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On ideal boundaries of some Coxeter groups

Saeko Yamagata

Abstract.

If a group acts geometrically (i.e., properly discontinuously, cocompactly and isometrically) on two geodesic spaces X and X', then an automorphism of the group induces a quasi-isometry $X \to X'$. We find a geometric action of a Coxeter group W on a CAT(0) space X and an automorphism ϕ of W such that the quasi-isometry $X \to X$ arising from ϕ can not induce a homeomorphism on the boundary of X as in the case of Gromov-hyperbolic spaces.

§1. Introduction

In the study of Gromov-hyperbolic spaces, it is well-known that for two proper Gromov-hyperbolic geodesic spaces X, X', if there exists a quasi-isometry $F : X \to X'$, then it induces a homeomorphism between their ideal boundaries ([BH, III.H.3.9]). We explain the homeomorphism between their ideal boundaries. For a geodesic ray γ in X there always exists a geodesic ray γ' such that the Hausdorff distance between $F(\gamma)$ and γ' is finite, therefore we define a map $\overline{F} : \partial X \ni \gamma(\infty) \mapsto \gamma'(\infty) \in \partial X'$. Here, we denote by $\gamma(\infty)$ the equivalence class of a geodesic ray γ. Then the map $\overline{F}$ is a homeomorphism between the ideal boundaries.

In the case of CAT(0) spaces, Croke–Kleiner [CK] proved that there exists a group acting geometrically on two CAT(0) spaces whose ideal boundaries are not homeomorphic to each other. Bowers–Ruane [BR] found two distinct geometric actions of $F_2 \times \mathbb{Z}$ on a CAT(0) space X and a quasi-isometry $F : X \to X$ (which is equivariant under the two actions) such that there exists a geodesic ray γ in X whose image $F(\gamma)$ does not have finite Hausdorff distance from any geodesic ray in X. Therefore, F can not induce a homeomorphism on ∂X in the same way as in the case of Gromov-hyperbolic spaces.

On the other hand, it is known that Coxeter groups act geometrically on some CAT(0) spaces ([M]). Let W be a Coxeter group having a

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presentation

$$W = \langle t_1, \dots, t_5 \,|\, t_i^2 = e \;\; (i = 1, \dots, 5),\, t_j t_k = t_k t_j \;\; (j = 1, 2, 3,\, k = 4, 5)\, \rangle,$$

and let (X, d) be the CAT(0) space defined in [M] on which W acts geometrically. Let ϕ be an automorphism on W defined by

$$t_i \mapsto t_i \quad (i \neq 3), \qquad t_3 \mapsto t_1 t_3 t_1.$$

We give W a word metric d_S associated to the generating set $S = \{t_1, t_2, \dots, t_5\}$. Then for any choice of a basepoint $x_0 \in X$, there exists a quasi-isometry $f : (W, d_S) \ni w \mapsto w \cdot x_0 \in (X, d)$ ([BH, I.8.19]), and the automorphism $\phi : W \to W$ is in fact a quasi-isometry $(W, d_S) \to (W, d_S)$. Therefore, $F = f \circ \phi \circ f^{-1} : (X, d) \to (X, d)$ is also a quasi-isometry. In this paper, we will prove the following theorem.

Theorem 1.1. *We have a geodesic ray γ in X such that there exist no geodesic rays in X whose Hausdorff distance from $F(\gamma)$ is finite.*

By Theorem 1.1 we know that the quasi-isometry $F : X \to X$ can not induce a homeomorphism $\partial X \to \partial X$ in the same way as in the case of Gromov-hyperbolic spaces.

§2. CAT(0) spaces and Coxeter groups

We shall recall terminologies about CAT(0) spaces and Coxeter groups. We refer to [BH] about CAT(0) spaces.

Definition 2.1. For a metric space (X, d), a *geodesic* from $x \in X$ to $y \in X$ is a map $\gamma : [0, l] \to X$ such that

$$l = d(x, y),\ \gamma(0) = x,\ \gamma(l) = y,$$

$$d(\gamma(t), \gamma(t')) = |\, t - t' \,| \quad (\forall t,\, t' \in [0, l]).$$

We denote the image in X of a geodesic from x to y by $[x, y]$ if we do not specify a choice of such geodesics joining x and y, and call it a *geodesic segment.* We call (X, d) a *geodesic space* if every two points in X can be joined by a (not necessarily unique) geodesic.

Definition 2.2. Given a geodesic space (X, d) and $a, b, c \in X$, we denote by $\triangle(a, b, c)$ a geodesic triangle whose vertexes are a, b, c, and sides are geodesic segments $[a, b]$, $[b, c]$, $[c, a]$.

For any geodesic triangle $\triangle(a, b, c)$ in X, we can construct a geodesic triangle $\overline{\triangle}(\overline{a}, \overline{b}, \overline{c})$ in the 2-dimensional Euclidean spase $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{a}, \overline{b}) = d(a, b)$, $d_{\mathbb{E}^2}(\overline{b}, \overline{c}) = d(b, c)$ and $d_{\mathbb{E}^2}(\overline{c}, \overline{a}) = d(c, a)$. Here, $d_{\mathbb{E}^2}$

is a standard metric on $\mathbb{E}^2$. We call $\overline{\triangle}(\overline{a}, \overline{b}, \overline{c})$ a *comparison triangle* of $\triangle(a, b, c)$.

Let x be a point in $[a, b]$. A point $\overline{x}$ in $[\overline{a}, \overline{b}]$ is called a *comparison point* of x if $d_{\mathbb{E}^2}(\overline{a}, \overline{x}) = d(a, x)$. In the case of $x \in [b, c]$ or $x \in [c, a]$, we define a comparison point of x in the same way.

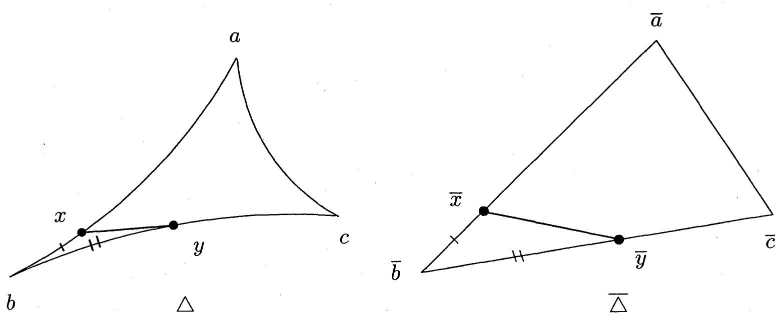

Fig. 1. A geodesic triangle and its comparison triangle

Definition 2.3. Let $\triangle$ be a geodesic triangle in a geodesic space (X, d), and $\overline{\triangle}$ a comparison triangle of $\triangle$. If for any $x, y \in \triangle$ and their comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, the inequality

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$$

holds, then we call (X, d) a *CAT(0) space.*

It is easy to see that for any points x, y in a CAT(0) space, there exists a unique geodesic joining x and y.

Definition 2.4. For a metric space (X, d), we call (X, d) a *proper* metric space if for every $x \in X$ and every $r > 0$, the closed ball $\overline{B}(x, r)$ is compact.

Let (X, d) be a proper CAT(0) space. If a map $\gamma : [0, \infty) \to X$ satisfies

$$d(\gamma(t), \gamma(t')) = |t - t'| \quad (\forall t, t' \in [0, \infty)), \quad \gamma(0) = x_0,$$

then γ is called a *geodesic ray* from x_0.

Two geodesic rays $\gamma, \gamma' : [0, \infty) \to X$ are said to be *asymptotic* if there exists a constant K such that $d(\gamma(t), \gamma'(t)) \leq K$ for all $t \geq 0$. We give an equivalence relation on the set of geodesic rays in X such that two geodesic rays are equivalent if and only if they are asymptotic. We denote by ∂X the set of equivalence classes of geodesic rays in X, and give the cone topology on ∂X (see [BH, II.8.6] for the definition of the topology).

Definition 2.5. Let (X_1, d_1) and (X_2, d_2) be complete CAT(0) spaces, X the product $X_1 \times X_2$, and define a metric d on X by $d = \sqrt{d_1^2 + d_2^2}$. Let $\gamma_1(\infty)$ (resp. $\gamma_2(\infty)$) be the equivalence class of a geodesic ray γ_1 in X_1 (resp. γ_2 in X_2).

If $\theta \in [0, \pi/2]$, we denote by $(\cos\theta)\gamma_1(\infty) + (\sin\theta)\gamma_2(\infty)$ the point of ∂X represented by the geodesic ray $\gamma(t) = (\gamma_1(t\cos\theta), \gamma_2(t\sin\theta))$ in X. The *spherical join* $\partial X_1 * \partial X_2$ is the quotient of the product $\partial X_1 \times [0, \pi/2] \times \partial X_2$ by the equivalence relation identifying $(\gamma_1(\infty), \theta, \gamma_2(\infty))$ with $(\gamma_1'(\infty), \theta', \gamma_2'(\infty))$ if and only if either of the following conditions are satisfied:

(1) $\gamma_1(\infty) = \gamma_1'(\infty), \theta = \theta'$ and $\gamma_2(\infty) = \gamma_2'(\infty)$;
(2) $\theta = \theta' = 0$ and $\gamma_1(\infty) = \gamma_1'(\infty)$;
(3) $\theta = \theta' = \pi/2$ and $\gamma_2(\infty) = \gamma_2'(\infty)$.

It is easy to see that the boundary ∂X is homeomorphic to the spherical join $\partial X_1 * \partial X_2$.

Definition 2.6. Let (X, d) be a metric space. For a subset $A \subset X$ and a positive number k, we denote the k-neighbourhood of A by

$$\mathcal{N}_k(A) = \{x \in X \mid \exists a \in A \quad \text{s.t. } d(x, a) \leq k\}.$$

For subsets $A, B \subset X$, the *Hausdorff distance* between A and B is defined by

$$d_H(A, B) = \inf\{k \mid A \subseteq \mathcal{N}_k(B),\ B \subseteq \mathcal{N}_k(A)\}.$$

Definition 2.7. Let (X, d) and (X', d') be metric spaces. If a map $f : X \to X'$ satisfies that there exist ε, $k \geq 0$, $\lambda \geq 1$ such that

$$\frac{1}{\lambda} d(x, y) - \varepsilon \leq d'(f(x), f(y)) \leq \lambda d(x, y) + \varepsilon \quad (\forall x,\ y \in X),$$

$$\mathcal{N}_k(\mathrm{Im} f) = X',$$

then f is called a (λ, ε)-*quasi-isometry.* If we do not specify the values λ, ε, then we call f a quasi-isometry simply.

We note that if there exists a (λ, ε)-quasi-isometry $f : X \to X'$, then there exists a (λ', ε')-quasi-isometry $f^{-1} : X' \to X$ (for some λ', ε') and a constant $k' \geq 0$ such that $d(f \circ f^{-1}(x'), x') \leq k'$ and $d(f^{-1} \circ f(x), x) \leq k'$ for all $x' \in X'$ and all $x \in X$. We call f^{-1} a *quasi-inverse* for f.

Finally, we recall the definition of Coxeter groups.

Definition 2.8. A *Coxeter group* W is a finitely presented group having the following presentation:

$$W = \langle\, S \mid (ss')^{m(s,s')} = e \quad \text{for} \quad \forall s, s' \in S \,\rangle,$$

where S is a non-empty finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,s) = 1$ for $\forall s \in S$;
(2) $m(s,s') = m(s',s)$ for $\forall s, s' \in S$;
(3) $m(s,s') \geq 2$ for $\forall s \neq s' \in S$.

Here, for $s, s' \in S$, $m(s,s') = \infty$ means that there exists no relation between s and s'.

§3. Proof of the main theorem

In the following context, let W be the Coxeter group whose presentation is given by

$$W = \langle t_1, \ldots, t_5 \mid t_i^2 = e \ \ (i = 1, \ldots, 5),\ t_j t_k = t_k t_j \ \ (j = 1, 2, 3,\ k = 4, 5)\,\rangle.$$

Let H be the subgroup of W generated by t_1, t_2 and t_3, and let H' be the subgroup of W generated by t_4, t_5.

By the presentation of W, we know that

$$\begin{aligned} W &= H \times H' \\ &\cong (\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2)\,. \end{aligned}$$

Define an automorphism ϕ of W by

$$t_i \mapsto t_i \quad (i \neq 3), \qquad t_3 \mapsto t_1 t_3 t_1.$$

(Especially, ϕ is an isomorphism of the Coxeter system.)

Let T be the Cayley graph of the group H with respect to the generating set $\{t_1, t_2, t_3\}$, which is a regular tree of valence 3. The Cayley graph of the group H' with respect to a generating set $\{t_4, t_5\}$ is isometric to $\mathbb{R}$ where the vertex set of this graph corresponds to $\mathbb{Z}$. Therefore, we call this graph $\mathbb{R}$.

Let X be the product $T \times \mathbb{R}$. Let d_T (resp. $d_{\mathbb{R}}$) be a metric on the Cayley graph T (resp. $\mathbb{R}$). A metric d on X is defined by

$$d((t,r),(t',r')) = \sqrt{d_T(t,t')^2 + d_{\mathbb{R}}(r,r')^2} \quad (\forall t,\ t' \in T,\ \forall r,\ r' \in \mathbb{R}).$$

Then X is a proper CAT(0) space and is called the Davis–Vinberg complex of W. The Coxeter group W acts geometrically (i.e., properly discontinuously, cocompactly and isometrically) on X ([M]).

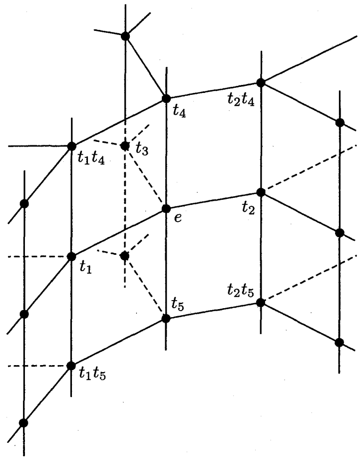

Fig. 2. $T \times \mathbb{R}$

We give W a word metric d_S with respect to the generating set $S = \{t_1, t_2, \dots, t_5\}$. Let $e \in X$ be the vertex corresponding to the unit element. Then there exists a quasi-isometry $f : (W, d_S) \ni w \mapsto w \cdot e \in X$ ([BH, I.8.19]). We can take a quasi-inverse $f^{-1} : X \to W$ satisfying that for any $w \in W$, $f^{-1}(w \cdot e) = w$.

The ideal boundary of T is a Cantor set and the ideal boundary of $\mathbb{R}$ consists of two points. Therefore, the ideal boundary of X is the spherical join of the Cantor set and the set of two points. Since the automorphism ϕ on W is in fact a quasi-isometry $(W, d_S) \to (W, d_S)$, and $f : (W, d_S) \to (X, d)$ is also a quasi-isometry, so is $F = f \circ \phi \circ f^{-1} : X \to X$.

Theorem 3.1. *We have a geodesic ray γ in X such that there exist no geodesic rays in X whose Hausdorff distance from $F(\gamma)$ is finite.*

Proof. Put $a = t_1t_2$, $b = t_3t_2$, $c = t_4t_5$ and $b' = t_1t_3t_1t_2$. We note that c commutes with a, b and b'. Then

$$F(a) = f \circ \phi \circ f^{-1}(a \cdot e) = f \circ \phi(a) = f(a) = a \cdot e = a,$$
$$F(b) = f \circ \phi \circ f^{-1}(b \cdot e) = f \circ \phi(b) = f(b') = b' \cdot e = b',$$
$$F(c) = f \circ \phi \circ f^{-1}(c \cdot e) = f \circ \phi(c) = f(c) = c \cdot e = c.$$

Let γ be a piecewise geodesic path in X such that

$$[e, ac] \cup [ac, abc^2] \cup [abc^2, abac^3] \cup [abac^3, ababc^4] \cup [ababc^4, abab^2c^5] \cup \dots$$

$$\cup[abab^2 \cdots ab^{n-1}c^{\frac{n(n+3)}{2}-1}, abab^2 \cdots ab^n c^{\frac{n(n+3)}{2}}] \cup \ldots.$$

The piecewise geodesic path γ is in fact a geodesic ray in X because the projection of γ onto T is a geodesic ray passing through e, a, ab, aba, $abab$, $abab^2$, ... ,$abab^2ab^3 \cdots ab^n$, ... , where the distance between successive two points is equal to 2, and the projection of γ onto $\mathbb{R}$ is also geodesic ray passing through e, c, c^2, ... ,c^n, ... , where the distance between successive two points is equal to 2.

Put $A_n = ab'ab'^2ab'^3 \cdots ab'^n c^{\frac{n(n+3)}{2}}$. Then $F(\gamma)$ passes through each A_n ($n \in \mathbb{N}$). We will deduce a contradiction under the assumption that there exists a geodesic ray γ' such that the Hausdorff distance between γ' and $F(\gamma)$ is finite.

For each $n \in \mathbb{N}$, the Hausdorff distance between γ' and a geodesic segment $[e, A_n]$ would be uniformly finite because $F(\gamma)$ passes through e and A_n.

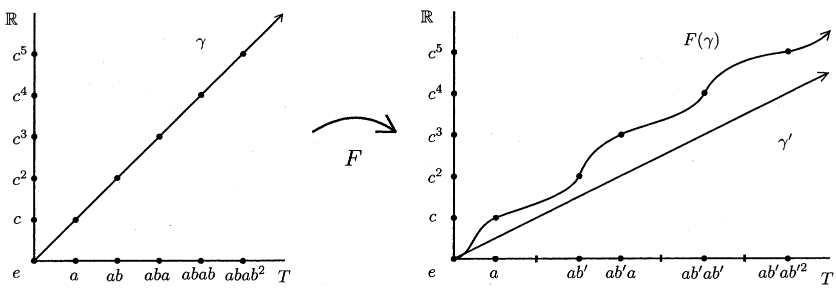

Fig. 3. γ and $F(\gamma)$

Next, we consider the slope of the geodesic segment $[e, A_n]$. Note that the projections of A_n onto T and $\mathbb{R}$ are equal to $ab'ab'^2ab'^3 \cdots ab'^n$ and $c^{\frac{n(n+3)}{2}}$, respectively. It is easy to see that

$$d_T(e, ab'ab'^2ab'^3 \cdots ab'^n) = 2n(n+2),$$

$$d_{\mathbb{R}}(e, c^{\frac{n(n+3)}{2}}) = n(n+3).$$

Hence the slope of the geodesic segment $[\,e,\, A_n\,]$ is $n(n+3)/2n(n+2)$. Then

$$\frac{n(n+3)}{2n(n+2)} \longrightarrow \frac{1}{2} \quad (n \to \infty).$$

Therefore, the slope of γ' should be $1/2$.

Finally, we calculate the distance between $A_n \in F(\gamma)$ and γ'. We take a geodesic ξ_n which passes through A_n and is orthogonal to γ'. The slope of ξ_n must be equal to -2. Let B_n be the intersection point of ξ_n and γ', which is the closest point on γ' to A_n. The distance between e and the projection of B_n onto T is equal to $2n(5n+11)/5$ and the distance between e and the projection of B_n onto $\mathbb{R}$ is equal to $n(5n+11)/5$. Therefore, the distance between A_n and B_n is equal to $2\sqrt{5}n/5$. Then

$$\frac{2\sqrt{5}}{5}n \longrightarrow \infty \quad (n \to \infty),$$

and therefore, the Hausdorff distance between γ' and $F(\gamma)$ must be infinite, which is a contradiction.

Consequently, we can not obtain a geodesic ray whose Hausdorff distance from $F(\gamma)$ is finite. Q.E.D.

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Mathematical Institute
Tohoku University
Aoba, Sendai, 980-8578
Japan

E-mail address: `sa1m28@math.tohoku.ac.jp`