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On manifolds which are locally modeled on the standard representation of a torus

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Abstract.

This is an expository article on manifolds which are locally modeled on the standard representation of a torus and their classifications.

§1. Introduction

This is an expository article based on the author's talk at short communications in MSJ-IHES Joint Workshop on Noncommutativity. Let S^1 be the unit circle in $\mathbb{C}$ and $T^n := (S^1)^n$ the n-dimensional compact torus. The T^n-action on $\mathbb{C}^n$ by coordinatewise complex multiplication is called the *standard representation of* T^n. Recently manifolds which are locally modeled on the standard representation of T^n attract a great deal of attention in toric topology [6, 4, 16]. In this note we shall report the classifications of such manifolds. A typical example is a nonsingular toric variety. T^n acts on an n-dimensional toric variety X as a subgroup of the n-dimensional complex torus $(\mathbb{C}^*)^n$. If X is nonsingular, then it is well-known that for each point $x \in X$, there exists a coordinate neighborhood (U, ρ, φ) of x, where U is a T^n-invariant open set of X, ρ is an automorphism of T^n, and φ is a ρ-equivariant diffeomorphism from U to some open subset in $\mathbb{C}^n$ invariant under the standard representation of T^n. The latter means that $\varphi(u \cdot x) = \rho(u) \cdot \varphi(x)$ for $u \in T^n$ and $x \in U$. In general, a T^n-action on a $2n$-dimensional manifold which has

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an atlas consisting of such coordinate neighborhoods, which is called a *standard atlas*, is said to be *locally standard.* This structure is one of the starting point of their pioneer work [6] of Davis–Januszkiewicz and now it plays a fundamental role in toric topology, see [6, 4]. In Section 2 we shall investigate locally standard torus actions. For a locally standard torus action an invariant called a *characteristic function* is defined in [6, 12]. We define another topological invariant called an *Euler class of the orbit map* and show that locally standard torus actions are classified by them.

There is a manifold which does not admit a torus action but which is locally modeled on the standard representation. Let $\omega_{\mathbb{C}^n} := \frac{1}{2\pi\sqrt{-1}}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$ be the standard symplectic structure on $\mathbb{C}^n$ (up to normalization). The standard representation of T^n preserves $\omega_{\mathbb{C}^n}$ and the map $\mu_{\mathbb{C}^n}: \mathbb{C}^n \to \mathbb{R}^n$ defined by

$$\mu_{\mathbb{C}^n}(z) = (|z_1|^2, \dots, |z_n|^2) \tag{1.1}$$

for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is a moment map of the standard representation of T^n. Notice that the image of $\mu_{\mathbb{C}^n}$ is the n-dimensional standard positive cone $\mathbb{R}^n_+ := \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n\colon \xi_i \geq 0,\ i = 1, \dots, n\}$. Let (X, ω) be a $2n$-dimensional symplectic manifold and B an n-dimensional smooth manifold with corners. A smooth map $\mu\colon (X, \omega) \to B$ is called a *locally toric Lagrangian fibration* if it is locally identified with $\mu_{\mathbb{C}^n}: \mathbb{C}^n \to \mathbb{R}^n_+$ (for the precise definition see Definition 3.1). In Section 3 we will see that a locally toric Lagrangian fibration has an underlying structure similar to a standard atlas, but which satisfies a weaker condition than that of a standard atlas. Locally toric Lagrangian fibrations are classified by Boucetta–Molino [3] up to fiber-preserving symplectomorphisms. We also recall their result. Finally, in Section 4, as a formulation of such an underlying structure of a locally toric Lagrangian fibration we define the notion of a local torus action modeled on the standard representation. We generalize a characteristic function and an Euler class of the orbit map for a locally standard torus actions to this case, and show that local torus actions are topologically classified by them. The last section and some part of Section 2 is an announcement of the forthcoming paper [16].

§2. Locally standard torus actions

Definition 2.1. Let T^n act smoothly on a $2n$-dimensional smooth manifold X. A *standard coordinate neighborhood* of X consists of a triple (U, ρ, φ), where U is a T^n-invariant connected open set of X, ρ is an automorphism of T^n, and φ is a ρ-equivariant diffeomorphism from U to some open subset of $\mathbb{C}^n$ which is invariant under the standard representation of T^n. The action of T^n on X is said to be *locally standard* if every point in X lies in some standard coordinate neighborhood.

Example 2.2. T^2 acts on a four-dimensional sphere $S^4 := \{(z,y) \in \mathbb{C}^2 \times \mathbb{R}\colon |z_1|^2+|z_2|^2+y^2=1\}$ by $u\cdot(z,y) := (u_1z_1, u_2z_2, y)$. This action is locally standard. More generally, an effective smooth T^2-action on a 4-dimensional smooth manifold X without nontrivial finite stabilizers are locally standard because of the slice theorem. See [2, Chapter 8] for the slice theorem. These actions has been studied by Orlik–Raymond in [14].

Example 2.3 (Nonsingular toric varieties). An n-dimensional complex toric variety is a normal complex algebraic variety X of dimension n with a $(\mathbb{C}^*)^n$-action having a dense orbit. T^n acts on X as a subgroup of $(\mathbb{C}^n)^*$. If X is nonsingular, the T^n-action on X is locally standard. In fact, the fundamental theorem of the toric theory says that there is a one-to-one correspondence between toric varieties and fans. Top-dimensional cones in the fan associated with X correspond to standard coordinate neighborhoods all of which covers X since all cones are nonsingular. For toric varieties, see [5, 9, 13].

Example 2.4 (Quasi-toric manifolds). A *quasi-toric manifold* is a smooth manifold equipped with a locally standard torus action whose orbit space is combinatorially isomorphic to a simple convex polytope. A quasi-toric manifold was first introduced by Davis–Januszkiewicz in their pioneer work [6] as a topological generalization of a projective toric variety. See [6, 4] for more details.

Let X be a $2n$-dimensional manifold equipped with a locally standard T^n-action. Let $B := X/T^n$ denote the orbit space and $\mu\colon X \to B$ the quotient projection.

Proposition 2.5. *B is a topological manifold with corners. Namely, on B there is a system of coordinate neighborhoods modeled on open subsets of $\mathbb{R}^n_+$ so that overlap maps are homeomorphisms which preserve the stratifications induced from the natural stratification of $\mathbb{R}^n_+$.*

In particular, B has a natural stratification. Let $\mathcal{S}^{(k)}B$ be the k-dimensional strata of B with respect to the natural stratification, namely, $\mathcal{S}^{(k)}B$ consists of those points which have exactly k nonzero components in a local coordinate. The closure of a connected component of the codimension one strata $\mathcal{S}^{(n-1)}B$ is called a *facet*. Let $B_1, \ldots,$ B_m be facets of B. By definition, for each i the preimage $\mu^{-1}(B_i)$ of B_i is fixed by a circle subgroup of T^n, say T_i. Let Λ be the lattice of integral elements of the Lie algebra $\mathfrak{t}$ of T^n, namely, $\Lambda := \{t \in \mathfrak{t}\colon \exp(t) = 1\}$. We denote by L_i the rank one sublattice of Λ spanned by the primitive vector in Λ which generates T_i. Hence we can obtain the map λ from the set of facets to the set of rank one sublattices in Λ. λ is called the *characteristic function* of X.

Example 2.6. Let X be a nonsingular toric variety. The one-dimensional cones in the fan associated with X corresponds one-to-one to the facets of B. Then λ can be defined by assigning to each facet of B the rank one sublattice spanned by the primitive vector generating the corresponding one-dimensional cone.

Definition 2.7. Let $\{L_1, \dots, L_m\}$ be an m-tuple of rank one sublattices of Λ. $\{L_1, \dots, L_m\}$ is said to be *unidomular*, if the sublattice $L_1 + \cdots + L_m$ generated by $L_1, \dots, L_m$ is a rank m direct summand of Λ as a free $\mathbb{Z}$-module.

The following lemma follows immediately from the local standardness of X.

Lemma 2.8. *If the intersection $B_{i_1} \cap \cdots \cap B_{i_k}$ is non-empty, then $\{\lambda(B_{i_1}), \dots, \lambda(B_{i_k})\}$ is unimodular.*

Given a point $b \in B$, suppose that b lies in $\mathcal{S}^{(k)}B$. Then there are exactly $n-k$ facets B_{i_1}, …, $B_{i_{n-k}}$ such that $b \in B_{i_1} \cap \cdots \cap B_{i_{n-k}}$. Let $T(b)$ be the subtorus of T^n generated by $\lambda(B_{i_1}), \dots, \lambda(B_{i_{n-k}})$. Notice that by Lemma 2.8 $T(b)$ is $(n-k)$-dimensional. Now introduce the identification space

$$X_\lambda := B \times T^n / \sim,$$

where $(b,u) \sim (b',u')$ if and only if $b' = b$ and $u'u^{-1} \in T(b)$. The natural T^n-action on $B \times T^n$ descends to an action of T^n on X_λ whose orbit space is B, and the natural projection $B \times T^n \to B$ also descends to the orbit map $\mu_\lambda\colon X_\lambda \to B$. It is easy to see that X_λ is a topological manifold and the T^n-action is locally standard. X_λ is called the *canonical model* of X.

By the construction, X_λ is locally equivariantly homeomorphic to X, namely, there is an open covering $\{U_\alpha\}$ of B such that $\mu^{-1}(U_\alpha)$ is equivariantly homeomorphic to $\mu_\lambda^{-1}(U_\alpha)$ for each α. We take an equivariant homeomorphism $h_\alpha\colon \mu^{-1}(U_\alpha) \to \mu_\lambda^{-1}(U_\alpha)$ for each α. Suppose that the overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$ of U_α and U_β is nonempty. Let $b \in U_{\alpha\beta}$. For any $x \in \mu_\lambda^{-1}(b)$, since h_α's are equivariant $h_\alpha \circ h_\beta^{-1}(x)$ lies in the same orbit of x by the T^n-action. This implies that there exists an element u of T^n such that

$$h_\alpha \circ h_\beta^{-1}(x) = u \cdot x.$$

u is unique modulo $T(b)$. By using the equivariantness of h_α's we can also show that u does not depend on the choice of x and depends on b. We denote u by $\theta_{\alpha\beta}(b)$. $\theta_{\alpha\beta}(b)$ induces a section $\theta_{\alpha\beta}$ of $\mu_\lambda\colon X_\lambda \to B$ on $U_{\alpha\beta}$. Let $\mathscr{S}_{X_\lambda}$ denote the sheaf of germs of continuous sections of $\mu_\lambda\colon X_\lambda \to B$. It is easy to see that the local sections $\theta_{\alpha\beta}$ form a Čech one-cocycle $\{\theta_{\alpha\beta}\}$ on $\{U_\alpha\}$ with values in $\mathscr{S}_{X_\lambda}$. Hence it defines a cohomology class $e_{orbit}(X) \in H^1(B; \mathscr{S}_{X_\lambda})$. It is easy to see that $e_{orbit}(X)$ does not depend on the choice of h_α's. $e_{orbit}(X)$ is called the *Euler class of the orbit map.*

Example 2.9. For a nonsingular toric variety X, $e_{orbit}(X)$ vanishes. See [16].

Example 2.10. For a quasi-toric manifold X, $e_{orbit}(X)$ vanishes. See [6].

Theorem 2.11 ([16]). *Let X_1 and X_2 be $2n$-dimensional manifolds equipped with locally standard T^n-actions. X_1 and X_2 are equivariantly homeomorphic if and only if the orbit spaces X_1/T^n and X_2/T^n are homeomorphic as manifolds with corners and under this identification, the characteristic functions and the Euler classes of the orbit maps are same.*

This is a generalization of the topological classification of quasi-toric manifolds by Davis–Januszkiewicz [6] and of effective T^2-actions on four-dimensional manifolds without nontrivial finite stabilizers by Orlik–Raymond [14].

The idea of the proof is as follows. The "only if" part is obvious. Suppose that X_1/T^n and X_2/T^n are homeomorphic as manifolds with corners and under this identification, X_1 and X_2 have the same characteristic functions. Then the canonical models are same. By definition, $e_{orbit}(X_i)$ measures the difference between X_i and its canonical model. So if $e_{orbit}(X_1) = e_{orbit}(X_2)$, then the differences are same. Hence, X_1 is equivariantly homeomorphic to X_2. For more details, see [16].

§3. Locally toric Lagrangian fibrations

Let $\mathrm{Aut}(T^n)$ be the group of automorphisms of T^n. $\mathrm{Aut}(T^n)$ can be identified with $\mathrm{GL}_n(\mathbb{Z})$ because of the decomposition $T^n = (S^1)^n$. Let (X, ω) be a $2n$-dimensional symplectic manifold and B an n-dimensional manifold with corners.

Definition 3.1 ([10]). A map $\mu\colon (X, \omega) \to B$ is called a *locally toric Lagrangian fibration* if there exists a system $\{(U_\alpha, \varphi_\alpha^B)\}$ of coordinate neighborhoods of B modeled on $\mathbb{R}^n_+$, and for each α there exists a symplectomorphism $\varphi_\alpha^X\colon (\mu^{-1}(U_\alpha),\omega) \to (\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha)),\omega_{\mathbb{C}^n})$ such that $\mu_{\mathbb{C}^n}\circ\varphi_\alpha^X = \varphi_\alpha^B\circ\mu$.

A locally toric Lagrangian fibration is a natural generalization of a moment map of a nonsingular projective toric variety. In the case of $\partial B = \emptyset$, it is a nonsingular Lagrangian fibration. Conversely, by the Arnold–Liouville theorem [1], a nonsingular Lagrangian fibration with closed connected fibers on a closed manifold is also such an example.

Let $\mu\colon (X,\omega) \to B$ be a locally toric Lagrangian fibration on an n-dimensional base B and $\{(U_\alpha,\varphi_\alpha^B,\varphi_\alpha^X)\}$ the atlas in Definition 3.1.

Lemma 3.2. *On each connected component of a nonempty overlap $U_{\alpha\beta} := U_\alpha\cap U_\beta$ there exists an automorphism $\rho_{\alpha\beta}\in \mathrm{Aut}(T^n)$ and there also exists a constant $c_{\alpha\beta}\in\mathbb{R}^n$ such that the overlap map $\varphi_\alpha^X\circ(\varphi_\beta^X)^{-1}$ on the total space X is $\rho_{\alpha\beta}$-equivariant with respect to the standard representation of T^n and the overlap map $\varphi_{\alpha\beta}^B := \varphi_\alpha^B\circ(\varphi_\beta^B)^{-1}$ on the base is of the form*

$$\varphi_{\alpha\beta}^B(\xi) = {}^t\rho_{\alpha\beta}^{-1}(\xi) + c_{\alpha\beta}, \tag{3.1}$$

where ${}^t\rho_{\alpha\beta}^{-1}$ is the inverse transpose of $\rho_{\alpha\beta}$.

For the proof, see [16] and see also [8, 15] for nonsingular Lagrangian fibrations.

Definition 3.3. The atlas $\{(U_\alpha^B,\varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ of B in Lemma 3.2 is called an *integral affine structure.*

By (3.1) the structure group of the cotangent bundle T^*B reduces to $\mathrm{GL}_n(\mathbb{Z})$ and the maps $\rho_{\alpha\beta}$ are nothing but the transition functions of T^*B. We denote the frame bundle of T^*B by $\pi_{P_X} : P_X \to B$ and also denote the associated Λ-bundle and T^n-bundle by $\pi_{\Lambda_X} : \Lambda_X \to B$ and $\pi_{T_X} : T_X \to B$, respectively. Then we have the following exact sequence of associated fiber bundles of P_X

$$0 \longrightarrow \Lambda_X \longrightarrow T^*B \longrightarrow T_X \longrightarrow 0.$$

As is well-known, T^*B is equipped with the standard symplectic structure, and it is easy to see that the standard symplectic structure on T^*B descends to the symplectic structure on T_X, which is denoted by ω_{T_X}, so that $\pi_{T_X} : (T_X,\omega_{T_X}) \to B$ is a nonsingular Lagrangian fibration.

For any point b of B, let $(U_\alpha,\varphi_\alpha^B)$ be a coordinate neighborhood of the integral affine structure which contains b. Suppose that b lies in $\mathcal{S}^{(k)}B$. Then the stabilizer of the T^n-action on $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(b))$ is an $(n-k)$-dimensional subtorus and by Lemma 3.2 it defines a unique

$(n-k)$-dimensional subtorus of the fiber $\pi_{T_X}^{-1}(b)$ of $\pi_{T_X}: T_X \to B$ at b which is denoted by Z_b. Notice that a fiber of $\pi_{T_X}: T_X \to B$ admits a group structure since its structure group is $\mathrm{GL}_n(\mathbb{Z})$. We define the equivalence relation $\sim$ on T_X by $t \sim t'$ if and only if $\pi_{T_X}(t) = \pi_{T_X}(t')$ and $t't^{-1} \in Z_{\pi_{T_X}(t)}$, and denote the quotient space with respect to $\sim$ by X_{can}. By the construction of X_{can} the bundle projection π_{T_X} descends to the projection $\mu_{can}: X_{can} \to B$.

Lemma 3.4 ([16]). *X_{can} becomes a $2n$-dimensional smooth manifold. Moreover, ω_{T_X} induces a symplectic structure ω_{can} on X_{can} so that $\mu_{can}: (X_{can}, \omega_{can}) \to B$ is a locally toric Lagrangian fibration.*

Roughly speaking, the proof is as follows. The integral affine structure defines a Hamiltonian action of a subtorus of T^n on each $\pi_{T_X}^{-1}(U_\alpha^B)$. (X_{can}, ω_{can}) can be obtained from (T_X, ω_{T_X}) by the symplectic cutting technique with respect to these Hamiltonian torus actions [11]. For more details, see [16].

By the construction of $\mu_{can}: (X_{can}, \omega_{can}) \to B$, it is locally isomorphic to the original one $\mu: (X, \omega) \to B$, namely, on each U_α there is a fiber-preserving symplectomorphism $h_\alpha: (\mu^{-1}(U_\alpha), \omega) \to (\mu_{can}^{-1}(U_\alpha), \omega_{can})$ covering the identity on U_α. By the similar argument used in Section 2, we can show that on each nonempty overlap $U_{\alpha\beta}$ the equation

$$h_\alpha \circ h_\beta^{-1}(x) = \theta_{\alpha\beta}(b) \cdot x$$

for $b \in U_{\alpha\beta}$ and $x \in \mu_{can}^{-1}(b)$ determines a section $\theta_{\alpha\beta}$ of $\pi_{T_X}: (T_X, \omega_{T_X}) \to B$ on $U_{\alpha\beta}$ such that $\theta_{\alpha\beta}^*\omega_{T_X}$ vanishes (see [16, Section 7] for more details). Such a section is called a *Lagrangian section.* Let $\mathscr{S}_{T_X}^{Lag}$ denote the sheaf of germs of Lagrangian sections of $\pi_{T_X}: (T_X, \omega_{T_X}) \to B_X$. It is easy to see that the local sections $\theta_{\alpha\beta}$ form a Čech one-cocycle $\{\theta_{\alpha\beta}\}$ on $\{U_\alpha\}$ with values in $\mathscr{S}_{T_X}^{Lag}$. Hence it defines a cohomology class in $H^1(B_X; \mathscr{S}_{T_X}^{Lag})$. We denote it by $\lambda(X)$. It is easy to see that $\lambda(X)$ does not depend on the choice of h_α's. $\lambda(X)$ is called a *Lagrangian class* of $\mu: (X, \omega) \to B$.

Theorem 3.5 ([3]). *Let $\mu_1: (X_1, \omega_1) \to B_1$ and $\mu_2: (X_2, \omega_2) \to B_2$ be locally toric Lagrangian fibrations. They are fiber-preserving symplectomorphic if and only if there is a diffeomorphism between B_1 and B_2 which preserves the integral affine structures and under this identification, $\lambda(X_1)$ and $\lambda(X_2)$ are same.*

For the proof, see [3, 16]. The idea of the proof is same as Theorem 2.11. It is a generalization of the classification of nonsingular Lagrangian fibrations by Duistermaat [8] and the classification of symplectic toric manifolds by Delzant [7]. See also [15, 17] for the classifications of Lagrangian fibrations.

§4. Local torus actions modeled on the standard representation of T^n

Lemma 3.2 says that the total space of a locally toric Lagrangian fibration has an atlas similar to a standard atlas, but which satisfies a weaker condition than that of a standard atlas. As a formulation of such an underlying structure, in [16] we introduced the following notion.

Definition 4.1. Let X be a paracompact, Hausdorff space. A *weakly standard C^r ($0 \leq r \leq \infty$) atlas* of X is an atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$ which satisfies the following properties

(1) for each α, φ_α^X is a homeomorphism from U_α^X to an open set of $\mathbb{C}^n$ invariant under the standard representation of T^n,
(2) for each connected component of a nonempty overlap $U_{\alpha\beta}^X := U_\alpha^X \cap U_\beta^X$,
  (a) $\varphi_\alpha^X(U_{\alpha\beta}^X)$ and $\varphi_\beta^X(U_{\alpha\beta}^X)$ are also invariant under the standard representation of T^n and
  (b) there exists an automorphism $\rho_{\alpha\beta} \in \mathrm{Aut}(T^n)$ such that the overlap map $\varphi_{\alpha\beta}^X := \varphi_\alpha^X \circ (\varphi_\beta^X)^{-1}$ is $\rho_{\alpha\beta}$-equivariant C^r diffeomorphic with respect to the restrictions of the standard representation of T^n to $\varphi_\alpha^X(U_{\alpha\beta}^X)$ and $\varphi_\beta^X(U_{\alpha\beta}^X)$.

Two weakly standard C^r atlases $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$ and $\{(V_\beta^X, \psi_\beta^X)\}_{\beta\in\mathcal{B}}$ of X^{2n} are *equivalent* if on each connected component of a nonempty overlap $U_\alpha^X \cap V_\beta^X$, there exists an automorphism ρ of T^n such that $\varphi_\alpha^X \circ (\psi_\beta^X)^{-1}$ is ρ-equivariant C^r diffeomorphic. We call an equivalence class of weakly standard C^r atlases a *C^r local T^n-action on X^{2n} modeled on the standard representation* or a local T^n-action on X if there are no confusions and denote it by $\mathcal{T}$.

Definition 4.2. Let $(X_i, \mathcal{T}_i)$ $(i = 1, 2)$ be a $2n$-dimensional manifold equipped with a C^r local T^n-action $\mathcal{T}_i$, and let $\{(U_\alpha^{X_1}, \varphi_\alpha^{X_1})\}_{\alpha\in\mathcal{A}} \in \mathcal{T}_1$ and $\{(U_\beta^{X_2}, \varphi_\beta^{X_2})\}_{\beta\in\mathcal{B}} \in \mathcal{T}_2$ be the maximal weakly standard atlases of X_1 and X_2, respectively. $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are said to be *C^r isomorphic* if there exists a C^r diffeomorphism $f_X\colon X_1 \to X_2$, and there exists an automorphism ρ of T^n on each nonempty overlap $U_\alpha^{X_1} \cap f_X^{-1}(U_\beta^{X_2}) \neq \emptyset$

such that $\varphi_\beta^{X_2} \circ f_X \circ (\varphi_\alpha^{X_1})^{-1}$ is ρ-equivariant. f_X is called a C^r *isomorphism* and we denote it by $f_X\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$.

Let $(X, \mathcal{T})$ be a $2n$-dimensional manifold X equipped with a C^r local T^n-action $\mathcal{T}$ and $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ a maximal weakly standard atlas of X which belongs to $\mathcal{T}$. For $(X, \mathcal{T})$ we can define the orbit space B_X by patching $\varphi_\alpha^X(U_\alpha^X)/T^n$s by the homeomorphisms induced by the overlap maps $\varphi_{\alpha\beta}^X$. The orbit map is defined by the obvious way and we denote it by $\mu_X\colon X \to B_X$. It is easy to see that $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ endows B_X with an n-dimensional topological manifold with corners.

A typical example of a manifold equipped with a local torus action is a locally standard torus action. But not all local torus actions are induced by locally standard torus actions. For any C^r local T^n-action $\mathcal{T}$ on a $2n$-dimensional manifold X, we take a weakly standard atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ belonging to $\mathcal{T}$. It is easy to see that on each $U_{\alpha\beta}^X$ the automorphisms $\rho_{\alpha\beta}$ in (2) of Definition 4.1 can be thought of as a map $\rho_{\alpha\beta}\colon \mu_X(U_{\alpha\beta}^X) \to \mathrm{Aut}(T^n)$ and $\rho_{\alpha\beta}$s define a cohomology class $[\{\rho_{\alpha\beta}\}]$ in the first Čech cohomology set $H^1(B_X; \mathrm{Aut}(T^n))$ of B_X with values in $\mathrm{Aut}(T^n)$.

Proposition 4.3 ([16]). *A C^r local T^n-action on X is induced by some C^r locally standard T^n-action if and only if $\{\rho_{\alpha\beta}\}$ and the trivial Čech one-cocycle are of the same equivalence class in $H^1(B_X; \mathrm{Aut}(T^n))$, where the trivial Čech one-cocycle is the one whose values on all open set are equal to the identity map of T^n.*

Another important example of a manifold equipped with a local torus action is a locally toric Lagrangian fibration. For a manifold $(X, \mathcal{T})$ equipped with a C^∞ local T^n-action $\mathcal{T}$, X becomes the total space of a locally toric Lagrangian fibration if and only if there is an atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}} \in \mathcal{T}$ such that the induced atlas of B_X by $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ is an integral affine structure and X satisfies an additional condition. See [16] for more details.

Finally we generalize the topological classification of locally standard torus actions to local torus actions. The Čech one-cocycle $\{\rho_{\alpha\beta}\}$ determines a principal $\mathrm{Aut}(T^n)$-bundle $\pi_{P_X}: P_X \to B_X$. Note that when $(X, \mathcal{T})$ is induced by a locally standard torus action, by Proposition 4.3 P_X is the trivial bundle $P_X = B_X \times \mathrm{Aut}(T^n)$ and when $(X, \mathcal{T})$ is an underlying structure of a locally toric Lagrangian fibration, P_X is nothing but the frame bundle of the cotangent bundle of the base. Let $\pi_{\Lambda_X}: \Lambda_X \to B_X$ and $\pi_{T_X}: T_X \to B_X$ be the associated Λ-bundle and T^n-bundle of P_X, respectively. In the case of $(X, \mathcal{T})$ T_X acts fiberwise on X, hence the characteristic function of a locally standard torus action is generalized to a rank one subbundle, called the *characteristic bundle* and denoted by $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B_X$, of the restriction of $\pi_{\Lambda_X} : \Lambda_X \to B_X$ to $\mathcal{S}^{(n-1)}B_X$. Notice that when $(X, \mathcal{T})$ is an underlying structure of a locally toric Lagrangian fibration, $\mathcal{L}_X$ is automatically determined by the integral affine structure. We also call the pair $(P_X, \mathcal{L}_X)$ of P_X and $\mathcal{L}_X$ the *characteristic pair*. By the same way as in the case of locally toric Lagrangian fibrations or locally standard torus actions we can construct the canonical model $X_{(P_X,\mathcal{L}_X)}$ from T_X by using $(P_X, \mathcal{L}_X)$. $X_{(P_X,\mathcal{L}_X)}$ is equipped with a C^0 local T^n-action whose orbit space is equal to B_X. By the construction of $X_{(P_X,\mathcal{L}_X)}$, X is locally C^0 isomorphic to $X_{(P_X,\mathcal{L}_X)}$ (for C^r isomorphisms see [16]). By the same way as before we can generalize the Euler class of the orbit map $e_{orbit}(X) \in H^1(B_X; \mathscr{S}_{X_{(P_X,\mathcal{L}_X)}})$ as a Čech one cohomology class of B_X with values in the sheaf of germs of continuous sections of the orbit map $\mu_{(P_X,\mathcal{L}_X)} : X_{(P_X,\mathcal{L}_X)} \to B_X$ of the canonical model.

Theorem 4.4 ([16]). *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two manifolds equipped with local torus actions. They are C^0 isomorphic if and only if B_{X_1} and B_{X_2} are homeomorphic as manifold with corners and under this identification, the characteristic pairs and the Euler classes of the orbit maps are same.*

The idea of the proof is same as Theorem 2.11.

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