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Upper curvature bounds and singularities

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Abstract.

The purpose of this note is to describe recent development on the study of the local structure of singular spaces with curvature bounded above, due to Lychak and Nagano[16], [17] for the characterization of topological manifolds, and due to Kleiner, Nagano, Shioya and Yamaguchi [15] for the general characterization of 2-dimensional spaces.

§1. Introduction

Let X be a geodesic metric space. By definition, every two points of X can be joined by a minimizing curve called a geodesic. Following A. D. Alexandrov, we say that X has *curvature bounded above* by a constant κ if and only if for any point $p \in X$ there is a small positive number r such that the metric r-ball $B(p, r)$ around p is convex and having the following properties: For any geodesic triangle $\triangle pqr$ in $B(p, r)$ with vertices p, q and r, we denote by $\tilde{\triangle} pqr$ a *comparison triangle* on the κ-plane, the simply connected complete surface of constant curvature κ, with the same side lengths as $\triangle pqr$. For every $x, y \in \triangle pqr$, let $\tilde{x}, \tilde{y} \in \tilde{\triangle} pqr$ be the points corresponding to x and y respectively. Then

$$d(x, y) \leq d(\tilde{x}, \tilde{y}). \tag{1.1}$$

A convex domain in X with this property is called a $CAT(\kappa)$*-domain.* Thus X has curvature $\leq \kappa$ if and only if every small geodesic triangle in X is thiner than the comparison triangle on the κ-plane.

For instance, a Riemannian manifold has curvature $\leq \kappa$ if and only if the sectional curvature is at most κ everywhere. Every metric graph has curvature $\leq \kappa$ for every κ.

There is a method to produce such spaces by Reshetnyak's gluing lemma ([20], see also [8]): suppose both X_1 and X_2 have curvature $\leq \kappa$

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and $D_1 \subset X_1$, $D_2 \subset X_2$ be closed convex subsets such that there is an isometry $f : D_1 \to D_2$, Then the result $X_1 \amalg_f X_2$ of gluing of X_1 and X_2 along f has curvature $\leq \kappa$.

One of the motivation of the study of this kind of singular spaces with curvature bounded above comes from geometric group theory. As a matter of fact, there is a lot of interesting discrete groups like Gromov's hyperbolic groups (see [11]), acting on such spaces. Another motivation is the importance of curvature itself. Historically, the notion of curvature has played crucial roles in geometry, even in mathematics. By studying this kind of singular spaces, we could know the essential feature of curvature being bounded above.

Here is our main

Problem 1.1. What can one say about the local structure of X? More precisely, for a given point $p \in X$, what can be said about the geometric structure of a small metric ball around p?

Here we assume that

(1) X is locally compact;
(2) X is geodesically complete, namely every geodesic in X can be locally extendable.

Those assumptions imply that X is finite dimensional. Actually, Otsu and Tanoue (see also Lytchak and Nagano's recent work [16]) proved that under those assumptions the Hausdorff dimension of every compact neighborhood of X must be an integer.

§2. Preliminaries

In this section, we present an example and some basic facts about spaces with curvature bounded above needed in the later sections. For details, we refer to [2], [7], [8] and [14].

Let us go back to Problem 1.1. If X is one-dimensional, then it is a graph and therefore locally a tree. If X is 2-dimensional however, there are non-triangulable such spaces, which was first found by Kleiner.

Example 2.1 (Kleiner, cf. [18]). First consider a smooth nonnegative function $f : \mathbb{R} \to \mathbb{R}_+$ with $\mathrm{supp}(f) \subset [0, 1]$ such that

(1) $f(x) > 0$ on $(0, 1) - \{x_n\}$ for a sequence $x_n \to 0$ of positive numbers;
(2) $f(x_n) = 0$;
(3) the absolute curvature of the graph of f is always less than a positive number ϵ;
(4) the length ℓ of the graph of $f|_{[0,1]}$ is less than $2\pi/\epsilon$.

Let $\Omega := \{(x,y) \mid |y| \le f(x), x \in \mathbb{R}\}$, equipped with the natural length metric induced from that of $\mathbb{R}^2$. We set $\partial_+\Omega := \{y = f(x)\}$, $\partial_-\Omega := \{y = -f(x)\}$, and let $\Gamma_+ \subset \mathbb{R}^2$ (resp. $\Gamma_- \subset \mathbb{R}^2$) be the graph of $+f|_{[0,1]}$ (resp. of $-f|_{[0,1]}$). One can take a closed domain H in $\mathbb{R}^2$ such that

(1) ∂H is smooth, connected and concave in the sense that the geodesic curvature from the side of H is nonpositive everywhere;
(2) the geodesic curvature of a subarc J of ∂H having length ℓ is less than $-\epsilon$ everywhere.

Take four copies $(H_1, J_1) \ldots, (H_4, J_4)$ of (H, J), and glue H_1, H_2 and Ω along their boundaries ∂H_1, ∂H_2 and $\partial_+\Omega$ in such a way that J_1 and J_2 are glued with Γ_+ in an obvious way. Similarly glue H_3, H_4 and Ω along their boundaries ∂H_3, ∂H_4 and $\partial H_-\Omega$. The result X of these gluings equipped with natural length metric is a 2-dimensional locally compact, geodesically complete $CAT(0)$-space. Let $\iota : \Omega \to X$ be the natural inclusion, and o the origin of Ω. Note that no neighborhood of $p := \iota(o)$ in X has polyhedral structure.

Let $\gamma : [0,a] \to X$ and $\sigma : [0,b] \to X$ be geodesics starting from $p \in X$. For each t consider the comparison triangle $\tilde{\triangle} p\gamma(t)\sigma(t) = \triangle \tilde{p}\tilde{\gamma}(t)\tilde{\sigma}(t)$, and denote by $\tilde{\theta}(t)$ the angle of $\tilde{\triangle} p\gamma(t)\sigma(t)$ at $\tilde{p}$. The curvature assumption implies that the function $t \to \tilde{\theta}(t)$ is monotone non-decreasing. The angle $\angle(\gamma, \sigma)$ between γ and σ is defined as

$$\angle(\gamma, \sigma) = \lim_{t \to 0} \tilde{\theta}(t).$$

Two geodesics γ and σ are defined to be equivalent if $\angle(\gamma, \sigma) = 0$. The *space of directions* of X at p, denoted by $\Sigma_p = \Sigma_p(X)$, is defined as the set of all equivalence classes of geodesics starting at p. Let $K_p = K_p(X)$ denote the Euclidean cone over $\Sigma_p(X)$, which is called the *tangent cone* at p: By definition, $K_p(X) = \Sigma_p \times [0,\infty)/\Sigma_p \times \{0\}$ equipped with the metric

$$d((\xi, t), (\eta, s)) = \sqrt{t^2 + s^2 - 2ts \cos \angle(\xi, \eta)}.$$

It is well known that the pointed space $(\frac{1}{r}X, p)$ converges to the pointed space $(K_p(X), o_p)$ with respect to the pointed Gromov–Hausdorff convergence (see Gromov [12]) as $r \to 0$, where $\frac{1}{r}X$ denotes the metric space $(X, \frac{1}{r}d)$, d is the original distance of X, and o_p is the vertex of the cone $K_p(X)$. Thus $K_p(X)$ as well as $\Sigma_p(X)$ provides infinitesimal information around p. Note that K_p is a $CAT(0)$-space and Σ_p is a $CAT(1)$-space.

In Example 2.1, $\Sigma_p(X)$ consists of four arcs of length π joining the two "singular directions". Note that the interior of $B(p,r)$ is not homeomorphic to $K_p(X)$ for any sufficiently small $r > 0$. This means that when we look at only the tangent cone, we miss some information in the metric ball.

At this stage, we would like to point out difference between the local properties of spaces with curvature bounded above and those of spaces with curvature bounded below. Here spaces with curvature $\geq \kappa$ are defined similarly: If we reverse the inequality (1.1), we get the notion of the spaces with curvature $\geq \kappa$. In [9], Burago, Gromov and Perelman established the basis of the study of those spaces with curvature bounded below. Further, Perelman's topological stability theorem ([19]. See also recent work [13] due to V. Kapovitch) implies that if X has curvature bounded below, then a small open metric ball around a given point $p \in X$ is homeomorphic to $K_p(X)$. This is no longer true for spaces with curvature bounded above.

§3. Homotopy types

In this section, we briefly discuss the properties concerning the homotopy type of a small metric ball $B(p,r)$ of a given point $p \in X$.

Because of the $CAT(\kappa)$-property of $B(p,r)$, every two points x and y in $B(p,r)$ can be joined by a unique minimizing geodesic. In particular, $B(p,r)$ is contractible.

Let $K_1(\Sigma_p) := \{(\xi,t) \in K_p \,|\, t \leq 1\}$ be the unit tangent cone at p. Recently by a more closer look at the convergence $(\frac{1}{r}B(p,r),p) \to (K_1(\Sigma_p), o_p)$, Lytchak and Nagano [16] have proved

Theorem 3.1 ([16]). *For every point $p \in X$, there is a positive number r_p such that for every $0 < r \leq r_p$, $\partial B(p,r)$ as well as $B(p,r) - \{p\}$ is homotopy equivalent to Σ_p.*

For the proof, one needs to construct a homotopy equivalence between $\partial B(p,r)$ and Σ_p. A natural map $\varphi : \partial B(p,r) \to \Sigma_p$ is defined by $\varphi(x) = \gamma'_{p,x}(0)$, where $\gamma_{p,x}$ is the unique shortest geodesic joining p to x. Since geodesics may branch however, the construction of the homotopy inverse of φ is non-trivial at all. They have actually showed that $\partial B(p,r)$ is locally contractible with control of the size of the contractible metric ball in $\partial B(p,r)$, and applied geometric topology (controlled topology) to the convergence $\partial B(p,r) \to \Sigma_p$.

The homotopy version of Problem 1.1 is now completely solved by Theorem 3.1. In later sections, we discuss the homeomorphism type of a small neighborhood of a given point.

§4. Characterization of topological manifolds

We now discuss a result due to Lytchak and Nagano [17] on a characterization for spaces with an upper curvature bound being topological manifolds.

Theorem 4.1 ([17]). *Let X be a locally compact, geodesically complete space with curvature bonded above. Then X is a topological n-manifold if and only if Σ_p is homotopy equivalent to S^{n-1} for each $p \in X$.*

In this result, one cannot replace "homotopy equivalent" by "homeomorphic" as the following example shows:

Example 4.2 ([17]). Let Σ^3 be the Poincaré homology three-sphere with positive constant curvature, and let X be the spherical double suspension over it:

$$X = \mathrm{Sus}(\mathrm{Sus}(\Sigma^3)).$$

By a celebrated Edward's double suspension theorem, X is a topological manifold. On the other hand, we may assume that Σ^3 is a $CAT(1)$-space. Therefore X as well as $\mathrm{Sus}(\Sigma^3)$ is again a $CAT(1)$-space. Let p_+, p_- be the vertices of X as a suspension over $\mathrm{Sus}(\Sigma^3)$. Then $\Sigma_{p_\pm} = \mathrm{Sus}(\Sigma^3)$ is not homeomorphic but homotopy equivalent to S^4.

To prove the theorem above, they developed a sort of "geometric deformation theory" to apply the so-called DDP-method, one of the typical method in geometric topology. Here DDP abbreviates the disjoint disk property that every two singular disks in the space can be slightly deformed to disjoint ones. DDP implies that the space must be a topological manifold if dimension of the space is at least five.

Problem 4.3. Let X be a locally compact, geodesically complete space with curvature bonded above. Is the following true? Every point $p \in X$ has a small *metric ball* $B(p, r)$ homeomorphic to n-disk if and only if Σ_p is homeomorphic to $(n-1)$-sphere for every $p \in X$.

§5. Homeomopshism types: The 2-dimensional case

In this section, we assume $\dim X = 2$ and consider the local structure of a small metric ball $B(p, r)$ of a given point $p \in X$. Note that there is a natural $1-1$ correspondence between the set of connected components of the punctured ball $B(p, r) - \{p\}$ and that of Σ_p. Since we are interested in the local structure of a 2-dimensional piece, we assume the most essential case when Σ_p is connected and one-dimensional. Since X

is geodesically complete, one can easily show that Σ_p is also geodesically complete. Note that any non-trivial geodesic loop in Σ_p has length $\geq 2\pi$. Thus Σ_p becomes a connected $CAT(1)$-graph having no endpoints.

Let $\mathcal{S}(X)$ denote the set of all points $x \in X$ any of whose neighborhoods is not homeomorphic to a disk. We call $\mathcal{S}(X)$ the singular locus of X.

We denote by D^2 the closed unit disk around the origin 0 on the plane $\mathbb{R}^2$. A map $f : D^2 \to B(p,r)$ is called *proper* if $f^{-1}(\partial B(p,r)) = \partial D^2$.

Theorem 5.1. *For every $p \in X$ such that Σ_p is a connected graph, there exists a positive number $r(p)$ such that for every $0 < r \leq r(p)$,*

(1) *$B(p,r)$ is the union of images $\mathrm{Im} f_i$ of finitely many proper Lipschitz immersions $f_i : D^2 \to B(p,r)$ with possible branch point $f_i^{-1}(p) = \{0\}$ satisfying the following:*
  (a) *With respect to the length metric induced from X, $\mathrm{Im} f_i$ are $CAT(\kappa)$-spaces;*
  (b) *Either f_i is an embedding or every multiple points are actually double points*

(2) *$\mathcal{S}(X) \cap B(p,r)$ consists of finitely many simple Lipschitz arcs C_α starting from p and reaching a point of $\partial B(p,r)$.*

Roughly speaking, Theorem 5.1 tells us that the metric ball is a union of finitely many $CAT(\kappa)$-singular disks, and the topological singular point set arises from the intersection (resp. the self-intersection) of two singular disks (resp. a singular disk), where we can see the information about those intersections as above.

Corollary 5.2. *$\mathcal{S}(X)$ has the structure of metric graph.*

It should be noted that the set of vertices of the graph $\mathcal{S}(X)$ can be uncountable. For instance, we have a variant of Example 2.1 such that the set of vertices of the graph $\mathcal{S}(X)$ is a Cantor set. However the orders of the vertices of $\mathcal{S}(X)$ are locally uniformly bounded. In that sense, the graph $\mathcal{S}(X)$ is not so bad one. In the next section, we analyze the geometric properties of the graph $\mathcal{S}(X)$ in detail in terms of the "geodesic curvature".

§6. Turn of $\mathcal{S}(X)$

The notion of *turn*, a generalization of geodesic curvature in the smooth case, of a curve in a 2-dimensional topological manifold with "bounded integral curvature" was introduced by Alexandrov and Zalgaller [3], where "bounded integral curvature" is a generalized notion of bounded total absolute curvature for a smooth surface. Burago and Buyalo [10] gave a complete characterization of 2-dimensional polyhedra with curvature bounded above, where the notion of turn of the singular locus played a crucial role. The discussion about the turn of $\mathcal{S}(X)$ in this and subsequent sections is related with that in [10].

Let $E(\mathcal{S})$ denote the set of open edges of the graph $\mathcal{S}$. From Local Structure Theorem 5.1, for each $e \in E(\mathcal{S})$, there are finitely many half disks F_α, $1 \le \alpha \le n(e)$, such that

(1) F_α bounds e;
(2) F_α, $1 \le \alpha \le n(e)$, form an open neighborhood of e in X;
(3) For every $\alpha \ne \beta$, $F_\alpha \cup F_\beta$ is a $CAT(\kappa)$-disk contained in some $CAT(\kappa)$-singular disk E_i.

We denote by τ_{F_α} the turn of e from the side F_α in the sense of Alexandrov and Zalgaller [4], which is a (signed) measure on e. For instance, $\tau_{F_\alpha}(e)$ is defined as follows: Let γ be a broken geodesic segment in $\bar{F}_\alpha$ joining the endpoints, say a and b, of e and approximating e. Let $\tilde{\tau}_{F_\alpha}(\gamma)$ be the sum of outer angles at the corners of γ measured from the side of F_α (from the different side of e). We suppose the angles between γ and e at a and b approach to zero. Then $\tau_{F_\alpha}(e)$ is defined as

$$\tau_{F_\alpha}(e) = \lim_{\gamma \to e} \tilde{\tau}_{F_\alpha}(\gamma).$$

By a result of [10], we know

$$|\tau_{F_\alpha}| < \infty, \quad \tau_{F_\alpha} + \tau_{F_\beta} \le 0, \tag{6.1}$$

for every $\alpha \ne \beta$.

We are now interested in the question: Is the converse to Local Structure Theorem 5.1 true? The answer is yes:

Theorem 6.1 ([15]). *Suppose that X is a geodesic space such that for every $p \in X$ there is an $r_0 > 0$ such that for every $0 < r \le r_0$*

(1) *$B(p,r)$ is the union of finitely many $CAT(\kappa)$-spaces E_i, where E_i is the image of some proper Lipschitz immersion $f_i : D^2 \to B(p,r)$ possibly with branch point $f_i^{-1}(p) = \{0\}$;*
(2) *$\mathcal{S}(X) \cap B(p,r)$ consists of finitely many simple Lipschitz arcs starting from p and reaching points of $\partial B(p,r)$, and therefore it has a graph structure;*
(3) *For each open edge e of the graph $\mathcal{S}(X) \cap B(p,r)$,* (6.1) *holds.*

Then X is an Alexandrov space with curvature $\le \kappa$.

Theorem 6.1 together with Local Structure Theorem 5.1 provides a complete local characterization of 2-dimensional Alexandrov spaces with curvature bounded above.

A consequence of Local Structure Theorem 5.1 is the following result on approximation of spaces via polyhedra, which yields an affirmative answer to a problem proposed by Burago and Buyalo [10]:

Theorem 6.2 ([15]). *Let (X,p) be a locally compact, geodesically complete, 2-dimensional pointed Alexandrov space with curvature $\leq \kappa$. Then there exists a sequence (X_n, p_n) of 2-dimensional pointed polyhedra with curvature $\leq \kappa$ such that*

(1) *X_n is homotopy equivalent to X;*
(2) *(X_n, p_n) converges to (X,p) with respect to the pointed Gromov–Hausdorff convergence.*

§7. Gauss–Bonnet formula

Finally we discuss the Gauss–Bonnet type formula for 2-dimensional Alexandrov spaces with curvature $\leq \kappa$. For singular surfaces of this kind, Alexandrov[1] and Alexandrov and Zalgaller [4] established the Gauss–Bonnet formula for topological 2-manifolds X with bounded integral curvature. Later Reshetnyak(see [21]) obtained it from a different view point using isothermal line elements.

They first defined the *excess* δ of a geodesic triangle domain $\blacktriangle$ in X homeomorphic to a disk as

$$\delta(\blacktriangle) = \alpha + \beta + \gamma - \pi,$$

where α, β and γ are the inner angles at the vertices of $\blacktriangle$. Then they extended it to a measure on X, denoted by ω_{AZ}, called the curvature measure of X, and obtained the Gauss–Bonnet formula in X as a natural consequence.

For polyhedra with curvature bounded above, Arshinova and Buyalo [5], and Burago and Buyalo[10] consider a natural curvature measure extending that of [4].

Let X be a 2-dimensional Alexandrov space with curvature bounded above. First we consider a curvature measure ω on X extending that of [10]. More explicitly, we let $\omega := \omega_{AZ}$ outside the singular locus $\mathcal{S}(X)$. For an open edge e of the singular graph $\mathcal{S}(X)$, Local Structure Theorem 5.1 ensures the existence of a neighborhood of e consisting of finitely many half disks F_α, $1 \leq \alpha \leq N$, bounding e. Then we let

$$\omega|_e := \sum_{\alpha=1}^{N} \tau_{F_\alpha}.$$

Finally for a vertex x of $\mathcal{S}(X)$, set

$$\omega(x) := \pi(2 - \chi(\Sigma_x)) - L(\Sigma_x),$$

which is the traditional definition, for instance found in [6].

Under this definition of the curvature measure ω, using Local Structure Theorem 5.1, we can prove that ω is actually a Radon measure on X. Namely it takes finite value on every compact subset of X.

Let D be an open bounded domain in X whose boundary meets $\mathcal{S}(X)$ transversally with finitely many points, say $p_1, \dots, p_k$. Let $L_1, \dots, L_k$ be the maximal subarcs of ∂D contained in $X - \mathcal{S}(X)$. The turn of ∂D from the side D is defined as

$$\omega_D(\partial D) = \sum_{i=1}^{k} (\tau_D(L_i) + \omega_D(p_i)),$$

where

$$\omega_D(p_i) = \pi(2 - \chi(\Sigma_{p_i}(D))) - L(\Sigma_{p_i}(D)),$$

is the "outer angle" of D at p_i.

Under this situation, the Gauss–Bonnet formula in X states

Theorem 7.1 ([15]).

$$\omega(D) = 2\pi\chi(D) - \tau_D(\partial D).$$

There are two approaches for the proof of Gauss–Bonnet Formula 7.1. One is to use approximation of X by polyhedra X_n, where we have the curvature measure $\omega_{BB,n}$ and the Gauss–Bonnet formula established in [10] for X_n. As $n \to \infty$, $\omega_{BB,n}$ weakly converges to the curvature measure ω on X, and we obtained the formula as the limit of the Gauss–Bonnet formula in X_n.

The other is a more direct method. Namely we first define the notion of excess for a certain family of basic subsets, which we call *fine* subsets, whose boundaries consist of broken geodesics meeting the singular locus in a nice way. We then extend it to a measure. It turns out that this measure is the same as ω, and we get the Gauss–Bonnet Formula as a natural consequence.

§8. Further problems

Let X be a locally compact space with curvature bonded above. Consider first the case when X is not geodesically complete. Then X has non-empty boundary ∂X. At this stage the structure of ∂X is unclear.

Problem 8.1. What can be said about the dimension of ∂X?

Finding local structure of singular spaces X with curvature bounded above is widely open in dimension ≥ 3. The method in Section 5 cannot be directly applied to higher dimensions. One of the difficulties is the complexity of Σ_p for $p \in X$. Of course, a good starting point is to study three-dimensional spaces X. To consider the ideal case when Σ_p is a 2-dimensional polyhedron with a nice structure like a spherical building will be a good test case for finding the local structure of a small neighborhood of p.

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