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A universal bivariant theory and cobordism groups

Dedicated to Professor Mutsuo Oka on the occasion of his 60th birthday

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Abstract.

This is a survey on a universal bivariant theory $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$, which is a prototype of a bivariant analogue of Levine–Morel's algebraic cobordism, and its application to constructing a bivariant theory $F\Omega(X \to Y)$ of cobordism groups. Before giving such a survey, we recall the genus such as signature, which is the main important invariant defined on the cobordism group, i.e, a ring homomorphism from the cobordism group to a commutative ring with a unit. We capture the Euler–Poincaré characteristic and genera as a drastic generalization of the very natural *counting of finite sets.*

§1. Introduction

The (oriented) cobordism group Ω_* was introduced by René Thom [Th] in 1950's and it was extended by Michael Atiyah [At] to the (oriented) cobordism theory $MSO^*(X)$ of a topological space X, which is a generalized cohomology theory. It is defined by Thom spectra $\{MSO(n)\}$. As a covariant or homology-like version of $MSO^*(X)$, Atiyah [At] introduced the bordism theory $MSO_*(X)$ geometrically in a quite simple manner, compared with the "spectral" definition of the cobordism theory. If we replace $SO(n)$ by the other groups $O(n)$, $U(n)$, $Spin(n)$, we get the corresponding cobordism and bordism theories.

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The cobordism theory $MSO^*(X)$ is not "geometrically defined" unlike singular cohomology, de Rham cohomology, K-theory, although the bordism theory $MSO_*(X)$ is very much geometrically simply defined. This kind of "drawback" pops up crucially when one deals with the so-called elliptic cohomology theory. Note that algebraic topologists have been looking for a more geometrically described definition of the elliptic cohomology. For a very recent survey on the elliptic cohomology, see Jacob Lurie's survey paper [Lu], and also see other papers and/or books [Hat, La2, Mi-Ra, Se, Ti].

Daniel Quillen introduced the notion of *(complex) oriented cohomology theory* on the category of differential manifolds [Qui] and this notion can be formally extended to the category of smooth schemes in algebraic geometry. Vladimir Voevodsky has introduced algebraic cobordism (now called *higher algebraic cobordism*), which was used in his proof of Milnor's conjecture [Voe]. Marc Levine and Fabien Morel constructed *the universal one* of oriented cohomology theories, which they also call *algebraic cobordism*, and have investigated furthermore (see [Le1, Le2, LM1, LM2, LM3] and also see [Mer] for a condensed review). Levine–Morel's algebraic cobordism is *the universal oriented Borel–Moore functor* satisfying some geometric axioms.

William Fulton and Robert MacPherson have introduced the notion of bivariant theory as a *categorical framework for the study of singular spaces*, which is the title of their AMS Memoir book [FM] (see also Fulton's book [Fu]). A bivariant theory is a unification of covariant functor such as a homology theory and a contravariant functor such as a cohomology theory. A typical example is Fulton–MacPherson's bivariant homology theory $\mathbb{H}$, whose associated contravariant functor $\mathbb{H}^*$ is the usual singular cohomology theory and whose associated covariant functor $\mathbb{H}_*$ is the Borel–Moore homology theory, *not* the usual singular homology theory. The main objective of [FM] is bivariant-theoretic Riemann–Roch's or bivariant analogues of various theorems of Grothendieck–Riemann–Roch type.

We have recently constructed (prototypes of) bivariant-theoretic analogues of the above two cobordism groups ([Yo1, Yo3]). There are two naïve motivations for this work:

(1) Levine–Morel's algebraic cobordism is a covariant theory, thus we want to know its contravariant version so that it is unified into its bivariant one.

(2) The definition of the cobordism theory $MSO^*(X)$ is not simple as that of the bordism theory $MSO_*(X)$, thus we want another contravariant version of $MSO_*(X)$ so that it is also unified into a bivariant-theoretic analogue of $MSO_*(X)$. (It would be hopefully related to the problem of a geometric description of the elliptic cohomology.)

A key ingredient is what we call *a universal bivariant theory*, which is the universal one among the bivariant theories with a nice canonical orientation, or what could be called *a "motivic" bivariant theory.*

In this paper we make a quick survey on these bivariant analogues after reviewing the Euler number and the genus, which is in a sense a drastic generalization of the Euler number.

§2. Euler–Poincaré characteristic and genera

The genus is a homomorphism from the cobordism ring to another ring; thus these two notions are two aspects of one thing in a sense. So, before going into the main parts of the paper in the next sections, in this section we recall or review how natural the notion of Euler number or Euler–Poincaré characteristic is and we see that the notion of genus is a "drastic" generalization of the Euler–Poincaré characteristic, although the Euler–Poincaré characteristic is *not* a genus. Also we see that the *ordinary (co)homology* or more correctly *the Borel–Moore homology theory* constructed from the ordinary (co)homology theory is in a sense for the Euler–Poincaré characteristic and *extraordinary or generalized (co)homology theories* are for genera, in particular the most interesting one is the elliptic genus and the elliptic cohomology.

The cardinality $c(S)$, a nonnegative integer, of a finite set S satisfies the following properties:

(1) $A \cong B$ (**set-isomorphism**) implies $c(A) = c(B)$,
(2) $c(A \sqcup B) = c(A) + c(B)$,
(3) $c(A \times B) = c(A) \cdot c(B)$,
(4) $c(pt) = 1$. (Here pt denotes one point space.)

Or one could say that the isomorphism classes $[S]$ of finite sets satisfies the above properties except the property (4). Setting $[\phi] := 0$ and $[pt] := 1$ gives us the nonnegative integers.

Now, let us consider the following similar problem for topological spaces:

Problem 2.1. *Can one define a cardinality $\chi_{top}(W)$ of a topological space W such that the above property (1) is replaced by the following "topological" one and the rest of the properties are unchanged ? Namely, does there exist χ_{top} with the following properties on topological spaces ?:*

(1) $A \cong B$ (**homeomorphism**) *implies* $\chi_{top}(A) = \chi_{top}(B)$,
(2) $\chi_{top}(A \sqcup B) = \chi_{top}(A) + \chi_{top}(B)$,
(3) $\chi_{top}(A \times B) = \chi_{top}(A) \cdot \chi_{top}(B)$,
(4) $\chi_{top}(pt) = 1$.

Since considering the cardinality $c(S)$ of a finite set S is the same as considering the cardinality $\chi_{top}(S)$ of the finite set S with the discrete topology, we just use the symbol χ_{top} even for finite sets.

Remark 2.2. In the case of finite sets (with discrete topology), the property (2) can be without any caution replaced by the usual one: "inclusion-exclusion formula"

$$\chi_{top}(A \cup B) = \chi_{top}(A) + \chi_{top}(B) - \chi_{top}(A \cap B).$$

However, in the general case of topological spaces one has to be a bit careful. In the property (2) $A \sqcup B$ *does not necessarily* mean a direct sum of two topological spaces A and B, or the disjoint union of two topological spaces. One should note that if it were so, the property (2) in general could not be replaced by the above "inclusion-exclusion formula." It really means that if a topological space $(X, \mathcal{T})$ is decomposed into $X = A \sqcup B$, then for the topological subspaces $(A, \mathcal{T}|_A)$ and $(B, \mathcal{T}|_B)$, where $\mathcal{T}_S$ is the relative topology of a subset $S \subset X$, we have the equality:

$$\chi_{top}((A \sqcup B, \mathcal{T})) = \chi_{top}((A, \mathcal{T}|_A)) + \chi_{top}((B, \mathcal{T}|_B)).$$

This of course already takes care of even the case when $A \sqcup B$ is the disjoint union of two topological spaces. Thus, the above "inclusion-exclusion formula" means that if $A \cup B$ is a topological space with a topology $\mathcal{T}$, then

$$\chi_{top}((A \cup B, \mathcal{T})) = \chi_{top}((A, \mathcal{T}|_A)) + \chi_{top}((B, \mathcal{T}|_B)) - \chi_{top}((A \cap B, \mathcal{T}|_{A \cap B})).$$

In this sense, even in the case of topological spaces the property (2) can be replaced by the above "inclusion-exclusion formula." However, we stick to the above simple "disjoint union" formula (2) for the sake of what follows later, i.e., *genera.*

Now, we consider the looked-for cardinality χ_{top} for the 1-dimensional Euclidean space $\mathbb{R}^1$ and apply the property (2) to the decomposition of $\mathbb{R}^1$:

$$\mathbb{R}^1 = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty).$$

Which implies that

$$\chi_{top}(\mathbb{R}^1) = \chi_{top}((-\infty, 0)) + \chi_{top}(\{0\}) + \chi_{top}((0, \infty)).$$

Hence we have

$$-\chi_{top}(\{0\}) = \chi_{top}((-\infty, 0)) + \chi_{top}((0, \infty)) - \chi_{top}(\mathbb{R}^1).$$

Since $\mathbb{R} \cong (-\infty, 0) \cong (0, \infty)$, it follows from (1) and (4) that

$$\chi_{top}(\mathbb{R}^1) = -\chi_{top}(\{0\}) = -1.$$

Hence, it follows from (3) that

$$\chi_{top}(\mathbb{R}^n) = (-1)^n.$$

So, in particular, if X is a finite CW-complex, i.e., a compact Hausdorff space X with a cellular structure, by applying the property (2) to the decomposition of X into all open cells, we get

$$\chi_{top}(X) = \sum_n (-1)^n \sharp_n(X)$$

where $\sharp_n(X)$ denotes the number of n-dimensional open cells, since each n-dimensional open cell is homeomorphic to $\mathbb{R}^n$. Namely, $\chi_{top}(X)$ is nothing but the so-called Euler–Poincaré characteristic of the CW-complex X. Of course we need to show that such a cardinality exists, in particular it is independent of the decomposition of a given topological space, and we know that the existence of such a cardinality is guaranteed at least by the (co)homology theory, to be more precisely, the *ordinary (co)homology theory.* Here we just recall an ordinary cohomology theory, since the homology version is a covariant one.

An ordinary cohomology theory is a sequence $\{H_n\}_{n \in \mathbb{Z}}$ of contravariant functors H_n from the category of pairs (X, A) of topological spaces to the category of abelian groups, together with a natural transformation $\partial : H^n(A) \to H^{n+1}(X, A)$ for each n. These are required to satisfy the following axioms:

Homotopy Axiom : If $f, g : (X, A) \to (Y, B)$ are homotopic, then for each n $f^* = g^* : H^n(Y, B) \to H^n(X, A)$.

Excision Axiom : If (X, A) is a pair and $U \subset X$ such that $\overline{U}$ is contained in the interior A° of A, then the inclusion map $j : (X - U, A - U) \to (X, A)$ induces an isomorphism ;

$$j^* : H^n(X, A) \cong H^n(X - U, A - U).$$

Exactness Axiom: For any pair (X, A) with inclusion maps $i : A \subset X$ and $j : X \subset (X, A)$ there is a long exact sequence:

$$\cdots \to H^i(X, A) \to H^n(X) \to H^n(A) \to H^{n+1}(X, A) \to \cdots.$$

Dimension Axiom: Let pt be the one-point space; then $H^n(pt) = 0$ for all $n \neq 0$.

$H^0(pt)$ is called the coefficient group of the given ordinary cohomology theory H^*.

For a given ordinary cohomology theory H^*, the H^*- Euler–Poincaré characteristic $\chi_{H^*}(X)$ of a topological space X is defined to be

$$\chi_{H^*}(X) := \sum_n (-1)^n \dim_{\mathbb{R}} H^n(X) \otimes \mathbb{R}$$

provided that it is well-defined. This H^*- Euler–Poincaré characteristic χ_{H^*} satisfies the following properties for the category of topological spaces:

(1) $A \cong B$ (**homeomorphism**) implies $\chi_{H^*}(A) = \chi_{H^*}(B)$,
(2) If $(A \sqcup B; A, B)$ is proper, i.e., $H^*(A) \cong H^*(A \sqcup B, B)$ and $H^*(B) \cong H^*(A \sqcup B, A)$, then $\chi_{H^*}(A \sqcup B) = \chi_{H^*}(A) + \chi_{H^*}(B)$,
(3) $\chi_{H^*}(A \times B) = \chi_{H^*}(A) \cdot \chi_{H^*}(B)$,
(4) $\chi_{H^*}(pt) = 1$.

The "inclusion-exclusion formula" version of (2) is the following: If $(A \cup B; A, B)$ is proper, i.e., $H^*(A, A \cap B) \cong H^*(A \cup B, B)$ and $H^*(B, A \cap B) \cong H^*(A \cup B, A)$, then $\chi_{H^*}(A \cup B) = \chi_{H^*}(A) + \chi_{H^*}(B) - \chi_{H^*}(A \cap B)$. Which follows from the Mayer–Vitoris exact sequence. Thus, the property (2) is quite delicate and does depend on how subspaces A and B are located in the ambient space $A \cup B$. With this definition there could be as many Euler–Poincaré characteristics χ_{H^*} as different ordinary cohomology theories χ_{H^*} exist.

In most cases objects treated in geometry and topology are finite CW-complexes or CW-complexes. For these objects, the ordinary cohomology theory turns out to be *unique*, that is to say, *if it is restricted to the category of pairs of finite CW complexes* any other ordinary (co)homology theory is in fact isomorphic to the singular cohomology theory, namely we have Eilenberg–Steenrod Theorem [ES1, ES2]:

Theorem 2.3. *On the category of pairs of finite CW complexes there exists a unique (up to isomorphism) ordinary (co)homology theory.*

The uniqueness in fact follows from the Dimension Axiom. Hence, if the Dimension Axiom is dropped, then there are many cohomology theories. A cohomology theory without Dimension Axiom being required is called *a generalized cohomology theory* or *an extraordinary cohomology theory*. Typical ones are K-theory and the cobordism theory and the latter is the main topic of this note.

Therefore, the Euler–Poincaré characteristic of a finite CW-complex is uniquely defined by any ordinary cohomology theory, e.g., the singular cohomology theory:

$$\chi(X) := \sum_{i \geq 0} (-1)^i \dim_{\mathbb{R}} H^i(X; Z) \otimes \mathbb{R}.$$

However, for non-finite CW-complexes this Euler–Poincaré characteristic is not necessarily uniquely defined and furthermore for any ordinary cohomology theory the H^*-Euler–Poincaré characteristic $\chi_{H^*}(X)$ of the n-dimensional Euclidean space $\mathbb{R}^n$ is always

$$\chi_{H^*}(\mathbb{R}^n) = 1 \quad \text{for any } n$$

since $\mathbb{R}^n$ is homotopic to one point space. Namely, this characteristic does not give rise to the above looked-for cardinality χ_{top} for open cells. To remedy this, we use the *cohomology with compact support* or the so-called *Borel–Moore homology group* for locally compact spaces, which can be made into compact spaces by adding just one point, i.e., one-point compactification. The Borel–Moore homology group $H_*^{BM}(X; R)$ of a locally compact Hasudorff space X is the relative singular homology of the one-point compactification X^+ with $*$ being the one point attached:

$$H_*^{BM}(X; \mathbb{Z}) := H_*(X^+, \{*\}; \mathbb{Z}).$$

With this we have that $H_n^{BM}(\mathbb{R}^n; \mathbb{Z}) = \mathbb{Z}$ and $H_k^{BM}(\mathbb{R}^n; \mathbb{Z}) = 0$ for $k \neq n$. Thus the Borel–Moore homological Euler–Poincaré characteristic

$\chi_{BM}(X)$ of a locally compact Hasudorff space X can be defined by

$$\chi_{BM}(X) := \sum_n (-1)^n \dim_{\mathbb{R}} H_n^{BM}(X) \otimes \mathbb{R}.$$

Then for a finite CW-complex X we have

$$\chi_{top}(X) = \chi(X) = \chi_{BM}(X) \quad \text{and} \quad \chi_{top}(\mathbb{R}^n) = \chi_{BM}(\mathbb{R}^n) = (-1)^n.$$

It is known (e.g., see [Ha, Corollary 2.24]) that in the category of finite CW-complexes, for any two subcomplexes A and B of a CW-complex $A \cup B$, the triple $(A \cup B; A, B)$ is proper, we get the following:

Theorem 2.4. *On the category of finite CW-complexes* $\chi_{top} = \chi_{BM}$ *satisfies the following properties:*

(1) $A \cong B$ (**homeomorphism**) *implies* $\chi_{top}(A) = \chi_{top}(B)$,
(2) $\chi_{top}(A \sqcup B) = \chi_{top}(A) + \chi_{top}(B)$,
(3) $\chi_{top}(A \times B) = \chi_{top}(A) \cdot \chi_{top}(B)$,
(4) $\chi_{top}(pt) = 1$.

Remark 2.5. Since A and B are subcomplexes of a CW-complex $A \sqcup B$, it is automatically that A and B are two connected components.

Summing up roughly:

(1) *If we stick to the fundamental/basic/essential properties which the way we count finite sets satisfies even when we "count" or "measure" topological spaces "topologically", we naturally ends up with the so-called Euler number for finite CW-spaces.*
(2) *Such a cardinality* χ_{top}, *which becomes the Euler–Poincaré characteristic on the category of finite CW-complexes, is just one aspect of the unique ordinary cohomology theory.*

The Euler–Poincaré characteristic is the simplest but most important invariant in modern geometry and topology. There are many important results concerning it, such as the Chern's Gauss–Bonnet Theorem and the Poincaré–Hopf Theorem relating vector fields on a manifold and The Euler–Poincaré characteristic of the manifold. Before going further on, we recall MacPherson's Theorem [Mac], which is a theorem of Grothendieck–Riemann–Roch type:

Theorem 2.6. *Let* $\mathcal{F}$ *be the covariant functor of constructible functions on the category of complex algebraic varieties with proper morphisms. There exists a unique natural transformation from the covariant functor* F *to the Borel–Moore homology theory*

$$c_* : \mathcal{F} \to H_*^{BM}$$

such that if X is a smooth variety then the value $c_*(1_X)$ of the characteristic function 1_X on X is the Poincaré dual of the total Chern cohomology class $c(TX)$ of the tangent bundle:

$$c_*(1_X) = c(TX) \cap [X].$$

A bivariant-theoretic analogue of the above transformation was conjectured in [FM] and it was solved affirmatively by J.-P. Brasselet under a certain condition [Br], and it was investigated furthermore in [BSY3, BSY4].

In the above we saw that one automatically obtains the notion of Euler–Poincaré characteristic just by changing the requirement of set isomorphism to that of the topological isomorphism in the property (1). Now the notion of *genus* is a "drastic" one along the same line of thinking. In the following part of this section, all manifolds are assumed to be smooth, compact and oriented.

René Thom [Th] made an epoc: he introduced the notion of cobordism or cobordant, i.e., he suggested two n-dimensional manifolds A, B to be "identified" if there exists an $(n+1)$-dimensional manifold W whose boundary ∂W is isomorphic to the disjoint union of A and $-B$, i.e., B with its orientation reversed:

$$\partial W = A \cup -B.$$

Then A and B are called "bordant" and denoted by

$$A \cong B.$$

And we consider the following problem of "cardinality" based on the bordism "isomorphism":

Problem 2.7. *On closed manifolds find a "cardinality" γ satsifying the following conditions:*

(1) $A \cong B$ (**bordant**) *implies* $\gamma(A) = \gamma(B)$,
(2) $\gamma(A \sqcup B) = \gamma(A) + \gamma(B)$,
(3) $\gamma(A \times B) = \gamma(A) \cdot \gamma(B)$,
(4) $\gamma(pt) = 1$.

In other words, if we let R be a commutative ring with unit, then γ is simply a ring homomorphism from the bordism ring Ω to R:

$$\gamma : \Omega \to R.$$

This "cardinality" is called a genus. Here we recall that $\Omega = \Omega^{SO}$ is the set of (co)bordism classes of smooth, compact and oriented manifolds, *which becomes a ring with addition by the disjoint union of manifolds and multiplication by Cartesian product of manifolds.*

Remark 2.8. (1) Any closed manifold cannot be decomposed into closed submanifolds, unless it is already the disjoint union of closed manifolds. Hence in the property (2) $A \sqcup B$ is *automatically* the disjoint union of closed manifolds. (2) In the case of the topological isomorphism, by decomposing the 1-dimensional Euclidean space $\mathbb{R}^1$ into $(-\infty, 0) \sqcup \{0\} \sqcup (0, \infty)$, we get the Euler–Poincaré characteristic. However, in the above problem we cannot do such a magic or trick.

In contrast to the Euler–Poincaré characteristic χ_{top} (which is certainly not a genus for oriented manifolds, but a genus only for stable complex manifolds), it is not uniquely determined. Indeed, the Hirzebruch's famous signature σ and $\hat{A}$ are the most typical and well-studied genera.

Here is a very simple problem on genera:

Problem 2.9. *Determine all genera.*

This problem is in a sense solved by the following fundamental theorem due to R.Thom:

Theorem 2.10. *On the category of closed oriented manifolds we have*

$$\Omega \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{P}^2, \mathbb{P}^4, \mathbb{P}^6, \cdots, \mathbb{P}^{2n}, \ldots].$$

So, if we consider a commutative ring R without torsion for a genus $\gamma : \Omega \to R$, then the genus γ is completely determined by the value $\gamma(\mathbb{P}^{2n})$ of the cobordism class of each even dimensional complex projective sapce $\mathbb{P}^{2n}$. Then using this value one could consider its generating "function" or formal power series such as $\sum_n \gamma(\mathbb{P}^{2n})x^n$, or $\sum_n \gamma(\mathbb{P}^{2n})x^{2n}$, and etc. In fact, a more interesting problem than determining "all genera" is to determine all *rigid* genera such as the above mentioned signature σ and $\hat{A}$; namely a genera satisfying the following multiplicativity stronger than the property (3) $\gamma(A \times B) = \gamma(A) \cdot \gamma(B)$:

$(3)_{rigid}$: $\gamma(M) = \gamma(F)\gamma(B)$ for a fiber bundle $M \to B$ with its fiber F being a spin-manifold and compact connected structural group.

For this rigidity problem on genera, one needs to consider its so-called "logarithmic" formal power series in $R[[x]]$:

$$\log_\gamma(x) := \sum_n \frac{1}{2n+1} \gamma(\mathbb{P}^{2n}) x^{2n+1}.$$

Now here is Ochanine–Bott–Taubes' epoc-making theorem:

Theorem 2.11. *The genus γ is rigid if and only if it is an elliptic genus, i.e., its logarithm $\log_\gamma$ is an elliptic integral, i.e.,*

$$\log_\gamma(x) = \int_0^x \frac{1}{\sqrt{1-2\delta t^2+\epsilon t^4}}dt$$

for some $\delta, \epsilon \in R$.

Remark 2.12. We note that if one allows its fiber F to be any manifold instead of a spin-manifold, then only the signature is rigid.

S. Ochanine [Oc] proved the "only if" part and later the "if part" was first "physically" proved by E. Witten [Wi] using the Dirac operator on the loop spaces and rigorously proved by C. Taubes [Ta] and also by R. Bott and C. Taubes [BT]. Also see B. Totaro's papers [To-1, To-2].

Given a ring homomorphism $\varphi : MSO^*(pt) \to R$, R is a $MSO^*(pt)$-module and

$$MSO^*(X) \otimes_{MSO^*(pt)} R$$

becomes "almost" a generalized cohomology theory in the sense that it does not necessarily satisfy the Exactness Axiom. P. S. Landweber [La1] gave an algebraic criterion (called the Exact Functor Theorem) for it to become a generalized cohomology theory, i.e., satisfy the Exactness Axiom too. Applying Landweber's Exact Functor Theorem, P. E. Landweber, D. C. Ravenel and R. E. Stong [LRS] showed the following theorem:

Theorem 2.13. *For the elliptic genus*

$$\gamma : MSO^*(pt) = MSO_*(pt) = \Omega \to \mathbb{Z}[\frac{1}{2}][\delta, \epsilon],$$

the following functors are generalized cohomology theories:

$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}[\frac{1}{2}][\delta, \epsilon][\epsilon^{-1}],$$

$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}[\frac{1}{2}][\delta, \epsilon][(\delta^2-\epsilon)^{-1}],$$

$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}[\frac{1}{2}][\delta, \epsilon][\Delta^{-1}],$$

where $\Delta = \epsilon(\delta^2-\epsilon)^2$.

More generally J. Franke [Fr] showed the following theorem:

Theorem 2.14. *For the elliptic genus*

$$\gamma : MSO^*(pt) = MSO_*(pt) = \Omega \to \mathbb{Z}[\frac{1}{2}][\delta, \epsilon],$$

the following functor is a generalized cohomology theory:

$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}[\frac{1}{2}][\delta, \epsilon][P(\delta, \epsilon)^{-1}],$$

where $P(\delta, \epsilon)$ *is a homogeneous polynomial of positive degree with* $\deg \delta = 4, \deg \epsilon = 8$.

The generalized cohomology theory

$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}[\frac{1}{2}][\delta, \epsilon][P(\delta, \epsilon)^{-1}]$$

is called an *elliptic cohomology theory.* It is defined in an algebraic manner, but not in a more topological or geometric manner as in K-theory, the bordism theory $MSO_*(X)$. So, people have been searching for a reasonable geometric or topological construction of the elliptic cohomology (e.g., see [KS] by M. Kreck and S. Stolz). In fact, the present work is motivated partly by this interesting geometric problem of the elliptic cohomology theory.

Remark 2.15. Before finishing this section, as to the notion of "cardinality" we would like to remark about a recent work of J. Baez and W. Dolan [BD] on what they call "homotopy cardinality" from the aspect of "categorification". The above Euler–Poincaré characteristic of a topological space X is the alternating sum of the dimension of (co)homology groups $H^*(X;\mathbb{R})$ or $H_*(X;\mathbb{R})$. Baez–Dolan's homotopy cardinality of a topological space X, denoted by $\chi^{\pi}_{top}(X)$ provisionally here, is a sort of "alternating product" of homotopy groups of X; to be more precise,

$$\chi^{\pi}_{top}(X) := \frac{\prod_{k\geq 0} |\pi_{2k}(X)|}{\prod_{k\geq 0} |\pi_{2k+1}(X)|}.$$

Where $|\pi_i(X)|$ denotes the cardinality of the group $\pi_i(X)$ considered as a category, i.e., a groupoid. This homotopy cardinality makes sense of course when it is well-defined, just as it is the case for the Euler–Poincaré characteristic. Baez and Dolan showed that this homotopy cardinality satisfies the same property as the Euler–Poincaré characteristic. So far it is a reasonable cardinality only when all the homotopy groups are finite groups and no other case has not been computed and their conjecture is that there would be no other case than finite homotopy groups. It would be an interesting problem whether there is a reasonable relation between the Euler–Poincaré characteristic and the Baez–Dolan homotopy cardinality. We hope to come back to this problem. For more details see their paper [BD] and also Baez's homepage (http://math.ucr.edu/home/baez/). In particular, one is recommended to read Baez's articles [Ba1, Ba2] and also T. Leinster's paper [Lein].

§3. Fulton–MacPherson's bivariant theory

We make a quick review of Fulton–MacPherson's bivariant theory [FM], where the reader also will find a lot of examples like algebraic K-theory and operational Chow groups in algebraic geometry, or bivariant versions of generalized cohomology theories in algebraic topology.

Let $\mathcal{V}$ be a category with a final object pt, a class of "indepedent squares" and a class $\mathcal{C}$ of "confined maps", which is closed under composition and base change in indepedent squares and contains all identity maps. Here we assume that all indepedent squares are fiber products (i.e. the corresponding fiber products exist in $\mathcal{V}$), which satisfy the following properties:

(i) if the two inside squares in

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f''} & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y \end{array}$$

or

$$\begin{array}{ccc} X' & \xrightarrow[h'']{} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[h']{} & Y \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ Z' & \xrightarrow[h]{} & Z \end{array}$$

are independent, then the outside square is also independent,

(ii) any square of the following forms are independent:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ f \downarrow & & \downarrow f \\ X & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \mathrm{id}_X \downarrow & & \downarrow \mathrm{id}_Y \\ X & \xrightarrow[f]{} & Y \end{array}$$

where $f : X \to Y$ is any morphism.

Example 3.1. Examples for "confined maps" are proper or projective maps in algebraic topology or geometry. In the category of smooth manifolds and maps, independent squares are by definition the squares with f and g *transversal* in the usual sense, i.e. $f \times g$ is transversal to the diagonal submanifold of $Y \times Y$ so that the fiber product $X' = Y' \times_Y X$ exists as a smooth manifold. Note that in this category not all fiber products exist.

A bivariant theory $\mathbb{B}$ on a category $\mathcal{V}$ with values in the category of graded abelian groups is an assignment to each morphism

$$X \xrightarrow{f} Y$$

in the category $\mathcal{V}$ a graded abelian group (in most cases we ignore the grading)

$$\mathbb{B}(X \xrightarrow{f} Y)$$

which is equipped with the following three basic operations. The i-th component of $\mathbb{B}(X \xrightarrow{f} Y)$, $i \in \mathbb{Z}$, is denoted by $\mathbb{B}^i(X \xrightarrow{f} Y)$.

Remark 3.2. One can also consider $\mathbb{Z}_2$-grading and $\{0\}$-grading, i.e., no grading.

Product operations: For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{B}^i(X \xrightarrow{f} Y) \otimes \mathbb{B}^j(Y \xrightarrow{g} Z) \to \mathbb{B}^{i+j}(X \xrightarrow{gf} Z)$$

is defined.

Pushforward operations: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f *confined*, the pushforward operation

$$f_* : \mathbb{B}^i(X \xrightarrow{gf} Z) \to \mathbb{B}^i(Y \xrightarrow{g} Z)$$

is defined.

Pullback operations: For an *independent* square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback operation

$$g^* : \mathbb{B}^i(X \xrightarrow{f} Y) \to \mathbb{B}^i(X' \xrightarrow{f'} Y')$$

is defined.

And these three operations are required to satisfy the seven compatibility axioms (see [FM, Part I, §2.2] for details):

(B-1) product is associative,
(B-2) pushforward is functorial,
(B-3) pullback is functorial,
(B-4) product and pushforward commute,
(B-5) product and pullback commute,
(B-6) pushforward and pullback commute, and
(B-7) projection formula.

We also assume that $\mathbb{B}$ has units:

Units: $\mathbb{B}$ has units, i.e., there is an element $1_X \in \mathbb{B}^0(X \xrightarrow{\mathrm{id}_X} X)$ such that $\alpha \bullet 1_X = \alpha$ for all morphisms $W \to X$, all $\alpha \in \mathbb{B}(W \to X)$; such that $1_X \bullet \beta = \beta$ for all morphisms $X \to Y$, all $\beta \in \mathbb{B}(X \to Y)$; and such that $g^* 1_X = 1_{X'}$ for all $g : X' \to X$.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category $\mathcal{V}$. Then a *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for a morphism $X \to Y$ in the category $\mathcal{V}$, which preserves the above three basic operations:

(i) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
(ii) $\gamma(f_* \alpha) = f_* \gamma(\alpha)$, and
(iii) $\gamma(g^* \alpha) = g^* \gamma(\alpha)$.

A bivariant theory unifies both a covariant theory and a contravariant theory in the following sense:

- $\mathbb{B}_*(X) := \mathbb{B}(X \to pt)$ becomes a covariant functor for *confined* morphisms and
- $\mathbb{B}^*(X) := \mathbb{B}(X \xrightarrow{\mathrm{id}_X} X)$ becomes a contravariant functor for *any* morphisms.

And a Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ induces natural transformations $\gamma_* : \mathbb{B}_* \to \mathbb{B}'_*$ and $\gamma^* : \mathbb{B}^* \to \mathbb{B}'^*$.

As to the grading, we set $\mathbb{B}_i(X) := \mathbb{B}^{-i}(X \xrightarrow{\mathrm{id}_X} X)$ and $\mathbb{B}^j(X) := \mathbb{B}^j(X \xrightarrow{\mathrm{id}_X} X)$.

A bivariant theory is called *commutative* (see [FM, §2.2]) if whenever both

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow[g]{} & Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W & \xrightarrow{f'} & Y \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & Z \end{array}$$

are independent squares, then for $\alpha \in \mathbb{B}(X \xrightarrow{f} Z)$ and $\beta \in \mathbb{B}(Y \xrightarrow{g} Z)$

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha.$$

If $g^*(\alpha)\bullet\beta = (-1)^{\deg(\alpha)\deg(\beta)} f^*(\beta)\bullet\alpha$, then it is called *skew-commutative.*

§4. Borel–Moore functor in the sense of Levine–Morel

For some special classes of morphisms an additive bivariant theory (see below) carries the so-called Borel–Moore functor with products, which is the basic ingredient for Levine–Morel's construction of algebraic cobordism [LM3].

Definition 4.1. ([FM, 2.6.2 Definition, Part I]) Let $\mathcal{S}$ be a class of maps in $\mathcal{V}$, which is closed under composition and containing all identity maps. Suppose that to each $f : X \to Y$ in $\mathcal{S}$ there is assigned an element $\theta(f) \in \mathbb{B}(X \xrightarrow{f} Y)$ satisfying that

(i) $\theta(g \circ f) = \theta(f) \bullet \theta(g)$ for all $f : X \to Y$, $g : Y \to Z \in \mathcal{S}$ and

(ii) $\theta(\mathrm{id}_X) = 1_X$ for all X with $1_X \in \mathbb{B}^*(X) := B(X \xrightarrow{\mathrm{id}_X} X)$ the unit element.

Then $\theta(f)$ is called a *canonical orientation* of f.

For a morphism $f : X \to Y \in \mathcal{S}$ the *Gysin homomorphism*

$$f^! : \mathbb{B}_*(Y) \to \mathbb{B}_*(X) \quad \text{defined by} \quad f^!(\alpha) := \theta(f) \bullet \alpha$$

is *contravariantly functorial.* And for the following independent square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y, \end{array}$$

where $f \in \mathcal{C} \cap \mathcal{S}$, the *Gysin homomorphism*

$$f_! : \mathbb{B}^*(X) \to \mathbb{B}^*(Y) \quad \text{defined by} \quad f_!(\alpha) := f_*(\alpha \bullet \theta(f))$$

is *covariantly functorial.* The notation should carry the information of $\mathcal{S}$ and the canonical orientation θ, but it will be usually omitted if it is not necessary to be mentioned.

Definition 4.2. (i) Let $\mathcal{S}$ be another class of maps in $\mathcal{V}$, called "specialized maps" (e.g., smooth maps in algebraic geometry), which is closed under composition and under base change in independent squares and containing all identity maps. Let $\mathbb{B}$ be a bivariant theory. If $\mathcal{S}$ has canonical orientations in $\mathbb{B}$, then we say that $\mathcal{S}$ is canonical $\mathbb{B}$-orientable and an element of $\mathcal{S}$ is called a canonical $\mathbb{B}$-orientable morphism. If we fix the orientation, then we say that $\mathcal{S}$ is canonical $\mathbb{B}$-oriented. (Of course $\mathcal{S}$ is also a class of confined maps, but since we consider the above extra condition of $\mathbb{B}$-orientability on $\mathcal{S}$, we give a different name to $\mathcal{S}$.)

(ii) Assume that $\mathcal{S}$ is canonical $\mathbb{B}$-oriented. Furthermore, if the orientation θ on $\mathcal{S}$ satisfies that for an independent square with $f \in \mathcal{S}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

the following condition holds

$$\theta(f') = g^*\theta(f),$$

(which means that the orientation θ preserves the pullback operation), then we call θ a *nice canonical orientation* and say that $\mathcal{S}$ is *nicely canonical* $\mathbb{B}$*-oriented* and an element of $\mathcal{S}$ is called *a nicely canonical* $\mathbb{B}$*-oriented morphism* .

Example 4.3. The following are typical examples of nice canonical orientations:

(1) Smooth morphisms in algebraic geometry for bivariant algebraic K-theory and operational Chow groups [FM, p.28–29].

(2) Smooth oriented submersions in the category of smooth manifolds for the bivariant versions of generalized cohomology theories, which have "canonical Thom-classes" for oriented vector bundles [FM, p. 46, p.49]. Examples of the generalized cohomology theories are usual cohomology H^*, real K-theory $KO[\frac{1}{2}]$ with 2 inverted, or oriented cobordism MSO^*.

In the following proposition we assume that for all morphism $f : X \to X'$ and $g : Y \to Y'$ in $\mathcal{V}$ any of the four small squares in the big diagrams below are independent (hence any square is independent):

$$\begin{array}{ccccc}
X \times Y & \xrightarrow{f \times Id_Y} & X' \times Y & \xrightarrow{p \times Id_Y} & Y \\
{\scriptstyle Id_X \times g}\downarrow & & \downarrow{\scriptstyle Id_{X'} \times g} & & \downarrow{\scriptstyle g} \\
X \times Y' & \xrightarrow{f \times Id_{Y'}} & X' \times Y' & \xrightarrow{p \times Id_{Y'}} & Y' \\
{\scriptstyle Id_X \times q}\downarrow & & \downarrow{\scriptstyle Id_{X'} \times q} & & \downarrow{\scriptstyle q} \\
X & \xrightarrow[f]{} & X' & \xrightarrow[p]{} & pt.
\end{array}$$

Proposition 4.4. *Let $\mathbb{B}$ be a bivariant theory and let $\mathcal{S}$ be a class of specialized maps which is nicely canonical $\mathbb{B}$-oriented.*

(1-i) For an independent square

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow[g]{} & Y
\end{array}$$

with $g \in \mathcal{C}$ and $f \in \mathcal{S}$, the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{B}_*(Y') & \xrightarrow{f'^!} & \mathbb{B}_*(X') \\
{\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g'_*} \\
\mathbb{B}_*(Y) & \xrightarrow[f^!]{} & \mathbb{B}_*(X),
\end{array}$$

(1-ii) The covariant functor $\mathbb{B}_*$ for confined morphisms and the contravariant functor $\mathbb{B}_*$ for morphisms in $\mathcal{S}$ are both compatible with the

exterior product

$$\times : \mathbb{B}(X \to pt) \otimes \mathbb{B}(Y \xrightarrow{\pi_Y} pt) \to \mathbb{B}(X \times Y \to pt)$$

defined by

$$\alpha \times \beta := \pi_Y^* \alpha \bullet \beta.$$

(2-i) For an independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

with $g \in \mathcal{C} \cap \mathcal{S}$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B}^*(Y') & \xrightarrow{f'^*} & \mathbb{B}^*(X') \\ {\scriptstyle g_!}\downarrow & & \downarrow{\scriptstyle g'_!} \\ \mathbb{B}^*(Y) & \xrightarrow[f^!]{} & \mathbb{B}^*(X), \end{array}$$

(2-ii) The contravariant functor $\mathbb{B}^*$ for any morphisms and the covariant functor $\mathbb{B}^*$ for morphisms in $\mathcal{C} \cap \mathcal{S}$ are both compatible with the exterior product

$$\times : \mathbb{B}(X \xrightarrow{\mathrm{id}_X} X) \otimes \mathbb{B}(Y \xrightarrow{\mathrm{id}_Y} Y) \to \mathbb{B}(X \times Y \xrightarrow{\mathrm{id}_{X \times X}} X \times X)$$

defined by

$$\alpha \times \beta := {p_1}^* \alpha \bullet {p_2}^* \beta$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projections.

Remark 4.5. The covariant and contravariant functors $\mathbb{B}_*$ and $\mathbb{B}^*$ are *almost* what Levine and Morel call *Borel–Moore functor with products* in [LM3, Mer]; namely they do not necessarily have the *additivity property*, which is, for example, the following one in the case of homology theory of topological spaces: for the disjoint union $X \sqcup Y$ of spaces

$$H_*(X \sqcup Y) = H_*(X) \oplus H_*(Y).$$

If we want a bivariant theory to have such an additivity property, we need a bit more requirements on the category, but here we do not go into details (see [Yo1] for more details), since the additivity is not an essential ingredient.

§5. A universal bivariant theory

We have constructed a universal or "motivic" bivariant one among certain bivariant theories [Yo1].

Theorem 5.1. *Let $\mathcal{V}$ be a category with independent squares, a class $\mathcal{C}$ of confined maps and a class $\mathcal{S}$ of specialized maps as before. Assume that any Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

with f confined is independent. We define

$$\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

to be the free abelian group generated by the set of isomorphism classes of confined morphisms $h : W \to X$ such that the composite of h and f is a specialized map:

$$h \in \mathcal{C} \quad \text{and} \quad f \circ h : W \to Y \in \mathcal{S}.$$

(1) The association $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ is a bivariant theory with the nice canonical orientation $\theta(f) := [id_X]$ for $f \in \mathcal{S}$, if the three operations are defined as follows:

Product operations: For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \otimes \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{gf} Z)$$

is defined by

$$\left(\sum_V m_V [V \xrightarrow{h_V} X]\right) \bullet \left(\sum_W n_W [W \xrightarrow{k_W} Y]\right) := \sum_{V,W} m_V n_W [V' \xrightarrow{h_V \circ k_W''} X],$$

where we consider the following fiber squares

$$\begin{array}{ccccccc} V' & \xrightarrow{h_V'} & X' & \xrightarrow{f'} & W & & \\ {\scriptstyle k_W''}\downarrow & & {\scriptstyle k_W'}\downarrow & & {\scriptstyle k_W}\downarrow & & \\ V & \xrightarrow[h_V]{} & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z. \end{array}$$

Pushforward operations: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f confined, the pushforward operation

$$f_* : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{gf} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z)$$

is defined by

$$f_*\left(\sum_V n_V[V \xrightarrow{h_V} X]\right) := \sum_V n_V[V \xrightarrow{f\circ h_V} Y].$$

Pullback operations: For an independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback operation

$$g^* : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X' \xrightarrow{f'} Y')$$

is defined by

$$g^*\left(\sum_V n_V[V \xrightarrow{h_V} X]\right) := \sum_V n_V[V' \xrightarrow{h_V{}'} X'],$$

where we consider the following fiber squares:

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h_V{}'}\downarrow & & \downarrow{\scriptstyle h_V} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

(2) Let $\mathbb{B}$ be a bivariant theory on $\mathcal{V}$ such that $\mathcal{S}$ is nicely canonical $\mathbb{B}$-oriented. Then there exists a unique Grothendieck transformation

$$\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{B}$$

such that for a specialized morphism $f : X \to Y \in \mathcal{S}$ *the homomorphism* $\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$ *satisfies the normalization condition that*

$$\gamma_{\mathbb{B}}([X \xrightarrow{\mathrm{id}_X} X]) = \theta_{\mathbb{B}}(f).$$

Corollary 5.2. *(1) The abelian group* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(X) := \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to pt)$ *is the free abelian group generated by the set of isomorphism classes*

$$[V \xrightarrow{h_V} X]$$

such that $h_V : V \to X \in \mathcal{C}$ *and* $V \to pt$ *is a specialized map in* $\mathcal{S}$*.*

(2) The abelian group $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^{*}(X) := \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$ *is the free abelian group generated by the set of isomorphism classes*

$$[V \xrightarrow{h_V} X]$$

such that $h_V : V \to X \in \mathcal{C} \cap \mathcal{S}$*.*

(3) Both functor $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}$ *and* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^{*}$ *are Borel–Moore functors with products, except for the additivity property.*

Remark 5.3. The assumptions of Theorem 5.1 are satisfied in the context of Example 4.3 with $\mathcal{S}$ the class of smooth morphisms or oriented submersions. So in Example 4.3 $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(X)$ (resp. $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^{*}(X)$) is the free abelian group generated by isomorphism classes of proper maps $V \to X$ with V smooth or an oriented manifold (resp. of proper maps $h : V \to X$ with h smooth or an oriented submersion).

Remark 5.4. A more subtle ring similar to the abelian group $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$ in algebraic geometry is the so-called relative Grothendieck ring or "motivic ring", which was introduced by E. Looijenga [Lo] and furthermore studied by F. Bittner [Bit]. This plays an important role in the motivic measure and integration (e.g. see [DL1, DL2, DL3, Ve]) and also for the motivic characteristic classes for singular varieties [BSY1, BSY2, SY] (also see [CMS, CLMS, CMSS, Sch1, Sch2, Yo4, Yo5]).

For more details and an abstract oriented bivariant theory see [Yo1] and in [Yo2] we will deal with a more *geometrical* oriented bivariant theory, i.e., a bivariant-theoretic version of Levine–Morel's algebraic cobordism.

§6. Bivariant bordism theory for smooth manifolds

Now in this section we want to apply our universal bivariant theory to cobordism theory. More precisely we want to get a bivariant bordism group in such a way that its associated covariant functor is supposed to be the bordism group $MSO_*(X)$. For that purpose we first introduce or recall the following definition of a parameterized family of smooth manifolds:

Definition 6.1. (i) In the topological context $\mathcal{V}$ is the category of locally compact Hausdorff spaces, which can be embedded as a closed subset into some $\mathbb{R}^n$ [FM, p. 32]. Confined maps and independent squares are defined to be proper maps and fiber products, respectively. A *family $h : M \to X$ of compact manifolds (with boundary)* is a proper continous map h, which locally on M is isomorphic to the projection onto the first factor

$$U \simeq h(U) \times V \mapsto h(U)\,,$$

with V an open subset of $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0\}$ (see [FM, p.65]).

(ii) In the smooth context $\mathcal{V}$ is the category of finite dimensional smooth manifolds and smooth maps between them. Confined maps and independent squares are defined to be proper maps and *transversal* fiber products (as before). A *family $h : M \to X$ of smooth compact manifolds (with boundary)* is a proper smooth map h, which locally on M is isomorphic to the projection onto the first factor

$$U \simeq h(U) \times V \mapsto h(U)\,,$$

with V an open subset of $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0\}$. In other words h is a submersion.

(iii) Moreover we assume that a consistent *orientation* of the fiber manifolds has been chosen, e.g. one has a coordinate covering $U \simeq h(U) \times V \mapsto h(U)$ of M such that the corresponding change of coordinates is given by orientation preserving homeomorphisms (or diffeomorphisms) between open subsets of $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | \, x_1 \geq 0\}$.

(iv) Note that the *fiberwise boundary of M*, denoted by $\partial_h(M)$ (corresponding to the points with $x_1 = 0$ in the above coordinates), with the induced map $\partial_h(M) \to X$ is again a "family of compact (smooth) oriented manifolds (without boundary)".

For a "family $h : M \to X$ of compact (smooth) oriented manifolds (with boundary)" its reverse oriented family is denoted by $-h : -M \to X$.

Remark 6.2. The above "family $h : M \to X$ of compact (smooth) oriented manifolds (with boundary)" is stable under base change (in independent squares) and composition (if at most one fiber of the two families may have a boundary). The usual notion of a fiber bundle over X, with fiber a compact (smooth) oriented manifold F (with boundary) and structure group the homeomorphisms (or diffeomorphisms) of the fiber F, is certainly stable under base change, but it is not clear whether or not it is also stable under composition. Originally we tried to formulate our bivariant bordism theory for smooth manifolds using fiber bundles and we came across to this stability problem. So to overcome or to avoid this unstability problem of fiber bundles we use such families above.

Definition 6.3. Let $h : M \to X, h' : M' \to X'$ be two families of compact (smooth) oriented manifolds without boundary. If there exists a family $H : W \to X$ of compact (smooth) oriented manifolds with boundary such that the induced family $H_{\partial_H(W)} : \partial_H(W) \to X$ is isomorphic to

$$h + (-h) : M \sqcup (-M') \to X ,$$

then the two families $h : M \to X$, $h' : M' \to X$ are called *fiberwise bordant*. The fiberwise bordism class of such a family $h : M \to X$ is denoted by $[M \xrightarrow{h} X]^{fib}$.

Lemma 6.4. *The fiberwise bordism is an equivalence relation for families over the same base space.*

Definition 6.5. Let a morphism $f : X \to Y$ and two morphisms $h : M \to X, h' : M' \to X'$ in the category $\mathcal{V}$ as above (i.e. either in the topological or smooth context) be given such that the composite $f \circ h$ and $f \circ h'$ are both a family of compact (smooth) oriented manifolds without boundary. Then these are called *fiberwise bordant with respect to f*, or simply *f-fiberwise bordant*, if there exists a morphism $H : W \to X$ such that $f \circ h$ is a family of compact (smooth) oriented manifolds with boundary with the induced family $f \circ H_{\partial_{f \circ H}(W)} : \partial_{f \circ H}(W) \to X$ isomorphic to

$$f \circ h + (-(f \circ h')) : M \sqcup (-M') \to X .$$

Lemma 6.6. *The fiberwise bordism with respect to a morphism f is an equivalence relation.*

This equivalence relation is called *f-fiberwise bordism* and the f-fiberwise bordism class of $h : M \to X$ with respect to a morphism f is denoted by $[M \xrightarrow{h} X]^{fib}_f$. For the identity map $id_X : X \to X$, the id_X-fiberwise bordism class $[M \xrightarrow{h} X]^{fib}_{id_X}$ is the fiberwise bordism class $[M \xrightarrow{h} X]^{fib}$. For a constant map $\pi_X : X \to pt$, the π_X-fiberwise bordism class $[M \xrightarrow{h} X]^{fib}_{\pi_X}$ is simply denoted by $[M \xrightarrow{h} X]$.

Definition 6.7. (*Relative Fiberwise Bordism Group*) For a morphism $f : X \to Y$ in $\mathcal{V}$,

$$F\Omega^{-n}(X \xrightarrow{f} Y)$$

denotes the set of f-fiberwise bordism classes $[M \xrightarrow{h} X]^{fib}_f$ such that $f \circ h : M \to Y$ is a family of compact oriented (smooth) manifolds without boundary, whose fibers are pure n-dimensional. Of course we assume that the fibers of a corresponding cobordism are pure $n+1$-dimensional compact oriented (smooth) manifolds with boundary. $F\Omega^{-n}(X \xrightarrow{f} Y)$ is an abelian group with the addition coming from disjoint union:

$$[M_1 \xrightarrow{h_1} X]^{fib}_f + [M_2 \xrightarrow{h_2} X]^{fib}_f := [M_1 \sqcup M_2 \xrightarrow{h_1+h_2} X]^{fib}_f .$$

The unit is $[\emptyset \to X]^{fib}_f$ and $[-M \xrightarrow{-h} X]^{fib}_f = -[M \xrightarrow{h} X]^{fib}_f$.

For a map $\pi_X : X \to pt$ to a point, the relative fiberwise bordism group is the classical bordism group of oriented topological (or smooth) n-dimensional manifolds proper over X:

$$F\Omega^{-n}(X \xrightarrow{\pi_X} pt) = MSO^{(top)}_n(X).$$

For the identity map $id_X : X \to X$, we set

$$FMSO^{-n}(X) := F\Omega^{-n}(X \xrightarrow{id_X} X).$$

It is obvious by definition that $MSO_*(X)$ is a covariant functor and that $FMSO^*(X)$ is a contravariant functor. $FMSO^*(X)$ shall be called the *fiberwise cobordism group of X*. The notation $FMSO^*$ is adopted to avoid some possible confusion with the usual cobordism group $MSO^*(X)$.

Theorem 6.8. *We consider either the topological or smooth context with the underlying category $\mathcal{V}$ as before. Then the* relative fiberwise bordism groups

$$F\Omega^{-n}(X \xrightarrow{f} Y)$$

define a (graded) bivariant theory if the three operations are defined as follows:

Product operations: For morphisms $f : M \to N$ and $g : N \to Z$, the product operation

$$\bullet : F\Omega^{-n}(M \xrightarrow{f} N) \otimes F\Omega^{-m}(N \xrightarrow{g} Z) \to F\Omega^{-n-m}(M \xrightarrow{gf} Z)$$

is defined by

$$\left(\sum_V m_V [V \xrightarrow{h_V} M]^{fib}_f\right) \bullet \left(\sum_W n_W [W \xrightarrow{k_W} N]^{fib}_g\right) := \sum_{V,W} m_V n_W [V' \xrightarrow{h_V \circ k_W''} M]^{fib}_{gf},$$

where we consider the following fiber squares

$$\begin{array}{ccccccc} V' & \xrightarrow{h'_V} & M' & \xrightarrow{f'} & W & & \\ {\scriptstyle k''_W}\downarrow & & {\scriptstyle k'_W}\downarrow & & {\scriptstyle k_W}\downarrow & & \\ V & \xrightarrow[h_V]{} & M & \xrightarrow[f]{} & N & \xrightarrow[g]{} & Z. \end{array}$$

Pushforward operations: For morphisms $f : M \to N$ and $g : N \to Z$ with f being proper, the pushforward operation

$$f_* : F\Omega^{-n}(M \xrightarrow{gf} Z) \to F\Omega^{-n}(N \xrightarrow{g} Z)$$

is defined by

$$f_*\left(\sum_V n_V [V \xrightarrow{h_V} M]^{fib}_{gf}\right) := \sum_V n_V [V \xrightarrow{f \circ h_V} N]^{fib}_g.$$

Pullback operations: For an independent square

$$\begin{array}{ccc} M' & \xrightarrow{g'} & M \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ N' & \xrightarrow[g]{} & N, \end{array}$$

the pullback operation

$$g^* : F\Omega^{-n}(M \xrightarrow{f} N) \to F\Omega^{-n}(M' \xrightarrow{f'} N')$$

is defined by

$$g^*\left(\sum_V n_V[V \xrightarrow{h_V} M]_f^{fib}\right) := \sum_V n_V[V' \xrightarrow{h'_V} N']_{f'}^{fib},$$

where we consider the following fiber squares:

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'_V}\downarrow & & \downarrow{\scriptstyle h_V} \\ M' & \xrightarrow{g'} & M \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ N' & \xrightarrow[g]{} & N. \end{array}$$

Remark 6.9. In other words the operations above are induced from the corresponding operations in Theorem 5.1 by dividing out the fiberwise bordism relation. The main point is then to show that these operations are welldefined, i.e. are independent of the choices for the representing morphisms. For example for the bivariant product

$$[V \xrightarrow{h_V} M]_f^{fib} \bullet [W \xrightarrow{k_W} N]_g^{fib} = [V' \xrightarrow{h_V \circ k''_W} M]_f^{fib}$$

one shows first that for fixed $k_W : W \to N$ the righthand side only depends on the *f-fiberwise bordism class* of $h_V : V \to M$. For this one looks at the correponding fiber squares with a corresponding bordism substituted for h_V. Then one shows in the same way that the righthand side only depends on the *g-fiberwise bordism class* of $k_W : W \to N$. Of course here we need the stability properties mentioned at the beginning of this section for our notion of a family of compact oriented (smooth) manifolds (with boundary).

Proposition 6.10. *Let $\mathcal{FM}$ be the class of families of compact oriented (smooth) manifolds (without boundary) in the underlying category $\mathcal{V}$. Then $\mathcal{FM}$ is nicely canonical $F\Omega$-oriented by*

$$\theta(f) := [M \xrightarrow{id_M} M]_f^{fib} \in F\Omega(M \xrightarrow{f} X)$$

for $f : M \to X$ a morphism in $\mathcal{FM}$.

Let $\mathbb{B}$ be a bivariant theory on $\mathcal{V}$ (i.e. in the topological or smooth context) such that $\mathcal{FM}$ is canonical $\mathbb{B}$-oriented. Then $\mathbb{B}$ together with this orientation θ is called a *fiberwise bordism invariant oriented bivariant theory*, if

$$h_*(\theta(f \circ h)) = h'_*(\theta(f \circ h'))$$

for any morphism $f : X \to Y$ and any two proper morphisms $h : M \to X, h' : M' \to X'$ in the category $\mathcal{V}$ such that the composite $f \circ h$ and $f \circ h'$ in $\mathcal{FM}$ are f-fiberwise bordant. In other words, the f-fiberwise bordism class

$$[M \xrightarrow{h} X]_f^{fib} \in F\Omega(X \xrightarrow{f} Y)$$

uniquely determines the element

$$h_*(\theta(f \circ h)) \in \mathbb{B}(X \xrightarrow{f} Y)\,.$$

Now we can state a universality theorem for the bivariant relative fiberwise bordism theory $F\Omega$.

Theorem 6.11. *Let $\mathbb{B}$ be a bivariant theory on $\mathcal{V}$ such that $\mathcal{FM}$ is nicely canonical $\mathbb{B}$-oriented. Assume that $\mathbb{B}$ together with this orientation θ is fiberwise bordism invariant oriented. Then there is a unique Grothendieck transformation*

$$\gamma_{\mathbb{B}} : F\Omega \to \mathbb{B}$$

such that for any $f : M \to N$ in $\mathcal{FM}$ the homomorphism $\gamma_{\mathbb{B}}$ satisfies the normalization

$$\gamma_{\mathbb{B}}([M \xrightarrow{\mathrm{id}_M} M]_f^{fib}) = \theta(f)\,.$$

Remark 6.12. Examples of fiberwise bordism invariant oriented bivariant theories and more work on the fiberwise cobordism group $FMSO^*(X)$, in particular related to Kreck–Stolz's work [KS], will be worked out somewhere else. On the corresponding *covariant* theories one has some natural transformations commuting with exterior products, coming from the fundamental class of an oriented (smooth) manifold:

(i) $F\Omega(X \to pt) = MSO^{top}(X) \to H_*(X;\mathbb{Z})$ in the topological context.

(ii) $F\Omega(X \to pt) = MSO(X) \to MSO^{top}(X) \to H_*(X;\mathbb{Z})$ in the smooth context.

(iii) $F\Omega(X \to pt) = MSO(X) \to KO(X)[\frac{1}{2}]$ in the smooth context.

And these can be extended at least to a suitable Grothendieck transformation on a bivariant subtheory of $F\Omega$ by the results of [BSY4].

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