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# Symmetries and the Riemann Hypothesis

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**Abstract.**

Associated to reductive groups and their maximal parabolic subgroups are genuine zeta functions. Naturally related to Riemann's zeta function and governed by symmetries, including that of Weyl, these zeta functions are expected to satisfy the Riemann Hypothesis.

## §1. Introduction

One of the central problems concerning the Riemann Hypothesis is to create a suitable framework for Riemann's zeta. In addition to the huge success of Weil's conjecture in arithmetic geometry and Selberg's zeta in analytic geometry, there are various very interesting approaches in literatures. In this paper, we initiate a totally new one. The start point is the so-called high rank zeta function for number fields [W1]. Being natural generalizations of Dedekind zetas, these functions satisfy all zeta properties. That is, they are well-defined meromorphic functions, satisfy standard functional equations, and admit only two singularities, all simple poles, at 0 and 1. Particularly, when rank is two, the corresponding zetas satisfy the RH. This then leads to the Riemann Hypothesis for all high rank zetas as well.

Defined using semi-stable lattices, these high rank zetas are supposed to expose non-commutative arithmetic aspects of number fields. As such, then the next step is to study high rank zetas in details. For this, by applying the Mellin transform, we can first write high rank zetas as integrations of Epstein (type) zetas over certain compact moduli spaces.

Note that Epstein zetas are indeed special kinds of Eisenstein series. Thus we are led to consider what we call Eisenstein periods. Furthermore, besides this geo-arithmetical interpretation of Eisenstein periods, these periods can also be understood analytically.

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Recall that due to the moderate growth constant terms, in general, Eisenstein series cannot be integrated over associated fundamental domains. To remedy this, Rankin, Selberg and Arthur introduced an analytic truncation for Eisenstein series, by cutting off these troubled constant terms near cusps, so as to get integrable functions over fundamental domains. Roughly speaking, analytically, Eisenstein periods are integrations of these truncated Eisenstein series.

With this, the analytic and geo-arithmetical interpretations of Eisenstein periods can be related by the follows. It can be shown, even though quite complicated, that moduli spaces of semi-stable lattices are identified with certain compact subsets of fundamental domains (for special linear groups), whose characteristic functions are simply the Arthur truncations of the constant function one (for the special parameter 0). (For details, see §3.) Moreover, with these identifications (and the help of basic properties of Arthur's analytic truncations), more or less, high rank zeta functions can be studied via Eisenstein periods analytically.

On the other hand, general Eisenstein periods are still too complicated to be precisely evaluated. Nevertheless, when the Eisenstein series involved are induced from cusp forms, we can have a satisfied answer, thanks to an advanced version of Rankin–Selberg & Zagier method ([JLR], see also [W0]).

Back to high rank zetas. Unfortunately, constant functions on maximal parabolic subgroups, used in Epstein type zetas, are far away from being cuspidal (except in the case of $SL(2)$). To overcome this difficulty, we then realize Epstein zetas as residues of Eisenstein series associated to constant functions on Borel subgroups which are then cuspidal.

As such, if we were able to interchange the operation of taking integration and the operation of taking residues, we would obtain high rank zetas by taking residues from special Eisenstein periods which we know how to evaluated. It is at this point that various aspects of mathematics involved show their strong characters: while high rank zetas are non-abelian in nature, residues of the latest Eisenstein periods (associated to the cuspidal constant function over the Borels) are essentially abelian as they reflect only properties of constant terms of Eisenstein series. Consequently, such an interchange of orders for two operations is not allowed. Accordingly, the study of high rank zetas breaks into two: the abelian part and the non-abelian part.

Currently, the non-abelian part still proves to be very complicated and hence quite difficult to study. In this work, we concentrate only on the abelian part.

For this, next, we offer an explicit realization of Epstein zetas in terms of residues of Siegel's Eisenstein series induced from the constant function on the Borel. This is first solved for $SL(3)$ with the help of Koecher's zeta and Siegel's zeta (see e.g. [W3]). As it turns out later, our work on $SL(3)$ is closely related with a more general result of Diehl [D]. In fact, based on [D], we are able to write down precisely Epstein zetas as residues of Siegel's Eisenstein series, with some extra efforts. This then leads to a natural definition of new abelian zetas for $SL(n)$.

With the success of $SL(n)$, note that Diehl's paper is in fact for $Sp(2n)$. So we naturally make a parallel study for $Sp(2n)$. As a result, we introduce abelian zetas for $Sp(2n)$.

Up to this point, we ask ourselves whether such a discussion works for all general reductive groups. This then leads to the problem whether an analogue of Epstein zeta for general reductive groups exists. Hence, we go back to check the role in the definition of our zetas played by special maximal parabolic subgroups, which are maximal parabolics $P_{n-1,1}$ (resp. the Siegel subgroups) for $SL(n)$ (resp. for $Sp(2n)$). Particularly, we raise then the following questions: what happens for other maximal parabolic subgroups? and what are singular hyperplanes along which residues are taken?

To answer these questions and to see structures involved clearly, we decide to test lower rank reductive groups. For the obvious reason, $G_2$ is chosen as the next target: Being a rank two reductive group, there are only two maximal parabolics, corresponding to the long and short roots; Moreover, the Eisenstein series associated to constant function on the Borel is of two variables. Thus the number of singular hyper-planes is reduced to one. This study of $G_2$ proves to be a big success. After the works for $G_2$, the role played by maximal parabolic subgroups becomes very clear: As the concrete calculation for $G_2$ shows, the singular hyperplanes can be obtained from the corresponding maximal parabolic. (For details, see §3.) This then leads to a general definition for the zetas associated to a pair $(G, P)$ consisting of reductive groups $G$ and their maximal parabolic subgroups $P$.

More precisely, first, for reductive groups $G$ over $\mathbb{Q}$, we start with the Eisenstein periods defined using Eisenstein series associated to the constant function one on the Borels. Then, using the advanced version of Rankin–Selberg & Zagier method mentioned above, we are able to write down these special Eisenstein periods, called the periods for reductive groups $G$, in terms of Riemann's zeta. These periods of $G$ are governed by the Weyl symmetry. To go further, for a fixed maximal parabolic $P$ (defined over $\mathbb{Q}$), we are able to find naturally singular hyper-planes associated to $P$. By taking residues along these singular hyper-planes, from the periods of $G$, we then obtain the periods of $(G, P)$ over $\mathbb{Q}$. Finally, normalizing them essentially by multiplying zeta factors appeared in the denominators of each terms, we obtain new abelian zetas for $(G, P)$ over $\mathbb{Q}$! This class of zetas are expected to be extremely nice: they (should) satisfy a standard functional equation and the Riemann Hypothesis. As such, we then offer a natural framework for the Riemann's zeta and for the RH.

This paper consists of three main parts: §2, a very short one, gives the basic constructions for the new zetas and formulates the corresponding conjectural FE and RH; §3, a very long one, explains in details all milestones mathematically from which we expose our new zetas; Finally an appendix is added to give examples for $SL(n)$, $n = 2, 3, 4, 5$, $Sp(4)$ and $G_2$.

The first version of this paper was circulated at the end of 2007. Detailed calculations for the above examples were included in the appendix there. However, soon after, Masatoshi Suzuki was able to prove the RH for new zetas associated to $G_2$ and $Sp(4)$. Because of this, we cut off the detailed calculation altogether even for $SL(n)$ in the appendix. Instead, we add a very important discussion on an additional symmetry for our zetas (in the case of $SL(3)$): In fact, our zetas are specializations at $T = 0$ of a more general $T$-version. For general $T$, these $T$-versions satisfy no functional equation themselves. But when $T$ lies on a special line (spanned by $\rho$, the half of all positive roots), the corresponding $T$-versions, i.e., the $\mathbb{C} \cdot \rho$-versions, then satisfy the functional equation. Furthermore, it appears that symmetries at this level is still not enough to guarantee the Riemann Hypothesis: To have the Riemann Hypothesis, we really need to take $T = 0$. That is to say, over the line $\mathbb{C} \cdot \rho$, even all corresponding functions satisfies the functional equation, it is the sole zeta corresponding to the special point 0 that satisfies the Riemann Hypothesis.

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## §2. Symmetries and the Riemann Hypothesis

Let $G$ be a reductive group defined over $\mathbb{Q}$, the field of rationals. As usual, for a fixed Borel subgroup $B/\mathbb{Q}$, denote by $\Delta_0$ the corresponding set of simple roots, $W$ its Weyl group, and $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha$ the so-called Weyl vector. For a positive root $\alpha$, denote by $\alpha^\vee$ its coroot.

**Definition 1.** The *period for $G$ over $\mathbb{Q}$* is defined by

$$\omega_{\mathbb{Q}}^{G}(\lambda) := \sum_{w\in W}\left(\frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\xi_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle)}{\xi_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle+1)}\right),$$
$$\operatorname{Re}\lambda\in\mathcal{C}^+\subset\mathfrak{a}_0^*\subset\mathfrak{a}_{0,\mathbb{C}}^* := \mathfrak{a}_0^*\otimes_{\mathbb{R}}\mathbb{C}$$

where $\mathcal{C}^+$ denotes the so-called positive chamber of $\mathfrak{a}_0^*$, the space of characters associated to $(G,B)$, and $\xi_{\mathbb{Q}}(s)$ the completed Riemann zeta function.

For a fixed maximal parabolic subgroup $P$, it is well known that (the conjugation class of) $P$ corresponds to a simple root $\alpha_P\in\Delta_0$. Denote by $\Delta_0\backslash\{\alpha_P\}=\{\beta_{1,P},\beta_{2,P},\dots,\beta_{r-1,P}\}$ with $r=r(G)$ the rank of $G$.

**Definition 2.** The *period for $(G,P)$ over $\mathbb{Q}$* is defined by

$$\omega_{\mathbb{Q}}^{G/P}(\lambda_P) := \operatorname{Res}_{\langle\lambda-\rho,\beta_{r(G)-1,P}^\vee\rangle=0,\,\cdots,\,\langle\lambda-\rho,\beta_{2,P}^\vee\rangle=0,\,\langle\lambda-\rho,\beta_{1,P}^\vee\rangle=0}\Big(\omega_{\mathbb{Q}}^{G}(\lambda)\Big),$$
$$\operatorname{Re}\lambda_P\gg 0$$

Here, starting from $r$-variable $\lambda\in\mathfrak{a}_{0,\mathbb{C}}^*$, after taking residues along with $(r-1)$ (independent) singular hyperplanes

$$\langle\lambda-\rho,\beta_{1,P}^\vee\rangle=0,\ \langle\lambda-\rho,\beta_{2,P}^\vee\rangle=0,\ \cdots,\ \langle\lambda-\rho,\beta_{r(G)-1,P}^\vee\rangle=0,$$

we are left with only one variable, which we call $\lambda_P$.

Clearly, there is a minimal integer $I(G/P) := I_{G/P}$ and finitely many factors

$$\xi_{\mathbb{Q}}\Big(a_1^{G/P}\lambda_P+b_1^{G/P}\Big),\ \xi_{\mathbb{Q}}\Big(a_2^{G/P}\lambda_P+b_2^{G/P}\Big),\ \cdots,\ \xi_{\mathbb{Q}}\Big(a_{I_{G/P}}^{G/P}\lambda_P+b_{I_{G/P}}^{G/P}\Big),$$

such that *no $\xi_{\mathbb{Q}}(a\lambda_P+b)$ factors appear in the denominators of (all terms of) the product* $\left[\prod_{i=1}^{I_{G/P}}\xi_{\mathbb{Q}}\Big(a_i^{G/P}\lambda_P+b_i^{G/P}\Big)\right]\cdot\omega_{\mathbb{Q}}^{G/P}(\lambda_P)$.

Similarly there is a minimal integer $J(G/P) := J_{G/P}$ and finitely many factors

$$\xi_{\mathbb{Q}}\Big(c_1^{G/P}\Big),\ \xi_{\mathbb{Q}}\Big(c_2^{G/P}\Big),\ \cdots,\ \xi_{\mathbb{Q}}\Big(c_{J_{G/P}}^{G/P}\Big),$$

such that *no factors of special $\xi_{\mathbb{Q}}$ values appear in the denominators of the product* $\Big[\prod_{i=1}^{J_{G/P}} \xi_{\mathbb{Q}}\Big(c_i^{G/P}\Big)\Big] \cdot \omega_{\mathbb{Q}}^{G/P}(\lambda_P)$.

**Definition 3.** (i) The *zeta function* $\xi_{\mathbb{Q};o}^{G/P}$ *for* $(G,P)$ *over* $\mathbb{Q}$ is defined by

$$\xi_{\mathbb{Q};o}^{G/P}\Big(s\Big) := \Bigg[\prod_{i=1}^{I(G/P)} \xi_{\mathbb{Q}}\Big(a_i^{G/P} s + b_i^{G/P}\Big) \cdot \prod_{j=1}^{J(G/P)} \xi_{\mathbb{Q}}\Big(c_j^{G/P}\Big)\Bigg] \cdot \omega_{\mathbb{Q}}^{G/P}\Big(s\Big), \qquad \mathrm{Re}\, s \gg 0$$

**Zeta Facts.** (1) $\xi_{\mathbb{Q};o}^{G/P}\Big(s\Big)$, $\mathrm{Re}\, s \gg 0$, *is a well-defined holomorphic function; admits a unique meromorphic continuation to the whole complex s-plane; and has only finitely many poles;* and
(2) (Conjectural **Functional Equation**) *There exists a constant* $c_{G/P} \in \mathbb{Q}$ *such that*

$$\xi_{\mathbb{Q};o}^{G/P}\Big(-s + c_{G/P}\Big) = \xi_{\mathbb{Q};o}^{G/P}\Big(s\Big)$$

Obviously, (1) stands. On the other hand, (2), offering an additional symmetry, is supposed to be rather complicated.

Classical symmetry $s \leftrightarrow 1-s$ for the standard functional equation then leads to the following normalization.
**Definition 3.** (ii) The *zeta function* $\xi_{\mathbb{Q}}^{G/P}\Big(s\Big)$ *for* $(G,P)$ *over* $\mathbb{Q}$ is defined by

$$\xi_{\mathbb{Q}}^{G/P}\Big(s\Big) := \xi_{\mathbb{Q};o}^{G/P}\Big(s + \frac{c_{G/P} - 1}{2}\Big)$$

With all this, we are now ready to make the following conjecture on the remarkable uniformity of their zeros (shared by all these newly introduced zetas).

**The Riemann Hypothesis$_{\mathbb{Q}}^{G/P}$.**

*All zeros of the zeta* $\xi_{\mathbb{Q}}^{G/P}\Big(s\Big)$ *lie on the central line* $\mathrm{Re}\, s = \frac{1}{2}$

## §3. Abelian parts of non-abelian zetas: discovery of zetas for $(G,P)/\mathbb{Q}$

In this section, we expose some of the landmarks leading to the discovery of these elegant zetas associated to reductive groups $G$ and their maximal parabolics $P$ over $\mathbb{Q}$. Consequently, we explain why the so-called abelian parts of high rank zetas are related to these general zetas.

Contents

## 3.1. High rank zeta functions

3.1.1. **High rank zeta functions** Let $F$ be a number field with $\mathcal{O}_F$ the ring of integers. Denote by $\Delta_F$ the discriminant of $F$. Fix a positive integer $r$. Then by definition, an $\mathcal{O}_F$-lattice $\Lambda$ of rank $r$ is a pair $(P,\rho)$ consisting of an $\mathcal{O}_F$-projective module $P$ of rank $r$ and a metric $\rho := (\rho_{\sigma:\mathbb{R}}, \rho_{\tau:\mathbb{C}})$ on $\Big(\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}\Big)^r = \Big(\mathbb{R}^r\Big)^{r_1}\times\Big(\mathbb{C}^r\Big)^{r_2}$. Here as usual, we denote by $r_1$ and $r_2$ the number of real embeddings $\sigma : F\hookrightarrow\mathbb{R}$ and the numbers of complex embeddings $\tau : F\hookrightarrow\mathbb{C}$, respectively. (Recall that by a standard result, $P$ is isomorphic to $\mathcal{O}_F^{\oplus(r-1)}\oplus\mathfrak{a}$ for a suitable fractional ideal $\mathfrak{a}$ of $F$. Thus via the natural inclusion $\mathcal{O}_F^{\oplus(r-1)}\oplus\mathfrak{a}\hookrightarrow F^r\hookrightarrow\Big(\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}\Big)^r$, we may view $P$ as a discrete subgroup of the matrized space $\Big(\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}\Big)^r$. Fix it.) It is well known that the quotient space $\Big(\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}\Big)^r/\Lambda$ is compact. Call its volume the (co-)volume of $\Lambda$ and denote it by $\mathrm{Vol}(\Lambda)$. By definition, a lattice $\Lambda$ is called *semi-stable* if for all $\mathcal{O}_F$-sublattices $\Lambda_1$, we have

$$\mathrm{Vol}(\Lambda_1)^{\mathrm{rk}(\Lambda)}\geq\mathrm{Vol}(\Lambda)^{\mathrm{rk}(\Lambda_1)}.$$

Denote by $\mathcal{M}_{F,r}$ the moduli space of semi-stable $\mathcal{O}_F$-lattices of rank $r$. (For details, see e.g., [W1-3], [Gr1,2], [St1,2].) This is the first ingredient needed to introduce high rank zetas for $F$. In particular, we know the following

**Fact A.** ([W1-3]) (1) *There is a natural decomposition* $\mathcal{M}_{F,r}=\cup_{T\in\mathbb{R}_{>0}}\mathcal{M}_{F,r}[T]$ *where,* $\mathcal{M}_{F,r}[T]$ *denotes the moduli space of semi-stable* $\mathcal{O}_F$*-lattices of rank r and of volume T;*
(2) $\mathcal{M}_{F,r}[T]$ *is compact;* and
(3) *There are natural measures* $d\mu$ *and* $d\mu_0$ *on* $\mathcal{M}_{F,r}$ *and on* $\mathcal{M}_{F,r}[|\Delta_F|^{\frac{r}{2}}]$ *respectively such that with respect to the decomposition (1), we have* $d\mu = d\mu_0\times\frac{dT}{T}$.

The second ingredient needed is a good geo-arithmetical cohomology. For this, we define the 0-th cohomology group $H^0(F,\Lambda)$ of an $\mathcal{O}_F$-lattice $\Lambda$ to be the the lattice $\Lambda$ itself, and the 1-st cohomology group $H^1(F,\Lambda)$ to be the compact quotient group $\Big(\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}\Big)^r/\Lambda$. Consequently, we have the following Pontryagin duality for them:

**Topological Duality.**

$$\widehat{H^1(F,\Lambda)}\simeq H^0(F,\omega_F\otimes\Lambda^\vee).$$

Here for a locally compact group $G$, denote by $\widehat{G}$ its Pontryagin dual, $\omega_F$ denotes the differential lattice, i.e., the lattice whose module part is simply the module of differentials of $F$, while whose metric is the standard one on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. As such, following Tate ([T] and [W]), we then can use Fourier analysis to count our $H^0(F,\Lambda)$ and $H^1(F,\Lambda)$. For example, each element $\mathbf{x} \in H^0(F,\Lambda)$ is counted with the weight of Gaussian distribution $\exp\Big(-\pi\sum_{\sigma:\mathbb{R}}\|\mathbf{x}\|^2_{\rho_\sigma} - 2\pi\sum_{\tau:\mathbb{C}}\|\mathbf{x}\|^2_{\rho_\tau}\Big)$ and accordingly define $h^0(F,\Lambda)$ to be the logarithm of this count. (See also [GS] and [Ne], where an interesting effectiveness view of point is adopted.) Particularly, with such $h^0$ and $h^1$ for a lattice $\Lambda$, by using the above topological duality and the Poisson summation formula, then we obtain the following

**Fact B.** ([W1-3]) *Let $\Lambda$ be an $\mathcal{O}_F$-lattice of rank $r$. Then*
(1) (**Duality**) $h^1(F,\Lambda) = h^0(F,\omega_F \otimes \Lambda^\vee)$; and
(2) (**Riemann–Roch Theorem**)
$$h^0(F,\Lambda) - h^1(F,\Lambda) = \deg(\Lambda) - \frac{r}{2}\log|\Delta_F|.$$
*Here* $\deg(\Lambda)$ *denotes the Arakelov degree of* $\Lambda$.
(For the reader who does not know Arakelov degree, recall then the following weak result
**Arakelov–Riemann–Roch Theorem:**
$$-\log \mathrm{Vol}(\Lambda) = \deg(\Lambda) - \frac{r}{2}\log|\Delta_F|.)$$

With all this, then we are ready to introduce the following

**Definition.** ([W1,3]) For an algebraic number field $F$ and a positive integer $r$, define its *rank $r$ zeta function* by

$$\xi_{F,r}(s) := \Big(|\Delta_F|\Big)^{\frac{r}{2}s}\int_{\mathcal{M}_{F,r}}\Big(e^{h^0(F,\Lambda)}-1\Big)\cdot\Big(e^{-s}\Big)^{\deg(\Lambda)}d\mu(\Lambda),\ \mathrm{Re}(s)>1.$$

From the definition, by Fact A for moduli spaces and Fact B on Duality and the Riemann–Roch for geo-arithmetic cohomologies, tautologically, we have the following

**Fact C.** ([W1,3]) (0) (Iwasawa) $\xi_{F,1}(s) \doteq \xi_F(s)$, *the completed Dedekind zeta function;*
(1) (**Meromorphic Continuation**) $\xi_{F,r}(s)$ *is well-defined and admits a meromorphic continuation to the whole complex $s$-plane;*
(2) (**Functional Equation**) $\xi_{F,r}(1-s) = \xi_{F,r}(s)$; and
(3) (**Singularities**) *There are only two singularities, i.e., simple poles at $s=0,1$ with the residue* $\mathrm{Res}_{s=1}\xi_{F,r}(s) = \mathrm{Vol}\,\mathcal{M}_{F,r}\Big(|\Delta_F|^{\frac{r}{2}}\Big)$.

3.1.2. **Relation with Eisenstein series** We next give a relation between our high rank zetas and what we call Eisenstein periods. The point here is instead of working over $\mathcal{M}_{F,r}$, we fix a volume so as to work over the *compact* subspace $\mathcal{M}_{F,r}[|\Delta_F|^{r/2}]$ and hence deduce the desired relation via Mellin transform. This goes as follows.

From now on, for simplicity, we work over the field $\mathbb{Q}$ of rationals. Accordingly, the rank $r$ zeta function $\xi_{\mathbb{Q},r}(s)$ of $\mathbb{Q}$ is given by

$$\xi_{\mathbb{Q},r}(s)=\int_{\mathcal{M}_{\mathbb{Q},r}}\Big(e^{h^0(\mathbb{Q},\Lambda)}-1\Big)\cdot\big(e^{-s}\big)^{\deg(\Lambda)}\,d\mu(\Lambda),\qquad \mathrm{Re}(s)>1.$$

Here $h^0(\mathbb{Q},\Lambda)=\log\big(\sum_{x\in\Lambda}\exp\big(-\pi|x|^2\big)\big)$ & $\deg(\Lambda)=-\log\mathrm{Vol}\big(\mathbb{R}^r/\Lambda\big)$.

Decompose according to their volumes, $\mathcal{M}_{\mathbb{Q},r}=\cup_{T>0}\mathcal{M}_{\mathbb{Q},r}[T]$, and there is a natural morphism $\mathcal{M}_{\mathbb{Q},r}[T]\to\mathcal{M}_{\mathbb{Q},r}[1]$, $\Lambda\mapsto T^{\frac{1}{r}}\cdot\Lambda$. Consequently,

$$\begin{aligned}\xi_{\mathbb{Q},r}(s)&=\int_{\cup_{T>0}\mathcal{M}_{\mathbb{Q},r}[T]}\Big(e^{h^0(\mathbb{Q},\Lambda)}-1\Big)\cdot\big(e^{-s}\big)^{\deg(\Lambda)}\,d\mu(\Lambda)\\&=\int_0^\infty T^s\frac{dT}{T}\int_{\mathcal{M}_{\mathbb{Q},r}[1]}\Big(e^{h^0(\mathbb{Q},T^{\frac{1}{r}}\cdot\Lambda)}-1\Big)\,d\mu(\Lambda).\end{aligned}$$

But $h^0(\mathbb{Q},T^{\frac{1}{r}}\cdot\Lambda)=\log\Big(\sum_{x\in\Lambda}\exp\big(-\pi|x|^2\cdot T^{\frac{2}{r}}\big)\Big)$. By applying the Mellin transform, we have

$$\xi_{\mathbb{Q},r}(s)=\frac{r}{2}\cdot\pi^{-\frac{r}{2}s}\Gamma\Big(\frac{r}{2}s\Big)\cdot\int_{\mathcal{M}_{\mathbb{Q},r}[1]}\Big(\sum_{x\in\Lambda\backslash\{0\}}|x|^{-rs}\Big)\,d\mu_0(\Lambda).$$

Accordingly, introduce the completed Epstein zeta function for $\Lambda$ by

$$\hat{E}(\Lambda;s):=\pi^{-s}\Gamma(s)\cdot\sum_{x\in\Lambda\backslash\{0\}}|x|^{-2s}.$$

We then arrive at

**Fact D.** ([W1-3]) (Eisenstein series and high rank zetas)

$$\xi_{\mathbb{Q},r}(s)=\frac{r}{2}\int_{\mathcal{M}_{\mathbb{Q},r}[1]}\widehat{E}(\Lambda,\frac{r}{2}s)\,d\mu_0(\Lambda),\qquad \mathrm{Re}(s)>1.$$

3.1.3. **SL(2): A toy model** To indicate basic ideas clearly, we first give some details on the rank two zeta $\xi_{\mathbb{Q},2}(s)$.

Consider the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half plane $\mathcal{H}(=SL(2,\mathbb{R})/SO(2))$. Then we obtain a standard 'fundamental domain'

$D = \{z = x + iy \in \mathcal{H} : |x| \le \frac{1}{2}, y > 0, x^2 + y^2 \ge 1\}$. Recall also the completed standard Eisenstein series

$$\hat{E}(z;s) := \pi^{-s}\Gamma(s) \cdot \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mz+n|^{2s}}.$$

Naturally, we are led to considering the integral $\int_D \hat{E}(z,s)\frac{dx\,dy}{y^2}$. However, this integration diverges. Indeed, near the only cusp $y = \infty$, by the Chowla–Selberg formula, $\hat{E}(z,s)$ has the Fourier expansion

$$\hat{E}(z;s) = \sum_{n=-\infty}^{\infty} a_n(y,s)e^{2\pi inx}$$

with

$$a_n(y,s) = \begin{cases} \xi(2s)y^s + \xi(2-2s)y^{1-s}, & \text{if } n = 0; \\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y), & \text{if } n \ne 0, \end{cases}$$

where $\xi(s)$ is the completed Riemann zeta function, $\sigma_s(n) := \sum_{d|n} d^s$, and $K_s(y) := \frac{1}{2}\int_0^\infty e^{-y(t+\frac{1}{t})/2}t^s\frac{dt}{t}$ is the K-Bessel function. Moreover,

$$|K_s(y)| \le e^{-y/2}K_{\mathrm{Re}(s)}(2), \text{ if } y > 4, \qquad \text{and} \qquad K_s = K_{-s}.$$

So $a_{n\ne0}(y,s)$ decay exponentially, and the constant term $a_0(y,s)$, being of slow growth, is problematic.

Therefore, to introduce a meaningful integration from the original ill-defined one, we need to cut off the slow growth part. There are two ways to do so: one is geometrical and hence rather direct and simple; the other is analytical, and hence rather technical and traditional, dated back to Rankin–Selberg.

(a) **Geometric truncation**

Draw a horizontal line $y = T \ge 1$ and set

$$D_T = \{z = x + iy \in D : y \le T\}, \qquad D^T = \{z = x + iy \in D : y \ge T\}.$$

Then $D = D_T \cup D^T$. Introduce a well-defined integration

$$I_T^{\mathrm{Geo}}(s) := \int_{D_T} \hat{E}(z,s)\,\frac{dx\,dy}{y^2}.$$

(b) **Analytic truncation**

Define a truncated Eisenstein series $\hat{E}_T(z;s)$ by

$$\hat{E}_T(z;s) := \begin{cases} \hat{E}(z;s), & \text{if } y \leq T; \\ \hat{E}(z,s) - a_0(y;s), & \text{if } y > T. \end{cases}$$

Introduce a well-defined integration

$$I_T^{\text{Ana}}(s) := \int_D \hat{E}_T(z;s)\,\frac{dx\,dy}{y^2}.$$

With this, from the Rankin–Selberg method, we have the following:

**Fact E.** (See e.g., [Z]) (Analytic truncation=Geometric truncation in Rank 2)

$$I_T^{\text{Geo}}(s) = \frac{\xi(2s)}{s-1}\cdot T^{s-1} - \frac{\xi(2s-1)}{s}\cdot T^{-s} = I_T^{\text{Ana}}(s).$$

Each of the above two integrations has its own merit: for the geometric one, we keep the Eisenstein series unchanged, while for the analytic one, we keep the original fundamental domain of $\mathcal{H}$ under $\mathrm{SL}(2,\mathbb{Z})$ as it is.

Note that a particular nice point about the fundamental domain is that it admits a modular interpretation. Thus it would be very nice if we could on the one hand keep the Eisenstein series unchanged, while on the other hand offer some integration domains which appear naturally in certain moduli problems. This is essential the idea of introducing $\mathcal{M}_{F,r}\left(|\Delta_F|^{\frac{r}{2}}\right)$, the first key ingredient for high rank zetas.

(c) **Arithmetic truncation**

Now we explain why above discussion and Rankin–Selberg method have anything to do with our high rank zeta functions. For this, we introduce yet another truncation, the geo-arithmetic one using stability.

So back to the moduli space of rank 2 lattices of volume 1 over $\mathbb{Q}$. Then classical reduction theory gives a natural map from this moduli space to the fundamental domain $D$ above: For any lattice $\Lambda$ in $\mathbb{R}^2$, fix $\mathbf{x}_1 \in \Lambda$ such that its length gives the first Minkowski minimum $\lambda_1$ of $\Lambda$. Then via rotation, we may assume that $\mathbf{x}_1 = (\lambda_1, 0)$. Further, from the reduction theory $\frac{1}{\lambda_1}\Lambda$ may be viewed as the lattice of the volume $\lambda_1^{-2} = y_0$ generated by $(1,0)$ and $\omega = x_0 + iy_0 \in D$. That is to say, the points in $D_T$ constructed in (a) above are in one-to-one corresponding to rank two lattices of volume one whose first Minkowski minimum $\lambda$ satisfy $\lambda_1^{-2} \leq T$, i.e, $\lambda_1 \geq T^{-\frac{1}{2}}$. Set $\mathcal{M}_{\mathbb{Q},2}^{\leq\frac{1}{2}\log T}[1]$ be the moduli space of rank 2 lattices $\Lambda$ of volume 1 over $\mathbb{Q}$ all of whose sublattices $\Lambda_1$ of rank 1 have degrees $\leq \frac{1}{2}\log T$. With this discussion, we have the following

**Fact F.** ([W1-3]) (Geometric truncation = Arithmetic truncation) *There is a natural (quasi) one-to-one, onto morphism*

$$\mathcal{M}_{\mathbb{Q},2}^{\leq \frac{1}{2}\log T}[1] \simeq D_T.$$

*In particular,*

$$\mathcal{M}_{\mathbb{Q},2}^{\leq 0}[1] = \mathcal{M}_{\mathbb{Q},2}[1] \simeq D_1.$$

Consequently, we have the following

**Example in Rank 2.** ([W1-3]) $\xi_{\mathbb{Q},2}(s) = \dfrac{\xi(2s)}{s-1} - \dfrac{\xi(2s-1)}{s}.$

### 3.2. General periods

3.2.1. **Arthur's truncation and Eisenstein periods** Recall that the upper half plane $\mathcal{H}$ admits the following group theoretic interpretation $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$. Thus for high rank zeta functions, we then naturally shift to $G = SL(n)$, or more generally, any split group $G$.

Fix a parabolic subgroup $P$ of $G$ with Levi decomposition $P = MN$, denote by $\mathfrak{a}_P^*$ the complexification of the space of characters associated to $P$. In particular, denote by $\mathfrak{a}_0^*$ the one for the Borel. Denote by $\Delta_0$ the associated collection of simple roots. By definition, an element $T \in \mathfrak{a}_0^*$ is said to be *sufficiently regular*, or sufficiently positive, and denoted by $T \gg 0$ if for all $\alpha \in \Delta_0$ $\langle \alpha, T\rangle \gg 0$ are large enough. Fix such a $T$.

Let $\phi : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$ be a smooth function where $K$ is a maximal compact subgroup of $G(\mathbb{R})$. We define *Arthur's analytic truncation* $\wedge^T\phi$ (for $\phi$ with respect to the parameter $T$) to be the function on $G(\mathbb{Z})\backslash G(\mathbb{R})/K$ given by

$$\Big(\wedge^T\phi\Big)(Z) := \sum_{P:\text{standard}} (-1)^{\text{rank}(P)} \sum_{\delta\in P(\mathbb{Z})\backslash G(\mathbb{Z})} \phi_P(\delta g)\cdot\hat{\tau}_P\big(H_P(\delta g)-T\big),$$

where $\phi_P := \int_{N(\mathbb{R})/N(\mathbb{R})\cap SL(n,\mathbb{Z})} f(xn)\,dn$ denotes the constant term of $\phi$ along with the standard parabolic subgroup $P$, $\hat{\tau}_P$ is the characteristic function of the so-called positive cone in $\mathfrak{a}_P^*$, and $H_P(g) := \log_M m_P(g)$ is an elelemnt in $\mathfrak{a}_P^*$. (For unknown notation, all standard, see e.g., [Ar1,2], [JLR], or [W-1,3].)

Fundamental properties of Arthur's truncation may be summarized as:

**Fact G.** ([Ar1,2] &/or [OW]) *For a sufficiently positive $T$ in $\mathfrak{a}_0^*$, we have*
(1) $\wedge^T\phi$ *is rapidly decreasing, if $\phi$ is an automorphic form on the space* $G(\mathbb{Z})\backslash G(\mathbb{R})/K$;
(2) $\wedge^T \circ \wedge^T = \wedge^T$;
(3) $\wedge^T$ *is self-adjoint;* and
(4) ([Ar3]) $\wedge^T\mathbf{1}$ *is a characteristic function of a certain compact subset of* $G(\mathbb{Z})\backslash G(\mathbb{R})/K$.

Denote by $\mathfrak{F}(T)$ the compact subset of $G(\mathbb{Z})\backslash G(\mathbb{R})/K$ whose characteristic function is given by $\wedge^T\mathbf{1}$ by (4).

**Corollary.** ([W1,3]) *Let $T \gg 0$ be a fixed element in $\mathfrak{a}_0^*$. If $\phi$ is an automorphic form on* $G(\mathbb{Z})\backslash G(\mathbb{R})/K$,

$$\int_{G(\mathbb{Z})\backslash G(\mathbb{R})/K} \wedge^T\phi(g)\,dg = \int_{\mathfrak{F}(T)} \phi(g)\,dg.$$

We call the above integration the *Arthur periods* associated to $\phi$. In most of applications, the following special class, called Eisenstein periods, plays a key role.

Recall that if $\varphi$ is an $M$-level automorphic form, then we may form the associated Eisenstein series $E^{G/P}(\varphi,\lambda)(g) = E(\varphi,\lambda)(g) = E(\varphi;\lambda;g)$ as follows:

$$E(\varphi,\lambda)(g) := \sum_{\delta\in P(\mathbb{Z})\backslash G(\mathbb{Z})} m_P(\delta g)^{\lambda+\rho_P}\cdot\phi(\delta g), \quad \mathrm{Re}\,\lambda \in \mathcal{C}_P^+$$

where $\mathcal{C}_P^+$ denotes the positive chamber in $\mathfrak{a}_P$. By definition, the *Eisenstein period* is the integration

$$\int_{G(\mathbb{Z})\backslash G(\mathbb{R})/K} \wedge^T E(\varphi,\lambda)(g)\,dg = \int_{\mathfrak{F}(T)} E(\varphi,\lambda)(g)\,dg.$$

(Here we use a normalization for the Eisenstein series as usual, i.e., shifting the variable from $\lambda$ to $\lambda+\rho_P$, so that the convergence region is simply the positive chamber.)

3.2.2. **Rankin–Selberg & Zagier method I: sufficiently positive case** In general, it is *very difficult*, in fact, *quite impossible,* to calculate Eisenstein period precisely. However, if the original automorphic function (in defining the Eisenstein series used) is *cuspidal*, this can be evaluated. This is a result due to Jaquet–Lapid–Rogowski (see e.g., [JLR]), which itself may be viewed as an advanced version of the so-called Rankin–Selberg & Zagier method. (See also section 4.2 [W0] for our own solution, which was quite similar and was independently written before we knew [JLR].) In particular, for constant function **1** over the Borel, and the associated Eisenstein series $E(\mathbf{1};\lambda;g)$, we have the following:

**Fact E′.** ([JLR], [W0]) *Assume that $T$ is sufficiently positive, then the Eisensetin period $\int_{G(\mathbb{Z})\backslash G(\mathbb{R})}\wedge^T E(\mathbf{1};\lambda;g)\,dg$ is given by*

$$\int_{G(\mathbb{Z})\backslash G(\mathbb{R})}\wedge^T E(\mathbf{1};\lambda;g)\,dg = v\sum_{w\in W}\frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot M(w,\lambda)$$

*where $v=\mathrm{Vol}\big(\{\sum_{\alpha\in\Delta_0}a_\alpha\alpha^\vee : 0\leq a_\alpha<1\}\big)$, $W$ denotes the Weyl group, $\Delta_0$ the set of simple roots, $\alpha^\vee$ the coroot associated to $\alpha$, and $M(w,\lambda)$ denotes the assosciated intertwining operator.*

3.2.3. **Geo-arithmetic truncation and analytic truncation** In algebraic geometry, or better in Geometric Invariant Theory ([M]), a fundamental principle, which we call *the Micro-Global Principle*, claims that if a point is not GIT stable then there exists a parabolic subgroup which destroys the corresponding stability.

Here even we do not have a proper definition of GIT stability for lattices, in terms of intersection stability, an analogue of the Micro-Global Principle holds. To see this, we go as follows (and for our own convenience, we adopt an adelic language when necessary).

For $g=g(\Lambda)\in G(\mathbb{A})$, denote its associated lattice by $\Lambda^g$, and its induced filtration from $P$ by

$$0=\Lambda_0^{g,P}\subset\Lambda_1^{g,P}\subset\cdots\subset\Lambda_{|P|}^{g,P}=\Lambda^g.$$

(Recall that all lattices can be obtained in this manner, and that for a fixed lattice, its associated fiber in $G(\mathbb{A})$ is compact.) Assume that $P$ corresponds to the partition $I=(d_1,d_2,\ldots,d_{n=:|P|})$. Consequently, we have

$$\mathrm{rk}(\Lambda_i^{g,P})=r_i:=d_1+d_2+\cdots+d_i,\qquad \text{for } i=1,2,\ldots,|P|.$$

Define the polygon $p_P^g=p_P^{\Lambda^g}:[0,r]\to\mathbb{R}$ of $\Lambda=\Lambda^g$ with respect to $P$ by
(1) $p_P^g(0)=p_P^g(r)=0$;
(2) $p_P^g$ is affine on $[r_i,r_{i+1}]$, $i=1,2,\ldots,|P|-1$; and
(3) $p_P^g(r_i)=\deg\big(\Lambda_i^{g,P}\big)-r_i\cdot\frac{\deg(\Lambda^g)}{r}$, $i=1,2,\ldots,|P|-1$.
Note that if the volume of $\Lambda$ is assumed to be one, then (3) is equivalent to
(3)′ $p_P^g(r_i)=\deg\big(\Lambda_i^{g,P}\big)$, $i=1,2,\ldots,|P|-1$.

Based on stability, we may introduce a more general geometric truncation for the space of lattices. For this we start with the following easy statement:
*For a fixed $\mathcal{O}_F$-lattice $\Lambda$, $\Big\{\mathrm{Vol}(\Lambda_1) : \Lambda_1 \subset \Lambda\Big\} \subset \mathbb{R}_{\geq 0}$ is discrete and bounded from below.*

As a direct consequence, we have the following
**Fact H.** ([W1,3]) (Canonical Filtration) *For an $\mathcal{O}_F$-lattice $\Lambda$, there exists a unique filtration, called the canonical filtration of $\Lambda$, of proper sublattices*

$$0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_s = \Lambda$$

*such that*
(1) *for all $i = 1, \cdots, s$, $\Lambda_i/\Lambda_{i-1}$ is semi-stable*; and
(2) *for all $j = 1, \cdots s-1$,*

$$\Big(\mathrm{Vol}(\Lambda_{j+1}/\Lambda_j)\Big)^{\mathrm{rk}(\Lambda_j/\Lambda_{j-1})} > \Big(\mathrm{Vol}(\Lambda_j/\Lambda_{j-1})\Big)^{\mathrm{rk}(\Lambda_{j+1}/\Lambda_j)}.$$

Accordingly, for an $\mathcal{O}_F$-lattice $\Lambda$ with the associated canonical filtration, (an analogue of the Harder–Narasimhan–Langton filtration for vector bundles over Riemann surfaces [HN],)

$$0 = \overline{\Lambda}_0 \subset \overline{\Lambda}_1 \subset \cdots \subset \overline{\Lambda}_s = \Lambda$$

define the associated *canonical polygon* $\overline{p}_\Lambda : [0, r] \to \mathbb{R}$ by the following conditions:

(1) $\overline{p}_\Lambda(0) = \overline{p}_\Lambda(r) = 0$;
(2) $\overline{p}_\Lambda$ is affine over the closed interval $[\mathrm{rk}\overline{\Lambda}_i, \mathrm{rk}\overline{\Lambda}_{i+1}]$; and
(3) $\overline{p}_\Lambda(\mathrm{rk}\overline{\Lambda}_i) = \deg(\overline{\Lambda}_i) - \mathrm{rk}(\overline{\Lambda}_i) \cdot \frac{\deg(\overline{\Lambda})}{r}$.

Let now $p, q : [0, r] \to \mathbb{R}$ be two polygons such that $p(0) = q(0) = p(r) = q(r) = 0$. Then, we say $q$ is *bigger than $p$ with respect to $P$* and denote it by $q >_P p$, if $q(r_i) - p(r_i) > 0$ for all $i = 1, \ldots, |P| - 1$. (See e.g., [Laf].) Introduce also the characteristic function $\mathbf{1}(\overline{p}^* \leq p)$ by

$$\mathbf{1}(\overline{p}^g \leq p) = \begin{cases} 1, & \text{if } \overline{p}^g \leq p; \\ 0, & \text{otherwise.} \end{cases}$$

Here $\overline{p}^g$ denotes the canonical polygon for the lattice corresponding to $g$. Recall that for a parabolic subgroup $P$, $p_P^g$ denotes the polygon induced by $P$ for (the lattice corresponding to) the element $g \in G(\mathbb{A})$.

**Fact I.** ([W1,3]) (Fundamental Relation) *For a fixed convex polygon* $p:[0,r]\to\mathbb{R}$ *such that* $p(0)=p(r)=0$, *we have*

$$\mathbf{1}(\overline{p}^g\leq p)=\sum_{P:\,\text{standard parabolic}}(-1)^{|P|-1}\sum_{\delta\in P(F)\backslash G(F)}\mathbf{1}(p_P^{\delta g}>_P p).$$

**Remarks.** (1) This is an arithmetic analogue of a result of Lafforgue ([Laf]) for vector bundles over function fields.
(2) The right hand side may be naturally decomposite into two parts according to whether $P=G$ or not. In such away, the right hand side becomes

$$\mathbf{1}_G-\sum_{P:\,\text{proper standard parabolic}}(-1)^{|P|-1}\cdots.$$

This then exposes two aspects of our geometric truncation: First of all, if a lattice is not stable, then there are parabolic subgroups which take the responsibility; Secondly, each parabolic subgroup has its fix role—Essentially, they should be counted only once each time. In other words, if more are substracted, then we need to add one fewer back to make sure the whole process is not overdone.

From (2) above, it is clear that the geo-arithmetical truncation defined using $\mathbf{1}(\overline{p}^g\leq p)$, or simply using stability, has the same strucrure as that for analytic truncations. Next, we want to give a precise relation between these two truncations, so that analytic methods created in the study of trace formula can be employed in the study of our zetas.

Recall that a polygon $p:[0,r]\to\mathbb{R}$ is called *normalized* if $p(0)=p(r)=0$. For a (normalized) polygon $p:[0,r]\to\mathbb{R}$, define the associated (real) character $T=T(p)\in\mathfrak{a}_0$ of $M_0$ (the Levi for the Borel) by the condition that

$$\alpha_i(T)=\Big[p(i)-p(i-1)\Big]-\Big[p(i+1)-p(i)\Big]$$

for all $i=1,2,\ldots,r-1$, where $\alpha_i=e_i-e_{i+1}\in\Delta_0$ denote simple roots. As such, one checks that

$$T(p)=\Big(p(1)-p(0),\cdots,\cdots,p(i)-p(i-1),\ldots,p(r)-p(r-1)\Big).$$

Set also $\mathbf{1}(p_P^*>_P p)$ to be the characteristic function of the subset of $g$'s such that $p_P^g>_P p$. Then we have the following

**Fact J.** (i) ([W1,3]) (Micro Bridge) *For a fixed convex normalized polygon* $p : [0, r] \to \mathbb{R}$, *and* $g \in SL_r(\mathbb{A})$, *with respect to any parabolic subgroup* $P$, *we have*

$$\hat{\tau}_P\Big( - H_0(g) - T(p)\Big) = \mathbf{1}\Big(p_P^g >_P p\Big).$$

With this micro bridge, we are ready to expose a beautiful intrinsic relation between our geo-arithmetic truncation using stability and analytic truncations.

**Fact J.** (ii) ([W1,3]) (Global Bridge) *For a fixed normalized convex polygon* $p : [0, r] \to \mathbb{R}$, *let*

$$T(p) = \Big(p(1), p(2)-p(1), \dots, p(i)-p(i-1), \dots, p(r-1)-p(r-2), -p(r-1)\Big)$$

*be the associated vector in* $\mathfrak{a}_0$. *If* $T(p)$ *is sufficiently positive, then*

$$\mathbf{1}(\overline{p}^g \leq p) = \Big(\wedge^{T(p)}\, \mathbf{1}\Big)(g).$$

In particular, by Facts G, I, and J, we arrive at the following analytic interpretation of the moduli space of semi-stable lattices.

**Fact G-I-J.** ([W1,3]) $\mathfrak{F}(0) = \mathcal{M}_{\mathbb{Q},r}[1]$.

3.2.4. **Rankin–Selberg & Zagier method II: semi-stable case** The Fact G-I-J proves to be very important: with this intrinsic relation between geo-arithmetical truncation and analytic truncation, instead of using geo-arithemtical method to study high rank zeta functions, which is rather new and less developed, we can equally use analytic technics and methods from trace formula, which is more systematic and rich, to help us. As an example, we here indicate how to evaluate the Eisenstein period $\int_{\mathcal{M}_{\mathbb{Q},r}[1]} E(\lambda; \mathbf{1}; g)\, dg$.

First, by Fact G-I-J, it is equal to $\int_{G(\mathbb{Z})\backslash G(\mathbb{R})/SO(n)} \Lambda^0 E(\lambda; \mathbf{1}; g)\, dg$. On the other hand, by Fact E$'$, we already know that when $T$ is sufficiently positive, $\int_{G(\mathbb{Z})\backslash G(\mathbb{R})/SO(n)} \Lambda^T E(\lambda; \mathbf{1}; g)\, dg$ can be evaluated. As such, then the only point here of course is to check whether the argument used for sufficiently positive $T$ are still valid when $T$ is taken to be 0.

By examining the proof in Arthur's fundamental works [Ar1,2], to take care of the change from sufficiently positive $T$ to smaller $T$, say $T = 0$, additional two main points must be checked. They are
(1) Fact G for smaller $T$. This now is replaced by Fact G-I-J. Cleared.
(2) The convergences of all integrations involved in the proof. This is indeed a very serious one. In a sense, modulo combinatorial technics,

establishing various convergences is really the technical heart of Arthur's trace formula (in its preliminary form as stated in [Ar1-3]). Fortunately, we can justify these convergences when $T$ is smaller, in particular when $T = 0$. Practically, this is carried out in two steps. First, for sufficiently positive $T$, we follow simply the original arguments in [Ar1-3] and [JLR]. Then for general $T \geq 0$, we use the fact that the difference for integral domains involved between sufficiently positive $T$ and rather small $T$, say, $T = 0$, is only up to a certain suitable compact subset in a fundamental domain—after all, over compact subsets, integrability becomes trivial for smooth functions. In this way, we then arrive at the following

**Fact E$''$.** ([W1,3]) *The Eisensetin period* $\int_{\mathcal{M}_{\mathbb{Q},r}[1]} E(\mathbf{1};\lambda;g)\,dg$ *is given by*

$$\int_{\mathcal{M}_{\mathbb{Q},r}[1]} E(\mathbf{1};\lambda;g)\,dg = v \sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot M(w,\lambda)$$

3.2.5. **Intertwining operator: Gindikin–Karpelevich formula** To go further, we need to write down also the intertwining operator $M(w,\lambda)$. This is now well known—by the Gindikin–Karpelevich formula, we have

**Fact K.** (See e.g., [La2]) *For every split, semi-simple group $G$, its associated intertwinging operator acting on constant function* **1** *over the Borel is given by*

$$M(w,\lambda) = \prod_{\alpha>0,w\alpha<0} \frac{\xi\big(\langle\lambda,\alpha^\vee\rangle\big)}{\xi\big(\langle\lambda,\alpha^\vee\rangle+1\big)}.$$

*Here $\xi(s)$ is the completed Riemann zeta with $\Gamma$-factor, namely, $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ with $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ the standard Riemann zeta function.*

3.2.6. **Periods for $SL(n)$ over $\mathbb{Q}$: Weyl symmetry** As usual, when $G = SL(n)$, we use $(z_1, z_2, \ldots, z_n)$ satisfying $z_1 + z_2 + \cdots + z_n = 0$ for the variables $\lambda$. By Facts E$''$, K, for sufficiently positive $T$, the associated Eisenstein period

$$\omega_{\mathbb{Q}}^{SL(n),T}(\lambda) := \int_{SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n)} \wedge^T E(\mathbf{1}; z_1, z_2, \ldots, z_n; M)\,d\mu(M)$$

is given by the following

**Fact L.** ([W1,3]) *Up to a constant factor,*

$$\omega_{\mathbb{Q}}^{SL(n),T}(\lambda) = \sum_{w\in W} \frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle} \cdot \prod_{\alpha>0,w\alpha<0} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}.$$

With this, by a close look at the right hand side, we conclude that now we may take $T = 0$, even the right hand only makes sense for sufficiently positive $T \gg 0$. This then leads to

**Definition 1.** *The period for $G$ over $\mathbb{Q}$ is defined by*

$$\omega_{\mathbb{Q}}^{G}(\lambda) := \sum_{w\in W} \left( \frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle} \cdot \prod_{\alpha>0,w\alpha<0} \frac{\xi_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle)}{\xi_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle+1)} \right), \qquad \mathrm{Re}\,\lambda \in \mathcal{C}^+$$

*where $\mathcal{C}^+$ denotes the standard positive Weyl chamber of $\mathfrak{a}_0^*$, the space of characters associated to the Borel $B$, and $\xi_{\mathbb{Q}}(s)$ the completed Riemann zeta function.*

Certainly this is exact the definition 1 in the previous section. As such, the most notable point in this definition is the huge symmetry created by the Weyl group.

### 3.3. New zetas for $SL(n)/\mathbb{Q}$

3.3.1. **Epstein, Koecher, Siegel zetas and Siegel–Eisenstein series** The reason why we care about Eisenstein periods $\int_{\mathcal{M}_{\mathbb{Q},r}[1]} E(\lambda;\mathbf{1};g)\,dg$, which are of several variables, is that this period can be evaluated and that Epstein zetas $E(\Lambda^g,s)$ appeared in the study of high rank zetas are residues of Eisenstein series $E(\mathbf{1};\lambda;g)$:

$$\xi_{\mathbb{Q},r}(s) = \frac{r}{2}\int_{\mathcal{M}_{\mathbb{Q},r}[1]} \hat{E}(\Lambda^g,\frac{r}{2}s)\,dg$$

where $\hat{E}(\Lambda^g,s) = \pi^{-s}\Gamma(s)\cdot E(\Lambda^g,s)$. To explain this, we go as follows.

Let $\mathfrak{N} := \{\mathrm{diag}(\pm1,\ldots,\pm1)\}\backslash SL(n,\mathbb{Z})$ and $\mathfrak{Q}_r$ the standard parabolic subgroup associated to the partition $n = r+1+1+\cdots+1$, that is, the parabolic subgroup $P_{r,1,\ldots,1}$ consisting of matrices in $SL(n,\mathbb{Z})$ of the form $\begin{pmatrix} H & & & \\ & 1 & * & \\ & 0 & \ddots & \\ & & & 1\end{pmatrix}$ with $H = H^{(r)}, |H| = 1$. Define the associated

*Siegel zeta functions* by

$$\xi_r^*(Y; s_r, \ldots, s_{n-1}) := \sum_{N \in \mathfrak{Q}_r \backslash \mathfrak{R}} \prod_{v=r}^{n-1} |Y[N]_v|^{-s_v}$$

for all $1 \leq r \leq n-1$, where, as usual, $Y[N] := N^t \cdot Y \cdot N$ and for a size $n$ matrix $A = (a_{ij})_{i,j=1}^n$, $A_v$ denotes the matrix $A_v = (a_{ij})_{i,j=1}^v$ . Then, from [D], we have the following

**Lemma 1.** ([D]) *There exists a constant c depending only on r such that*

$$\mathrm{Res}_{s_r = \frac{r+1}{2}} \xi_r^*(Y; s_r, \ldots, s_{n-1}) = c_r \cdot \xi_{r+1}^*(Y; s_{r+1} + \frac{r}{2}, s_{r+2}, \ldots, s_{n-1}).$$

(Please correct a misprint in [D] for this formula.) Consequently, taking $r = 1$ and repeating this process, we obtain the following

$$\mathrm{Res}_{s_{n-1}=1} \cdots \mathrm{Res}_{s_2=1} \mathrm{Res}_{s_1=1} \Big( \xi_1^*(Y; s_1, s_2, \ldots, s_{n-1}) \Big) = |Y|^{-\frac{n-1}{2}},$$

up to a constant factor.

Similarly, for $Y$ a positive definite symmetric $n \times n$ real matrix, and the standard parabolic group $P = P_{n_1, n_2, \ldots, n_q}$ corresponding to the partition $n = n_1 + n_2 + \cdots + n_q$, define the associated *Siegel's Eisenstein series* by

$$\begin{aligned} E_P(\mathbf{s}|Y) :=& E_{n_1,n_2,\ldots,n_q}(\mathbf{s}|Y) \\ :=& \sum_{(A_j *) = A \in \Gamma_n / P, A_j \in \mathbb{Z}^{n \times N_j}} \prod_{j=1}^q |Y[A_j]|^{-s_j}, \quad \mathrm{Res}_j > \frac{n_j + n_{j+1}}{2} \end{aligned}$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_q)$, $N_j = n_1 + n_2 + \cdots + n_j$, $\Gamma_n := GL(n, \mathbb{Z})$ and we identify $\mathbb{Z}^{n \times m}$ with $M_{n \times m}(\mathbb{Z})$. Define also *Koecher's zeta function* by

$$Z_{m,n-m}(Y, s) := \sum_{A \in \mathbb{Z}^{n \times m} / GL(m, \mathbb{Z}), \mathrm{rk} A = m} |Y[A]|^{-s}, \qquad \mathrm{Re}(s) > \frac{n}{2}.$$

**Lemma 2.** (See e.g. [Te]) (1) *$E_{n_1,n_2,\ldots,n_q}(\mathbf{s}|Y)$ and $Z_{m,n-m}(X, s)$ are well-defined in the above indicated regions and admit meromorphic continuation to the whole parameter spaces;* and
(2) *They satisfy the following relations:*

$$|Y|^{-s} \cdot E_{n-1,1}(\mathbf{1}; s|Y^{-1}) = E_{1,n-1}(\mathbf{1}; s|Y) = Z_{1,n-1}(Y; s)/Z_{1,0}(I; s)$$

*and*

$$Z_{n,0}(X, s) = |X|^{-s} \cdot \prod_{j=0}^{n-1} \zeta(2s - j).$$

In parallel, for a positive definite matrix $Y$ with $|Y|=1$ and $\mathbf{s}=(s_1,s_2,\dots,s_n)$, introduce as usual the power function

$$p_{-\mathbf{s}}(Y):=\prod_{j=1}^{n}|Y_j|^{-s_j}.$$

Then associated Siegel's Eisenstein series for the Borel $B=P_{1,1,\dots,1}$ is defined as

$$E_{(n)}(\mathbf{s}|Y):=\sum_{\gamma\in\Gamma_n/P_{1,1,\dots,1}}p_{-\mathbf{s}}(Y[\gamma]),\quad \mathrm{Re}(s_j)>1,\ j=1,2,\dots,n-1.$$

**Lemma 3.** (See e.g. [Te]) *We have*

$$\xi_1^*(Y;s_1,s_2,\dots,s_{n-1})=E_{(n)}(s_1,s_2,\dots,s_n|Y),$$

*and*

$$E_{(n)}(\mathbf{s}|Y^{-1})=E_{(n)}(\mathbf{s}^*|Y)$$

*where* $\mathbf{s}^*:=(s_{n-1},s_{n-2},\dots,s_2,s_1,-(s_1+s_2+\dots+s_n))$. *Consequently,*

$$\xi_1^*(Y^{-1};t_1,t_2,\dots,t_{n-1})=\xi_1^*(Y;t_{n-1},\dots,t_2,t_1).$$

Thus, in particular, for the Siegel Eisenstein series corresponding to the maximal parabolic subgroup $P_{n-1,1}$, i.e., for

$$\begin{aligned}E_{n-1,1}(s_1,s_2|Y):=&E_{n-1,1}(\mathbf{1};s_1,s_2|Y)\\:=&\sum_{(A_1*)=A\in\Gamma_n/P_{n-1,1},A_1\in\mathbb{Z}^{n\times(n-1)}}|Y[A_1]|^{-s_1}|Y[A]|^{-s_2},\end{aligned}$$

we have, by Lemma 3,

$$\begin{aligned}E_{n-1,1}(s,t|Y)=&|Y|^{-t}\cdot\sum_{(A_1*)=A\in\Gamma_n/P_{n-1,1}}|Y[A_1]|^{-s}\\=&|Y|^{-t}\cdot\sum_{A\in\Gamma_n/P_{n-1,1}}|Y[A]_{n-1}|^{-s}=\xi_{n-1}^*(Y,s).\end{aligned}$$

Here, we used the fact that the group involved is $SL(n)$.

Consequently, by Lemmas 1 and 2, we obtain the following

**Fact M.** (1) $\xi_{n-1}^*(Y;s)$ *and* $E(\Lambda;s)$ *are related by*

$$\xi_{n-1}^*(Y^{-1};s)=\frac{1}{\zeta(2s)}\cdot\sum_{\mathbf{x}\in\mathbb{Z}^n\backslash\{0\}}|Y[\mathbf{x}]|^{-s}=\frac{1}{\zeta(2s)}\cdot E\Big(\Lambda^g;\frac{s}{n/2}\Big)$$

*where* $Y := g^t \cdot g$ *and* $\Lambda$ *denotes the lattice* $(\mathbb{Z}^n; \rho(g))$ *with the metric* $\rho(g)$ *on* $\mathbb{R}^n$ *induced by the positive definite matrix* $Y = g^t \cdot g$; and
(2) $\xi_1^*(Y; s_1, s_2, \dots, s_{n-1}) = E_{(n)}(s_1, s_2, \dots, s_n|Y)$.
*In particular,* $E(\Lambda^g; s) =$

$$\mathrm{Res}_{t_{n-2}=1,\, t_{n-3}=1,\, \dots,\, t_2=1,\, t_1=1}\, \xi_1^*(Y; ns - \tfrac{n-2}{2}, t_{n-2}, t_{n-3}, \dots, t_2, t_1).$$

3.3.2. **Siegel's Eisenstein versus Langlands' Eisenstein** To apply Fact E$''$ directly, we still need to write Siegel's Eisenstein series introduced using classical language in terms of Langlands' Eisenstein series introduced using a language which is more convenient for theoretical purpose. The point of course is about the power function $p$ and the function $m_B$. For this, write a positive definite $Y$ (with $|Y| = 1$) as $Y = \mathbf{a}[\mathbf{n}]$ with $\mathbf{a} = \mathrm{diag}(a_1, a_2, \dots, a_n)$ and $\mathbf{n}$ upper triangular unipotent (with diagonal entries 1). Then $a_i = |Y_i|/|Y_{i-1}|, i = 1, 2, \dots, n$.

Consequently, by definition,

$$\begin{aligned}
p_{-\mathbf{s}}(Y) &= \prod_{j=1}^{n} |Y_j|^{-s_j}\\
&=|Y_1|^{-(s_1+s_2+\cdots+s_n)}\Big(|Y_2|/|Y_1|\Big)^{-(s_2+s_3+\cdots+s_n)}\\
&\quad\cdots\Big(|Y_{n-1}|/|Y_{n-2}|\Big)^{-(s_{n-1}+s_n)}\cdot\Big(|Y_n|/|Y_{n-1}|\Big)^{-s_n}\\
&=a_1^{-(s_1+s_2+\cdots+s_n)}a_2^{-(s_2+s_3+\cdots+s_n)}\cdots a_{n-1}^{-(s_{n-1}+s_n)}\cdot\Big(a_1a_2\cdots a_{n-1}\Big)^{-s_n}\\
&=a_1^{-(s_1+s_2+\cdots+s_{n-1})}a_2^{-(s_2+s_3+\cdots+s_{n-1})}\cdots a_{n-1}^{-s_{n-1}}
\end{aligned}$$

since $\prod_{j=1}^n a_j = |Y| = 1$.

On the other hand, if $Y = g^t g$ with $T(g) = \mathrm{diag}(t_1, t_2, \dots, t_n)$, then we have $a_j = t_j^2$ and

$$m_B(g)^{\lambda+\rho_B} = T(g)^{\lambda+\rho_B}$$

where as usual, we let $\lambda = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, $\sum_{j=1}^n z_j = 0$ so that

$$\rho = \rho_B = \Big(\frac{n-1}{2}, \frac{n-1}{2} - 1, \dots, 1 - \frac{n-1}{2}, -\frac{n-1}{2}\Big).$$

Hence, by a direct calculation, we get

$$\begin{aligned}
m_B(g)^{\lambda+\rho_B} =&t_1^{-[(n-1)+(2z_1+z_2+\cdots+z_{n-1})]}\\
&\cdot t_2^{-[(n-2)+(z_1+2z_2+\cdots+z_{n-1})]}\cdots t_{n-1}^{-[1+(z_1+z_2+\cdots+2z_{n-1})]}.
\end{aligned}$$

Recall also that the Langlands Eisenstein series associated to the constant function $\mathbf{1}$ on the Borel $B = P_{1,1,\dots,1}$ (related to $SL(n)/B$) is given by

$$E(\mathbf{1};\lambda)(g) := \sum_{\gamma\in SL(n,\mathbb{Z})/P_{1,1,\dots,1}=B} m_B(\delta g)^{\lambda+\rho_B}.$$

So if we make the variable transformation from $\lambda$ to $\mathbf{s}$ by

$$\begin{cases} 2s_1 = & 1+(z_1-z_2) \\ 2s_2 = & 1+(z_2-z_3) \\ \dots & \dots \\ 2s_{n-1} = & 1+(z_{n-1}-z_n) \end{cases}$$

Then we arrive at the

**Fact M′.** (1) *$E(\mathbf{1};\lambda)(g) = E_{(n)}(\mathbf{s}|Y)$, where $\lambda = (z_1, z_2, \dots, z_n)$ with $\sum_{j=1}^n z_j = 0$ and $\mathbf{s} = (s_1, s_2, \dots, s_{n-1})$ satisfying*

$$\begin{cases} 2s_1 = & 1+(z_1-z_2) \\ 2s_2 = & 1+(z_2-z_3) \\ \dots & \dots \\ 2s_{n-1} = & 1+(z_{n-1}-z_n). \end{cases}$$

(2) *Introduce the variable $s$ via $2ns-n+1 = z_1-z_2$, then we have the following realization of the Epstein zeta function in terms of the residues of Siegel's Eisenstein series:*

$$E\Big(\Lambda(g);s\Big) = \mathrm{Res}_{z_2-z_3=1,\, z_3-z_4=1,\, \cdots,\, z_{n-1}-z_n=1} E(\mathbf{1}; z_1, z_2, \dots, z_n)(g).$$

3.3.3. **New zetas: genuine but different** Recall that, by Fact D, high rank zetas are given by

$$\xi_{\mathbb{Q},r}(s) = \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \widehat{E}(\Lambda;\frac{r}{2}s)\, d\mu_0(\Lambda).$$

Thus by Facts G-I-J and M, to offer a close formula, it suffices to evaluate the integration

$$\int_{\mathfrak{F}(0)} \mathrm{Res}_{z_2-z_3=1,\, z_3-z_4=1,\, \cdots,\, z_{r-1}-z_r=1} \Big(E(\mathbf{1}; z_1, z_2, \dots, z_r)(g)\Big)\, d\mu(g).$$

Thus, if we were able to freely make an interchange between
(i) the operation of taking integration $\int_{\mathfrak{F}(0)}$ and

(ii) the operation of taking residues $\mathrm{Res}_{z_2-z_3=1,\, z_3-z_4=1,\, \cdots,\, z_{r-1}-z_r=1}$, it would be sufficient for us to evaluate

$$\mathrm{Res}_{z_2-z_3=1,\, z_3-z_4=1,\, \cdots,\, z_{r-1}-z_r=1}\Big(\int_{\mathfrak{F}(0)} E(\mathbf{1}; z_1, z_2, \ldots, z_r)(g)\, d\mu(g)\Big),$$

or better, to evaluate the expression

$$\mathrm{Res}_{z_2-z_3=1,\, z_3-z_4=1,\, \cdots,\, z_{r-1}-z_r=1} \left(\sum_{w\in\Omega} \frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot \prod_{\alpha>0,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right)$$

where $\lambda=(z_1,z_2,\ldots,z_r)$ with $z_1+z_2+\cdots+z_r=0$, since by Fact G,

$$\begin{aligned}\int_{\mathfrak{F}(T)} & E(\mathbf{1}; z_1, z_2, \ldots, z_r)(g)\, d\mu(g)\\ &=\int_{SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)} \Lambda^T E(\mathbf{1}; z_1, z_2, \ldots, z_r)(g)\, d\mu(g)\\ &=\sum_{w\in\Omega} \frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot \prod_{\alpha>0,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\end{aligned}$$

by Fact L.

Unfortunately, this interchange of orders of two operations is not allowed in general. As examples, one can observe this by working on $SL(n)$ and by comparing the poles for the resulting expressions. (For details, see the remark at the end of A.3.4 below.)

On the other hand, even with the existence of such discrepancies, the function

$$\mathrm{Res}_{z_2-z_3=1}\cdots \mathrm{Res}_{z_3-z_4=1}\cdots \mathrm{Res}_{z_{r-1}-z_r=1} \left(\sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot \prod_{\alpha>0,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right)$$

proves to be extremely *natural and nice*. This then leads to

**Definition 2.** The single variable period $Z_{\mathbb{Q}}^{SL(r)}(z_1)$ associated to $SL(r)$ over $\mathbb{Q}$ is defined by

$$Z_{\mathbb{Q}}^{SL(r)}(z_1) :=\mathrm{Res}_{z_2-z_3=1}\cdots \mathrm{Res}_{z_3-z_4=1}\cdots \mathrm{Res}_{z_{r-1}-z_r=1} \left(\sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot \prod_{\alpha>0,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right),$$

where $\lambda = (z_1, z_2, \dots, z_r)$ with $z_1 + z_2 + \cdots + z_r = 0$.

Clearly, there are some factors $\xi(ax+b)$'s left in the denominator even after all cancelations made. To clear them, we make the following observations:
(i) there is a minimal integer $I = I(SL(r))$ and finitely many factors

$$\xi\Big(a_1^{SL(r)} z_1 + b_1^{SL(r)}\Big),\ \xi\Big(a_2^{SL(r)} z_1 + b_2^{SL(r)}\Big),\ \cdots,\ \xi\Big(a_I^{SL(r)} \lambda_P + b_I^{SL(r)}\Big),$$

such that *the product* $\Big[\prod_{i=1}^{I(SL(r))} \xi\Big(a_i^{SL(r)} z_1 + b_i^{SL(r)}\Big)\Big] \cdot Z_{\mathbb{Q}}^{SL(r)}(z_1)$ *admits only finitely many singularities.*
(ii) there is a minimal integer $J(SL(r))$ and finitely many factors

$$\xi\Big(c_1^{SL(r)}\Big),\ \xi\Big(c_2^{SL(r)}\Big),\ \cdots,\ \xi\Big(c_{J(SL(r))}^{SL(r)}\Big),$$

such that *there are no factors of special* $\xi$ *values appearing at the denominators in the product* $\Big[\prod_{i=1}^{J(SL(r))} \xi\Big(c_i^{SL(r)}\Big)\Big] \cdot Z_{\mathbb{Q}}^{SL(r)}(z_1)$.

**Definition 3.** The *zeta function* $\xi_{\mathbb{Q};o}^{SL(r)}$ *for* $SL(r)$ *over* $\mathbb{Q}$ is defined by

$$\xi_{\mathbb{Q};o}^{SL(r)}\Big(s\Big) := \left[\prod_{i=1}^{I(SL(r))} \xi\Big(a_i^{SL(r)} s + b_i^{SL(r)}\Big) \cdot \prod_{j=1}^{J(SL(r))} \xi\Big(c_j^{SL(r)}\Big)\right] \cdot Z_{\mathbb{Q}}^{SL(r)}\Big(s\Big), \qquad \mathrm{Re}\, s \gg 0$$

Clearly, Definitions 2 and 3 here are special cases of Definitions 2 and 3 in the previous section. In fact here implicitly the maximal parabolic subgroup $P_{r-1,1}$ is used.

**Remark.** We remind the reader that the version with parameter $T$ is in fact also very important. In rank two case, one can show that for $T$ non-negative, the associated period also satisfies the functional equation and the RH. For general cases, the structure is more complicated on one hand, and beautiful on the other: Say the functional equation for $\xi_{\mathbb{Q}}^{SL(n)/P_{m,n-m};T}$ is related with a different function $\xi_{\mathbb{Q}}^{SL(n)/P_{n-m,m};T}$ (for a different maximal parabolic subgroup), based on another type of symmetry between $E_{m,n-m}$ for $Y$ and $E_{n-m,n}$ for $Y^{-1}$ stated above (for classical Siegel Eisenstein series). However, when $T=0$, $\xi_{\mathbb{Q}}^{SL(n)/P_{m,n-m};0}$ is essentially the function $\xi_{\mathbb{Q}}^{SL(n)/P_{n-m,m};0}$. All this then leads to the functional equation for $\xi_{\mathbb{Q}}^{SL(n)/P_{m,n-m}}(s)$.

3.3.4. **Functional equation & the Riemann Hypothesis** Just as high rank zetas, we certainly expect that these new zetas introduced in the previous subsection satisfy the functional equation and an analogue of the Riemann Hypothesis. For this we have the following

**Conjecture.** (**Functional Equation**) *There exists a constant* $c_{SL(r)}$ *depending on* $r$ *only such that*

$$\xi_{\mathbb{Q};o}^{SL(r)}\Big(c_{SL(r)}-s\Big)=\xi_{\mathbb{Q};o}^{SL(r)}\Big(s\Big).$$

To make the functional equation canonical, i.e., reflecting the standard symmetry $s\leftrightarrow 1-s$ for the standard functional equation, we make the following normalization.

**Definition 3′** The *zeta function* $\xi_{\mathbb{Q}}^{SL(r)}\Big(s\Big)$ *for* $SL(r)$ *over* $\mathbb{Q}$ is defned by

$$\xi_{SL(r);\mathbb{Q}}\Big(s\Big):=\xi_{\mathbb{Q};o}^{SL(r)}\Big(s+\frac{c_{SL(r)}-1}{2}\Big)$$

As such then we have the following

**Conjecture′.** (**Functional Equation**) $\xi_{SL(r);\mathbb{Q}}(1-s)=\xi_{SL(r);\mathbb{Q}}(s)$.

The most remarkable property shared by all these newly introduced zetas is the following Zeta Fact about the uniformity of their zeros. That is to say, we expect the following

**The Riemann Hypothesis**$_{\mathbb{Q}}^{G/P}$.

*All zeros of the zeta* $\xi_{SL(r);\mathbb{Q}}(s)$ *lie on the central line* $\mathrm{Re}\, s=\frac{1}{2}$.

After making these conjectures, we felt that more examples should be provided at least numerically. This then led to the problem of finding precise expressions for '$\xi_{\mathbb{Q},r}(s)$' with $r=4,5$. Limited progress had been made after the work on $SL(3)$ until the summer of 2007, when Henry Kim brought us the paper of Diehl [D]. With [D], we were able to write down precisely Epstein zetas as residues of certain Siegel's Eienstein series. Being compatible with our old approach for the rank 3 zeta by taking residues in [W3], as a continuation of our works on high rank zeta functions, we then were able to obtain precise expressions for what we called the abelian part for the high rank zetas. As for the examples for $SL(4)$ and $SL(5)$, accordingly, we did some painful calculations:
a) For rank 4, totally $24\times 6=144$ cases were discussed, from which we obtained the final zeta consisting of 12 terms;

b) For rank 5, totally $120 \times 10 = 1200$ cases were discussed, from which we obtained the final zeta consisting of 28 terms.

For details, see the Appendix on Examples. Consequently, we have the following

**Fact N.** (1) (**Functional Equation**$_{\leq 5}$)

$$\xi_{SL(r);\mathbb{Q}}(1-s) = \xi_{SL(r);\mathbb{Q}}(s) \qquad \textit{when } r = 2,\, 3,\, 4,\, 5;$$

(2) ([LS]) (**Riemann Hypothesis**$_{SL(2);\mathbb{Q}}$)
*All zeros of* $\xi_{SL(2);\mathbb{Q}}(s)$ *lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

### 3.4. Zetas for $(G,P)/\mathbb{Q}$

3.4.1. **From SL to Sp: analytic method adopted & periods chosen** For quite sometime, we want to use geo-arithmetic method to find an analogue of high rank zetas for other reductive groups. The first natural target is $Sp$. However, this proves to be a bit complicated, since for the completed theory, we should start with what might be called principal lattices associated to $Sp$ and establish all the $Sp$ properties corresponding to Facts listed above for $SL$.

Fortunately, for the purpose of finding corresponding abelian zeta functions $\xi_{Sp(2n);\mathbb{Q}}(s)$ for $Sp$, with our success for $SL$ discussed above and the paper of Diehl [D], which in fact deals with $Sp$ instead of $SL$, we realize that instead of the approach using geo-arithmetic method, an alternative way using pure analytic methods is sufficient. This goes as follows.

Let $G = Sp(2n)$ with $G(\mathbb{R}) = Sp(2n,\mathbb{R})$ the symplectic group of degree $n$ over $\mathbb{R}$. For any $Z \in \mathfrak{S} = \mathfrak{S}_n$, the Siegel upper half space of rank $n$, write $Z = X + \sqrt{-1}Y$ according to its real and imaginary parts. By definition, $Y = \mathrm{Im}\, Z > 0$ and $Z^t = Z$ is symmetric. For an $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{R})$, as usual, set $M\langle Z\rangle := (AZ+B)\cdot(CZ+D)^{-1}$ and write $Y(M) := \mathrm{Im} M\langle Z\rangle$. Note that the action is transitive and the stablizer in $Sp(2n,\mathbb{R})$ for $\sqrt{-1}I$ is given by $Sp(2n,\mathbb{R}) \cap SO(2n)$. Consequently, we obtain a natural isomorphism $Sp(2n,\mathbb{R})/SO(2n) \cap Sp(2n,\mathbb{R}) \simeq \mathfrak{S}_n$.

Introduce also $\Gamma_n := \big\{\mathrm{diag}(\pm 1, \pm 1, \cdots, \pm 1)\big\}\backslash Sp(2n,\mathbb{Z})$ the Siegel modular group, and $\mathfrak{B} = \mathfrak{P}_n := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma \right\}$ the associated maximal parabolic subgroup.

Fix $Z \in \mathfrak{S}$, define then the associated *Siegel–Maaß* Eisenstein series, or better, the *Siegel–Epstein zeta function* by

$$E_n(Z;s) := \sum_{\gamma \in \mathfrak{B}\backslash\Gamma} \frac{|Y|^s}{\|CZ+D\|^{2s}}.$$

Motivated by our study on high rank zetas associated to $SL(n)$, for sufficiently positive $T$, we define a *principal period for $Sp(n)$ over* $\mathbb{Q}$ by

$$\zeta^T_{Sp(n),\mathbb{Q}}(s) := \int_{\Gamma\backslash\mathfrak{S}_n} \wedge^T E_n(Z;s)\, d\mu(Z).$$

This is then a function on $s$ depending also on the parameter $T$. It is then an open problem whether we can evaluate this expression at $T=0$ since the corresponding Fact G-I-J for $Sp$ is still missing. Assume that the answer to this is affirmative, then

$$\zeta_{Sp(n),\mathbb{Q}}(s) := \zeta^0_{Sp(n),\mathbb{Q}}(s) := \zeta^T_{Sp(n),\mathbb{Q}}(s)|_{T=0}$$

may be viewed as an $Sp$-analogue of the high rank zeta functions, call it the *principal zeta function for $Sp(n)$ over* $\mathbb{Q}$.

As for the case of $SL(n)$, it is, for the time being, *very difficult*, in fact, *quite impossible,* to offer a precise formula for the Eisenstein period $\zeta^T_{Sp(n),\mathbb{Q}}(s)$. However, motivated by our study for $SL(n)$, we want to introduce an analogue for the new type of abelian zeta functions $\xi_{SL(r);\mathbb{Q}}(s)$. For this (a bit changed yet very meaningful) purpose, we make the following preparations.

a) **Siegel Eisenstein series.** As usual, corresponding to the partition $n = r+1+1+\cdots+1$, introduce the standard parabolic subgroup $\mathfrak{P}_r := \left\{ \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} \in \Gamma \right\}$ where $A = \begin{pmatrix} H^t & & & \\ & 1 & 0 & \\ & * & \ddots & \\ & & & 1 \end{pmatrix}$, $B = \begin{pmatrix} H^{-1} & & & \\ & 1 & * & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$ with $H = H^{(r)}, |H| = 1$. Accordingly, define the associated *Siegel Eisenstein series* by

$$E_r(Z; s_r, \ldots, s_n) := \sum_{\gamma \in \mathfrak{P}_r\backslash\Gamma} \prod_{v=r}^{n} |Y(\gamma)_v)|^{-s_v}.$$

It is known that these Siegel Eiesnetsin series are naturally related to the Siegel zeta functions associated to the standard parabolic subgroup $\mathfrak{Q}_r$ of $SL(n)$, used already in our study for zetas associated to $SL(n)$. Recall that, if $\mathfrak{N} := \{\mathrm{diag}(\pm 1, \dots, \pm 1)\}\backslash SL(n, \mathbb{Z})$ and $\mathfrak{Q}_r$ is the standard parabolic subgroup associated to the partition $n = r + 1 + 1 + \cdots + 1$, then the associated *Siegel zeta functions* are defined by

$$\xi_r^*(Y; s_r, \dots, s_{n-1}) := \sum_{N \in \mathfrak{Q}_r \backslash \mathfrak{N}} \prod_{v=r}^{n-1} |Y[N]_v|^{-s_v}$$

for all $1 \leq r \leq n-1$.

**Lemma 1.** ([D]) We have
(i)

$$E_r(Z; s_r, \dots, s_n) = \sum_{\gamma \in \mathfrak{B} \backslash \Gamma} |Y(\gamma)|^{-s_n} \cdot \xi_r^*\Big(Y(\gamma); s_r, \dots, s_{n-1}\Big);$$

(ii) *There exists a constant c depending only on r such that*

$$\mathrm{Res}_{s_r = \frac{r+1}{2}} \xi_r^*(Y; s_r, \dots, s_{n-1}) = c_r \xi_{r+1}^*(Y; s_{r+1} + \frac{r}{2}, s_{r+2}, \dots, s_{n-1}).$$

Consequently,

$$\mathrm{Res}_{s_{n-1}=1} \cdots \mathrm{Res}_{s_2=1} \mathrm{Res}_{s_1=1} \Big(\xi_1^*(Y; s_1, s_2, \dots, s_{n-1})\Big) = |Y|^{-\frac{n-1}{2}}$$

up to a constant factor. Therefore, up to constant factors,

$$\begin{aligned}
&\mathrm{Res}_{s_{n-1}=1} \cdots \mathrm{Res}_{s_2=1} \mathrm{Res}_{s_1=1} E_r(Z; s_r, \dots, s_n) \\
= &\sum_{\gamma \in \mathfrak{B} \backslash \Gamma} |Y(\gamma)|^{-s_n} \cdot \mathrm{Res}_{s_{n-1}=1} \cdots \mathrm{Res}_{s_2=1} \mathrm{Res}_{s_1=1} \xi_r^*(Y(\gamma); s_r, \dots, s_{n-1}) \\
= &\sum_{\gamma \in \mathfrak{B} \backslash \Gamma} |Y(\gamma)|^{-s_n} \cdot |Y(\gamma)|^{-\frac{n-1}{2}} = E_n(Z; s_n + \frac{n-1}{2}).
\end{aligned}$$

b) **Siegel Eisenstein series and Langlands Eisenstein series.** As for the case of $SL(n)$, we next write the classical Siegel Eisenstein series in terms of Langlands' language. This is given by the following formula: Let $\lambda = (z_1, z_2, \dots, z_n) \in \mathfrak{a}_0^*$, then by defintion,

$$\mathbf{a}^\lambda(Z) = \prod_{v=1}^n a_v^{-z_v} \qquad \text{with} \qquad a_v = |Y_v|/|Y_{v-1}|.$$

Thus, the so-called power function

$$\mathbf{p}_{-\mathbf{s}}(Y) := \prod_{\mu=1}^{n} |Y_\mu|^{-s_\mu}$$

is given by

$$\begin{aligned}\prod_{\mu=1}^{n} |Y_\mu|^{-s_\mu} =&\mathbf{p}_{-\mathbf{s}}(Y) = \mathbf{a}^{\lambda}(Y) = \prod_{v=1}^{n} a_v^{-z_v} \\ =&|Y_1|^{-z_1+z_2}|Y_2|^{-z_2+z_3}\cdots|Y_{n-1}|^{-z_{n-1}+z_n}|Y_n|^{-z_n}.\end{aligned}$$

That is to say, we need to make the following change of variables

$$s_1 = z_1 - z_2, s_2 = z_2 - z_3, \ldots, s_{n-1} = z_{n-1} - z_n, s_n = z_n.$$

Consequently, we obtain the following
**Fact M″**. (1) $E(\mathbf{1};\lambda;Y) = E_1(Z; s_1, s_2, \ldots, s_n)$, and
(2) *Up to a suitable constant factor,*

$$\begin{aligned}&E_n(Z, z_n + \frac{n-1}{2}) \\ =&\mathrm{Res}_{z_{n-1}-z_n=1}\cdots\mathrm{Res}_{z_2-z_3=1}\mathrm{Res}_{z_1-z_2=1}E(\mathbf{1}; z_1, z_2, \ldots, z_n; Y).\end{aligned}$$

In particular, when $n = 2$, i.e, for $Sp(4)$, we have

$$\mathrm{Res}_{z_1-s=1}E(\mathbf{1}; z_1, s; Y) = E_n(Z, s + \frac{1}{2}).$$

c) **The Siegel–Maaß–Eisenstein period.** Note that the constant function one on the Borel is *cuspidal*, by the result of [JLR] cited above, and using the corresponding Gindikin–Karpelevich formula for the associated intertwining operator, we have the following:

**Fact E$^{(3)}$**. *Up to a constant factor,*

$$\begin{aligned}&\int_{Sp(n,\mathbb{Z})\backslash\mathfrak{S}_n} \wedge^T E(\mathbf{1};\lambda;M)\,d\mu(M) \\ =&\sum_{w\in W}\frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}.\end{aligned}$$

With all this, we are now ready to introduce our new zeta for $Sp(2n)$: first define (*not-yet-normalized*) *zeta* as the residue

$$\mathrm{Res}_{z_{n-1}-z_n=1,\cdots,z_2-z_3=1,z_2-z_1=1}$$

$$\sum_{w\in W}\frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)},$$

since $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta_0$, where $\lambda = (z_1, z_2, \ldots, z_n) \in \mathfrak{a}_0^*$, corresponding to Definition 2; then, make certain normalizations following Definition 3. As such, we finally obtain a new series natural zetas $\xi_{Sp(2n),\mathbb{Q}}(s)$ for $Sp(2n)$ over $\mathbb{Q}$, which in fact coincide with $\xi_{\mathbb{Q}}^{Sp(2n)/\mathfrak{P}_n}(s)$ defined in the main text.

As concrete examples, we worked out all the details for $n = 2$. Similarly, we have the functional equation

$$\xi_{Sp(4),\mathbb{Q}}(1-s) = \xi_{Sp(4),\mathbb{Q}}(s).$$

For details, see the Appendix below.

In summary, what we have done for $Sp$ is as follows:

(i) First, motivated by our study for high rank zeta functions associated to $SL(n)$, we introduce a principal zeta for $Sp(2n)$ by evaluating the integration

$$\int_{Sp(2n,\mathbb{Z})\backslash\mathfrak{S}_n} \wedge^T E_n(Z; s)\, d\mu(Z)$$

at $T = 0$: in assuming that Fact G-I-J for $Sp$ can be established, even in the integration $T$ is supposed to be sufficiently positive, an evaluation at $T = 0$ is allowed;

(ii) By b), we know that, up to constant factors,

$$\begin{aligned}&E_n(Z; z_n)\\ =&\mathrm{Res}_{z_{n-1}-z_n=1,\cdots,z_2-z_3=1,z_2-z_1=1} E(\mathbf{1}; z_1, z_2, \ldots, z_{n-1}, z_n + \frac{n-1}{2}; Y).\end{aligned}$$

So it suffices to evaluate

$$\int_{Sp(2n,\mathbb{Z})\backslash\mathfrak{S}_n} \mathrm{Res}_{z_{n-1}-z_n=1,\cdots,z_2-z_3=1,z_2-z_1=1} \Big(\wedge^T E(\mathbf{1}; z_1, z_2, \ldots, z_{n-1}, z_n + \frac{n-1}{2}; Y)\Big)\, d\mu(Y);$$

(iii) Even an interchange of $\int_{Sp(2n,\mathbb{Z})\backslash\mathfrak{S}_n}$ and $\mathrm{Res}_{z_{n-1}-z_n=1,\cdots,z_2-z_3=1,z_2-z_1=1}$ is not allowed, we, motivated by our success for $SL(n)$, still decide to study the period

$$\mathrm{Res}_{z_{n-1}-z_n=1,\cdots,z_2-z_3=1,z_2-z_1=1} \int_{Sp(2n,\mathbb{Z})\backslash\mathfrak{S}_n} \Big(\wedge^T E(\mathbf{1}; z_1, z_2, \ldots, z_{n-1}, z_n + \frac{n-1}{2}; Y)\Big)\, d\mu(Y);$$

(iv) Now by c), for sufficiently positive $T$, the integration $\int_{Sp(2n,\mathbb{Z})\backslash\mathfrak{S}_n} \wedge^T E(\mathbf{1};\lambda;Y)\,d\mu(Y)$ is simply

$$\sum_{w\in W} \frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot \prod_{\alpha>0,w\alpha<0} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}.$$

(v) Evaluate the latest period at $T=0$ using the expression appeared in the right hand side and further take the residue. This then leads to the not yet normalized new zeta function for $Sp(2n)$ over $\mathbb{Q}$:

$$\mathrm{Res}_{z_{n-1}-z_n=1,\cdots,z_2-z_3=1,z_2-z_1=1}$$
$$\Big(\sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot \prod_{\alpha>0,w\alpha<0} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\Big).$$

(vi) Suitably normalized, we obtain a new type of zeta function $\xi_{Sp(2n)\mathbb{Q}}(s)$ for which we have the following

**Conjecture.** (1) (**Functional Equation**) $\xi_{Sp(2n);\mathbb{Q}}(1-s)=\xi_{Sp(2n);\mathbb{Q}}(s)$;
(2) (**The Riemann Hypothesis** $_{Sp(2n);\mathbb{Q}}$)
*All zeros of the zeta* $\xi_{Sp(2n);\mathbb{Q}}(s)$ *lie on the central line* $\mathrm{Re}\,s=\frac{1}{2}$.

Up to this point, the importance of the period

$$\omega_{\mathbb{Q}}^G(\lambda) := \sum_{w\in W}\Big(\frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot \prod_{\alpha>0,w\alpha<0} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\Big)$$

has been fully exposed and the huge symmetry induced from the Weyl group $W$ is noticed.

3.4.2. **$\mathbf{G_2}$: maximal parabolics discovered** The success of introducing natural zetas for $Sp(n)$ which are supposed to satisfying the Riemann Hypothesis proves to be very crucial. Passing this point, we then seriously try to find natural zetas for other types of classical groups.

Practically, to be able to find such zetas, we still need to solve two main technical problems:

1) how to introduce an analog of Epstein zeta function for other groups? Such a function should at least satisfy the property that it can be obtained as the residue along certain singular hyperplanes of the (relative) Eisenstein series $E^{G/B}(\mathbf{1};\lambda)(g)$ associated to constant function one on the Borel; and

2) what are singular hyperplanes along which the residues should be taken?

However, by reviewing what has been done for $SL(n)$ and $Sp(2n)$, for the purpose of introducing abelian zetas, we realize that the completed theory for (1) is not really needed absolutely: What matters (for introducing our new zetas) is not Epstein type zeta, but the period

$$\omega_{\mathbb{Q}}^{G}(\lambda) := \sum_{w\in W}\frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\xi\big(\langle\lambda,\alpha^\vee\rangle\big)}{\xi\big(\langle\lambda,\alpha^\vee\rangle+1\big)}.$$

With (1) solved, we then shift to (2). At the very beginning, we had no idea on how to deal it—to solve this problem we first need to understand where are singularities for $E^{G/B}(\mathbf{1};\lambda)(g)$; more importantly, even if knowing the singularities, we still need to figure out along which singular hyperplanes we take the residues, as there are many many possible choices.

As such, at this preliminary stage of our study, we decide to be more practical. That is, not trying to solve the problem completely, but to work with examples with the hope to expose hidden structures: After all, the most important points are to introduce new zetas, and once introduced to check whether they satisfy the functional equation and further the Riemann Hypothesis.

For such a limited practical purpose, then clearly, among all classical groups, we need to test these groups which are with relatively smaller ranks and with reasonable smaller sizes of Weyl groups. By looking at $B_n$, $D_n$, $E_{6,7,8}$, $F_4$ and $G_2$, it is obvious why we decide to focus on $G_2$—$G_2$, being exceptional and interesting, is of rank two and with only 12 Weyl elements. This is extremely nice: rank two should make our study more like to be successful—after all, the period $\omega_{\mathbb{Q}}^{G_2}(z_1,z_2)$, that is,

$$\sum_{w\in W}\frac{1}{\langle w\lambda-\rho,\alpha_{\text{short}}^\vee\rangle\cdot\langle w\lambda-\rho,\alpha_{\text{long}}^\vee\rangle}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\xi\big(\langle\lambda,\alpha^\vee\rangle\big)}{\xi\big(\langle\lambda,\alpha^\vee\rangle+1\big)},$$

is a function with two variables $(z_1,z_2)=\lambda\in\mathfrak{a}_0^*$, where $\Delta_0:=\{\alpha_{\text{short}},\alpha_{\text{long}}\}$ with $\alpha_{\text{short}}$ the short root and $\alpha_{\text{short}}$ the long root. Consequently, we only need to find a single singular line $az_1+bz_2+c=0$.

At this point, then by recall what has happened for $SL$ and $Sp$, we conclude that in fact *all singular hyper-planes appeared there are factors of the denominator of the term in* $\omega_{\mathbb{Q}}^{G}(\lambda)$ *corresponding to the identity*

*Weyl element* Id. Applying this to $G_2$, we are led to

$$\frac{1}{\langle \lambda - \rho, \alpha_{\text{short}}^\vee \rangle \cdot \langle \lambda - \rho, \alpha_{\text{long}}^\vee \rangle}.$$

Now it is crystal clear that we should do—There are two possibilities for the choice of a single singular line:
(1) $\langle \lambda - \rho, \alpha_{\text{short}}^\vee \rangle = 0$ or
(2) $\langle \lambda - \rho, \alpha_{\text{long}}^\vee \rangle = 0$.

In this way, then we obtain two new zetas for $G_2$. Now recall that by Lie theory, there is a one-to-one and onto correspondence between maximal parabolic subgroups and simple roots, it is then only natural for us to name the corresponding zeta functions $\xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(s)$ and $\xi_{\mathbb{Q}}^{G_2/P_{\text{short}}}(s)$ respectively, where $P_{\text{short}}$ and $P_{\text{long}}$ correspond to $\alpha_{\text{long}}$ and $\alpha_{\text{short}}$ respectively. The precise calculation was carried out in the Appendix previously but now moved to [SW]. In particular, the result confirms that we have *the functional equation*

$$\xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(1-s) = \xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(s) \quad \text{and} \quad \xi_{\mathbb{Q}}^{G_2/P_{\text{short}}}(1-s) = \xi_{\mathbb{Q}}^{G_2/P_{\text{short}}}(s).$$

3.4.3. **Zetas for $(G,P)/\mathbb{Q}$: singular hyper-planes found** With the discovery of importance played by the period $\omega_{\mathbb{Q}}^G(\lambda)$ in our study of zeta functions, and the success of the discussion on $G_2$, we next want to systematically understand how singular hyperplanes are chosen in the process of taking residues. For this we go back to examine the examples of $SL(n), Sp(2n)$ and $G_2$ (with standard choices of the Borels).
a) For $SL(n)$, a rank $(n-1)$ group, as usual,

$$\Delta_0 = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n\},$$

with

$$\lambda = (z_1, z_2, \dots, z_n) \in \mathfrak{a}_0 \subset \mathbb{C}^n, \qquad \sum_{i=1}^n z_i = 0,$$

where $e_i$'s are the standard orthonormal basis for $\mathbb{C}^n$. In the definition of $\xi_{SL(n),\mathbb{Q}}(s)$, the $(n-2)$-singular hyperplanes are chosen to be

$$z_1 - z_2 = 1, z_2 - z_3 = 1, \dots, z_{n-2} - z_{n-1} = 1;$$

b) For $Sp(2n)$, a rank $n$ group, as usual,

$$\Delta_0 = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, 2e_n\}$$

with $\lambda = (z_1, z_2, \dots, z_n) \in \mathfrak{a}_0^* = \mathbb{C}^n$. In the definition of $\xi_{Sp(2n),\mathbb{Q}}(s)$, the $(n-1)$-singular hyperplanes are chosen to be

$$z_1 - z_2 = 1, z_2 - z_3 = 1, \dots, z_{n-1} - z_n = 1;$$

c) For $G_2$, a rank two group, as usual

$$\Delta_0 = \{\alpha_{\mathrm{short}}, \alpha_{\mathrm{long}}\}.$$

In this case, we decided to use $\lambda = z_1(2\alpha_{\mathrm{short}}+\alpha_{\mathrm{long}})+z_2(\alpha_{\mathrm{short}}+\alpha_{\mathrm{long}})$. As said above, two different choices of a single singular line are chosen: $z_1 - z_2 = 1$ and $z_2 = 0$.

As such, by looking at these singular hyperplanes more carefully, we conclude that

a) For $SL(n)$, they are given by

$$\langle \lambda-\rho, e_1-e_2\rangle = 0,\ \langle \lambda-\rho, e_2-e_3\rangle = 0, \ldots, \langle \lambda-\rho, e_{n-2}-e_{n-1}\rangle = 0,$$

or better, are given by

$$\langle \lambda-\rho, \alpha^\vee\rangle = 0, \qquad \alpha\in\Delta\backslash\{e_{n-1}-e_n\};$$

b) For $Sp(2n)$, they are given by

$$\langle \lambda-\rho, e_1-e_2\rangle = 0,\ \langle \lambda-\rho, e_2-e_3\rangle = 0, \ldots, \langle \lambda-\rho, e_{n-1}-e_n\rangle = 0,$$

or better, are given by

$$\langle \lambda-\rho, \alpha^\vee\rangle = 0, \qquad \alpha\in\Delta\backslash\{2e_n\};$$

c) For $G_2$, easily with the choice $\lambda = z_1(2\alpha_{\mathrm{short}}+\alpha_{\mathrm{long}})+z_2(\alpha_{\mathrm{short}}+\alpha_{\mathrm{long}})$, the line $z_1 - z_2 = 1$ corresponds to $\langle \lambda-\rho, \alpha_{\mathrm{short}}^\vee\rangle = 0$, while line $z_2 = 1$ corresponds to $\langle \lambda-\rho, \alpha_{\mathrm{long}}^\vee\rangle = 0$. Or better put, the line $z_1 - z_2 = 1$ is given by

$$\langle \lambda-\rho, \alpha^\vee\rangle = 0, \qquad \alpha\in\Delta_0\backslash\{\alpha_{\mathrm{long}}\};$$

while the line $z_2 = 1$ is given by

$$\langle \lambda-\rho, \alpha^\vee\rangle = 0, \qquad \alpha\in\Delta_0\backslash\{\alpha_{\mathrm{short}}\}.$$

Recall now that, to introduce new zetas, we are determined to use

$$\omega_{\mathbb{Q}}^G(\lambda) = \sum_{w\in W}\left(\frac{1}{\prod_{\alpha\in\Delta_0}(\langle\lambda, w^{-1}\alpha^\vee\rangle - 1)}\cdot\prod_{\alpha>0, w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right),$$

a special period governed by huge symmetries. Recall also that, for finding singular hyper-planes, our success for $SL$ and $Sp$ led to the term corresponding to $w = 1$. Namely,

$$\frac{1}{\prod_{\alpha\in\Delta_0}(\langle\lambda,\alpha^\vee\rangle - 1)}\cdot 1 = \frac{1}{\prod_{\alpha\in\Delta_0}(\langle\lambda,\alpha^\vee\rangle - 1)}.$$

With such a focus, it is then not too difficult for us to detect that *all $(r-1)$-singular hyperplanes are taken from the total $r$-factors in the denominator of this term, where $r$ is the rank of the group.*
Once this is observed, then it is extremely clear what we have done so far: a special choice of the $(r-1)$-singular hyperplanes correspond to a fixed choice of certain special maximal parabolic subgroup. More precisely, for a fixed standard maximal parabolic subgroup $P$, by Lie theory, there exists a single simple root $\alpha_P$ such that $P$ corresponding to $\Delta_0\backslash\{\alpha_P\}$. As such, *the $(r-1)$ singular hyperplanes chosen may be understood as these given by $\langle\lambda-\rho,\alpha^\vee\rangle=0$ for $\alpha\in\Delta_0,\alpha\neq\alpha_P$.*

Upon this point, we are quite sure how a new type of zetas for $(G,P)$ should be introduced. And more importantly, we understand the importance of the role played by the symmetry. This then leads to Definition 2 of periods of $(G,P)/\mathbb{Q}$:

$$\omega_{\mathbb{Q}}^{G/P}(\lambda_P):=\mathrm{Res}_{\{\langle\lambda-\rho,\alpha^\vee\rangle=0\,:\,\alpha\in\Delta_0\backslash\{\alpha_P\}\}}\Big(\omega_{\mathbb{Q}}^{G}(\lambda)\Big)$$

where $\alpha_P$ is the simple root corresponds to the maximal parabolic $P$. With suitable normalization as done in Definition 3, we then finally obtain our new zetas $\xi_{\mathbb{Q}}^{G/P}(s)$ for $(G,P)$ over $\mathbb{Q}$, whose importance can be read from the following

**Conjecture.** (1) (**Functional Equation**) $\xi_{\mathbb{Q}}^{G/P}(1-s)=\xi_{\mathbb{Q}}^{G/P}(s)$;

(2) (**The Riemann Hypothesis**$_{\mathbb{Q}}^{G/P}$)

*All zeros of the zeta $\xi_{\mathbb{Q}}^{G/P}(s)$ lie on the central line* $\mathrm{Re}\,s=\dfrac{1}{2}$.

To support this new approach, we start working on more examples (for these new zetas) associated to other type of standard maximal subgroups (of $SL(3)$, $SL(4)$, $SL(5)$, $Sp(4)$ and $G_2$). The details are given in the Appendix.

### 3.5. Conclusion remarks

3.5.1. **Analogue of high rank zetas** We here propose an approach aiming at introducing genuine zeta functions for $(G,P)/F$, as a natural generalization of high rank zeta functions.

Denote by $\mathbb{A}_F$ the adelic ring of $F$. Let $G$ be a reductive group defined over $F$, and $P$ a maximal parabolic subgroup. Then for the constant function $\mathbf{1}$ on $P$, we form the relative Eisenstein series $E(\mathbf{1};\lambda_{G/P};g)$ $=E^{G/P}(\mathbf{1};\lambda_{G/P};g)$. For a fixed sufficiently positive $T\in\mathfrak{a}_0$, the space of characters of the Borel $B$ of $G$, introduce a single variable period $\omega_{G/P;F}^{T}$

by setting

$$\omega^T_{G/P;F}(\lambda_{G/P}) := \int_{Z_{G(\mathbb{A}_F)}G(F)\backslash G(\mathbb{A}_F)} \Lambda^T E^{G/P}(\mathbf{1};\lambda_{G/P};g)\,d\mu(g).$$

We expect that an analogue of Fact G-I-J for $G$-principal lattices holds.

If so, then it makes sense to introduce

$$\begin{aligned}\omega_{G/P;F}(\lambda) :=& \omega^T_{G/P;F}(\lambda)|_{T=0}\\ =& \int_{\mathfrak{F}_G(0)\subset Z_{G(\mathbb{A}_F)}G(F)\backslash G(\mathbb{A}_F)} E^{G/P}(\mathbf{1};\lambda_{G/P};g)\,d\mu(g).\end{aligned}$$

In particular, from $\omega_{G/P;F}(\lambda)$, a suitable normalization will then finally lead to an analogue of high rank zetas for $(G,P)/F$.

**Questions.** (1) Is it possible to get $E^{G/P}(\mathbf{1};\lambda_{G/P};g)$ from $E^{G/B}(\mathbf{1};\lambda;g)$, the relative Eisenstein series associated to the constant function $\mathbf{1}$ on the Borel, by taking residues along with suitable $\mathrm{rank}(G)-1$ singular hyper-planes?
(2) Can we take these singular hyper-planes simply as $\langle\lambda-\rho,\alpha^\vee\rangle = 0$, $\alpha\in\Delta_0\backslash\{\alpha_P\}$? Here, as usual, $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha$ denotes the Weyl vector.
(3) Is it possible to introduce a completed Eisenstein series $\widehat{E}^{G/P}(\mathbf{1};\lambda_{G/P};g)$ from $E^{G/P}(\mathbf{1};\lambda_{G/P};g)$ so that the resulting zeta function admits only finite many singularities, satisfies a simple functional equation, and the Riemann Hypothesis?

3.5.2. $T$**-version** In our discussion above, by adapting an analytic method, we can extend our discussion for periods defined originally for sufficiently positive $T$ to these for $T=0$. This makes the theory more canonical and elegant. However the use of $T$-version proves to be quite helpful—as example for $SL(3,4,5)$ shows, such a $T$-version can be used to help us to understand the additional symmetry for our new zeta functions. For example, we know that

$$\xi_{\mathbb{Q}}^{SL(3)/P_{2,1}}(s) = \xi_{\mathbb{Q}}^{SL(3)/P_{1,2}}(s), \qquad \xi_{\mathbb{Q}}^{SL(4)/P_{3,1}}(s) = \xi_{\mathbb{Q}}^{SL(4)/P_{1,3}}(s),$$

and

$$\xi_{\mathbb{Q}}^{SL(5)/P_{4,1}}(s) = \xi_{\mathbb{Q}}^{SL(5)/P_{1,4}}(s), \qquad \xi_{\mathbb{Q}}^{SL(5)/P_{2,3}}(s) = \xi_{\mathbb{Q}}^{SL(5)/P_{3,2}}(s).$$

On surface, these relations may be viewed as a reflection of the symmetry between the Eisenstein series $E_{r-m,m}$ associated to the maximal parabolic $P_{r-m,m}$ and the Eisenstein series $E_{m,r-m}$ associated to the maximal parabolic $P_{m,r-m}$. (See 3.3.1 for details.) More deeply, it roots into the symmetry between $P_{r-m,m}$ and $P_{m,r-m}$ for maximal parabolic subgroups of $SL(r)$.

Put this in concrete term, for $SL(3)$, we can further introduce $T$-version zeta functions $\xi_{\mathbb{Q}}^{SL(3)/P_{2,1};T}(s)$ and $\xi_{\mathbb{Q}}^{SL(3)/P_{1,2};T}(s)$, analogues of $\xi_{\mathbb{Q}}^{SL(3)/P_{2,1}}(s)$ and $\xi_{\mathbb{Q}}^{SL(3)/P_{1,2}}(s)$ respectively, starting from the $T$-version period $\omega_{\mathbb{Q}}^{SL(3);T}(\lambda)$ in 3.2.6. Then one checks that with $T \in \mathbb{C} \cdot \rho$, i.e., with $T$ specialized as points on the line spanned by $\rho$, we have

$$\xi_{\mathbb{Q}}^{SL(3)/P_{2,1};T}(1-s) = \xi_{\mathbb{Q}}^{SL(3)/P_{1,2};T}(s).$$

This is then the root of the equality

$$\xi_{\mathbb{Q}}^{SL(3)/P_{2,1}}(s) = \xi_{\mathbb{Q}}^{SL(3)/P_{1,2}}(s).$$

We expect that holds for all zetas related to $(SL(r), P_{r-m,m})/\mathbb{Q}$.

Along with this line, then we also expect that the symmetry, or better, the duality, between types $B_n$ and $C_n$ groups will have similar impact to our new zetas. In a sense, various symmetries are the main reason why our new zetas satisfy the functional equations and the Riemann Hypothesis.

We end this $T$-version discussion by pointing out that the Riemann Hypothesis does not hold for $\xi_{\mathbb{Q}}^{SL(3)/P_{2,1};T}(s)$ if $T$ is not 0. So our new zetas $\xi_{\mathbb{Q}}^{G/P}(s)$, being specialization of $T$-version zetas $\xi_{\mathbb{Q}}^{G/P;T}(s)$ to the ground zero and hence delicate, are quite canonical, hence absolutely beautiful.

3.5.3. **Where leads to** It is hard to predict, being new and rich. In general terms, two aspects are worth being mentioned. One is for the zetas themselves, the other is for possible applications.

For zetas themselves, the first and the up-most task is then concentrated on the (proof of) functional equations and the corresponding Riemann Hypothesis. Examples listed in the Appendix for $SL(2, 3, 4, 5)$, $Sp(4)$ and $G_2$ show that the associated zetas satisfy the Functional Equation. This is beautiful, reflecting additional symmetry, and supposedly doable even expected to be very complicated. On the other hand, for the Riemann Hypothesis associated to new zetas, responding to our inquires ([W4]), Suzuki first made several crucial numerical tests on zeros of zetas $\xi_{SL(4);\mathbb{Q}}(s)$, $\xi_{SL(5);\mathbb{Q}}(s)$ and $\xi_{Sp(4);\mathbb{Q}}(s)$ ([S2]). Shortly after, in January 2008, he was able to theoretically verify the Riemann Hypothesis for zetas $\xi_{Sp(4);\mathbb{Q}}(s)$ and $\xi_{\mathbb{Q}}^{G_2/P}(s)$ ([S3, 4]), by strengthening a method used for establishing the RH of $\xi_{SL(2);\mathbb{Q}}(s)$ ([LS]) and of $\xi_{SL(3);\mathbb{Q}}(s)$ ([S]). (In fact, this method can also be used to show that outside a certain finite box, all zeros of $\xi_{\mathbb{Q}}^{Sp(4)/P_{2e_2}}(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$ as well.)

The third is about a generalization to all reductive groups. Even physically, this can be done simply since all the framework works in this generality. But we are somehow a bit hesitated feeling that time is not ripe to make such a move, even we know that, up to a constant factor,

$$\xi_F^{G_1\times G_2/P_1\times G_2}(s) = \xi_F^{G_1/P_1}(s)$$

and that the RH holds for all rank 2 groups (modulo the finite box mentioned above for $\xi_{\mathbb{Q}}^{Sp(4)/P_{2e_2}}(s)$).

For applications, an obvious is about the relation between new zetas and the classical Riemann zeta function. Problems likely to be asked here are: what should be the relations between their zeros? This can be put more precisely, for example, as: if we just consider a series, e.g., the series for $SL(r)/P_{r-1,1}$, or a collection, e.g., the collection of rank $r$ groups, what should be the sequence of the $n$-th zeros for a fixed $n$? what about the distributions of these zeros, the gaps between ordered pairs of zeros? etc. For this, a related interesting point should be mentioned: the completed Riemann zeta function can be written as a difference between two entire functions which both satisfy the RH. This is a new structure emerged in our understanding of $\xi_{SL(3);\mathbb{Q}}(s)$. (See also [S3] for $\xi_{Sp(4);\mathbb{Q}}(s)$.)

We end this section by proposing a bit indirect, but quite speculating use of our new zetas. We call this a 'wonderful idea'—the final goal is to replace the original Riemann Hypothesis in the study of distribution of primes, of classical problems such as the Goldbach conjecture, etc., with the RH for our zetas, some of which have been established.

Added in July, 2009: Much progress has been made since the paper was written in December, 2007:
(1) In April, 2008, Henry Kim in a joint effort with the author obtained a proof of the functional equation for $\xi_{\mathbb{Q}}^{SL(n)/P_{n-1,1}}(s)$;
(2) Independently, in June, 2009, Yasushi Komori found an elegant proof of the functional equation for all zetas $\xi_{\mathbb{Q}}^{G/P}(s)$; and
(3) In May, 2009, Haseo Ki gave a uniform proof of the RH for all 10 examples listed in the Appendix. His method is different from that of (Jeffrey C. Lagarias and) Masatoshi Suzuki.

# Appendix: Examples

We here list zetas $\xi_{\mathbb{Q}}^{G/P}$ for $G = SL(2,\,3,\,4,\,5)$, $Sp(4)$ and $G_2$. Consequence, all these zetas satisfy the FE: $\xi_{\mathbb{Q}}^{G/P}(1-s) = \xi_{\mathbb{Q}}^{G/P}(s)$. (Detailed calculations were given in version 2007 of this paper, but are omitted here as zetas for $SL(2,\,3)$, $Sp(4)$ and $G_2$ are now available in [W1, 3, 4] and [SW] respectively).

## Contents



## A.1 SL(n)

**A.1.1 SL(2)** A degenerate case, since $P = B$, the Borel. We have

$$\boxed{\xi_{\mathbb{Q}}^{SL(2)/B}(s) = \xi_{\mathbb{Q},2} : (s) = \frac{\xi_{\mathbb{Q}}(2s)}{s-1} - \frac{\xi_{\mathbb{Q}}(2s-1)}{s}} \tag{1}$$

It is the first natural example exposed that satisfies the RH ([W1,2,3], [LS]).

**A.1.2 SL(3)** Two maximal parabolic subgroups $P$, corresponding to partitions $3 = 2+1 = 1+2$. They share the same zetas:

$$\boxed{\begin{aligned}\xi_{\mathbb{Q}}^{SL(3)/P}(s) =&\xi_{\mathbb{Q}}(2) \cdot \frac{1}{3s-3} \cdot \xi_{\mathbb{Q}}(3s) \\ &- \xi_{\mathbb{Q}}(2) \cdot \frac{1}{3s} \cdot \xi_{\mathbb{Q}}(3s-2) \\ &- \frac{1}{3} \cdot \frac{1}{3s-3} \cdot \xi_{\mathbb{Q}}(3s-1) \\ &+ \frac{1}{3} \cdot \frac{1}{3s} \cdot \xi_{\mathbb{Q}}(3s-1) \\ &+ \frac{1}{2} \cdot \frac{1}{3s-1} \cdot \xi_{\mathbb{Q}}(3s-2) \\ &- \frac{1}{2} \cdot \frac{1}{3s-2} \cdot \xi_{\mathbb{Q}}(3s)\end{aligned}} \tag{2}$$

Contradicting to Ch. 9 of [W3], $\xi_{\mathbb{Q},3}(s) \neq \xi_{\mathbb{Q}}^{SL(3)/P_{2,1}}(s)$. Komori pointed out that there were sign mistakes for $\xi_{\mathbb{Q}}^{SL(3)/P_{2,1}}(s)$ there.

**A.1.3 SL(4)** Three maximal parabolics $P_{3,1}, P_{2,2}, P_{1,3}$, corresponding to $4 = 3+1 = 2+2 = 1+3$ respectively. $P_{1,3}$ and $P_{3,1}$ share the same zetas, while the zeta for $P_{2,2}$ is different. More precisely, we have

$$\begin{aligned}
\xi_{\mathbb{Q}}^{SL(4)/P_{3,1}}(s) =&\xi_{\mathbb{Q}}^{SL(4)/P_{1,3}}(s)\\
=&\frac{1}{4s-4}\xi(2)\xi(3)\cdot\xi(4s) - \frac{1}{4s}\xi(2)\xi(3)\cdot\xi(4s-3)\\
&+\frac{1}{4}\frac{1}{4s-2}\cdot\xi(4s) - \frac{1}{4}\frac{1}{4s-2}\cdot\xi(4s-3)\\
&+\frac{1}{3}\Big[\frac{1}{4s-1}+\frac{1}{4s-2}\Big]\xi(2)\cdot\xi(4s-3)\\
&-\frac{1}{3}\Big[\frac{1}{4s-2}+\frac{1}{4s-3}\Big]\xi(2)\cdot\xi(4s)\\
&+\frac{1}{2}\frac{1}{(4s)(4s-3)}\cdot\xi(4s-1)\\
&+\frac{1}{2}\frac{1}{(4s-1)(4s-4)}\cdot\xi(4s-2)\\
&-\frac{1}{(4s)(4s-4)}\xi(2)\cdot\xi(4s-1)\\
&-\frac{1}{(4s)(4s-4)}\xi(2)\cdot\xi(4s-2)
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\xi_{\mathbb{Q}}^{SL(4)/P_{2,2}}(s)\\
:=&\frac{1}{2s-3}\xi(2)^2\cdot\xi(2s)\xi(2s+1) - \frac{1}{2s+1}\xi(2)\cdot\xi(2s-2)\xi(2s-1)\\
&+\frac{1}{2s-1}\cdot\frac{1}{4}\cdot\xi(2s)\xi(2s+1) - \frac{1}{2s-1}\cdot\frac{1}{4}\cdot\xi(2s-2)\xi(2s-1)\\
&+\frac{1}{(2s)^2(2s-3)}\cdot\xi(2s-1)^2 - \frac{1}{(2s-2)^2(2s+1)}\cdot\xi(2s)^2\\
&-\frac{1}{2s-2}\xi(2)\cdot\xi(2s)\xi(2s+1) + \frac{1}{2s}\xi(2)\cdot\xi(2s-2)\xi(2s-1)\\
&+\frac{1}{(2s-2)(2s)}\cdot\xi(2s-1)\xi(2s)\\
&-\frac{2}{(2s-3)(2s+1)}\xi(2)\cdot\xi(2s-1)\xi(2s)
\end{aligned} \tag{4}$$

**A.1.4 SL(5)** Four maximal parabolic subgroups correspond to the partitions $5 = 4+1 = 3+2 = 2+3 = 1+4$. Denote the associated standard maximal parabolic subgroups by $P_{4,1}, P_{3,2}, P_{2,3}, P_{1,4}$ respectively. Then we know that the zeta for $P_{4,1}$ is the same as that for $P_{1,4}$, while the zeta for $P_{3,2}$ is the same as that for $P_{2,3}$.

More precisely, the new zeta functions $\xi_{\mathbb{Q}}^{SL(5)/P_{4,1}}(s) = \xi_{\mathbb{Q}}^{SL(5)/P_{1,4}}(s)$ are given by

$$
\boxed{
\begin{aligned}
&\xi_{\mathbb{Q}}^{SL(5)/P_{4,1}}(s) = \xi_{\mathbb{Q}}^{SL(5)/P_{1,4}}(s) = \xi_{SL(5);\mathbb{Q}}(s) := \\
&\Big[\frac{1}{5s-5}\xi(5s) - \frac{1}{5s}\xi(5s-4)\Big]\xi(2)\xi(3)\xi(4) \\
&+ \frac{1}{4}\Big\{\Big[\frac{1}{5s-1}\xi(5s-4) - \frac{1}{5s-4}\xi(5s)\Big] \\
&+ \Big[\frac{1}{5s-3}\xi(5s-4) - \frac{1}{5s-2}\xi(5s)\Big]\Big\}\xi(2)\xi(3) \\
&+ \frac{1}{9}\Big[\frac{1}{5s-2}\xi(5s) - \frac{1}{5s-3}\xi(5s-4)\Big]\xi(2) \\
&+ \frac{1}{6}\Big\{\Big[\frac{1}{5s-3}\xi(5s) - \frac{1}{5s-2}\xi(5s-4)\Big] \\
&+ \Big[\frac{1}{5s-2}\xi(5s) - \frac{1}{5s-3}\xi(5s-4)\Big]\Big\}\xi(2) \\
&+ \Big\{\frac{1}{3}\Big[\frac{1}{5s(5s-4)}\xi(5s-1) + \frac{1}{(5s-5)(5s-1)}\xi(5s-3)\Big] \\
&+ \frac{1}{2}\Big[\frac{1}{(5s-1)(5s-5)}\xi(5s-2) + \frac{1}{(5s-4)(5s)}\xi(5s-2)\Big] \\
&+ \frac{1}{3}\Big[\frac{1}{(5s-2)(5s-5)}\xi(5s-3) + \frac{1}{(5s-3)(5s)}\xi(5s-1)\Big]\Big\}\xi(2) \\
&+ \frac{1}{8}\Big[\frac{1}{5s-3}\xi(5s-4) - \frac{1}{5s-2}\xi(5s)\Big] \\
&+ \frac{1}{4}\Big[\frac{1}{5s-2}\xi(5s-4) - \frac{1}{5s-3}\xi(5s)\Big]\xi(2)^2 \\
&- \frac{1}{4}\Big[\frac{1}{(5s-3)(5s)}\xi(5s-1) + \frac{1}{(5s-2)(5s-5)}\xi(5s-3)\Big] \\
&- \Big[\frac{1}{(5s)(5s-5)}\xi(5s-1) + \frac{1}{(5s)(5s-5)}\xi(5s-3)\Big]\xi(2)\xi(3) \\
&- \frac{1}{4}\frac{1}{(5s-1)(5s-4)}\xi(5s-2) - \frac{1}{(5s)(5s-5)}\xi(5s-2)\xi(2)^2
\end{aligned}
}
\tag{5}
$$

(which, as well as the next, is quite complicated to obtain: totally 1200 cases should be discussed from which further residues should be taken,)

and
(6)

$$
\begin{aligned}
&\xi_{\mathbb{Q}}^{SL(5)/P_{3,2}}(s+1) := \xi_{\mathbb{Q};o}^{SL(5)/P_{3,2}}(s) = \xi_{\mathbb{Q};o}^{SL(5)/P_{2,3}}(s)\\
&=: \frac{1}{5s}\xi(2)^2\xi(3)\cdot\xi(5s+4)\xi(5s+5) + \frac{1}{4(5s+2)}\xi(2)\cdot\xi(5s+4)\xi(5s+5)\\
&\quad + \frac{1}{(5s+4)^2(5s)}\xi(2)\cdot\xi(5s+2)\xi(5s+3) - \frac{1}{2(5s+1)}\xi(2)\xi(3)\cdot\xi(5s+4)\xi(5s+5)\\
&\quad - \frac{1}{3(5s+1)}\xi(2)^2\cdot\xi(5s+4)\xi(5s+5) - \frac{1}{3(5s+2)}\xi(2)^2\cdot\xi(5s+4)\xi(5s+5)\\
&\quad - \frac{1}{4(5s+2)(5s+4)}\cdot\xi(5s+3)\xi(5s+4) + \frac{1}{3(5s+3)}\xi(2)^2\cdot\xi(5s+1)\xi(5s+2)\\
&\quad - \frac{1}{2(5s+1)(5s+3)(5s+4)}\cdot\xi(5s+2)\xi(5s+3) + \frac{1}{8(5s+2)}\cdot\xi(5s+1)\xi(5s+2)\\
&\quad - \frac{1}{(5s+1)^2(5s+5)}\xi(2)\cdot\xi(5s+3)\xi(5s+4) - \frac{1}{6(5s+3)}\xi(2)\cdot\xi(5s+1)\xi(5s+2)\\
&\quad - \frac{1}{2(5s)(5s+3)^2}\cdot\xi(5s+2)^2 - \frac{1}{4(5s+2)(5s+3)}\cdot\xi(5s+2)\xi(5s+4)\\
&\quad + \frac{1}{2(5s+4)}\xi(2)\xi(3)\cdot\xi(5s+1)\xi(5s+2) + \frac{1}{2(5s)(5s+4)}\xi(2)\cdot\xi(5s+2)\xi(5s+3)\\
&\quad + \frac{1}{2(5s+1)(5s+4)}\xi(2)\cdot\xi(5s+2)\xi(5s+3) + \frac{1}{6(5s+2)}\xi(2)\cdot\xi(5s+4)\xi(5s+5)\\
&\quad + \frac{1}{6(5s+3)}\xi(2)\cdot\xi(5s+4)\xi(5s+5) + \frac{1}{2(5s+1)(5s+4)}\xi(2)\cdot\xi(5s+3)\xi(5s+4)\\
&\quad + \frac{1}{(5s+1)^2(5s+4)^2}\cdot\xi(5s+3)^2 + \frac{1}{3(5s+2)(5s+4)}\xi(2)\cdot\xi(5s+2)\xi(5s+4)\\
&\quad - \frac{1}{8(5s+3)}\cdot\xi(5s+4)\xi(5s+5) - \frac{1}{6(5s+2)}\xi(2)\cdot\xi(5s+1)\xi(5s+2)\\
&\quad - \frac{1}{4(5s+3)}\xi(2)\cdot\xi(5s+1)\xi(5s+2) + \frac{1}{2(5s+1)(5s+5)}\xi(2)\cdot\xi(5s+3)\xi(5s+4)\\
&\quad + \frac{1}{2(5s+2)^2(5s+5)}\cdot\xi(5s+4)^2 - \frac{1}{(5s)(5s+5)}\xi(2)\xi(3)\cdot\xi(5s+2)\xi(5s+4)\\
&\quad - \frac{1}{(5s)(5s+5)}\xi(2)^2\cdot\xi(5s+3)\xi(5s+4) - \frac{1}{(5s+1)(5s+2)(5s+5)}\xi(2)\cdot\xi(5s+4)^2\\
&\quad + \frac{1}{3(5s+1)(5s+3)}\xi(2)\cdot\xi(5s+2)\xi(5s+4)\\
&\quad + \frac{1}{2(5s+1)(5s+2)(5s+4)}\cdot\xi(5s+3)\xi(5s+4)\\
&\quad - \frac{1}{4(5s+1)(5s+3)}\cdot\xi(5s+2)\xi(5s+3) - \frac{1}{(5s+5)}\xi(2)^2\xi(3)\cdot\xi(5s+1)\xi(5s+2)\\
&\quad - \frac{1}{(5s)(5s+5)}\xi(2)^2\cdot\xi(5s+2)\xi(5s+3) + \frac{1}{(5s)(5s+3)(5s+4)}\xi(2)\cdot\xi(5s+2)^2\\
&\quad + \frac{1}{3(5s+4)}\xi(2)^2\cdot\xi(5s+1)\xi(5s+2)
\end{aligned}
$$

**A.2 Sp(4)** Two maximal parabolic subgroups corresponding to simple roots $\{e_1 - e_2\}$ and $\{2e_2\}$ respectively. Their zetas read as follows:

$$\boxed{\begin{aligned}&\xi_{\mathbb{Q}}^{Sp(4)/P_{e_1-e_2}}(s)\\ =&\frac{1}{s-2}\xi(2)\cdot\xi(s+1)\xi(2s)-\frac{1}{s+1}\xi(2)\cdot\xi(s-1)\xi(2s-1)\\ &-\frac{1}{2s-2}\cdot\xi(s+1)\xi(2s)+\frac{1}{2s}\cdot\xi(s-1)\xi(2s-1)\\ &-\frac{1}{(2s-2)(s+1)}\cdot\xi(s)\xi(2s)-\frac{1}{(2s)(s-2)}\cdot\xi(s)\xi(2s-1)\end{aligned}} \tag{7}$$

and
(8)

$$\boxed{\begin{aligned}\xi_{\mathbb{Q}}^{Sp(4)/P_{2e_2}}(s)=&\frac{1}{2s-3}\xi(2)\cdot\xi(2s+1)-\frac{1}{2s+1}\xi(2)\cdot\xi(2s-2)\\ &-\frac{1}{2(2s-1)}\cdot\xi(2s+1)+\frac{1}{2(2s-1)}\cdot\xi(2s-2)\\ &-\frac{1}{(2s+1)(2s-2)}\cdot\xi(2s)-\frac{1}{(2s)(2s-3)}\cdot\xi(2s-1).\end{aligned}}$$

The RH for $\xi_{\mathbb{Q}}^{Sp(4)/P_1}(s)$ is confirmed ([S2]), whose method, a generalization of ([S] and/or [SW]), can also be used to show that outside a finite box, all zeros of $\xi_{\mathbb{Q}}^{Sp(4)/P_2}(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$.

**A.3 $\mathbf{G_2}$** Two maximal parabolic subgroups corresponding to the long and the short root respectively. Their zetas read as follows:

$$\boxed{\begin{aligned}&\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s)\\ =&\frac{1}{s-3}\xi(2)\cdot\xi(s+2)\xi(2s)-\frac{1}{s+2}\xi(2)\cdot\xi(s-2)\xi(2s-1)\\ &+\frac{1}{2s-2}\cdot\xi(s-2)\xi(2s-1)-\frac{1}{2s}\cdot\xi(s+2)\xi(2s)\\ &-\frac{1}{s(s-3)}\cdot\xi(s-1)\xi(2s-1)-\frac{1}{(s-1)(s+2)}\cdot\xi(s+1)\xi(2s)\\ &-\frac{1}{(2s-2)(s+1)}\cdot\xi(s)\xi(2s)-\frac{1}{(2s)(s-2)}\cdot\xi(s)\xi(2s-1)\end{aligned}} \tag{9}$$

and

$$
\xi_{\mathbb{Q}}^{G_2/P_{\rm long}}(s) = \frac{1}{s-2}\xi(2)\cdot\xi(s+1)\xi(2s)\xi(3s)
- \frac{1}{s+1}\xi(2)\cdot\xi(s-1)\xi(2s-1)\xi(3s-2)
- \frac{1}{2s-2}\cdot\xi(s+1)\xi(2s)\xi(3s)
+ \frac{1}{2s}\cdot\xi(s-1)\xi(2s-1)\xi(3s-2)
- \frac{1}{(3s)(2s-2)}\cdot\xi(s)\xi(2s)\xi(3s-1)
- \frac{1}{(3s-1)(s-2)}\cdot\xi(s)\xi(2s-1)\xi(3s-2)
- \frac{1}{(3s-3)(2s)}\cdot\xi(s)\xi(2s-1)\xi(3s-1)
- \frac{1}{(3s-2)(s+1)}\cdot\xi(s)\xi(2s)\xi(3s) \tag{10}
$$

The RH for both $\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s)$ and $\xi_{\mathbb{Q}}^{G_2/P_{\rm long}}(s)$ are confirmed by Suzuki ([SW]).

**A.4 $T$-version for SL(3)** In this subsection, we indicate how functional equation for our zetas can be obtained from a general $T$-construction. This, in turn, will expose a hidden symmetry. For simplicity, we consider $G = SL(3)$ only.

By definition,

$$
\omega_{\mathbb{Q}}^{G;T}(s) = \sum_{w\in W}\left(\frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right).
$$

In particular, for $G = SL(3)$, we may take $\lambda = (z_1, z_2, z_3)$ with $z_1 + z_2 + z_3 = 0$, $T = (x, y, -x-y)$, $\rho = (1, 0, -1)$. Note that the Weyl group is simply $W = S_3$ and $w \in W = S_3$ acts via the corresponding permutation on lower indices.

Thus by taking residue along the singular-plane $z_1 - z_2 = 1$ and renaming $z_2 = t$, we have $z_1 = t+1, z_3 = -2t-1$. Consequently, we, by a detailed calculation which we omit, get the following explicit expression for the $T$-version period $\omega_{\mathbb{Q}}^{SL(3)/P_{1,2};T}(t)$ associated to $(SL(3), P_{1,2})$ over the field of rationals $\mathbb{Q}$:

$$\begin{aligned}\omega_{\mathbb{Q}}^{SL(3)/P_{1,2};T}(t) =& \frac{1}{3t}\xi(2)\cdot\xi(3t+3)\cdot e^{3tx+3ty+4x+2y} \\ &- \frac{1}{2}\frac{1}{3t+1}\cdot\xi(3t+3)\cdot e^{(3t+3)(x+y)} \\ &+ \frac{1}{2}\frac{1}{3t+2}\cdot\xi(3t+1)\cdot e^{-3tx} + 0 \\ &- \frac{1}{3t+3}\xi(2)\cdot\xi(3t+1)\cdot e^{-3ty+x-y} \\ &- \frac{1}{3t}\frac{1}{3t+3}\cdot\xi(3t+2)\cdot e^{-3tx+x+2y}.\end{aligned}$$

Similarly, by taking residue along $z_2 - z_3 = 1$ and assuming $z_3 = s, z_2 = s+1, z_1 = -2s-1$, we get

$$\begin{aligned}\omega_{\mathbb{Q}}^{SL(3)/P_{2,1};T}(s) =& -\frac{1}{3s+3}\xi(2)\cdot\xi(3s+1)\cdot e^{-3sx+x+2y} + 0 \\ &- \frac{1}{2}\frac{1}{3s+1}\cdot\xi(3s+3)\cdot e^{(3s+3)(x+y)} \\ &+ \frac{1}{2}\frac{1}{3s+2}\cdot\xi(3s+1)\cdot e^{-3sx} \\ &- \frac{1}{3s}\frac{1}{3s+3}\cdot\xi(3s+2)\cdot e^{3sx+3sy+4x+2y} \\ &+ \frac{1}{3s}\xi(2)\cdot\xi(3s+3)\cdot e^{-3sy+x-y}.\end{aligned}$$

Clearly, there is no functional equation at this stage. However, if we set $y = 0$ in $T = (x, y, -x-y)$ so that $T = (x, 0, -x)$, that is to say, $T = x\rho \in \mathbb{C}\cdot\rho$ sitting on the line spanned by $\rho$, then we have

$$\boxed{\begin{aligned}\omega_{\mathbb{Q}}^{SL(3)/P_{1,2};x\rho}(t) =& \frac{1}{3t}\xi(2)\cdot\xi(3t+3)\cdot e^{3tx+4x} \\ &- \frac{1}{2}\frac{1}{3t+1}\cdot\xi(3t+3)\cdot e^{(3t+3)x} \\ &+ \frac{1}{2}\frac{1}{3t+2}\cdot\xi(3t+1)\cdot e^{-3tx} \\ &- \frac{1}{3t+3}\xi(2)\cdot\xi(3t+1)\cdot e^{x} \\ &- \frac{1}{3t}\frac{1}{3t+3}\cdot\xi(3t+2)\cdot e^{-3tx+x}\end{aligned}}$$

and

$$\boxed{\begin{aligned}\omega_{\mathbb{Q}}^{SL(3)/P_{2,1};x\rho}(s) =& -\frac{1}{3s+3}\xi(2)\cdot\xi(3s+1)\cdot e^{-3sx+x}\\ &-\frac{1}{2}\frac{1}{3s+1}\cdot\xi(3s+3)\cdot e^{(3s+3)x}\\ &+\frac{1}{2}\frac{1}{3s+2}\cdot\xi(3s+1)\cdot e^{-3sx}\\ &-\frac{1}{3s}\frac{1}{3s+3}\cdot\xi(3s+2)\cdot e^{3sx+4x}\\ &+\frac{1}{3s}\xi(2)\cdot\xi(3s+3)\cdot e^{x}\end{aligned}}$$

In particular, we have the functional equation

$$\boxed{\omega_{\mathbb{Q}}^{SL(3)/P_{1,2};x\rho}(-1-s) = \omega_{\mathbb{Q}}^{SL(3)/P_{2,1};x\rho}(s)}$$

Or put it in a better form, we set

(11)

$$\boxed{\begin{aligned}\xi_{\mathbb{Q};\mathbf{T}}^{SL(3)/P_{1,2}}(s) :=& \frac{1}{3s-3}\xi(2)\cdot\xi(3s)\cdot\mathbf{T}^{3s+1} - \frac{1}{3s}\xi(2)\cdot\xi(3s-2)\cdot\mathbf{T}\\ &-\frac{1}{2}\frac{1}{3s-2}\cdot\xi(3s)\cdot\mathbf{T}^{3s} + \frac{1}{2}\frac{1}{3s-1}\cdot\xi(3s-2)\cdot\mathbf{T}^{-3s+3}\\ &-\frac{1}{3s-3}\frac{1}{3s}\cdot\xi(3t-1)\cdot\mathbf{T}^{-3s+4}\end{aligned}}$$

and

(12)

$$\boxed{\begin{aligned}\omega_{\mathbb{Q};\mathbf{T}}^{SL(3)/P_{2,1}}(s) =& -\frac{1}{3s}\xi(2)\cdot\xi(3s-2)\cdot\mathbf{T}^{-3s+4} + \frac{1}{3s-3}\xi(2)\cdot\xi(3s)\cdot\mathbf{T}\\ &-\frac{1}{2}\frac{1}{3s-2}\cdot\xi(3s)\cdot\mathbf{T}^{3s} + \frac{1}{2}\frac{1}{3s-1}\cdot\xi(3s-2)\cdot\mathbf{T}^{-3s+3}\\ &-\frac{1}{3s-3}\frac{1}{3s}\cdot\xi(3s-1)\cdot\mathbf{T}^{3s+1}\end{aligned}}$$

Then we get

$$\boxed{\xi_{\mathbb{Q};\mathbf{T}}^{SL(3)/P_{1,2}}(1-s) = \xi_{\mathbb{Q};\mathbf{T}}^{SL(3)/P_{2,1}}(s)} \tag{13}$$

This exposes a new symmetry for our zetas.

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