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Rough path conditions for smooth paths

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Abstract.

We show that there exist non-trivial smooth paths that have rough path property. The applications and the related open problems are also discussed.

§1. Introduction

This article is based on a joint work with T. J. Lyons (Oxford University), especially on Hara–Lyons [2] (2007), but includes new remarks and some open problems with the partial answers.

Rough path theory has been applied to many fields since T. J. Lyons established the foundation in [5] (1998). It is worthy to note that rough path theory had already the general and abstract framework at the very early stage. However, the targets of rough path theory have been mainly limited to "rough" objects that do not have regularity, like Brownian paths, solutions of stochastic differential equations, or more irregular paths, etc. In this article, we like to emphasize the power of rough path theory to study smooth objects. More precisely, we will show that there exist non-trivial smooth "rough paths".

Let us see the essence of our idea here. The best way to do that is to see the first level of rough path theory, that is, analysis with p-variation. First let us recall the definition of p-variation.

Let $F : I \to \mathbb{R}^d$ be a continuous path in d-dimensional Euclidean space defined on an interval I. For a fixed real number $p \geq 1$, we define

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the p-variation $\|F\|_{p\text{-var}}$ of F as follows.

$$\|F\|_{p\text{-var}} = \left\{ \sup_{\mathcal{D}\subset I} \sum_j |F(t_{j+1}) - F(t_j)|^p \right\}^{\frac{1}{p}},$$

where sup takes over all the possible finite partitions $\mathcal{D} = \{t_0 < t_1 < \cdots < t_n : t_j \in I\}$. If $p = 1$, this is nothing but the total variation of F. It is easy to see that it measures how much the path F oscillates on the interval I. We especially concern finiteness of p-variations. Considering the meaning of pth power of the difference, we see that the path F oscillates more heavily on I if we need a bigger p to get the finite p-variation of F.

Usually we consider p-variations on a compact interval. It is because we are interested in the local oscillation of an irregular path. On the other hand, a smooth path has a good estimate for the differences like

$$\|F(t_{j+1}) - F(t_j)\| \leq \max_{t_{j+1}\leq u \leq t_j} \|F'(u)\| |t_{j+1} - t_j|.$$

Therefore, it automatically has the finite total variation, which is estimated by $\max_{u\in I} \|F'(u)\|$ times the length of I. Hence it is almost nonsense to study the finiteness of the p-variation.

However, once we consider a non-compact interval like the whole real line $\mathbb{R}$, the situation changes. It no longer has such an estimate with the length of the interval I. The sum of the differences $\|F(t_{j+1}) - F(t_j)\|^p$ for any p can be infinite even if the path F is infinitely smooth. Now the finiteness of p-variation measures how much the smooth path oscillates on the real line just like the situation of irregular paths defined on compact intervals. Therefore, we can use the concept of p-variation to study the behaviour of smooth paths near the infinity.

Since rough path theory is a nonlinear generalization of analysis with p-variation, we expect that we can apply rough path theory to study the asymptotic behaviour of smooth paths in the same sense above. More precisely, we will be interested in not only the difference of the path itself, i.e.,

$$F(t) - F(s) = \int_{s<u<t} dF_u,$$

but also the iterated integrals

$$\int \cdots \int_{s<u_1<\cdots<u_n<t} dF_{u_1} \otimes \cdots \otimes dF_{u_n}, \quad (n = 2, 3, 4, \dots)$$

where $\otimes$ is the tensor product. It is natural to expect that analysis with p-variation of these iterated integrals will give us much more information to study how smooth paths oscillate and tangle near the infinity.

In short, rough path theory, which is a generalization of p-variation analysis, gives us information how smooth paths oscillate globally on the real line in the same way as irregular paths oscillate locally on compact intervals.

In the next section, we will prepare the basics of rough path theory. It is much easier than usual because we limit ourselves to working with smooth paths. In Section 3, we will show the fundamental theorems for smooth rough paths with the sketch of the proofs. In the last section, we will see some open problems. Some of them are related to our smooth rough paths, but there will be more general problems. The author would like to emphasize here that they include an interesting problem related to Fourier analysis.

§2. Smooth rough paths

In this section, we prepare the concept of smooth rough paths. This is much easier than usual rough path theory, because we work with differentiable paths. The trickiest part of rough path theory is how to define "iterated integrals". The rough path X is defined through the concept of "signature", that is, the ensemble of the iterated integrals

$$\mathbb{X}_{s,t} = (1, X^1_{s,t}, X^2_{s,t}, ...) \in \bigoplus_{n \geq 0} \mathbb{R}^{\otimes n},$$

where

$$X^n_{s,t} = \int \cdots \int_{s<u_1<\cdots<u_n<t} dX_{u_1} \otimes \cdots \otimes dX_{u_n} \quad (n = 1, 2, 3, \ldots).$$

However, we cannot define the iterated integrals for irregular paths without the concept of rough paths. The truth is that the signature is not an ensemble of iterated integrals but an object in $\bigoplus_{n\geq 0} \mathbb{R}^{\otimes n}$ whose methods ensure the same operation as the iterated integrals in the both of algebraic and analytic sense. But we do not need to care about such subtle points because we can define the iterated integrals as usual Riemann integrals for smooth paths.

First we define a control function, which is a very useful tool to study p-variation. Please note that we add a new "global control" to the usual definition for the case of infinite intervals.

Definition 1 (a control function). *A control function (or a control) on an interval I is a continuous function ω defined on $\Delta_I = \{(s,t) \in I \times I; s \le t\}$ satisfying the following two conditions:*

(i) *For any $s \le u \le t \in I$,*

$$\omega(s,u) + \omega(u,t) \le \omega(s,t).$$

(ii) *ω is non-negative and $\omega(t,t) = 0$ for any $t \in I$.*

When I is an infinite interval, we call a control function ω a global control if ω is bounded on I.

It is easy to see that a continuous function F defined on I has finite p-variation if and only if there exists a control such that

$$\|F(t) - F(s)\|^p \le \omega(s,t)$$

for any $(s,t) \in \Delta_I$.

Now we can define a smooth rough path.

Definition 2 (a smooth rough path). *Let $p \ge 1$ be a real number. Let $F : \mathbb{R} \to \mathbb{R}^d$ be an infinitely differentiable function and $F^i_{s,t}$ be the ith iterated integral:*

$$F^i_{s,t} = \int \cdots \int_{s<u_1<\cdots<u_i<t} dF_{u_1} \otimes \cdots \otimes dF_{u_i} \quad (i = 1,2,3,\dots),$$

and let $\|\cdot\| = \|\cdot\|_i$ be the norm in the ith tensor product space of $\mathbb{R}^d$. If there exists a global control ω such that

$$\|F^i_{s,t}\|^{p/i} \le \omega(s,t) \quad (i = 1,2,\dots,\lfloor p \rfloor)$$

for any $(s,t) \in \Delta_{\mathbb{R}}$, we call F a (p-)smooth rough path with a global control ω.

Remark 3. The definition of smooth rough path needs the p-variation estimates only for $1 \le i \le \lfloor p \rfloor$. But the same estimates hold for any $i \ge 1$ by the Extension Theorem of rough paths, which is the cornerstone of rough path theory. (See the extension theorem in Lyons [5] (1998), Lyons and Z. Qian [8] (2002), Lyons, M. J. Caruana, and T. Lévy [6] (2007).)

Remark 4. The definition above states only the analytic condition. A very important property of rough paths is Chen's identity, which says that the signature must satisfy the following algebraic identity:

$$\mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t} = \mathbb{X}_{s,t}$$

for any $s \leq u \leq t$. However, in our smooth setting, this condition is automatically satisfied by $\mathbb{X}_{s,t} = (1, F^1_{s,t}, F^2_{s,t}, \dots)$. (Actually this statement is nothing but Chen's theorem.) Therefore, we did not include this identity as a condition in the definition above.

§3. Basic theorems on smooth rough paths

In this section, we look for conditions such that a smooth path F becomes a smooth rough path. First we must suppose some reasonable assumptions for F.

Let F be an infinitely differentiable path in $\mathbb{R}^d$ defined on $\mathbb{R}$. Though we can easily extend our result to paths in a Banach space, we choose this setting for simplicity. It seems natural to suppose the integrability for both of F itself and the derivative F' because we want to estimate the global oscillation. Therefore, we suppose that

$$\|F\|_p = \left(\int_{\mathbb{R}} \|F(t)\|^p dt\right)^{\frac{1}{p}} < \infty$$

and

$$\|F'\|_q = \left\|\frac{d}{dt}F(t)\right\|_q = \left(\int_{\mathbb{R}} \left\|\frac{d}{dt}F(t)\right\|^q dt\right)^{\frac{1}{q}} < \infty$$

for some $p, q \geq 1$. We want to get the condition of p and q such that F is a smooth rough path.

As the first level estimate, we have the following theorem.

Theorem 5 (K. H. and T. J. Lyons (2007) [2]). *Let $1 < p, q < \infty$. If $F \in L^p(\mathbb{R}^n)$ and $F' \in L^q(\mathbb{R}^n)$, then there exists a control function ω on $\Delta_{\mathbb{R}}$ such that*

$$\left|\int_{s<u<t} dF(u)\right|^r \leq \omega(s,t), \quad \textit{for} \quad r = p\left(1 - \frac{1}{q}\right) + 1.$$

Therefore specially, $\|F\|_{r\text{-var}} < \infty$. It holds also for $r = 2$ if $p = \infty$ and $q = 1$, and for $r = p + 1$ if $q = \infty$.

Moreover, if $r < 2$, i.e., if $\frac{1}{p} + \frac{1}{q} > 1$, F is a $\lfloor r \rfloor$-rough path, i.e., a 1-rough path.

Since the proof is standard, we only show the essential estimate and skip the detail.

If G is a smooth positive functions, we have

$$\begin{aligned}|G(t)^r - G(s)^r| &= \left|\int_{s<u<t} d\{G(u)^r\}\right| \\ &= \left|\int_{s<u<t} r\,G(u)^{r-1}\,G'(u)\,du\right| \\ &\leq r\int_{s<u<t} G(u)^{r-1}\,|G'|(u)\,du.\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}&|G(t)^r - G(s)^r| \\ &\leq r\left(\int_{s<u<t} G(u)^{(r-1)\alpha}\,du\right)^{1/\alpha}\left(\int_{s<u<t} |G'(u)|^{\beta}\,du\right)^{1/\beta}\end{aligned}$$

for $1/\alpha + 1/\beta = 1$. Now choose $p = (r-1)\alpha$ and $q = \beta$ to get the estimate in the theorem above with the L^p norm of G and the L^q norm of G'.

In our case that F is not necessarily positive, so we need to divide it into the positive part F^+ and the negative one F^-, and use Jensen's inequality:

$$\begin{aligned}|F(t) - F(s)|^r &\leq (|F^+(t) - F^+(s)| + |F^-(t) - F^-(s)|)^r \\ &\leq 2^{r-1}\left(|F^+(t) - F^+(s)|^r + |F^-(t) - F^-(s)|^r\right).\end{aligned}$$

The rest part of the proof is easy.

Next we want to show the higher level estimates. However, we do not have such an estimate in general if $1/p + 1/q < 1$. The following simple example clearly shows the reason.

Consider the function $H : \mathbb{R} \to \mathbb{R}^2$:

$$H(t) = (H_1(t),\, H_2(t)) = (R(t)\cos t,\, R(t)\sin t),$$

where the both of the radius $R(t) : \mathbb{R} \to \mathbb{R}$ and the derivative $R'(t)$ are in L^p for some $p > 2$, but $R(t)$ is not in L^2. Then, we can easily see that H satisfies our condition, i.e., $\|H\|_p + \|H'\|_p < \infty$. But the area explodes to infinity:

$$\left|\int_{-\infty}^{\infty}(H_2 dH_1 - H_1 dH_2)\right| = \int_{-\infty}^{\infty} R(t)^2 dt = \infty.$$

So we can not have finite r-variation for any r.

Therefore, considering this essential example and the theorem above (i.e., the first level estimate), our only chance should be to estimate the area just when $1/p+1/q=1$. Fortunately we have the following theorem.

Theorem 6 (K. H. and T. J. Lyons (2007) [2]). *Let $1/p+1/q=1$, $p>1$. If $F \in L^p(\mathbb{R}^n)$ and $F' \in L^q(\mathbb{R}^n)$, then the estimate for the area*

$$\left| \int\int_{s<u<v<t} dF(u) \otimes dF(v) \right| \leq C\omega(s,t), \quad (-\infty \leq s < t \leq \infty)$$

holds for a constant C and the same control function ω as the Theorem 5. Therefore, the function F is a 2-rough path.

The proof is not as easy as in the first theorem. We need to estimate carefully the area enclosed by the path $\{F(u)\}_{s\leq u\leq t}$. The main idea is to write this area as the difference of the two areas, one of which is $\int_{s<u<t} F(u) \times dF(u)$ and the other one is the triangle $F(s) \times (F(t)-F(s))$. The former area is easily estimated by Hölder's inequality. The difficulty lies in the area of the triangle. Actually we need a delicate study of the trigonometric geometry, which is elementary, but tricky. The detail is shown in [2] (2007).

In this section, we showed that there exist non-trivial smooth rough paths, though they are only 1-rough paths or 2-rough paths. This is enough to apply our theory to smooth objects. But it might be an interesting problem to seek other settings that we can get the essentially higher rough paths, 3-rough paths, 4 rough paths, etc. (see Section 4.1).

We also like to remark a good property of our smooth rough paths, which was pointed out by T. J. Lyons during the session of MSJ-SI.

Remark 7. Our class of smooth rough paths is an algebra. More precisely, if F_1 and F_2 satisfy the condition that $\|F_i\|_p + \|F_i'\|_q < \infty$ $(i=1,2)$, then the product F_1F_2 also satisfies the same condition. The proof is easy but we need the fact that F_1 and F_2 are bounded. Since this uniform bound is a consequence of the rough path property, this remark is not trivial.

§4. Related problems and others

In this section, we propose the related problems and others that are not necessarily related to smooth rough paths. Smooth rough paths are really rough paths and they are non-trivial. But it is much easier to study because we can use the usual calculus. Therefore smooth rough paths are not only interesting and useful tools for global analysis, but also give good examples to study rough path theory itself.

4.1. Yet another class of smooth rough paths

In our setting to seek smooth rough paths, we supposed that a path itself and the derivative have the each integrability. This framework seems natural. But unfortunately, we have only 1-rough paths and 2-rough paths. Of course there exist higher smooth rough paths, which satisfy

$$\|X^i_{s,t}\|^{p/i} \leq \omega(s,t) \quad (i = 1, 2, \ldots, \lfloor p \rfloor)$$

for some $p \geq 3$ and a global control ω.

The problem is to look for a natural framework, or a set of the conditions, to study such higher rough paths. It may possibly need the integrability of the higher derivatives of F.

4.2. Preserving of rough path property

In Hara–Lyons [2] (2007), a motivation to establish the concept of smooth rough paths is to study the nonlinear Fourier analysis (Tao–Thiele [9]) in the framework of rough path theory. In that paper we saw that we need a precise estimate of the classical Fourier transform as a rough path to study the nonlinear Fourier transform. For example, we want to show something like a statement that the indefinite integral

$$\tilde{F}_t = \int_{\cdot}^{t} e^{i\theta u} dF_u$$

is also a rough path if F is a rough path. It means that we need to estimate the iterated integrals $\tilde{F}^1_{s,t}, \tilde{F}^2_{s,t}, \ldots$ in the sense of p-variation. We have a partial answer for a 1-rough path F (a private communication with T. J. Lyons), but we need to study much more. The proof for 1-rough paths already need a careful application of a refinement of Young's theory (Lyons [4] (1994)) and Hambly–Lyons's dyadic argument ([1] (1998)).

A trivial generalization of this problem is the following question. Let T be a transform of a path F. Then, does the transform T preserve the rough path property of F? Though this question seems fundamental, there is almost no study at the present.

4.3. Trivialization of rough path theory

The essence of rough path theory is to require a path to satisfy the p-variation type estimate not only for the path itself but also the iterated integrals. Of course we generally need all the conditions. But it might be interesting to consider how we really need such higher estimates.

For example, let us think about a 2-rough path $X(t)$, which is required to satisfy the following two conditions:

$$\|X(t) - X(s)\| \le \omega(s,t)^{1/2},$$

$$\left\| \int_{s<u<v<t} dX(u) \otimes dX(v) \right\| \le \omega(s,t)$$

for a control ω. The second estimate means essentially that the area enclosed by the path satisfies the variational estimate. Since the ratio of $\omega(s,t)^{1/2}$ to $\omega(s,t)$ is natural for the length to the area, it is not so strange that we can deduce the second estimate from only the first estimate in special cases. Actually it is a subtle problem to construct such a path that satisfies the first condition but does not satisfy the second one.

Though such an example in an infinite dimensional space is shown in N. Victoir [10] (2004), this question is not so easy even in the Euclidean space. Therefore it might be interesting to ask when we can deduce the higher estimates from the first one, in other saying, when rough path theory is trivialized. This study should deepen our understanding of the rough path property.

4.4. Probabilistic rough path theory

Rough path theory is basically a deterministic theory without any measure on the path space. But sometimes we like to give an almost sure statement for a path although we do not consider the probabilistic situation. For example, if we cannot show the usual rough path condition, we might have to satisfy with a weaker one in the probabilistic sense. Therefore it might be useful if we can establish a weaker theory than usual rough path theory. More precisely, if we have only the following weak type condition:

$$E^{\theta}\left[\|X^i_{s,t}(\theta)\|^{p/i}\right] \le \omega(s,t) \quad (i = 1, 2, \dots, \lfloor p \rfloor)$$

for a path $X(t;\theta)$ that has another parameter θ, is it possible to say something useful?

Of course, if we want to answer this question, we need to modify the Extension Theorem of rough paths, or the so-called "First Theorem" of rough path theory in the probabilistic sense. At the present, we do not have such probabilistic theory that can be used for solving any interesting problems.

4.5. Discrete rough path theory

Just in the same meaning of smooth rough paths, we can consider a "discrete rough path" $X(n)$ on the discrete time $\mathbb{N} = 1, 2, 3, \ldots$. This might have another importance because it should be related to numerical analysis of rough path theory. Of course we can study only discretized data in the computer and if we consider a global analysis of time series, we should naturally study discrete rough paths. The author likes to refer the paper on a data compression scheme with rough path theory by T. J. Lyons and N. Sidorova [7] (2005) although the relation is not clear.

4.6. Neoclassical inequality

The cornerstone of rough path theory is the extension theorem of rough paths. And the theorem relies upon the following lemma, which is called the neoclassical inequality by T. J. Lyons [5] (1998).

Lemma 8 (T. J. Lyons). *Let n be any natural number, x and y any positive real numbers. Then the following inequality holds for $0 \le \alpha \le 1$:*

$$\alpha^2 \sum_{j=0}^{n} \frac{(\alpha n)!}{(\alpha j)!(\alpha(n-j))!} x^{\alpha j} y^{\alpha(n-j)} \le (x+y)^{\alpha n},$$

where $z!$ means the Gamma function $\Gamma(z+1)$.

If $\alpha = 1$, the equality holds because this is nothing but the classical binomial theorem. In that sense, this is a generalization of the binomial theorem. Though this inequality seems easy, only proof that we know is tricky. Moreover the proof is not elementary because it essentially uses the maximum principle of partial differential equations.

After his proof in [5], Lyons conjectured that the parameter α^2 of the left hand side of the inequality should be replaced by α. Though many numerical experiments support this conjecture, this is still an open problem. As the only progress, E. R. Love [3] (1998) proved that the conjecture is true if α is $1/2^n$ $(n = 1, 2, 3, \ldots)$. His proof is surprisingly easy and elementary. However, it does not seem that we can extend his idea to solve the full conjecture in a simple way.

Personally the author thinks that this problem is quite hard though it seems innocent.

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