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Estimates on the number of the omitted values by meromorphic functions

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Abstract.

We give some Nevanlinna's theorems for value distribution of meromorphic functions on general Kähler manifolds using stochastic calculus. We also give some examples where we can give explicit estimates on the number of the omitted values.

§1. Introduction

The classical little Picard's theorem says that the number of the omitted values of nonconstant meromorphic functions can be at most two. It is the origin of the theory of the value distribution theory of meromorphic maps. Namely to discuss the number of the omitted values of nonconstant meromorphic functions is a basic problem on this subject. After Picard's theorem Nevanlinna theory appeared. This is a strong tool on this subject and it has greatly progressed, especially in generalization on the target manifolds of maps. However it seems that there has been less progress for generalization on the source manifolds even in the case of meromorphic functions (namely, the target is one-dimensional projective space) after the work of W. Stoll[11]. We are interested in this generalization of source manifolds based on a simple relationship between Nevanlinna theory and stochastic calculus. In particular, we are interested in giving estimates on the number of the omitted values by meromorphic functions. In this note we first give probabilistic expressions of classical Nevanlinna theory in terms of Brownian motion. These expressions can be easily generalized in the case of meromorphic functions on general Kähler manifolds. It leads us naturally

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to a general version of Nevanlinna's theorems for these functions in the section 3. We will see that it works well specially in case of meromorphic functions on algebraic manifolds in the following section. There are some problems when we apply this theorem to get estimates of the omitted values in more general cases. To avoid these difficulties we consider another version of Nevanlinna theory based on heat semigroups in the last section.

§2. Probabilistic expression of Nevanlinna's functions

In the proof of classical Nevanlinna's first main theorem Jensen's formula and Green's formula play important roles. Roughly speaking integrated Ito's formula or Dynkin's formula is a probabilistic counterpart of Green's formula. As seen below Jensen's formula is related to locality of martingale and this gives a probabilistic expression of counting functions of Nevanlinna. A relationship between classical Nevanlinna theory and Brownian motion was first considered by Carne[5] and see also [6] about related probabilistic problems to our expression of Jensen's formula.

Jensen's formula implies:
f : a holomorphic function on $\mathbf{C}$ with $f(o) \neq a (\in \mathbf{C})$.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta}) - a|d\theta = \sum_{f(\zeta)=a,\ |\zeta|<r} \log\frac{r}{|\zeta|}, \tag{1}$$

where the sum in the right hand side counts the number of roots of $f(\zeta) = a$ with multiplicity. Let Z_t be a complex Brownian motion with $Z_0 = o$ and $\tau_r = \inf\{t > 0 : |Z_t| > r\}$. We set $M_t := \log|f(Z_{t\wedge\tau_r}) - a|$ which is a local martingale. By Tanaka's formula([12]) the positive part of M_t takes a form as

$$M_t^+ - M_0^+ = \text{ a martingale } + \frac{1}{2}L_t,$$

where L_t is the local time of M_t at 0. This is a bounded submartingale. On the other hand the negative part can be decomposed as

$$M_t^- - M_0^- = \text{ a } \textit{local} \text{ martingale } + \frac{1}{2}L_t.$$

This is a *local* submartingale. Taking expectations of both sides we have $E[M_T^+] - M_0^+ = \frac{1}{2}E[L_T]$ and

$$E[M_T^-] - M_0^- + \lim_{\lambda\to\infty} \lambda P(\sup_{0<s<T} M_s^- > \lambda) = \frac{1}{2}E[L_T]$$

for any stopping time T.

The left hand side of (1) equals $E[M^+_{\tau_r}] - E[M^-_{\tau_r}]$. Then we have

$$\sum_{f(\zeta)=a,\ |\zeta|<r} \log \frac{r}{|\zeta|} = \lim_{\lambda\to\infty} \lambda P(\sup_{0<s<\tau_r} \log |f(Z_s) - a|^{-1} > \lambda).$$

The quantity in the left hand side is called Nevanlinna's counting function denoted by $N(r,a)$. Thus this gives a probabilistic expression of $N(r,a)$.

For a meromorphic function f we use $\log[f(z),a]^{-1}$ instead of $\log|f(z)-a|^{-1}$ where $[w,a]$ is the chordal distance on $\mathbf{P}^1(\mathbf{C})$. Namely

$$[w,a] = \begin{cases} \frac{|w-a|}{\sqrt{|w|^2+1}\sqrt{|a|^2+1}} & (\text{ if } a \neq \infty), \\ \\ \frac{1}{\sqrt{|w|^2+1}} & (\text{ if } a = \infty). \end{cases}$$

Note that $\Delta_{\mathbf{P}^1(\mathbf{C})} \log[w,a]^{-1} = 1$ where $\Delta_{\mathbf{P}^1(\mathbf{C})}$ is the Laplacian defined from Fubini–Study metric on $\mathbf{P}^1(\mathbf{C})$. Lévy's conformal invariance implies that $f(Z_t)$ is a time-changed Brownian motion on $\mathbf{P}^1(\mathbf{C})$. Namely

$$(2) \qquad f(Z_t) = W_{\rho_t}, \quad \text{with} \quad \rho_t = \int_0^t \frac{2|f'(Z_s)|^2}{(1+|f(Z_s)|^2)^2} ds,$$

where W_t is Brownian motion on $\mathbf{P}^1(\mathbf{C})$ whose generator is $\frac{1}{2}\Delta_{\mathbf{P}^1(\mathbf{C})}$. Then Ito's formula with the previous argument implies

$$E[\log[f(Z_{\tau_r}),a]^{-1}] - \log[f(o),a]^{-1} + \lim_{\lambda\to\infty} \lambda P(\sup_{0<s<\tau_r} \log[f(Z_s),a]^{-1} > \lambda) = E[\int_0^{\tau_r} \frac{|f'(Z_s)|^2}{(1+|f(Z_s)|^2)^2} ds].$$

Then we have a relationship among these functions which is called First Main Theorem in classical Nevanlinna theory:

$$m(r,a) - m(0,a) + N(r,a) = T(r),$$

where

$$m(r,a) = \int_0^{2\pi} \log[f(re^{i\theta}),a]^{-1}\frac{d\theta}{2\pi} \text{ (proximity function)},$$

$$N(r,a) = \sum_{f(\zeta)=a,|\zeta|<r} \log\frac{r}{|\zeta|}$$

(counting with multiplicity, counting function),

$$T(r) = \int_{|z|<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} g_r(o,z)dv(z)$$

(Ahlfors–Shimizu characteristic function),

where $g_r(x,y)$ is the Green's function on $\{|z| < r\}$ of Laplacian with Dirichlet condition on $\{|z| = r\}$ and dv is Lebesgue measure.

The following inequality, called Second Main Theorem, is important and nontrivial in classical Nevanlinna theory. Our main interest is to get a sort of Second Main Theorem in general case.

Theorem 1 (Second Main Theorem(cf. [10],[8])). *Let $a_1, a_2, \ldots, a_q \in \mathbf{C} \cup \{\infty\}$ be distinct points and f a nonconstant meromorphic function on $\mathbf{C}$. Then there exists $E \subset [0,\infty)$ such that $|E| < \infty$ and*

$$\sum_{k=1}^{q} m(r,a_i) + N_1(r) \le 2T(r) + O(\log T(r) + \log r) \tag{3}$$

holds for $r \notin E$.

Define a quantity called defect by

$$\delta(a) := \liminf_{r\to\infty} \frac{m(r,a)}{T(r)}.$$

$\delta(a) = 1$ if f omits a by First Main Theorem. Hence a bound of $\sum_{a\in\mathbf{P}^1}\delta(a)$ gives a bound of the number of omitted values of meromorphic function. In case of meromorphic function on $\mathbf{C}$ we can get the following called defect relation:

$$\sum_{a\in\mathbf{P}^1} \delta(a) \le 2$$

from Second Main Theorem since $O(\log T(r) + \log r)$ term in (3) can be reduced into $o(T(r))$ for any nonconstant f. This implies Picard's little theorem.

§3. General case

We give an inequality like Second Main Theorem for meromorphic functions on general complete Kähler manifolds. Our result given in this section is a natural extension of the classical Nevanlinna theory.

We first give our setting. Let M be a complete Kähler manifold with Kähler form ω, $dim_{\mathbf{C}} M = m$ and v a nonnegative, smooth and subharmonic exhaustion function on M. Note that such a function always exists due to Greene–Wu [7]. (X_t, P_x) denotes Brownian motion on M defined from the Kähler metric. Set $\tau_r = \inf\{t > 0 : v(X_t) > r\}$. Fix $o \in M$ as a reference point. Note that a nonconstant meromorphic function f on M can be regarded as a nonconstant holomorphic map from M to $\mathbf{P}^1(\mathbf{C})$. Let $||df||^2$ denote the energy density of f. From our observation in the classical case we define our Nevanlinna functions as follows.

Definition 3.1. *Assume $a \in \mathbf{P}^1(\mathbf{C})$ and $f(o) \neq a$.*

$$m(r,a) = E_o[\log[f(X_{\tau_r}), a]^{-2}], \tag{4}$$

$$N(r,a) = \lim_{\lambda \to \infty} \lambda P_o(\sup_{0<s<\tau_r} \log[f(X_s), a]^{-2} > \lambda), \tag{5}$$

$$T(r) = \frac{1}{2} E_o[\int_0^{\tau_r} ||df||^2(X_s)ds]. \tag{6}$$

We remark that these functions have the following analytic expressions

$$m(r,a) = \int_{\partial B(r)} \log[f(z), a]^{-2} d\pi_r^o(z)$$

$$T(r) = c_m \int_{B(r)} g_r(o,z) f^* \omega_o \wedge \omega^{m-1}$$

where $B(r) = \{x \in M : v(x) < r\}$, $d\pi_r^o$ is the harmonic measure on $\partial B(r)$ with respect to o, $g_r(o,z)$ is Green's function on $B(r)$ with Dirichlet boundary condition on $\partial B(r)$, ω_o is Fubini–Study metric on $\mathbf{P}^1(\mathbf{C})$ and $c_m = 2\pi^m/(m-1)!$. Since $\log[f(z), a]^{-2}$ is a δ-subharmonic function, $\Delta_M \log[f(z), a]^{-2}$ can be regarded as a signed measure denoted by $d\mu$. This signed measure $d\mu$, which is called a Riesz charge of $\log[f(z), a]^{-2}$, has a unique Jordan decomposition $d\mu = d\mu_1 - d\mu_2$ where $d\mu_1, d\mu_2$ are the smallest ones for which such a decomposition holds (cf. [9]). We note that μ_2 is supported by $f^{-1}(a)$. One can see that our counting function satisfies

$$N(r,a) = \frac{1}{2} \int_{B(r) \cap f^{-1}(a)} g_r(o,z) d\mu_2(z).$$

Since M is a Kähler manifold, $f(X_t)$ is a time-changed Brownian motion on $\mathbf{P}^1(\mathbf{C})$ defined from Fubini–Study metric similarly to (2):

$$f(X_t) = W_{\eta_t}, \quad \eta_t = \frac{1}{2}\int_0^t ||df(X_s)||^2 ds.$$

As the classical case, we immediately have the following formula like First Main Theorem: Assume $f(o) \neq a$.

$$m(r,a) - m(0,a) + N(r,a) = T(r).$$

We remark that

Proposition 2 ([1]). *If M has Liouville property and f is non-constant, then $T(r) \to \infty$ $(r \to \infty)$ and logarithmic capacity of $f(M)^c$ vanishes.*

To state our Second Main Theorem we need some quantities. Let

$$R(x) = \inf_{\xi \in T_x M,\ ||\xi||=1} Ric(\xi, \xi).$$

Define

$$N(r, Ric) = -E_o[\int_0^{\tau_r} R(X_s)ds].$$

We also introduce an analogy of a counting function of critical points of f:

$$N_1(r) := \lim_{\lambda \to \infty} \lambda P_o(\sup_{0 \leq t \leq \tau_r} \log^- ||df||^2(X_t) > \lambda).$$

For each regular value r of v there exists $r' < r$ such that $\inf_{r'<t<r} \inf_{x \in B(t)} ||\nabla v||(x) > 0$. We define the following quantities using r'. Fix $\alpha > 0$ and let $\alpha < r$ and $x \in B(\alpha)$. Define

$$C(x,r,\epsilon) = \frac{C_1(x,r)C_3(x,r,\epsilon)}{C_2(x,r)^{(1+\epsilon)^2}},$$

where $C_1(x,r), C_3(x,r,\epsilon), C_2(x,r)$ are defined as follows.

$$C_1(x,r) = \sup_{z \in \partial B(\alpha)} g_r(x,z)/(r-\alpha).$$

$$C_2(x,r) = \inf_{y \in \partial B(r')} g_r(x,y)(\int_{v(x)}^r e^{-\int_{v(x)}^t 2\mu(z)dz} dt)^{-1},$$

where $\mu(t)$ is defined by

$$\mu(t) = \begin{cases} 0 & \text{for } 0 \leq t < r', \\ \mu^{(0)}(t) & \text{for } r' \leq t < r \end{cases}$$

and

$$\mu^{(0)}(t) = \frac{1}{2} \sup_{x \in \partial B(t)} \frac{\Delta_M v}{||\nabla v||^2}(x).$$

$$C_3(x, r, \epsilon) = \exp(2(1+\epsilon) \int_{v(x)}^{r} \mu(z) dz).$$

Our Second Main Theorem is as follows.

Theorem 3 ([2]). *Let $a_1, a_2, \ldots, a_q \in \mathbf{P}^1(\mathbf{C})$ distinct points and f a nonconstant meromorphic function on M. For any $\epsilon > 0$, there exists $E_\epsilon \subset [0, \infty)$ such that $|E_\epsilon| < \infty$ and*

$$(7) \quad \sum_{j=1}^{q} m(r, a_j) + N_1(r) \leq 2T(r) + 2N(r, Ric) + \log C(o, r, \epsilon) + E[\log ||\nabla v||^2 (X_{\tau_r})] + O(\log T(r))$$

holds for $r \notin E_\epsilon$.

It turns out that we need good estimates on $\log C(o, r, \epsilon)$ to get bounds of the number of the omitted values. We give a simple case where this quantity can be easily computed in the next section.

§4. Algebraic hypersurfaces in $\mathbf{C}^n$

Let M be an algebraic hypersurface of degree k nonsingular at infinity in $\mathbf{C}^n$. i.e. $M = \{h = 0\}$ s.t. $h = h^{(k)} + h^{(k-1)} + \cdots + h^{(0)}$ where $h^{(j)}$ is a homogeneous polynomial of degree j and $\{h^{(k)} = 0\}$ is with no singularities in $\mathbf{P}^{n-1}(\mathbf{C})$.

Consider the induced metric from $\mathbf{C}^n$ as the Kähler metric. Take the Euclidean distance between x and o denoted by $r(x)$ as the exhaustion function $v(x)$. Let $B(R) = \{r(x) < R\}$ and $g_R(x, y)$ denotes Green's function on $B(R)$ with Dirichlet condition.

Proposition 4 ([3]).

$$c(x_o) \log \frac{R}{r(x)} \leq g_R(x_o, x) \leq c'(x_o) \log \frac{R}{r(x)} \ (n = 2),$$

$$c(x_o)(r(x)^{4-2n} - R^{4-2n}) \leq g_R(x_o, x) \leq c'(x_o)(r(x)^{4-2n} - R^{4-2n}) \ (n \geq 3).$$

Remark. i) If Green's functions of a complex hypersurface N in $\mathbf{C}^n$ have estimates as above, then N is algebraic.
ii) Any algebraic submanifold has Liouville property.

We have a Second Main Theorem in this case:

Theorem 5 ([3]). *For any $\epsilon > 0$ there exists $E_\epsilon \subset [0, \infty)$ such that $|E_\epsilon| < \infty$ and*

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \leq 2T(r) + (2(k-1) + \epsilon(2n-3)) \log r + O(1)$$

holds for $r \notin E_\epsilon$.

We can also see

$$T(r) \geq const. \log r$$

if f is nonconstant.

Define $c(f) = \liminf_{r\to\infty} \frac{T(r)}{\log r}$ $(\leq \infty)$. Note that $c(f) > 0$ if f is nonconstant.

Then we have

$$\sum_{i=1}^{q} \delta(a_i) \leq 2 + \frac{2(k-1)}{c(f)}.$$

§5. Another estimate: Use heat kernels instead of Green's functions

In Theorem 3 it is important to have good estimates on Green's functions for our application. It does not so easy to do that in case of transcendental submanifolds. Then we ask if there is another method without Green's functions. We also ask if the term of $\log C(o, r, \epsilon)$ in (7) can be eliminated.

For these problems we employ another method. Our simple idea is to replace Green's functions by heat kernels.

In this section we assume that M is a complete and stochastically complete Kähler manifold.

Definition 5.1. *Assume $a \in \mathbf{P}^1(\mathbf{C})$ and $f(x) \neq a$.*

$$\tilde{m}_x(t, a) = E_x[\log[f(X_t), a]^{-2}],$$
$$\tilde{N}_x(t, a) = \lim_{\lambda\to\infty} \lambda P_x(\sup_{0\leq s\leq t} \log[f(X_s), a]^{-2} > \lambda),$$
$$\tilde{T}_x(t) = \frac{1}{2} E_x[\int_0^t ||df||^2(X_s) ds].$$

We remark that $\tilde{T}_x(t)$ has the following analytic expression

$$\tilde{T}_x(t) = c_m \int_0^t \int_M p(s, x, z) f^* \omega_o \wedge \omega^{m-1} ds,$$

where $p(t,x,y)$ is the heat kernel of a half of Laplacian. Differently from the previous case, one cannot always say that $\tilde{T}_x(t) < +\infty$ $(t > 0)$, and $\tilde{N}_x(t,a) = 0$ if f omits a.

Remark. $u(X_t)$ is not always a *pure* submartingale even if u is a smooth subharmonic function.

We put the following assumptions **A1** and **A2**. $r(x)$ denotes the Riemannian distance function from a reference point o to x and $B(r)$ is the geodesic ball with center o of radius r.

A1: The energy density $||df||^2$ of f satisfies

$$\int_1^\infty e^{-\epsilon r^2}\left(\int_{B(r)} ||df||^2 dv\right)dr < \infty \text{ for any } \epsilon > 0.$$

A2: There exists a nonnegative moderately increasing function k on $[0,\infty)$ such that

$$R(x) \geq -k(r(x)^2) \text{ and } k(t) = o(t) \text{ as } t \to \infty,$$

where $R(x) = \inf_{\xi \in T_x M,\ ||\xi||=1} Ric(\xi,\xi)$.

Then we can see that **A1** with **A2** implies that $\tilde{T}_x(t) < +\infty$ for $t > 0$ and also that $\tilde{N}_x(t,a) = 0$ if f omits a.

Remark. **A2** implies stochastic completeness of M.

From the following Second Main Theorem we can see that the growth of Ricci curvature is essential for the bounds of the number of omitted values under **A1** and **A2**.

Theorem 6 ([4]). *Let f be a nonconstant meromorphic function on a complete Kähler manifold M and $a_1, \ldots, a_q$ distinct points in $\mathbf{C} \cup \{\infty\}$. Assume* **A1** *and* **A2**. *Then*

$$\sum_{j=1}^{q} \tilde{m}_x(t,a_j) + \tilde{N}_1(t,x)$$
$$\leq 2\tilde{T}_x(t) + 2\tilde{N}_x(t,Ric) + (5+\delta)\log \tilde{T}_x(t) + O(1)$$

holds for any t outside an exceptional set of finite length and a.e. $x \in M$.

Define

$$\tilde{\delta}_x(a) = 1 - \limsup_{r\to\infty} \frac{\tilde{N}_x(r,a)}{\tilde{T}_x(r)}.$$

A1 with **A2** implies that $\tilde{\delta}_x(a) = 1$ if f omits a.

Corollary 7. *Assume* **A1** *and* **A2**. *If*

$$\lim_{t\to\infty} \frac{\tilde{T}_x(t)}{\int_0^t k(s)ds} = +\infty,$$

then

$$\sum_{i=1}^{q} \tilde{\delta}_x(a_i, f) \leq 2.$$

We have a similar estimate on the sum of defects in cases of algebraic submanifolds to the previous section. We here give a simple example of transcendental manifold but the following is a very partial result. Consider $M = \{e^x + e^y = 1\} \subset \mathbf{C}^2$. Then Ricci curvature of M is bounded. Then **A2** is satisfied.

Proposition 8. *Assume that f satisfies* **A1**.
Set

$$C_x(f) := \liminf_{t\to\infty} \frac{T_x(t)}{\sqrt{t}}.$$

We have

$$\sum_{i=1}^{q} \tilde{\delta}_x(a_i, f) \leq 2 + \frac{2}{C_x(f)}.$$

Remark. The above proposition does not make sense for the functions of polynomial growth since $C_x(f) = 0$, although they satisfy **A1**.

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