L$ denotes the maximal idea of $\mathcal{O}_L$. Consequently, it makes sense to talk about

$$\Lambda_K(t) := \left(\pi_* \left(\Lambda \otimes \mathcal{O}_L\left(- [\deg(\pi) \cdot t]\mathfrak{m}_L \right) \right) \right)^{\operatorname{Gal}(L/K)}.$$

Or equivalently, in pure algebraic language,

$$\Lambda_K(t) := \left(\Lambda \otimes \mathfrak{m}_L^{\left[t \cdot \# G_{L/K} \right]} \right)^{\operatorname{Gal}(L/K)}.$$

Even we can read ramification information involved from this decreasing filtration consisting of invariant $\mathcal{O}_K$-lattices, unfortunately, we have not yet been able to obtain its relation with ω-structure wanted.

On the other hand, to go back from filtered $(\varphi, N; \omega)$-modules to filtered $(\varphi, N; G_{L/K})$-modules, a solution to the inverse Galois problem for p-adic number fields is needed. (Alternatively, as pointed by Hida, we can first use an independent geometric approach to be explained below to establish the conjectural MRL and hence the general CFT for p-adic number fields and then turn back as an application of our CFT to solve the inverse Galois problem for p-adic number fields.)

§45. Tate–Sen theory and its generalizations

From now on, we explain what is involved in our second approach to the conjectural Micro Reciprocity Law. As said, this approach is an arithmetic-geometrical one, with the main aim to characterize *p-adic representations* of fundamental groups with finite monodromy around marks of algebraic curves defined over finite fields of characteristic p in terms of what we call semi-stable parabolic rigid F-bundles on the logarithmic rigid analytic spaces associated to logarithmic formal schemes whose special fibers are the original marked curves. For this purpose, also for the completeness, we start with some preparations.

45.1. Sen's method

Consider then the natural action of G_K on $\widehat{\overline{K}} = \mathbb{C}$. For a closed subgroup H of G_K, clearly, $\overline{K}^H \subset \mathbb{C}^H$, which implies in particular that $\widehat{\overline{K}^H} \subset \mathbb{C}^H$. In fact much strong result holds:

Ax–Sen–Tate Theorem. *For every closed subgroup H of G_K, we have* $\widehat{\overline{K}^H} = \mathbb{C}^H$. *In particular,* $\widehat{K_\infty} = \mathbb{C}^{H_K}$.

With this, to understand the action of G_K on $\mathbb{C}$, we are led to the study of the residual action of Γ_K on $\widehat{K_\infty}$. By using the so-called Tate–Sen decompletion process, this can be reduced to the study of the action of Γ_K on K_∞, which is known to be given by the cyclotomic character.

Motivated by this, following Sen, for a general $\mathbb{C}$-representation of G_K, we first concentrate on its H_K-invariant part, which offers a natural $\widehat{K_\infty}$-representation of Γ_K; then by the decomposition technique just mentioned, we are led to a K_∞-representation of Γ_K. Γ_K is a rather simple p-adic Lie group, namely, abelian of rank 1 over $\mathbb{Z}_p$. This final residual K_∞-representation of Γ_K can be described via its infinitesimal action of $\mathrm{Lie}\,\Gamma_K$, which in turn is controlled by a single differential operator (modulo a certain finite extension):

Theorem. (Sen) *(1)* $H^1(H_K, GL_d(\mathbb{C})) = 1$;
(2) The natural map $H^1(\Gamma_K, GL_d(K_\infty)) \to H^1(\Gamma_K, GL_d(\widehat{K_\infty}))$ *induced by the natural inclusion* $K_\infty \hookrightarrow \widehat{K_\infty}$ *is a bijection;*
(3) Denote by $\mathbb{D}_{\mathrm{Sen}}(V)$ *the union of all K_∞-vector subspaces of $(\mathbb{C} \otimes_{\mathbb{Q}_p} V)^{H_K}$ which are Γ_K-stable and finite dimensional (over K_∞). Then for $\gamma \in \Gamma_K$ close enough to 1, the series operator on* $\mathbb{D}_{\mathrm{Sen}}(V)$ *defined by*

$$\Theta := -\frac{1}{\log_p \chi_{\mathrm{cyc}}(\gamma)} \cdot \sum_{n \geq 1} \frac{(1-\gamma)^n}{n}$$

converges and is independent of the choice of γ.

Consequently, for a $\mathbb{C}$-representation V of G_K of dimension d, we have the following associated structures:

(1) The H_K-invariants $(\mathbb{C} \otimes_{\mathbb{Q}_p} V)^{H_K}$ is a $\widehat{K_\infty}$-vector space of dimension d;

(2) $\mathbb{D}_{\mathrm{Sen}}(V)$ is a K_∞-vector space of dimension d;

Therefore, the natural map

$$\widehat{K_\infty} \otimes_{K_\infty} \mathbb{D}_{\mathrm{Sen}}(V) \to (\mathbb{C} \otimes_{\mathbb{Q}_p} V)^{H_K}$$

is an isomorphism and we have a natural residual action of Γ_K on $\mathbb{D}_{\mathrm{Sen}}(V)$.

(3) The action of $\mathrm{Lie}(\Gamma_K)$ on $\mathbb{D}_{\mathrm{Sen}}(V)$ is given by the operator $\Theta := \frac{\log(\gamma)}{\log_p \chi_{\mathrm{cyc}}(\gamma)}$ (where $\gamma \in \Gamma_K$ is chosen to be close enough to 1) defined as above, which is K_∞-linear.

Due to the fact that Θ is defined only for γ close enough to 1, the Lie action is only defined for a certain open subgroup of Γ_K. This is why in literature quite often we have to shift our discussion from K-level to K_n-level for a certain n.

45.2. Sen's theory for $\mathbb{B}_{\mathrm{dR}}$

The above result of Sen is based on the so-called Sen–Tate method. This method has been generalized by Colmez to a much more general context. (See e.g., [Col3], [FO].) This then leads Fontaine to obtain Sen's theory for $\mathbb{B}_{\mathrm{dR}}$ and Cherbonnier–Colmez to the theory of overconvergence, both of which play key roles in Berger's solution to Fontaine's Monodromy Conjecture for p-adic Galois representations.

Theorem. (Fontaine) *Let V be a p-adic representation of G_K of dimension d. Then we have the following associated structures:*

(i) There is a maximal element $\mathbb{D}_{\mathrm{Fon}}^+(V)$ in the set of finitely generated Γ_K-stable $\mathbb{K}_\infty[[t]]$-submodules of $(\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$;

(ii) The $\mathbb{K}_\infty[[t]]$-submodule $\mathbb{D}_{\mathrm{Fon}}^+(V)$ is a free $\mathbb{K}_\infty[[t]]$ of rank d equipped with a natural residual Γ_K-action whose infinitesimal action via $\mathrm{Lie}(\Gamma_K)$ is given by a differential operator ∇_V;

(iii) V is de Rham if and only if ∇_V has a full set of solutions in $\mathbb{D}_{\mathrm{Fon}}^+(V)$;

(iv) Natural residue map $\theta : \mathbb{B}_{\mathrm{dR}}^+ \to \mathbb{C}$ when applying to $(\mathbb{D}_{\mathrm{Fon}}^+(V), \nabla_V)$ gives rise naturally to $(\mathbb{D}_{\mathrm{Sen}}(V), \Theta_V)$.

45.3. Overconvergency

By the work of Fontaine, for a p-adic representation V of G_K, we can associate it to an etale (φ, Γ)-module $\mathbb{D}(V) := (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{H_K}$. While

useful, this etale (φ, Γ)-module $\mathbb{D}(V)$ is only a first approximation to the Galois representation V since $\mathbb{B}$ is too rough. Thus, certain refined structures should be introduced. This leads to the theory of overconvergence.

Let $\mathbb{B}^{\dagger,r}$ be the subring of $\mathbb{B}$ defined by

$$\mathbb{B}^{\dagger,r} := \Big\{ x \in \mathbb{B} : x = \sum_{k \gg -\infty} p^k[x_k],$$

$$x_k \in \widetilde{\mathbb{E}}, \lim_{k \to \infty} \Big(k + \frac{p-1}{p} \cdot \frac{1}{r} \cdot v_E(x_k) \Big) = +\infty \Big\}.$$

One checks that

$$\mathbb{B}_K^{\dagger,r} := (\mathbb{B}^{\dagger,r})^{H_K} = \Big\{ \sum_{k=-\infty}^{\infty} a_k \pi_K^k : a_k \in K_\infty \cap F^{\mathrm{ur}},$$

$$\sum_{k=-\infty}^{\infty} a_k X^k \text{convergent and bounded on } p^{-1/e_K r} \le |X| < 1 \Big\}$$

where e_K denotes the ramification index of $K_\infty/K_{0,\infty}$.

We say that a p-adic representation V of G_K is *overconvergent* if, for some $r \gg 0$, $\mathbb{D}(V) := \big(\mathbb{B} \otimes_{\mathbb{Q}_p} V \big)^{H_K}$ has a basis consisting of elements of $\mathbb{D}^{\dagger,r}(V) := (\mathbb{B}^{\dagger,r} \otimes_{\mathbb{Q}_p} V)^{H_K}$. In other words, there exists a basis of $\mathbb{D}(V)$ whose corresponding matrix $\mathrm{Mat}(\varphi)$ for the Frobenius φ belongs to $M(d, \mathbb{B}^{\dagger,r})$ for some $r \gg 0$.

Theorem. (Cherbonnier‖Cherbonnier–Clomez) *Every p-adic representation of G_K is overconvergent.*

§46. *p*-adic monodromy theorem

Now we are ready to recall Berger's proof of Monodromy Theorem for p-adic Representations.

Let V be a p-adic Galois representation of G_K. Following Fontaine, we obtain an etale (φ, Γ)-module $\mathbb{D}(V)$. This, together with the overconvergence of $\mathbb{D}(V)$, naturally gives raise to the question whether the differential operator $\nabla (:= \log(\gamma)/\log_p(\chi(\gamma))$ for $\gamma \in \Gamma_K$ close enough to 1, reflecting the Lie action of Γ_K,) makes sense on the overconvergent subspace $\mathbb{D}^{\dagger}(V) := \big(\mathbb{B}^{\dagger} \otimes_{\mathbb{Q}_p} V \big)^{H_K}$. Thus, we need to check how ∇ acts on the periods $\mathbb{B}_K^{\dagger} := \cup_{r \gg 0} \mathbb{B}_K^{\dagger,r}$. Unfortunately, $\mathbb{B}_K^{\dagger}$ is not ∇-closed: Easily one finds that

$$\nabla\big(f(\pi)\big) = \log(1 + \pi) \cdot (1 + \pi) \cdot df/d\pi.$$

In particular, with the appeerence of the factor $\log(1 + \pi)$, boundness condition for the elements involved in the definition of $\mathbb{B}_K^\dagger$ becomes clearly too restricted and hence should be removed. To remedy this, we make the following extension of periods (from $\mathbb{B}_K^\dagger$) to

$$\mathbb{B}_{\mathrm{rig},K}^{\dagger,r} := \Big\{ f(\pi_K) = \sum_{k=-\infty}^{\infty} a_k \pi_K^k : a_k \in \mathrm{Fr}\, W\big(k_{K_\infty}\big)$$

$$\& \, f(X) \text{ convergent on } p^{-1/e_K r} \leq |X| < 1 \Big\}$$

to include $\log(1 + \pi)$. In fact, much more has been achieved, namely, we now have a natural geometric interpretation for the periods: The union $\cup_{r \gg 0} \mathbb{B}_{\mathrm{rig},K}^{\dagger,r} =: \mathbb{B}_{\mathrm{rig},K}^{\dagger}$ is exactly the so-called *Robba ring* used in the theory of p-adic differential equations. Consequently, $\mathbb{B}_K^\dagger$ is the subring of $\mathbb{B}_{\mathrm{rig},K}^{\dagger}$ consisting of those functions which are bounded; and ∇ naturally acts on the periods

$$\mathbb{D}_{\mathrm{rig}}^{\dagger}(V) := \mathbb{B}_{\mathrm{rig},K}^{\dagger} \otimes_{\mathbb{B}_K^\dagger} \mathbb{D}^{\dagger}(V).$$

For general p-adic representations V, the differential operators ∇ do not behave nicely. However, for de Rham representations, the situation changes dramatically:

Theorem. (Berger) *Let V be a p-adic Galois representation of dimension d. Then*
(i) V is de Rham if and only if there exists a free $\mathbb{B}_{\mathrm{rig},K}^{\dagger}$-submodule $\mathbb{N}_{\mathrm{BW}}(V)$ of rank d of $\mathbb{D}_{\mathrm{rig}}^{\dagger}(V)\big[\frac{1}{t}\big]$ which is stable under the differential operator $\partial_V := \frac{1}{\log(1+\pi)} \cdot \nabla_V$ and the Frobenius operator φ such that $\varphi^* \mathbb{N}_{\mathrm{BW}}(V) = \mathbb{N}_{\mathrm{BW}}(V)$;
(ii) V is semi-stable if and only if $\big(\mathbb{B}_{\log,K}^{\dagger}[\frac{1}{t}] \otimes_{\mathbb{B}_K^\dagger} \mathbb{D}^{\dagger}(V)\big)^{\Gamma_K}$ is a K_0-vector space of dimension d, where, as usual, $\mathbb{B}_{\log,K}^{\dagger} := \mathbb{B}_{\log,K}^{\dagger}[\log \pi]$; and
(iii) V is crystalline if and only if $\big(\mathbb{B}_{\mathrm{rig},K}^{\dagger}[\frac{1}{t}] \otimes_{\mathbb{B}_K^\dagger} \mathbb{D}^{\dagger}(V)\big)^{\Gamma_K}$ is a K_0-vector space of dimension d.
In fact, (ii) and (iii) may be obtained by using Sen's method, that is, the so-called regularization and decompletion processes.

Examples. (1) When V is crystalline, we have $\mathbb{N}_{\mathrm{BW}}(V) = \mathbb{B}_{\mathrm{rig},K}^{\dagger} \otimes_F \mathbb{D}_{\mathrm{crys}}(V)$, a result essentially due to Wach [Wa1,2];
(2) When V is semi-stable, we have $\mathbb{N}_{\mathrm{BW}}(V) = \mathbb{B}_{\mathrm{rig},K}^{\dagger} \otimes_F \mathbb{D}_{\mathrm{st}}(V)$.

Berger–Wach modules $\mathbb{N}_{\mathrm{BW}}(V)$ above are examples of the so-called p-adic differential equation with Frobenius structure. In this language, Berger's theorem claims that V is de Rham if and only if there exists a p-adic differential equation $\mathbb{N}_{\mathrm{BW}}(V) \subset \mathbb{D}^{\dagger}_{\mathrm{rig}}(V)\left[\frac{1}{t}\right]$ with Frobenius structure.

Remark. We say that a *p-adic differential equation* is a free module M of finite rank over the Robba ring $\mathbb{B}^{\dagger}_{\mathrm{rig},K}$ equipped with a connection $\partial_M : M \to M$; M is equpied with a *Frobenius structure* if there is a semi-linear Frobenius $\varphi_M : M \to M$ which commutes with ∂_M; and M is called *quasi-unipotent* if there exists a finite extension L/K such that ∂_M has a full set of horizontal solutions in $\mathbb{B}^{\dagger}_{\mathrm{rig},L}\left[\log(\pi)\right] \otimes_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}} M$.

With all this, then we are in a position to recall the following fundamental result on p-adic differential equations.

p-adic Monodromy Thm. (Crew, Tsuzuki‖Andre, Kedlaya, Mebkhout) *Every p-adic differential equation with a Frobenius structure is quasi-unipotent.*

Consequently, if V is a de Rham representation of dimension d, then following Berger, we obtain a p-adic differential equation $\mathbb{N}_{\mathrm{BW}}(V)$ equipped with a Frobenius structure. Thus there exists a finite extension L/K such that $\left(\mathbb{B}^{\dagger}_{\mathrm{rig},L}\left[\log(\pi)\right] \otimes_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}} \mathbb{N}_{\mathrm{BW}}(V)\right)^{G_K}$ is a K_0-vector space of dimension d. Therefore, by Theorem (ii), V is a semi-stable representation of G_L. In other words, V itself is a potentially semi-stable representation of G_K. This is nothing but the statement of Fontaine‖Berger's Monodromy Theorem for p-adic Galois Representations.

§47. Infinitesimal, local and global

In this section, we briefly recall how micro arithmetic objects of Galois representations are naturally related with global geometric objects of the so-called overconvergent F-isocrystals.

47.1. From arithmetic to geometry

The shift from arithmetic to geometry, as said, is carried out via Fontaine–Wintenberger's fields of norms.

Let K be a p-adic number field with $\overline{K}$ a fixed algebraic closure and $K_\infty = \cup_n K_n$ with $K_n := K(\mu_{p^n})$ the cyclotomic extension of K by adding p^n-th root of unity. Denote by k its residue field, and $K_0 := \mathrm{Fr}\, W(k)$ the maximal unramified extension of $\mathbb{Q}_p$ contained in K. Set $\varepsilon := (\varepsilon^{(n)})$ with $\varepsilon^{(n)} \in \mu_{p^n}$ satisfying $\varepsilon^{(1)} \neq 1$, $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$, and introduce the base field $E_{K_0} := k_K((\varepsilon - 1))$. Then, from the theory

of fields of norms, associated to K, there exists a finite extension E_K of E_{K_0} in a fixed separable closure $E_{K_0}^{\mathrm{sep}}$ such that we have a canonical isomorphism

$$H_K := \mathrm{Gal}\left(\overline{K}/K_\infty\right) \simeq \mathrm{Gal}\left(E_K^{\mathrm{sep}}/E_K\right),$$

where $E_K^{\mathrm{sep}} := \bigcup_{L/K:\text{finite Galois}} E_L$ is a separable closure of E_K. In this way, the arithmetically defined Galois group H_K for p-adic field K_∞ is tranformed into the geometrically defined Galois group $\mathrm{Gal}\left(E_K^{\mathrm{sep}}/E_K\right)$ for the field E_K of power series defined over finite field.

47.2. From infinitesimal to global

Let $\rho : G_K \to GL(V)$ be a p-adic representation of G_K. Then, following Fontaine, we obtain an etale (φ, Γ)-module $\mathbb{D}(V)$. Moreover, by a result of Cherbonnier–Colmez [CC], $\mathbb{D}(V)$ is an overconvergent representation. Note that now Γ_K, being the Galois group of K_∞/K, is abelian and may be viewed as an open subgroup of $\mathbb{Z}_p^*$ via cyclotomic character. This, following Sen, leads naturally to a certain connection. In this way, we are able to transform our initial arithmetic objects of Galois representations into the corresponding structures in geometry, namely, that of p-adic differential equations with Frobenius structure, following Berger [B1]. However, despite of this successful transformation, we now face a new challenge—In general, the p-adic differential equations obtained have singularities. This finally leads to the category of de Rham representations: thanks to the works of Fontaine and Berger, for de Rham representations, there are only removable singularities.

On the other hand, contrary to this infinitesimal theory, thanks to the works of Levelt and Katz ([Le], [Ka2]), we are led to a corresponding global theory, the framework of which was first built up by Crew ([Cre]) based on Berthelot's overconvergent isocrystals ([B2], [BO1,2], [O]). For more details, see the discussion below. Simply put, the up-shot is the follows: If $X^0 \hookrightarrow X$ is a marked regular algebraic curve defined over $\mathbb{F}_q$, then, Crew (for rank one) ([Cre]) and Tsuzuki (in general) ([Ts1]) show that there exists a canonical one-to-one correspondence between p-adic representations of $\pi_1(X^0, *)$ with finite monodromy along $Z = X\backslash X^0$ and the so-called unit-root F-isocrystals on X^0 overconvergent around Z. This result is an arithmetic-geometric analogue of the result of Weil recalled in Part A on correspondence between complex representations of fundamental groups and flat bundles over compact Riemann surfaces, at least when Z is trivial.

Conversely, to go from global overconvergent isocrystals to micro p-adic Galois representations, aiming at establishing the conjectural

MRL relating de Rham representations to semi-stable filtered $(\varphi, N; \omega)$-modules, additional works should be done. We suggest the reader to go to the papers [B3], [Ts2] and [Mar].

§48. Convergent F-isocrystals and rigid stable F-bundles

Recall that the p-adic Monodromy Theorem is built up on Crew and Tsuzuki's works on overconvergent unit-root F-isocrystals. To understand it, in this section, we make some preparations following [Cre]. Along with this same line, we also offer a notion called semi-stable rigid F-bundles of slope zero in rigid analytic geometry, which is the key to our algebraic characterization of p-adic representations of fundamental groups of complete, regular, geometrically irreducible curves defined over finite fields.

48.1. Rigid analytic spaces

Let R be a complete DVR of characteristic zero with perfect residue field k of characteristic p and fraction field K. Let $\mathfrak{X}/R$ be a flat p-adic formal R-scheme with closed fiber $X = \mathfrak{X} \otimes k$ and generic fiber the rigid analytic space $\mathfrak{X}^{\mathrm{an}}/K$. Following Raynaud, the points of $\mathfrak{X}^{\mathrm{an}}$ then are naturally in bijection corresponding to the set of closed subschemes of $\mathfrak{X}$ which are integral, finite and flat over R. Therefore, we have the so-called *specialization map* sp : $\mathfrak{X}^{\mathrm{an}} \to X$ sending a point of $\mathfrak{X}^{\mathrm{an}}$, viewed as a subscheme $\mathfrak{z} \subset \mathfrak{X}$, to its support $\mathfrak{z} \otimes k$, which is a closed point of X. Define, for any subscheme of X (or of $\mathfrak{X}$), its tube $]Z[_{\mathfrak{X}}:=]Z[:= \mathrm{sp}^{-1}(Z)$. One can easily check that

(i) if $Z \subset X$ is open then $]Z[_{\mathfrak{X}} = \mathfrak{z}^{\mathrm{an}}$ where $\mathfrak{z}$ is a flat lifting of Z over R. In particular, $]X[= \mathfrak{X}^{\mathrm{an}}$;

(ii) if $Z \subset X$ is closed, say, defined by $f_1, \ldots, f_n \in \Gamma(\mathcal{O}_{\mathfrak{X}})$, then

$$]Z[_{\mathfrak{X}} = \{x \in \mathfrak{X}^{\mathrm{an}} : |f_i(x)| < 1 \ \forall i\}.$$

48.2. Convergent F-isocrystals

Let X/k be a separated k-scheme of finite type, and $X \hookrightarrow \mathfrak{Y}$ a closed immersion into a flat p-adic formal R-scheme that is formally smooth in a neighborhood of X. Then the diagonal embedding gives us two natural projections $p_1, p_2 :]X[_{\mathfrak{Y} \times \mathfrak{Y}} \to]X[_{\mathfrak{Y}}$. Following Berthelot ([Ber2]), a *convergent isocrystal* on $(X/K, \mathfrak{Y})$ is a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_{]X[_{\mathfrak{Y}}}$-modules endowed with an isomorphism

$$p_1^* \mathcal{E} \simeq p_2^* \mathcal{E} \qquad (*)$$

restricting to the identity on the image of the diagonal and satisfying the usual compatibility conditions (for more involved copies). A morphism

of convergent isocrystals on $(X/K, \mathfrak{Y})$ is just a morphism of locally free sheaves compatible with $(*)$.

Theorem. (Berthelot) *The category of convergent isocrystals on* $(X/K, \mathfrak{Y})$ *is*
(i) independent, up to canonical equivalence, of the choice of $X \hookrightarrow \mathfrak{Y}$;
(ii) functorial in X/K; *and*
(iii) of local nature on X.

Consequently, since every separated X/k of finite type always admits such embeddings locally on X, we obtain the category of convergent isocrystals on a general X/k by glueing.

48.3.   Integrable and convergent connections

Let $\mathcal{E}$ be a locally free $\mathcal{O}_{]X[}$-sheaf. Then an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{]X[}$ on $\mathcal{E}$ may be obtained via an isomorphismn

$$q_1^* \mathcal{E} \to q_2^* \mathcal{E}, \qquad q_{1,2} : \Delta_1 \to]X[$$

where Δ_1 is the *first* infinitesimal neighborhood Δ_1 of the diagonal $]X[\subset]X[\times]X[$, satisfying the usual cocycle conditions (above). Motivated by this, an integrable connection ∇ on $\mathcal{E}$ is called *convergent* if the associated isomorphism above can be extended to $(*)$, i.e., from the first infinitesimal neighborhood to all levels of infinitesmial neighborhoods.

48.4.   Frobenius structure

Now assume that $k \supset \mathbb{F}_q$ and let $F = F_q$ be a fixed power of the absolute Frobenius of k. Choose once and for all a homomorphism $\sigma : K \to K$ extending the p-adic lifting of F_q on $W(k)$ and fixing a uniformizer π of R. Then by the functorial property of categories of convergent isocrystals, the pair (F_q, σ) gives rise to a semi-linear functor F_σ^*. An *F-isocrystal* on X/K is defined to be a convergent isocrystal $\mathcal{E}$ equipped with an isomorphism

$$\Phi : F_\sigma^* \mathcal{E} \to \mathcal{E}.$$

We can see that if ∇ is the integral connection with $\nabla(s) =: \sum_i s_i \otimes \eta_i$, $\eta_i \in \Omega^1_{]X[}$, then

$$\nabla\big(\Phi(s)\big) = \sum_i \Phi(s_i) \otimes \sigma^* \eta_i.$$

48.5.   Unit-root F-isocrystals

In the case when $X = \mathrm{Spec}(k)$, an F-isocrystal on X/K is simply a finite-dimensional K-vector space endowed with a σ-linear automorphism $\Phi : \sigma^* V \simeq V$. Since we assume that $\sigma(\pi) = \pi$, following

Dieudonne (see e.g., [Man], [Dem]), there is a natural decomposition of F-isocrystals $V = \oplus_l V_l$ indexed by a finite set of $l \in \mathbb{Q}$, where, if $l = a/b$ with $a, b \in \mathbb{Z}, (a, b) = 1$ and $b > 0$, $V_l \otimes_{W(k)} W(\overline{k})$ is simply a π^a-eigenspace of Φ^b. We call the number $\frac{l}{\dim V}$ the *Dieudonne slope* of Φ in V. If all slopes are the same, V is *pure*; moreover, V is called a *unit-root isocrystal*, if it is pure of slope zero.

More generally, if $(\mathcal{E}, \Phi)$ is an F-isocrystal on X/R, then for any point $x \to X$ with values in a perfect field, there exists a formal covering $\mathrm{Spf}(R') \to \mathrm{Spf}(R)$ for $\mathrm{Spf} W(k(x)) \to \mathrm{Spf}(W(k))$. Denote by $\sigma' : R' \to R'$ a compatible lift of F. Then the pull-back of $(\mathcal{E}, \Phi)$ to x/R' is an F-isocrystal on x/R', the so-called *fiber* of $(\mathcal{E}, \Phi)$ at x. We say that an F-isocrystal $(\mathcal{E}, \Phi)$ is called *unit-root* if all its fibers are.

Theorem. (Crew) *Let X/k be a smooth k-scheme and suppose that $\mathbb{F}_q \subset k$. Then there exists a natural equivalence of categories*

$$\mathbb{G} : \mathbb{Rep}_K\big(\pi_1(X)\big) \simeq \mathrm{Isoc}^{F;\mathrm{ur}}(X/K)$$

where $\mathbb{Rep}_K\big(\pi_1(X)\big)$ denotes the category of K-representations of the fundamental group $\pi_1(X)$ of X, and $\mathrm{Isoc}^{F;\mathrm{ur}}(X/K)$ denotes the category of unit-root F-isocrystals on X/K.

This result is based on Katz's work on the correspondence between R-representations of $\pi_1(X)$ and the so-called unit-root F-lattices on $\mathfrak{X}/R$ ([Ka1]). Here, as usual, by an *F-lattice* on $\mathfrak{X}/(R, \phi)$, we mean a locally free $R \otimes \mathcal{O}_{\mathfrak{X}}$-modules $\mathbb{E}$ equipped with a map $\Phi : \phi^* \mathbb{E} \to \mathbb{E}$ such that $\Phi \otimes \mathbb{Q}$ is an isomorphism ($\phi : \mathfrak{X} \to \mathfrak{X}$ a lifting of the absolute Frobenius of X).

The key to Crew's proof is the following Langton type result:

Lemma. ([Cre]) *Let X/k be a smooth affine k-scheme and $(\mathcal{E}, \Phi)$ be a unit-root F-isocrystal on X/K. Then there is a unit-root F-lattice $(\mathbb{E}, \Pi)$ on $\mathfrak{X}/R$ such that $(\mathcal{E}, \Phi) = (\mathbb{E}, \Pi)^{\mathrm{an}}$.*

48.6. Stability of rigid F-bBundles

The above result of Crew may be viewed as an arithmetic analogue of Weil's result on the correspondence between representations of fundamental groups and flat bundles over compact Riemann surfaces. However now the context is changed to curves defined over finite fields of characteristic p, the representations are p-adic, and, accordingly the flat bundles are replaced by unit-root F-isocrystals. In fact, the arithmetic result is a bit more refined: since the associated fundamental group is pro-finite, the actural analogue in geometry is better to be understood as the one for unitary representations and unitary flat bundles.

With this picture in mind, it is then very naturally to ask whether an arithmetic structure in parallel with Narasimhan–Seshadri correspondence between unitary representations and semi-stable bundles of slope zero can be established in the current setting. This is our next topic.

With the same notations as above, assume in addition that X is completed. Then it makes sense to talk about locally free F-sheaves $\mathcal{E}$ of $\mathcal{O}_{]X[}$-modules. If $X = \mathrm{Spec}(k)$, then $\mathcal{E}$ is nothing but a finite-dimensional K-vector space V endowed with a σ automorphism Φ : $\sigma^*V \simeq V$. Similarly, we have its associated Dieudonne slope. Consequently, for general X, if $\mathcal{E}$ is a locally free F-sheaves $\mathcal{E}$ of $\mathcal{O}_{]X[}$-modules, then we can talk above its fibers at points of X with values in a perfect field. We say that a locally free F-sheaf $\mathcal{E}$ of $\mathcal{O}_{]X[}$-modules is of *slope* $s \in \mathbb{Q}$, denoted by $\mu(\mathcal{E}) = s$, if all its fibers have slope s; and $\mathcal{E}$ is called *semi-stable* if for all saturated F-submodules $\mathcal{E}'$, we have all slopes of the fibers of $\mathcal{E}'$ is at most $\mu(\mathcal{E})$. As usual, if the slopes satisfy the strict inequalities, then we call $\mathcal{E}$ *stable*. For simplicity, we call such locally free objects semi-stable (resp. stable) rigid F-bundles on X/K of slope s.

Conjectural MRL in Rigid Analytic Geometry. *Let X be a regular projective curve defined over k. There is a natural one-to-one correspondence between absolutely irreducible K-representations of $\pi_1(X)$ and stable rigid F-bundles on X/K of slope zero.*

Remark. It is better to rename the above as a Working Hypothesis: There are certain points here which have not yet been completed understood due to lack of time. (For example, in terms of intersection, the so-called Hodge polygon is better than Newton polygon adopted here....) See however [Ked1,2].

§49. Overconvergent F-isocrystals, log geometry and stability

49.1.   Overconvergent isocrystals

Suppose that $j : X \hookrightarrow \bar{X}$ is an open immersion, $\bar{X} \hookrightarrow \mathfrak{Y}$ is a closed immersion with $\mathfrak{Y}/R$ smooth in a neighborhood of X and let $Z := \bar{X} - X$. If Z is locally defined by $f_1, \ldots, f_n \in \Gamma(\mathcal{O}_{\mathfrak{Y}})$, set, for $\lambda < 1$,

$$Z_\lambda := \big\{ x \in]\bar{X}[: |f_i(x)| < \lambda \ \forall i \big\}, \qquad X_\lambda :=]\bar{X}[-Z_\lambda,$$

and let $j_\lambda : X_\lambda \hookrightarrow]\bar{X}[$ be the natural inclusion. It is well-known that the pro-object $\{X_\lambda\}_{\lambda \to 1}$ does not depend on the choice of f_i. So, for any coherent sheaf $\mathcal{E}$ on $]\bar{X}[$, it makes sense to talk about $j^\dagger \mathcal{E} := \varinjlim (j_\lambda)_* j_\lambda^* \mathcal{E}$. For example, the sheaf $j^\dagger \mathcal{O}_{]\bar{X}[} \subset \mathcal{O}_{]X[}$ is the ring of germs of functions on $]X[$ extending into the tube $]Z[$. Denote by p_1^*, p_2^* the two functors from

the category of $j^\dagger \mathcal{O}_{]\bar{X}[\mathfrak{Y}}$-modules to the category of $j^\dagger \mathcal{O}_{]\bar{X}[\mathfrak{Y} \times \mathfrak{Y}}$-modules. An *overconvergent isocrystal* $\mathcal{E}$ on $(X/K, \mathfrak{Y}, Z)$ is defined to be a locally free sheaf of $j^\dagger \mathcal{O}_{]\bar{X}[\mathfrak{Y}}$-module $\mathcal{E}$ endowed with an isomorphism $p_1^* \mathcal{E} \simeq p_2^* \mathcal{E}$ satisfying the standard cocyle conditions.

Theorem. (Berthelot) *The category of overconvergent isocrystals on $(X/K, \mathfrak{Y}, Z)$ is*
(i) independent of $\mathfrak{Y}$, up to canonical equivalence;
(ii) of local nature on $\bar{X}$; and
(iii) functorial in the pair $(X \subset \bar{X})$.

Consequently, we define a category of overconvergent isocrystals on $(X/K, Z)$ for any $X \subset \bar{X}$ with $\bar{X}/k$ separated of finite type by glueing. In fact, much stronger result holds:

Theorem. (Berthelot) *If X/k is separated and of finite type and $X \subset \bar{X}$ is a compactification of X, then the category of overconvergent isocrystals on $(X/K, \bar{X})$ (i) depends, up to canonical equivalence, on X/K only; (ii) is of local nature on X; and (iii) is functorial in X/K.*

Due to this, we often call it the category of overconvergent isocrystals on X/K simply.

Similarly, an *overconvergent F-isocrystal* on X/K is defined to be an overconvergent isocrystal $\mathcal{E}$ equipped with an isomorphism $\Phi : F_\sigma^* \mathcal{E} \simeq \mathcal{E}$. Denote by $\mathrm{OIsoc}^{F;\mathrm{ur}}(X/K)$ the category of unit-root overconvergent F-isocrystals on X/K.

49.2. *p*-adic reps with finite local monodromy

From now on assume that X/k is a regular geometrically connected curve with regular compatification $\bar{X}$. Let $Z := \bar{X} - X$. We say that a *p*-adic representation $\rho : \pi_1(X) \to GL(V)$ is having *finite (local) monodromy around Z* if for each $x \in Z$, the image under ρ of the inertia group at x is finite. Denote by $\mathbb{Rep}_K\big(\pi_1(X)\big)^{\mathrm{fin}}$ the associated Tannakian category.

Theorem. (Crew‖Crew for rank one, Tsuzuki in general) *The restriction of the Crew equivalence $\mathbb{G}$ induces a natural equivalence*

$$\mathbb{G}^\dagger : \mathbb{Rep}_K\big(\pi_1(X)\big)^{\mathrm{fin}} \to \mathrm{OIsoc}^{F;\mathrm{ur}}(X/K).$$

More generally, instead of unit-root condition, there is a notion of quasi-unipotency. In this language, then the *p*-adic Monodromy Theorem is nothing but the following

***p*-adic Monodromy Theorem.** (Crew, Tsuzuki‖Crew, Tsuzuki, Andre, Kedlaya, Mebkhout) *Every overconvergent F-isocrystal is quasi-unipotent.*

In addition, quasi-unipotent overconvergent F-isocrystal has been beautifully classified by Matsuda ([Mat]). Simply put, we now have the following structural

Theorem. (Crew, Tsuzuki, MA(C)K, Matsuda) *Every overconvergent F-isocrystal is Matsudian, i.e., admits a natural decomposition to the so-called Matsuda blocks defined by tensor products of etale and unipotent objects.*

In a certain sense, while unit-root objects are coming from representations of fundamental groups, quasi-unipotent objects are related with representations of central extension of fundamental groups. Finally, we would like to recall that overconvergent isocrystals have been used by Shiho to define crystalline fundamental groups for high dimensional varieties ([Sh1,2]).

49.3. Logarithmic rigid analytic geometry

The above result of Crew & Tsuzuki is built up from the open part X of $\bar{X}$, a kind of arithmetic analogue of local constant systems over $\mathbb{C}$. As we have already seen, in Part A, to have a complete theory, it is even better if such a theory can be studied over the whole $\bar{X}$: After all, for representation side, $\mathbb{R}\mathrm{ep}_K\big(\pi_1(X)\big)^{\mathrm{fin}}$ is nothing but $\mathbb{R}\mathrm{ep}_K\big(\pi_1(X)\big)^Z$, that is, p-adic representations of $\pi_1(X)$ with finite local monodromy around every mark $P \in Z$. For doing so, we propose two different approaches, namely, analytic one and algebraic one.

Let us start with the analytic approach. As said, the analytic condition of unit-root F-isocrystals on X overconvergent around Z is defined over (infinitesmal neighborhood of) X. We need to extend it to the total space $\bar{X}$. As usual, this can be done if we are willing to pay the price, i.e., allowing singularities along the boundary. Certainly, in general term, singularities are very hard to deal with. However, with our experience over $\mathbf{C}$, particularly, the work of Deligne on local constant systems ([De1]), for the case at hands, fortunately, we can expect that singularities involved are very mild—There are only logarithmic singularities appeared. This leads to the notion of logarithmic convergent F-isocrystals $\mathcal{E}$ over $(\bar{X}, Z)$: Simply put, it is an overconvergent F-isocrystal that can be extended and hence realized as a locally free sheaf of $\mathcal{O}_{]\bar{X}[}$-module $\mathcal{E}$, endowed with an integral connection ∇ with logarithmic singularities along Z

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{]\bar{X}[}(\log Z),$$

not only defined over the first infinitesimal neighborhood but over all levels of infinitesimal neighborhoods.

Let us next turn to algebraic approach. With the notion of semi-stable rigid F-bundles introduced previously, it is not too difficult to introduce the notion of what should be called semi-stable parabolic rigid F-bundles.

Even we understand that additional work has to be done here using what should be called logarithmic formal, rigid analytic geometry, but with current level of understanding of mathematics involved, we decide to leave the details to the ambitious reader. Nevertheless, motivated also by the recent works of Shiho, we would like to single out the following

Correspondence I. *There is a natural one-to-one correspondence between tamely unit-root F-isocrystals on X overconvergent around $Z :=$ $\bar{X} - X$ and what should be called unit-root logarithmic overconvergent F-isocrystals on $(X, Z)/K$.*

Correspondence II. *There is a natural one-to-one correspondence between tamely unit-root F-isocrystals on X overconvergent around $Z :=$ $\bar{X} - X$ and what should be called poly-semi-stable parabolic rigid F-bundles of slope zero on $\left(\mathfrak{X}^{\mathrm{an}}, \mathfrak{Z}^{\mathrm{an}}\right)$. Here $(\mathfrak{X}, \mathfrak{Z})$ denotes a logarithmic formal scheme associated to (X, Z).*

Moreover, by comparing the theory to be developed here with that for π-bundles of algebraic geometry for Riemann surfaces recalled in Part A, for a fixed finite Galois covering $\pi : Y \to X$ ramified at Z, branched at $W := \pi^{-1}(Z)$, it is also natural for us to expect the following

Correspondence III. *There is a natural one-to-one correspondence between orbifold rigid F-bundles on $\left(\mathfrak{Y}^{\mathrm{an}}, \mathfrak{W}^{\mathrm{an}}\right)$ and rigid (fat) parabolic F-bundles on $\left(\mathfrak{X}^{\mathrm{an}}, \mathfrak{Z}^{\mathrm{an}}\right)$ satisfying the following compatibility conditions:*
(i) it induces a natural correspondences among saturated sub-objects;
(ii) it scales the slopes by a constant multiple $\deg(\pi)$.

Assuming all this, then we can obtain the following
Micro Reciprocity Law in Log Rigid Analytic Geometry. *There is a natural one-to-one and onto correspondence*

$$\left\{ \text{irreducible } p\text{-adic representations of } \pi_1(X, *) \right.$$
$$\left. \text{with finite monodromy along } Z := \bar{X} \backslash X \right\}$$

$$\Updownarrow$$

$$\left\{ \text{stable rigid (fat) parabolic } F\text{-bundles of slope } 0 \text{ on } \left(\mathfrak{X}^{\mathrm{an}}, \mathfrak{Z}^{\mathrm{an}}\right) \right\}.$$

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