Advanced Studies in Pure Mathematics 57, 2010
Probabilistic Approach to Geometry
pp. 171–188

A fixed-point property of finitely generated groups and an energy of equivariant maps

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Abstract.

The purpose of this article is to present sufficient conditions for an isometric action of a finitely generated group to have a fixed point in terms of an energy of equivariant maps (Proposition 2.1 and Theorem 3.1), which turn out to be useful in proving that groups with fixed-point property form a large family in the set of finitely presented groups. Most of the results are included in [7] and [8].

§1. Fixed-point property of finitely generated groups

Let Γ be a finitely generated group, and Y a metric space. We say Γ has **fixed-point property for** Y, abbreviated as Γ has $F(Y)$, if every isometric action of Γ on Y admits a fixed point. In other words, Γ has $F(Y)$ if, for every homomorphism $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$, there is a point $p \in Y$ such that $\rho(\gamma)p = p$ for all $\gamma \in \Gamma$. In what follows we assume that a metric space Y to be complete.

As we will see in the following subsections, fixed-point property of a finitely generated group is related to other interesting properties of the group.

1.1. Fixed-point property for Hilbert spaces

Take Y to be a Hilbert space $\mathcal{H}$. According to theorems due to Delorme and Guichardet, Γ has $F(\mathcal{H})$ if and only if Γ has Kazhdan's property (T), which means that the trivial representation is isolated in the space of irreducible unitary representations of Γ. For a good reference on this subject, we refer the reader to [2].

Received January 24, 2009.
Revised March 4, 2009.
2000 *Mathematics Subject Classification.* 58E20, 20F65.
Key words and phrases. Fixed-point property, finitely generated groups, harmonic maps.

Example 1. Let Γ be a finite group and $\rho : \Gamma \longrightarrow \mathrm{Isom}(\mathcal{H})$ a homomorphism. Here $\mathrm{Isom}(\mathcal{H})$ is the group of affine isometry of $\mathcal{H}$. Take any point $v \in \mathcal{H}$, and let v_0 be the barycenter of the $\rho(\Gamma)$-orbit of v: $v_0 = (\sum_{\gamma\in\Gamma} \rho(\gamma)v)/\#\Gamma$. Since the orbit is obviously invariant under the action of $\rho(\Gamma)$ and the action is isometric, v_0 must be fixed by the action. This shows that any finite group Γ has $F(\mathcal{H})$.

Example 2. On the other hand, infinite amenable groups, including abelian, nilpotent, and solvable groups, do not have $F(\mathcal{H})$.

Example 3. Lattices in noncompact, simple Lie groups with rank greater than 1 are known to have $F(\mathcal{H})$. As for rank 1 case, lattices in $\mathrm{Sp}(n, 1)$ $(n > 1)$ and F_4^{-20} have $F(\mathcal{H})$.

Until recently, known examples of an infinite group with $F(\mathcal{H})$ other than that in Example 3 were quite few. However, in the last decade, several results as in [1] and [22] came up and suggested that the class of discrete groups with $F(\mathcal{H})$ might be rich. As a matter of fact, it has been shown that such groups are "generic" among a certain class of groups, that is, certain random groups have property $F(\mathcal{H})$ with overwhelming probability. See [5], [23], [19], and [17].

1.2. Fixed-point property for a family of metric spaces

Take $\mathcal{Y}$ to be a family of metric spaces consisting of Hilbert spaces, Riemannian symmetric spaces $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, and Bruhat–Tits buildings $\mathrm{PGL}(n, \mathbb{Q}_p)/\mathrm{PGL}(n, \mathbb{Z}_p)$ (for all $n \in \mathbb{N}$ and all prime numbers p). Suppose that Γ has $F(\mathcal{Y})$, namely, Γ has $F(Y)$ for all $Y \in \mathcal{Y}$. Then Γ has an interesting property: for any $m \in \mathbb{N}$ and homomorphism $\rho : \Gamma \longrightarrow \mathrm{GL}(m, \mathbb{C})$, $\rho(\Gamma)$ is a finite group. This fact follows from a modification of a part of a proof of Margulis' arithmeticity theorem via superrigidity theorem (cf. [21, Chapter 6]). Note that, if an infinite group Γ has this property, then Γ cannot be a subgroup of $\mathrm{GL}(m, \mathbb{C})$ for any $m \in \mathbb{N}$.

Example 4. For any finitely generated infinite simple group Γ and any $m \in \mathbb{N}$, $\rho : \Gamma \longrightarrow \mathrm{GL}(m, \mathbb{C})$ has trivial image; such Γ has the property derived from $F(\mathcal{Y})$. This follows from the fact that any finitely generated subgroup of $\mathrm{GL}(m, \mathbb{C})$ is residually finite, in particular, such a group has a lot of normal subgroups.

One might expect that certain random groups have $F(\mathcal{Y})$, which is a very strong fixed-point property; if such random groups are also infinite groups, then this implies that certain random groups cannot be subgroups of $\mathrm{GL}(m, \mathbb{C})$ for any $m \in \mathbb{N}$. In order to prove this, one needs good sufficient conditions for a given action to have a fixed point. We will give a basic criterion in terms of the energy of equivariant maps (Proposition 2.1) in the next section, and will give an application of this criterion (Theorem 3.1). Proposition 2.1 is expected to be useful, since it is close to a necessary condition for an action to have a fixed point (see Theorem 4.1).

The metric spaces in $\mathcal{Y}$ are CAT(0) spaces, which are metric spaces with nonpositive curvature in a certain sense. We will focus on fixed-point property for CAT(0) spaces in the rest of this article. A basic reference on CAT(0) spaces is [3]. We note here that, for a CAT(0) space Y, the existence and the uniqueness of barycenter are known, and hence any finite group has $F(Y)$ as in Example 1. Therefore, we have nothing to do for finite groups, and we assume that groups under consideration are infinite groups in what follows unless otherwise stated.

§2. The energy of equivariant maps

Let Γ be a finitely generated group, and Y a CAT(0) space. Suppose a homomorphism $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$ is given. We give a sufficient condition for $\rho(\Gamma)$ to have a fixed point (i.e., there exists a point $p \in Y$ such that $\rho(\Gamma)p = p$) in terms of the *energy functional* of ρ-equivariant maps.

We say a map $f : \Gamma \longrightarrow Y$ is *ρ-equivariant* if f satisfies $f(\gamma\gamma') = \rho(\gamma)f(\gamma')$ for all γ, $\gamma' \in \Gamma$. (Here we regard Γ itself as a space with left Γ-action.) Let μ be a Γ-invariant, irreducible, and symmetric random walk on Γ with finite-support property:

- $\mu : \Gamma \times \Gamma \longrightarrow [0, 1]$.
- $\mu(\gamma\gamma', \gamma\gamma'') = \mu(\gamma', \gamma'')$ for any γ, γ', and $\gamma'' \in \Gamma$.
- For any $\gamma \in \Gamma$, $\mu(\gamma, \gamma') = 0$ for all but finitely many $\gamma' \in \Gamma$.
- For any $\gamma \in \Gamma$, $\sum_{\gamma' \in \Gamma} \mu(\gamma, \gamma') = 1$.
- $\mu(\gamma, \gamma') = \mu(\gamma', \gamma)$ for any $\gamma, \gamma' \in \Gamma$.
- For any $\gamma, \gamma' \in \Gamma$, there exists $\gamma_0, \gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\gamma = \gamma_0$, $\gamma' = \gamma_n$ and $\mu(\gamma_i, \gamma_{i+1}) \neq 0$ for $i = 0, 1, \ldots, n-1$.

The last condition is called the *irreducibility* of a random walk, and the third condition is referred as *finite-support property* in this article. Suppose Γ is a finitely generated group with a finite generating set $S \subset \Gamma$. We may assume that S is symmetric, namely $s \in S$ implies $s^{-1} \in S$. Then, with respect to S, we can define a random walk μ by

$$(2.1) \qquad \mu(\gamma, \gamma') = \begin{cases} \frac{1}{\#S} & \text{if } \exists s \in S \text{ such that } \gamma' = \gamma s, \\ 0 & \text{otherwise,} \end{cases}$$

and this turns out to be a Γ-invariant, irreducible, and symmetric random walk with finite-support property. A random walk μ defined by (2.1) is called the *standard random walk* with respect to a generating set S, which can be understood as the standard random walk on the Cayley graph of Γ with respect to S. (See Remark 4 for the definition of the standard random walk on a graph.)

Given a Γ-invariant, irreducible, and symmetric random walk μ with finite-support property, we can define the energy $E_\rho(f)$ of a ρ-equivariant map f by

$$E_\rho(f) = \frac{1}{2}\sum_{\gamma\in\Gamma} \mu(e,\gamma) d_Y(f(e), f(\gamma))^2, \tag{2.2}$$

where e denotes the identity element in Γ.

Note that the image of a ρ-equivariant map $f : \Gamma \longrightarrow Y$ is the $\rho(\Gamma)$-orbit of the point $f(e)$, and f is determined by the choice of $f(e)$. Therefore, the set of all ρ-equivariant maps from Γ to Y can be identified with Y. Then the energy functional E_ρ becomes a convex continuous function on Y. Let $-\mathrm{grad}E_\rho(f)$ be the negative of the gradient of the energy functional E_ρ at f. When Y is a Riemannian manifold this should be understood as the negative of the ordinary gradient. In general, one can give a reasonable definition of $-\mathrm{grad}E_\rho(f)$ as an element of the tangent cone of Y at a point f ([7], see also Remark 1), where Y is identified with the space of ρ-equivariant maps. The following proposition gives a sufficient condition for the existence of a fixed point of $\rho(\Gamma)$ in terms of the energy functional.

Proposition 2.1 ([7]). *Let Γ be a finitely generated group with a Γ-invariant, irreducible, symmetric random walk with finite-support property. Let Y be a* CAT(0) *space and $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$ a homomorphism. Suppose there is a positive constant C such that $|-\mathrm{grad}E_\rho(f)|^2 \geq CE_\rho(f)$ holds for every ρ-equivariant map f. Then $\rho(\Gamma)$ has a fixed point.*

In fact, under the assumption, $|-\mathrm{grad}E_\rho(f_t)|$ decays rapidly along the Jost–Mayer's gradient flow f_t of E_ρ, and f_t converges to a constant map, which must be ρ-equivariant. The image of a ρ-equivariant constant map is a $\rho(\Gamma)$-orbit consisting of a single point. Thus the point is fixed by $\rho(\Gamma)$. (See [10] and [15] for the Jost–Mayer's gradient flow.)

A useful tool to show the existence of a positive constant C in Proposition 2.1 is so-called Bochner technique for our discrete setting. Let us briefly recall corresponding argument in smooth case.

Let X be the universal covering space of a compact manifold M with $\pi_1(M) \cong \Gamma$. Suppose a homomorphism $\rho : \Gamma \longrightarrow \mathbb{R}$ is given. Then ρ gives the action of Γ on $\mathbb{R}$ by translations. Note that, in this case, having a fixed point means that the action is trivial.

Take any equivariant map $f : X \longrightarrow \mathbb{R}$. Since the action of $\rho(\Gamma)$ is by translations, we have $f(\gamma(x)) = f(x)+\rho(\gamma)$ for any $x \in X$ and $\gamma \in \Gamma$. Thus the difference $f(\gamma(x))-f(x) = \rho(\gamma) \in \mathbb{R}$ is constant and this makes a 1-form df on X to be Γ-invariant. Hence df descends to a 1-form α_f on M. It is easy to see that, once a homomorphism ρ is fixed, the de Rham cohomology class $[\alpha_f] \in H^1(M)$ defined by a ρ-equivariant map f is independent of the choice of a map f. Therefore one obtains a map which assigns a cohomology class $[\alpha_f] \in H^1(M)$ to ρ. This map turns out to be a bijection between $\mathrm{Hom}(\Gamma, \mathbb{R})$ and $H^1(M)$, where $\mathrm{Hom}(\Gamma, \mathbb{R})$ denotes the set of homomorphisms from Γ to $\mathbb{R}$.

Take any Riemannian metric g on M, and denote its volume form by dv_g. Recalling the definition of the energy of an equivariant map from Γ, one sees that the energy $E(f)$ for a ρ-equivariant map $f : X \longrightarrow \mathbb{R}$ should be defined as

$$E(f) = \frac{1}{2}\|\alpha_f\|_{L^2}^2 = \frac{1}{2}\int_M g(\alpha_f, \alpha_f)\ dv_g,$$

where we use the same symbol g for the metric on the space of 1-forms. One easily sees that the gradient of the energy functional defined in this way can be written as

$$|-\mathrm{grad}E(f)|^2 = \int_M g(\Delta\alpha_f, \alpha_f)\ dv_g,$$

where Δ denotes the Laplacian acting on the space of 1-forms. Thus an inequality corresponding to that in Proposition 2.1 takes the following form:

$$\int_M g(\Delta\alpha_f, \alpha_f)\ dv_g \geq C\|\alpha_f\|_{L^2}^2.$$

Thus the existence of a positive constant C implies the vanishing of $H^1(M)$, since any harmonic 1-form must be trivial. Equivalently, it implies that, for any homomorphism $\rho : \Gamma \longrightarrow \mathbb{R}$, the action of $\rho(\Gamma)$ on $\mathbb{R}$ has a fixed point.

To show the existence of $C > 0$, differential geometers often rely on so-called Bochner technique, where the local structure, actually the curvature, of M is involved. For example, a well-known theorem due to Bochner says that if the Ricci curvature of M is everywhere positive, then we can take $C > 0$, and hence $H^1(M) = 0$. Another important result along this line is due to Matsushima ([14]), which says that

$H^1(M) = 0$ for a compact, irreducible locally symmetric space M with rank greater than 1 and with nonpositive sectional curvature. (See also [18] for a comprehensible proof.)

It is possible, in principle, to apply the argument above to a homomorphism $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$, where Y is a Riemannian manifold; indeed, some superrigidity theorems were obtained in this way by Corlette ([4]), Mok–Siu–Yeung ([16]) and Jost–Yau ([11], [12]). In particular, a theorem due to Mok–Siu–Yeung and Jost–Yau can be regarded as a nonlinear version of the Matsushima's theorem above. Also Gromov–Schoen extended this technique to maps between a manifold and a certain singular space ([6]). Moreover, a similar argument also works even when we consider an equivariant map from a discrete space, such as a finitely generated group, to a CAT(0) space. Here, instead of the curvature, a local invariant of a discrete space, Wang's invariant of a link, is involved. (The definition of Wang's invariant can be found in [9] and [20]. See also Remark 4.) Here a link is a finite graph describing the shape (the adjacency relation) of a neighborhood of a point of a discrete space. Fixed-point theorems for CAT(0) spaces obtained by such a method can be seen, for example, in [20], [5], [9], and [7]. These results tells us that the approach described as above is so efficient that, for example, a fixed-point theorem for random groups given by the triangular model can be shown ([7], see also [13]); in the triangular model, all relators have length three and hence appear as edges in the link of the identity element in the Cayley graph.

If one wishes to obtain a fixed-point theorem for random groups given by another model, one needs a sufficient condition for the existence of $C > 0$ described in terms of a large scale (not local) structure of discrete spaces in order to handle long loops in the Cayley graph of a group arising from long relators. We will introduce such a condition in the next section.

§3. The n-step energy of equivariant maps

Let Γ be a finitely generated group with a Γ-invariant, symmetric, and irreducible random walk μ with finite-support property. Suppose that Y is a CAT(0) space, and that a homomorphism $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$ is given. Denote by μ^n the nth convolution of μ:

$$\mu^n(\gamma, \gamma') = \sum_{\gamma_1 \in \Gamma} \cdots \sum_{\gamma_{n-1} \in \Gamma} \mu(\gamma, \gamma_1) \dots \mu(\gamma_{n-1}, \gamma').$$

We define *n-step energy* $E_{\rho,n}(f)$ of a ρ-equivariant map f by

$$E_{\rho,n}(f) = \frac{1}{2}\sum_{\gamma\in\Gamma} \mu^n(e,\gamma)d_Y(f(e),f(\gamma))^2.$$

Then we can show

Theorem 3.1 ([5], [19], [8]). *Let Γ be a finitely generated group with a Γ-invariant, irreducible, symmetric random walk with finite-support property, and Y a* CAT(0) *space. Suppose that $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$ is a homomorphism, and that, for some n, there exists a positive constant ε such that $E_{\rho,n}(f) \leq (n-\varepsilon)E_\rho(f)$ holds for every ρ-equivariant map f. Then there exists a positive constant C as in Proposition 2.1. In particular, $\rho(\Gamma)$ has a fixed point in Y.*

This claim was first pointed out by Gromov in [5]. In [19], a similar result was proved for the case $Y = \mathcal{H}$ by using a discrete heat flow introduced by Gromov in [5]. In the next subsection, we present a detailed proof of Theorem 3.1, in which we use Proposition 2.1, for the case $Y = \mathcal{H}$.

3.1. The case $Y = \mathcal{H}$

Let Γ be a finitely generated group and $\rho : \Gamma \longrightarrow \mathrm{Isom}(\mathcal{H})$ a homomorphism, where $\mathcal{H}$ is a Hilbert space. Define $-\Delta f(\gamma)$ by

$$-\Delta f(\gamma) = \sum_{\gamma'\in\Gamma} \mu(\gamma,\gamma')(f(\gamma') - f(\gamma)).$$

Thus $-\Delta f(\gamma)$ is the average of $f(\gamma') - f(\gamma)$ with respect to μ. In this subsection, we denote the energy by E_μ, dropping ρ and indicating the random walk μ chosen to define the energy.

Remark 1. Using the notion of the barycenter, we can define $-\Delta f(\gamma)$ for a map from Γ to a general CAT(0) space Y as an element of $TC_{f(\gamma)}Y$, the tangent cone of Y at $f(\gamma)$ (see [9, Definition 2.6]). It turns out that $-\mathrm{grad}E(f) = 2(-\Delta f(e)) \in T_{f(e)}Y$ holds, where we regard the set of all equivariant maps as Y by the identification $f \mapsto f(e) \in Y$. According to a result of Mayer together with [7, Proposition 4.2], we see that f minimizes the energy E_μ among all ρ-equivariant maps if and only if $-\Delta f(e) = 0$. We call a map minimizing the energy E_μ to be a (μ-) *harmonic map.*

Let $\mathcal{U}(\mathcal{H})$ denote the group of unitary transformations on $\mathcal{H}$. Then any homomorphism $\rho : \Gamma \longrightarrow \mathrm{Isom}(\mathcal{H})$ is decomposed as $\rho(\gamma) = \rho_0(\gamma) + b(\gamma)$ according to the semi-direct product decomposition of $\mathrm{Isom}(\mathcal{H})$, where $\rho_0 : \Gamma \longrightarrow \mathcal{U}(\mathcal{H})$ is a homomorphism and $b : \Gamma \longrightarrow \mathcal{H}$ is a map (actually a cocycle). We note that $-\Delta f : \Gamma \longrightarrow \mathcal{H}$ is ρ_0-equivariant, in fact,

$$\begin{aligned}
&-\Delta f(\gamma'\gamma)\\
=&\sum_{\gamma''\in\Gamma}\mu(\gamma'\gamma,\gamma'')(f(\gamma'')-f(\gamma'\gamma))\\
=&\sum_{\gamma''\in\Gamma}\mu(\gamma,\gamma'^{-1}\gamma'')(\rho(\gamma')f(\gamma'^{-1}\gamma'')-\rho(\gamma')f(\gamma))\\
=&\sum_{\gamma''\in\Gamma}\mu(\gamma,\gamma'^{-1}\gamma'')(\rho_0(\gamma')f(\gamma'^{-1}\gamma'')+b(\gamma')-\rho_0(\gamma')f(\gamma)-b(\gamma'))\\
=&\sum_{\gamma''\in\Gamma}\mu(\gamma,\gamma'^{-1}\gamma'')\rho_0(\gamma')(f(\gamma'^{-1}\gamma'')-f(\gamma))\\
=&\rho_0(\gamma')(-\Delta f(\gamma)).
\end{aligned}$$

Note that we have used the Γ-invariance of μ and the linearity of $\rho_0(\gamma')$.

Proposition 3.2. *Let μ and μ' be Γ-invariant, irreducible, and symmetric random walks with finite-support property on Γ. Then, for any ρ-equivariant map f,*

$$E_{\mu*\mu'}(f)=E_\mu(f)+E_{\mu'}(f)-\langle-\Delta_\mu f(e),-\Delta_{\mu'}f(e)\rangle \tag{3.1}$$

holds, where $\mu*\mu'$ denotes the convolution of μ and μ'.

Proof. By a straightforward computation

$$\begin{aligned}
&E_{\mu*\mu'}(f)\\
=&\frac{1}{2}\sum_{\gamma,\gamma'}\mu'(e,\gamma)\mu(\gamma,\gamma')|f(e)-f(\gamma')|^2\\
=&\frac{1}{2}\sum_{\gamma,\gamma'}\mu'(e,\gamma)\mu(\gamma,\gamma')\big(|f(e)-f(\gamma)|^2+|f(\gamma')-f(\gamma)|^2\\
&-2\langle f(e)-f(\gamma),f(\gamma')-f(\gamma)\rangle\big)\\
=&\frac{1}{2}\sum_{\gamma}\mu'(e,\gamma)|f(e)-f(\gamma)|^2\\
&+\frac{1}{2}\sum_{\gamma}\mu'(e,\gamma)\sum_{\gamma'}\mu(\gamma,\gamma')|f(\gamma')-f(\gamma)|^2\\
&-\sum_{\gamma,\gamma'}\mu'(e,\gamma)\mu(\gamma,\gamma')\langle f(e)-f(\gamma),f(\gamma')-f(\gamma)\rangle.
\end{aligned}\tag{3.2}$$

Since μ is Γ-invariant, we can rewrite

$$\begin{aligned}
&\text{(the 1st and 2nd terms in the last expression in (3.2))}\\
=&\frac{1}{2}\sum_{\gamma}\mu'(e,\gamma)|f(\gamma)-f(e)|^2\\
&+\frac{1}{2}\sum_{\gamma}\mu'(e,\gamma)\sum_{\gamma'}\mu(e,\gamma^{-1}\gamma')|f(\gamma^{-1}\gamma')-f(e)|^2\\
=&E_{\mu'}(f)+\sum_{\gamma}\mu'(e,\gamma)E_{\mu}(f)\\
=&E_{\mu'}(f)+E_{\mu}(f).
\end{aligned}$$

On the other hand, we see

$$\begin{aligned}
&\text{(the 3rd terms in the last expression in (3.2))}\\
=&-\sum_{\gamma}\mu'(e,\gamma)\langle f(e)-f(\gamma),-\Delta_{\mu}f(\gamma)\rangle\\
=&-\sum_{\gamma}\mu'(\gamma^{-1},e)\langle\rho_0(\gamma)(f(\gamma^{-1})-f(e)),\rho_0(\gamma)(-\Delta_{\mu}f(e))\rangle\\
=&-\sum_{\gamma}\mu'(e,\gamma^{-1})\langle f(\gamma^{-1})-f(e),-\Delta_{\mu}f(e)\rangle\\
=&-\langle-\Delta_{\mu'}f(e),-\Delta_{\mu}f(e)\rangle.
\end{aligned}\tag{3.3}$$

This completes the proof. Q.E.D.

Corollary 3.3. *For any Γ-invariant, irreducible, symmetric random walk μ with finite-support property, ρ-equivariant map f, and an integer $n\geq 2$,*

$$E_{\mu^n}(f)=nE_{\mu}(f)-\sum_{i=1}^{n-1}\langle-\Delta_i f(e),-\Delta f(e)\rangle$$

holds, where $-\Delta_i f(e)$ denotes the average of $\{f(\gamma)-f(e)\mid\gamma\in\Gamma\}$ with respect to μ^i.

Proof. When $n = 2$, this follows from the proposition above. Suppose this is true for $n-1$. Then

$$\begin{aligned}
&E_{\mu^n}(f)\\
=&E_{\mu^{n-1}}(f) + E_\mu(f) - \langle -\Delta_{n-1} f(e), -\Delta f(e)\rangle\\
=&(n-1)E_\mu(f) - \sum_{i=1}^{n-2}\langle -\Delta_i f(e), -\Delta f(e)\rangle\\
&+ E_\mu(f) - \langle -\Delta_{n-1} f(e), -\Delta f(e)\rangle\\
=&nE_\mu(f) - \sum_{i=1}^{n-1}\langle -\Delta_i f(e), -\Delta f(e)\rangle,
\end{aligned}$$

where the first equality follows from the proposition above, and the second follows from the assumption of the induction. Q.E.D.

Let $\mathcal{M}_{\rho_0}$ denote the set of ρ_0-equivariant maps from Γ to $\mathcal{H}$. As in the case of ρ, the correspondence $\mathcal{M}_{\rho_0} \ni \varphi \mapsto \varphi(e) \in \mathcal{H}$ allows us to identify $\mathcal{M}_{\rho_0}$ with $\mathcal{H}$. Then an inner product on $\mathcal{M}_{\rho_0}$ can be defined in a natural way:

$$\langle \varphi, \psi\rangle_{\mathcal{M}_{\rho_0}} := \langle \varphi(e), \psi(e)\rangle = \langle \rho_0(\gamma)\varphi(e), \rho_0(\gamma)\psi(e)\rangle = \langle \varphi(\gamma), \psi(\gamma)\rangle.$$

We define an averaging operator M by

$$M\varphi(\gamma) = \sum_{\gamma'\in\Gamma} \mu(\gamma, \gamma')\varphi(\gamma'), \quad \varphi \in \mathcal{M}_{\rho_0}.$$

Since $\rho_0(\gamma)$ is linear, we see that $M\varphi \in \mathcal{M}_{\rho_0}$:

$$\begin{aligned}
M\varphi(\gamma''\gamma) &= \sum_{\gamma'\in\Gamma} \mu(\gamma''\gamma, \gamma')\varphi(\gamma''\gamma) = \sum_{\gamma'\in\Gamma} \mu(\gamma, \gamma''^{-1}\gamma')\rho_0(\gamma'')\varphi(\gamma)\\
&= \rho_0(\gamma'')M\varphi(\gamma).
\end{aligned}$$

Again by the linearity of $\rho_0(\gamma)$, M becomes linear, and hence M is a linear operator acting on $\mathcal{M}_{\rho_0} \cong \mathcal{H}$. Since μ is symmetric and Γ-invariant, M is selfadjoint:

$$\begin{aligned}
\langle M\varphi, \psi\rangle_{\mathcal{M}_{\rho_0}} &= \langle M\varphi(e), \psi(e)\rangle = \sum_\gamma \mu(e, \gamma)\langle \rho_0(\gamma)\varphi(e), \psi(e)\rangle\\
&= \sum_\gamma \mu(\gamma^{-1}, e)\langle \varphi(e), \rho_0(\gamma^{-1})\psi(e)\rangle\\
&= \sum_\gamma \mu(e, \gamma^{-1})\langle \varphi(e), \rho_0(\gamma^{-1})\psi(e)\rangle\\
&= \langle \varphi, M\psi\rangle_{\mathcal{M}_{\rho_0}}.
\end{aligned}$$

Also note that, for any $\varphi \in \mathcal{M}_{\rho_0}$,

$$
\begin{aligned}
|\langle M\varphi(e), \varphi(e)\rangle| &= \left|\sum_{\gamma} \mu(e,\gamma)\langle \rho_0(\gamma)\varphi(e), \varphi(e)\rangle\right| \\
&\leq \sum_{\gamma} \mu(e,\gamma)|\rho_0(\gamma)\varphi(e)||\varphi(e)| \\
&= |\varphi(e)|^2
\end{aligned}
\tag{3.4}
$$

holds, since $\rho_0(\gamma) \in \mathcal{U}(\mathcal{H})$. Using this operator M, we can rewrite $-\Delta_n f$ as follows.

$$
\begin{aligned}
-\Delta_n f(\gamma) &= \sum_{\gamma' \in \Gamma} \mu^n(\gamma,\gamma')(f(\gamma') - f(\gamma)) \\
&= \sum_{\gamma_1,\gamma' \in \Gamma} \mu(\gamma,\gamma_1)\mu^{n-1}(\gamma_1,\gamma')(f(\gamma') - f(\gamma_1) + f(\gamma_1) - f(\gamma)) \\
&= \sum_{\gamma_1} \mu(\gamma,\gamma_1)\left(-\Delta_{n-1} f(\gamma_1) + f(\gamma_1) - f(\gamma)\right) \\
&= M(-\Delta_{n-1} f)(\gamma) + (-\Delta f)(\gamma).
\end{aligned}
$$

Since $\gamma \in \Gamma$ is arbitrary, we obtain

$$-\Delta_n f = M(-\Delta_{n-1} f) + (-\Delta f).$$

Proceeding inductively, we see that

$$
\begin{aligned}
-\Delta_n f &= M(-\Delta_{n-1} f) + (-\Delta f) \\
&= M\left(M\left(-\Delta_{n-2} f\right) + (-\Delta f)\right) + (-\Delta f) \\
&= M^2\left(-\Delta_{n-2} f\right) + M\left(-\Delta f\right) + (-\Delta f) \\
&= M^2\left(M\left(-\Delta_{n-3} f\right) + (-\Delta f)\right) + M\left(-\Delta f\right) + (-\Delta f) \\
&\;\;\vdots \\
&= (M^{n-1} + M^{n-2} + \cdots + M + I)(-\Delta f).
\end{aligned}
\tag{3.5}
$$

In particular, we see that if f is μ-harmonic, then f must be μ^n-harmonic.

Corollary 3.4. *For any Γ-invariant, irreducible, symmetric random walk μ with finite-support property, ρ-equivariant map f, and an integer n,*

$$E_{\mu^n}(f) \leq nE_\mu(f)$$

holds. The equality holds if and only if f is harmonic.

Proof. By Corollary 3.3, it suffices to show $\langle -\Delta_i f(e), -\Delta f(e)\rangle \geq 0$ for each i in order to prove the inequality. When $i=1$, this is obvious, since $-\Delta_1 f = -\Delta f$. Suppose $i = 2n+2$, $n \geq 0$. By the expression of $-\Delta_i f(\gamma)$ above, we get

$$
\begin{aligned}
&\langle -\Delta_i f(e), -\Delta f(e)\rangle \\
&= \sum_{k=0}^{n} \langle (M^{2k} + M^{2k+1})(-\Delta f)(e), -\Delta f(e)\rangle \\
&= \sum_{k=0}^{n} \langle (M^{k} + M^{k+1})(-\Delta f)(e), M^k(-\Delta f)(e)\rangle \\
&= \sum_{k=0}^{n} \langle (I + M)M^k(-\Delta f)(e), M^k(-\Delta f)(e)\rangle,
\end{aligned} \tag{3.6}
$$

since M is selfadjoint. Recalling (3.4), we see that the operator $I+M$ is nonnegative:

$$\langle (I+M)\varphi, \varphi\rangle_{\mathcal{M}_{\rho_0}} = \sum_{\gamma\in\Gamma} \mu(e,\gamma)\langle \varphi(e) + \rho_0(\gamma)\varphi(e), \varphi(e)\rangle \geq 0.$$

Applying this to (3.6), we obtain $\langle -\Delta_i f(e), -\Delta f(e)\rangle \geq 0$.

Suppose $i = 2n+3$, $n \geq 0$. Then, by the computation above,

$$
\begin{aligned}
&\langle -\Delta_i f(e), -\Delta f(e)\rangle \\
&= \sum_{k=0}^{n} \langle (I + M)M^k(-\Delta f(e)), M^k(-\Delta f(e))\rangle \\
&\qquad + \langle M^{2n+2}(-\Delta f(e)), -\Delta f(e)\rangle \\
&= \sum_{k=0}^{n} \langle (I + M)M^k(-\Delta f(e)), M^k(-\Delta f(e))\rangle \\
&\qquad + \langle M^{n+1}(-\Delta f(e)), M^{n+1}(-\Delta f(e))\rangle \\
&\geq 0.
\end{aligned}
$$

Now suppose $E_{\mu^n}(f) = nE_\mu(f)$. Then $\langle -\Delta f(e), -\Delta f(e)\rangle = 0$, and hence f is harmonic. The converse is obvious. Q.E.D.

Corollary 3.5. *For any Γ-invariant, irreducible, symmetric random walk μ with finite-support property, ρ-equivariant map f, and an integer n,*

$$E_{\mu^n}(f) \geq nE_\mu(f) - \frac{n(n-1)}{2}|\Delta f(e)|^2$$

holds.

Proof. By Corollary 3.3, (3.5), and (3.4), we obtain

$$\begin{aligned} E_{\mu^n}(f) &= nE_\mu(f) - \sum_{i=1}^{n-1} \langle -\Delta_i f(e), -\Delta f(e)\rangle \\ &\geq nE_\mu(f) - \sum_{i=1}^{n-1} i|-\Delta f(e)|^2 \\ &= nE_\mu(f) - \frac{n(n-1)}{2}|-\Delta f(e)|^2. \end{aligned}$$

Q.E.D.

This implies Theorem 3.1 for a Hilbert space as follows.

Proof of Theorem 3.1 for a Hilbert space. Suppose there is an integer n and a positive real number ε such that, for any ρ-equivariant map f,

$$E_{\mu^n}(f) \leq (n-\varepsilon)E_\mu(f)$$

holds. Then, by Corollary 3.5, we obtain

$$(n-\varepsilon)E_\mu(f) \geq nE_\mu(f) - \frac{n(n-1)}{2}|-\Delta f(e)|^2,$$

which implies

$$\frac{n(n-1)}{2}|-\Delta f(e)|^2 \geq \varepsilon E_\mu(f).$$

Recalling Remark 1 and applying Proposition 2.1, we get the desired result. Q.E.D.

3.2. General case.

Though the discussion in the preceding subsection cannot be applied directly to a general CAT(0) space Y, it is still possible to show the following inequality.

Proposition 3.6 ([8]). *Let Γ be a finitely generated group with a Γ-invariant, irreducible, symmetric random walk μ with finite-support property, and Y a* CAT(0) *space. Suppose that $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$ is a homomorphism, and that f is a ρ-equivariant map. Then*

$$E_{\mu^n}(f) \geq nE_\mu(f) - \frac{n(n-1)}{2}\sqrt{2E_\mu(f)}\,|-\Delta_\mu(f)| \tag{3.7}$$

holds.

Inequality (3.7) is different from the one in Corollary 3.5, which we have used to prove Theorem 3.1 for a Hilbert space. However, this is sufficient for proving Theorem 3.1.

Proof of Theorem 3.1. Suppose there is an integer n and a positive real number ε such that, for any ρ-equivariant map f

$$E_{\mu^n}(f) \leq (n-\varepsilon)E_\mu(f)$$

holds. We may assume $E_\mu(f) \neq 0$. By Proposition 3.6, we see

$$nE_\mu(f) - \frac{n(n-1)}{2}\sqrt{2E_\mu(f)}|-\Delta_\mu f(e)| \leq (n-\varepsilon)E_\mu(f),$$

and hence

$$\frac{\sqrt{2}\varepsilon}{n(n-1)}\sqrt{E_\mu(f)} \leq |-\Delta_\mu f(e)|$$

holds. Therefore, for any ρ-equivariant map f, we get

$$|-\Delta_\mu f(e)|^2 \geq \frac{2\varepsilon^2}{n^2(n-1)^2}E_\mu(f).$$

Theorem 3.1 follows from this inequality and Proposition 2.1. Q.E.D.

Remark 2. As we have seen in Corollary 3.4, for a Hilbert space, the ratio $E_{\mu^n}(f)/E_\mu(f)$ is at most n, and equal to n if and only if f is harmonic. In other words, an equivariant map is harmonic if and only if its image spreads in the most efficient way. Expressing in this way, one might expect the same is true also for a general CAT(0) space, although it is not clear from inequality (3.7).

Remark 3. When Y is not so singular, then, for a given homomorphism $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$, the condition in Theorem 3.1 becomes a necessary condition for ρ to have a fixed point as Proposition 3.7 below claims. This suggests that the condition should be useful in detecting finitely generated groups with fixed-point property.

Proposition 3.7 ([8]). *Let Y be either a* CAT(0) *Riemannian manifold or an $\mathbb{R}$-tree, and $\rho : \Gamma \longrightarrow \mathrm{Isom}(Y)$ a homomorphism. Suppose $\rho(\Gamma)$ admits a fixed point. Then there exists a positive constant C_ρ such that $E_{\mu^n}(f) \leq C_\rho E_\mu(f)$ for any $n \in \mathbb{N}$ and ρ-equivariant map f. In particular, taking $n > C_\rho$, we obtain n, ε in Theorem 3.1.*

Remark 4. Let G be a finite graph. We denote by $G(0)$ the set of vertices of G and $G(1)$ the set of (unoriented) edges of G. The degree of $x \in G(0)$, denoted by $\deg(x)$, is the number of edges one of whose vertices is x. Consider a random walk μ defined as follows:

$$\mu(x,y) = \begin{cases} 0 & \text{if } \{x,y\} \notin G(1), \\ \frac{1}{\deg(x)} & \text{if } \{x,y\} \in G(1). \end{cases}$$

The random walk μ defined in this way is called the *standard random walk* of G. Although μ is not symmetric in general, it is symmetric with respect to a measure ν on $G(0)$ defined by $\nu(x) = \deg(x)/(2\#G(1))$, which means that

$$\nu(x)\mu(x,y) = \nu(y)\mu(y,x).$$

Using the standard random walk μ, we can define the energy $E_\mu(f)$ of a map $f : G(0) \longrightarrow Y$, where Y is a CAT(0) space:

$$E_\mu(f) = \frac{1}{2} \sum_{x \in G(0)} \nu(x) \sum_{y \in G(0)} \mu(x,y) d_Y(f(x), f(y))^2,$$

where $d_Y(\cdot,\cdot)$ denotes the metric of Y. In the same way, by using nth convolution μ^n of μ, we can define the n-step energy $E_{\mu^n}(f)$. Let $\lambda_1(G,Y)$ be the Wang's invariant of G with respect to Y, namely,

$$\lambda_1(G,Y) = \inf \frac{E_\mu(f)}{\sum_{x \in G(0)} d_Y(f(u), \overline{f})^2},$$

where $\overline{f}$ denotes the barycenter of $f(G(0)) \subset Y$ with respect to a measure $f_*\nu$ on $f(G(0))$ and the infimum is taken over all nonconstant map $f : G(0) \longrightarrow Y$. Then one can show that, for any map $f : G(0) \longrightarrow Y$,

$$E_{\mu^n}(f) \leq \frac{2}{\lambda_1(G,Y)} E_\mu(f)$$

holds. Therefore, one could expect that if a finite graph G with large $\lambda_1(G,Y)$ is embedded nicely in the Cayley graph of a finitely generated group Γ, then it implies the inequality in Theorem 3.1. In fact, this observation leads us to a fixed-point theorem for random groups given by the graph model ([8]), which is a generalization of a result proved in [19]. See also [13].

§4. Constant C as a Kazhdan's constant

If we consider a family of CAT(0) spaces satisfying a certain assumption, then the existence of a positive constant C as in Proposition 2.1 turns out to be equivalent to fixed-point property for such a family.

In this section, we denote the energy of ρ-equivariant map $f : \Gamma \longrightarrow Y$ by $E_{Y,\rho}(f)$.

Theorem 4.1 (I–Kondo–Nayatani [7]). *Let Γ be a finitely generated group equipped with a Γ-invariant, irreducible, symmetric random walk with finite-support property. Let $\mathcal{Y}$ be a family of* CAT(0) *spaces such that, for any $\{(Y_n, d_n) \mid n \in \mathbb{N}\} \subset \mathcal{Y}$, $\{o_n \in Y_n \mid n \in \mathbb{N}\}$, and $\{r_n > 0 \mid n \in \mathbb{N}\}$, there exists a non-principal ultrafilter ω on $\mathbb{N}$ such that $\omega\text{-}\lim_n (Y_n, r_n d_n, o_n)$ belongs to $\mathcal{Y}$. Then Γ has the fixed-point property for all $Y \in \mathcal{Y}$ if and only if there exists a positive constant C such that*

$$|-\mathrm{grad}E_{Y,\rho}(f)|^2 \geq C E_{Y,\rho}(f) \tag{4.1}$$

holds for any $Y \in \mathcal{Y}$, $\rho\colon \Gamma \longrightarrow \mathrm{Isom}(Y)$, and ρ-equivariant map f. The constant C should be independent of Y, ρ and f.

If we take $\mathcal{Y}$ to be a family consisting of the Hilbert spaces, then the constant C in (4.1), which depends only on the presentation P of $\Gamma = \Gamma(P)$, is closely related to the so-called Kazhdan constant of a finitely generated group Γ. The Kazhdan constant of a finitely generated group Γ is defined in terms of unitary representations of Γ and depends on the choice of a generator set of Γ. It becomes positive if and only if Γ has property (T). In [23], Żuk defined a geometric constant that bounds our C from below, and obtained Kazhdan constants for certain groups. These facts suggest that we may regard the constant C in (4.1) as a Kazhdan constant of Γ for the family $\mathcal{Y}$ of CAT(0) spaces. The existence of a finitely generated group having the fixed-point property for a family $\mathcal{Y}$ of certain CAT(0) spaces has been shown in [9] and [7]; while an example of such a family $\mathcal{Y}$ can be given by bounding an invariant $\delta(Y)$ of a CAT(0) space Y. For the definition of $\delta(Y)$, see [9].

Acknowledgments. The author would like to thank the referee for a careful reading of the manuscript.

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