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Partial regularity and its application to the blow-up asymptotics of parabolic systems modelling chemotaxis with porous medium diffusion

Yoshie Sugiyama

§1. Introduction

We consider the following reaction-diffusion equation:

$$(\mathrm{KS})_m \begin{cases} \partial_t u = \Delta u^m - \nabla\cdot\left(u^{q-1}\nabla v\right), & x\in\mathbb{R}^N,\ t>0, \\ 0 = \Delta v - \gamma v + u, & x\in\mathbb{R}^N,\ t>0, \\ u(x,0) = u_0(x), & x\in\mathbb{R}^N. \end{cases}$$

Throughout this article, we assume that $N \geq 3$, and that m, q, and γ are the constants satisfying

$$m > 1,\ q \geq 2,\ \gamma > 0.$$

The initial data u_0 is a non-negative function satisfying

$$u_0 \in L^1 \cap L^\infty(\mathbb{R}^N) \quad \text{with } u_0^m \in H^1(\mathbb{R}^N).$$

This equation is often called the Keller–Segel model describing the motion of the chemotaxis molds, where $u(x,t)$ and $v(x,t)$ denote the density of amoebae and the concentration of the chemo-attractant, respectively. (we refer to Keller–Segel [6], Horstman [4], Suzuki [17].)

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In this article, we summarize our results in [22], [23] and give the outline of the proof. In [25], we have already obtained an extension criterion such that if the solution satisfies

$$\sup_{0<t<T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} < \infty, \tag{1.1}$$

then u can be continued to the solution on $\mathbb{R}^N \times [0, T')$ for some $T' > T$. The aim of this article is to relax the condition (1.1) by means of the assumption on the local behavior of u in the space variable, *i.e.,* that to establish the so-called ε-regularity theorem for the weak solutions of $(\mathrm{KS})_m$. Indeed, for the critical case of $q = m + \frac{2}{N}$, we show that there is a positive constant $\varepsilon_0 = \varepsilon_0(N, m)$ depending only on N and m such that if

$$\sup_{0<t<T} \int_{B(x_0,2\rho)} u(x,t)\, dx \quad < \quad \varepsilon_0 \tag{1.2}$$

for some $x_0 \in \mathbb{R}^N$ and $\rho > 0$, then it holds

$$\sup_{(x,t)\in B(x_0,\rho)\times(0,T)} u(x,t) \quad < \quad \infty,$$

where $B(x, \rho)$ is the ball in $\mathbb{R}^N$ centered at x with the radius ρ. This kind of result is called a partial regularity theorem, which has been studied for many other equations, *e.g.,* the Navier–Stokes equations by Caffarelli–Kohn–Nirenberg [1], the harmonic maps by Schoen–Uhlenbeck [14], the heat flow of an H-surface by Struwe [16], and the weak flows of harmonic maps by Chen–Struwe [2]. Our result corresponds to that for the Keller–Segel system $(\mathrm{KS})_m$ in the critical case of $q = m + \frac{2}{N}$.

As an application of our ε-regularity theorem, we observe that the number of blow-up points is finite, which can be controlled in terms of the mass of initial data and ε_0 in (1.2). In addition, the mass concentration of solution to $(\mathrm{KS})_m$ enables us to prove that the blow-up solution behaves like the delta function at the blow-up points. See Definition 3, below.

In the 2-D semi-linear case *i.e.,* $m = 1$, and $N = 2$, it was shown in Nagai–Senba–Suzuki [12], Senba–Suzuki [15] that the solution $u(x, t)$ of $(\mathrm{KS})_1$ before the blow-up time T is so regular that

$$u(\cdot, t) \in C^2\Big(\mathbb{R}^2 \setminus \bigcup_{i=1}^{k} \{x_i\}\Big), \quad 0 < t < T$$

with

$$\int_B u(x,t)\varphi(x)dx \in W^{1,1}(0,T)$$

for all $\varphi \in C_0^\infty(B)$ and for all balls B in $\mathbb{R}^2$, where $\{x_i\}_{i=1}^k$ are k-blow-up points of u. To obtain this property, they made use of the regularity such as $\partial_t u \in C(B \times (0,T))$ and the fact that u satisfies $(\mathrm{KS})_1$ on $[0,T)$ in the classical sense. On the other hand, in our quasi-linear case *i.e.,* $m > 1$, we do not have any information on the time derivative of u in the classical sense. Hence we need to treat the weak solution but not the classical solution, which is an essential difference between the semi-linear and quasi-linear cases. Without the regularity on $\partial_t u$ in the classical sense, assuming some additional integrability conditions such as (2.5)–(2.7) below, we can show that our weak solution $u(\cdot,t)$ becomes weakly continuous in $L^1_{loc}(\mathbb{R}^N)$ on $[0,T]$, which yields the finiteness of blow-up points of u. Our assumptions (2.5)–(2.7) are not so restrictive because it is a larger class than that of solutions with the scaling invariance associated with $(\mathrm{KS})_m$ (See Remarks 1 and 2, below.). In addition, we can construct the blow-up solution of $(\mathrm{KS})_m$ which satisfies integrability condition such as (2.5). (See Sugiyama–Velázquez [26].)

Furthermore, for investigation of asymptotic profile at the blow-up time T, it is necessary to determine the regular part $f(x)$ of $u(x,t)$ as $t \to T$. To this end, instead of u itself, we deal with u^m and show that

$$\partial_t u^m \in L^2\big(0,T; W^{1,2}(\Omega_r)^*\big), \qquad \Omega_r := B \setminus \bigcup_{i=1}^k B(x_i, r)$$

for sufficiently large ball B, which states that $u^m(\cdot,t)$ is a continuous function on $[0,T]$ with values in $L^2(\Omega_r)$. This continuity of $u^m(\cdot,t)$ at $t = T$ together with the L^1-conservation law yields the limiting function $f \in L^1(B)$ such that $u(x,t)$ converges to $f(x)$ for almost all $x \in B$ as $t \to T$. This procedure includes an essential difference between ours and the 2-D semi-linear case $(\mathrm{KS})_1$, because such higher regularity as $u \in C^{2,1}(\mathbb{R}^2 \setminus \bigcup_{i=1}^k B(x_i,r) \times [0,T])$ can be obtained from the standard argument in the latter case.

Throughout this article, we impose the following assumption:

Assumption. The space dimension $N \geq 3$ and the coefficient $\gamma > 0$. Moreover, $m > 1$ and $q \geq 2$ satisfy

$$q = m + \frac{2}{N}.$$

The initial data u_0 is a non-negative function satisfying

$$u_0 \in L^1 \cap L^\infty(\mathbb{R}^N) \quad \text{with } u_0^m \in H^1(\mathbb{R}^N).$$

Our definition of a weak solution now reads:

Definition 1. *Let the Assumption hold. A pair (u, v) of non-negative functions defined in $\mathbb{R}^N \times [0, T)$ is called a weak solution of* (KS)$_m$ *on* $[0, T)$ *if*

(i) $u \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(0, T'; L^\infty(\mathbb{R}^N))$,

(ii) $\nabla u^m \in L^2(0, T'; L^2(\mathbb{R}^N))$,

(iii) $v \in L^\infty(0, T'; H^1(\mathbb{R}^N))$ *for all* T' *with* $0 < T' < T$;

(iv) (u, v) *satisfies the following identities:*

$$\int_0^T \int_{\mathbb{R}^N} \left(\nabla u^m \cdot \nabla\varphi - u^{q-1}\nabla v \cdot \nabla\varphi - u \cdot \partial_t\varphi\right) \, dxdt$$
$$= \int_{\mathbb{R}^N} u_0(x)\varphi(x, 0) \, dx,$$

and $$\int_{\mathbb{R}^N} (\nabla v \cdot \nabla\psi + v \cdot \psi - u \cdot \psi) \, dx = 0 \qquad a.a. \ t \in [0, T)$$

for all $\varphi \in H^1(0, T; L^2(\mathbb{R}^N)) \cap L^2(0, T; H^1(\mathbb{R}^N))$ *satisfying* $\varphi(\cdot, t) = 0$ *for all* $t \in [T', T]$ *with some* $0 < T' < T$, *and all* $\psi \in H^1(\mathbb{R}^N)$.

Concerning the time local existence of weak solutions to (KS)$_m$, the following result can be shown by a slight modification of the argument developed by the author [19, Theorem 1.1].

Proposition 1.1. (Local existence of weak solution and its uniform L^∞-bound).
Let the Assumption hold. Then there exist T_0 and a weak solution (u, v) of (KS)$_m$ on $[0, T_0)$ in Definition 1 with the following additional properties:

$$\|u(t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)} \qquad \textit{for all } 0 \leq t < T_0; \tag{1.3}$$

$$\partial_t(u^{\frac{m+1}{2}}) \in L^2(0, T_0; L^2_{loc}(\mathbb{R}^N)). \tag{1.4}$$

Such an interval T_0 of local existence can be taken as

$$T_0 = \left(\|u_0\|_{L^\infty(\mathbf{R}^N)} + 2\right)^{-q},$$

and the weak solution $u(t)$ above satisfies the following estimate:

$$\|u(t)\|_{L^\infty(\mathbf{R}^N)} \leq \|u_0\|_{L^\infty(\mathbf{R}^N)} + 2 \qquad \text{for all } t \in [0, T_0).$$

§2. Main results

Let us state the main theorem on the ε-regularity for the weak solutions of $(\mathrm{KS})_m$.

Theorem 2.1. ([22], **ε-regularity theorem**) *Let the Assumption hold. Then there exists a positive number ε_0 depending only on N and m with the following property:*

Suppose that (u, v) is an arbitrary weak solution of $(KS)_m$ on $[0, T)$ in Definition 1 with the additional properties (1.3)–(1.4) with $T = T_0$. If u satisfies

$$\sup_{0<t<T} \int_{B(x_0, 2\rho_0)} u(x,t)\, dx \leq \varepsilon_0 \tag{2.1}$$

for some $x_0 \in \mathbb{R}^N$ and $\rho_0 > 0$, then it holds that

$$\sup_{(x,t)\in B(x_0, \frac{\rho_0}{2})\times(0,T)} u(x,t) < C,$$

where $C = C(N, m, \gamma, \|u_0\|_{L^1 \cap L^\infty}, T, \rho_0)$ is a constant independent of x_0.

Remark 1. It should be noted that the quantity

$$\sup_{0<t<\infty} \|u(t)\|_{L^{\frac{N(q-m)}{2}}(\mathbf{R}^N)}$$

is invariant under the change of scaling associated with $(\mathrm{KS})_m$ with $\gamma = 0$. In fact, if (u, v) solves $(\mathrm{KS})_m$ with $\gamma = 0$, then (u_λ, v_λ) is also a solution for all $\lambda > 0$, where

$$\begin{cases} u_\lambda(x,t) &:= \lambda^2 u\left(\lambda^{q-m}x, \lambda^{2(q-1)}t\right), \\ v_\lambda(x,t) &:= \lambda^{2(m-q+1)} v\left(\lambda^{q-m}x, \lambda^{2(q-1)}t\right). \end{cases} \tag{2.2}$$

The scaling invariance in $L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)$ means that, for all $\lambda > 0$,

$$\sup_{0<t<\infty} \|u_\lambda(t)\|_{L^{\frac{N(q-m)}{2}}(\mathbf{R}^N)} = \sup_{0<t<\infty} \|u(t)\|_{L^{\frac{N(q-m)}{2}}(\mathbf{R}^N)}. \tag{2.3}$$

In particular, for $q = m + \frac{2}{N}$, the above (2.3) is equivalent to

$$\sup_{0<t<\infty} \|u_\lambda(t)\|_{L^1(\mathbf{R}^N)} = \sup_{0<t<\infty} \|u(t)\|_{L^1(\mathbf{R}^N)} \qquad \text{for all } \lambda > 0$$

since $\frac{N(q-m)}{2} = 1$. Therefore, we may say that (2.1) is a reasonable condition concerning the theorem on the ε-regularity of weak solutions to $(\mathrm{KS})_m$.

As an application of the ε-regularity theorem as Theorem 2.1, we characterize the asymptotic behavior of blow-up solutions to $(\mathrm{KS})_m$. For that purpose, let us introduce definitions for the *blow-up time* and the *blow-up point.*

Definition 2. *Let (u, v) be the weak solution of $(KS)_m$ on $[0, T)$ in Definition 1.*

(i) (blow-up time) We say that u blows up at the time $T < \infty$ if

$$\limsup_{t \to T-0} \|u(t)\|_{L^\infty(\mathbf{R}^N)} = \infty.$$

Such a T is called a blow-up time of u.

(ii) (blow-up point) Let T be a blow-up time of u. We call $x_0 \in \mathbb{R}^N$ a blow-up point of u at the time T if there exists $\{(x_n, t_n)\}_{n=1}^\infty \subset \mathbb{R}^N \times (0, T)$ such that

$$x_n \to x_0, \quad t_n \to T, \quad \text{and} \quad u(x_n, t_n) \to \infty \quad \text{as } n \to \infty.$$

We denote by S_u the set of all blow-up points of u at the time T.

An immediate consequence of Theorem 2.1 is the following characterization of both the blow-up point x_0 and the time T.

Corollary 2.2. ([22]) *Let the Assumption hold. Suppose that (u, v) is the weak solution of $(KS)_m$ on $[0, T)$ with the additional properties (1.3)–(1.4) with $T = T_0$. Let T be the blow-up time of the weak solution u of $(KS)_m$. Then, for any $x_0 \in S_u$, it holds that*

$$\sup_{0<t<T} \int_{B(x_0,\rho)} u(x,t)\, dx \;>\; \varepsilon_0 \qquad \text{for all } \rho > 0,$$

where ε_0 is the same constant given by Theorem 2.1.

Furthermore, under some additional assumptions on u, we can show the finiteness of the blow-up points of u. To this end, we introduce the Lyapunov function $W(t)$ of u as

$$\begin{aligned} W(t) &= \frac{m}{(m-q+1)(m-q+2)} \int_{\mathbf{R}^N} u(t)^{m-q+2}\, dx \\ &\quad - \int_{\mathbf{R}^N} u(x,t)v(x,t)\, dx \\ &\quad + \frac{1}{2}\Big(\|\nabla v(t)\|^2_{L^2(\mathbf{R}^N)} + \|v(t)\|^2_{L^2(\mathbf{R}^N)} \Big). \end{aligned}$$

By Corollary 2.2, we establish the finiteness of the number of blow-up points. Indeed, it holds

Theorem 2.3. ([23], **Finiteness of the blow-up points**) *Let the Assumption hold. Suppose that (u, v) is the weak solution of $(KS)_m$ on $[0, T)$ with the additional properties (1.3)–(1.4) with $T = T_0$. Let T be the blow-up time of the weak solution u of $(KS)_m$. Suppose that S_u is the set of blow-up points of u at the time T in Definition 2.*

We define the positive integer k_0 by

$$k_0 := \left[\frac{\|u_0\|_1}{\varepsilon_0} \right] + 1, \tag{2.4}$$

where $[\cdot]$ denotes the Gauss symbol and where ε_0 is the same constant as in (2.1).

(1) We have the following alternative (i) or (ii):

(i) $^\sharp S_u \le k_0 - 1$;

(ii) $^\sharp S_u = \infty$ and S_u does not have more than $k_0 - 1$ isolated points, or generally S_u does not have more than $k_0 - 1$ isolated cluster points.

(2) We consider the following three conditions (i), (ii) and (iii) on q and u:

(i) $q = m + \frac{2}{N} = 2$ and u has the property that

$$u \in L^m(\mathbb{R}^N \times (0, T)); \tag{2.5}$$

(ii) $q = m + \frac{2}{N} \ge 2 + \frac{2}{N}$ and u has the property that

$$u \in L^{m+\frac{m'}{N}}(0, T; L^m(\mathbb{R}^N)) \quad \text{with} \quad m' = \frac{m}{m-1}; \tag{2.6}$$

(iii) $q = m + \frac{2}{N} \geq 2$ *and* u *has the property that*

(2.7) $u \in L^{m+\frac{2}{N}-1}(B \times (0,T))$ *for all balls* B *in* $\mathbb{R}^N$

and that

$$\inf_{0<t<T} W(t) \; > \; -\infty;$$

If one of these three conditions (i), (ii) and (iii) is satisfied, then we have $^\sharp S_u \; \leq \; k_0 - 1$.

Remark 2. As we stated in the Introduction, our assumptions (i)–(iii) in Theorem 2.3 are not so restrictive because the blow-up solution of $(\mathrm{KS})_m$ with the integrability condition in (2.5) can be constructed for an arbitrary initial data u_0 in the Assumption. (See [26].) Moreover, each of (2.5)–(2.7) gives a larger class than that of solutions with the scaling invariance associated with $(\mathrm{KS})_m$. Indeed, it follows from a direct calculation of (2.2) that

$$\|u_\lambda\|_{L^s(0,\infty;L^p(\mathbb{R}^N))} \;=\; \lambda^{2\left(1-(\frac{1}{p}+\frac{q-1}{s})\right)}\|u\|_{L^s(0,\infty;L^p(\mathbb{R}^N))}$$

for all $\lambda > 0$ and for all $1 \leq p, s \leq \infty$. Hence, the space $L^s(0,\infty;L^p(\mathbb{R}^N))$ is called the scaling invariant class associated with $(\mathrm{KS})_m$ provided $\frac{1}{p} + \frac{q-1}{s} = 1$. In (i), (ii) and (iii), the pair (p,s) of exponent for $u \in L^s(0,T;L^p(\mathbb{R}^N))$ are taken as $(p,s) = (m,m)$, $(p,s) = (m, m+\frac{m'}{N})$ and $(p,s) = (q-1, q-1)$, respectively. In all of these cases, we have

$$\frac{1}{p} + \frac{q-1}{s} \; > \; 1.$$

Next, we give a definition that $u(x,t)$ forms the *δ-function* singularity.

Definition 3. *Let* T *be a blow-up time of the weak solution* u *of* $(\mathrm{KS})_m$. *Let* $\{x_i\}_{i=1}^k \subset S_u$. *We say that* u *forms the δ-function singularity at* $\{x_i\}_{i=1}^k$ *and at the time* T *with the mass* $\{M_i\}_{i=1}^k$ *if the following property holds:*

There exist a function f *in* $L^1(\mathbb{R}^N)$ *and a sequence* $\{t_n\}_{n=1}^\infty \subset (0,T)$ *with* $\lim_{n\to\infty} t_n = T$ *such that, in the sense of distributions in* $\mathbb{R}^N$,

$$u(\cdot, t_n) \;\longrightarrow\; \sum_{i=1}^k M_i \delta_{x_i}(\cdot) + f(\cdot) \quad \textit{as} \;\; n \to \infty$$

i.e., that, for all $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty}\int_{\mathbf{R}^N} u(x,t_n)\psi(x)\,dx \;=\; \sum_{i=1}^{k} M_i\psi(x_i) \;+\; \int_{\mathbf{R}^N} f(x)\psi(x)\,dx.$$

As an application of Corollary 2.2, the structure of asymptotics of blow-up solution is clarified. Indeed, we show that $u(x,t)$ forms the δ-*function* singularity at $\{x_i\}_{i=1}^k$ and at the time T with the mass $\{M_i\}_{i=1}^k$.

Theorem 2.4. ([22], **δ-function singularity**) *Let the Assumption hold. Suppose that (u,v) is the weak solution of $(KS)_m$ on $[0,T)$ with the additional properties (1.3)–(1.4) with $T = T_0$. Let T be the blow-up time of a weak solution u of $(KS)_m$. Suppose that $^\sharp S_u < \infty$, say, $^\sharp S_u = k$. Let $\{x_i\}_{i=1}^k = S_u$. Suppose that ε_0 is the constant given by Theorem 2.1. Then, there exist k constants $M_i \geq \varepsilon_0$ $(1 \leq i \leq k)$ such that u forms the δ-function singularity at $\{x_i\}_{i=1}^k$ and at the time T with the mass $\{M_i\}_{i=1}^k$.*

We next investigate the size of the set of blow-up points. To this end, we recall the definition of *Hausdorff* dimension and we estimate the Hausdorff dimension of the set of blow-up points of weak solutions u.

Definition 4. *For any $X \subset \mathbb{R}^N$ and $s \geq 0$, we define the Hausdorff measure $H^s(X)$ as*

$$H^s(X) \;:=\; \lim_{\delta\to+0} H^s_\delta(X),$$

$$H^s_\delta(X) \;:=\; \inf\Big\{\sum_{i=1}^{\infty}\rho_i^s;\; X \subset \bigcup_i B_{\rho_i},\; \rho_i < \delta\Big\},$$

where B_{ρ_i} is an arbitrary closed subset of $\mathbb{R}^N$ of diameter at most ρ_i. We define the Hausdorff dimension $D_H(X)$ as

$$D_H(X) \;:=\; \inf\{s;\; H^s(X) = 0\}.$$

Theorem 2.5. ([23], **Hausdorff dimension**) *Let the Assumption hold. Suppose that (u,v) is the weak solution of $(KS)_m$ on $[0,T)$ with the additional properties (1.3)–(1.4) with $T = T_0$. Let T be a blow-up time of the weak solution u of $(KS)_m$. If u satisfies*

$$\text{(2.8)} \qquad \int u(x,t)\psi(x)\,dx \text{ is a continuous function on } [0,T]$$

for every $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi(x) = \psi(|x-x_0|)$ for some $x_0 \in \mathbb{R}^N$, then the Hausdorff dimension $D_H(S_u)$ is zero. In particular, if u

satisfies

$$u \in C_w([0,T]; L^1(\mathbb{R}^N)), \tag{2.9}$$

then the Hausdorff dimension $D_H(S_u)$ *is zero.*

Remark 3. For the estimate of $D_H(S_u)$, the assumption (2.9) is too strong. In fact, we need only to assume the weaker continuity such as (2.8). In Lemma 4.1 below, we will see that if u satisfies one of the assumptions among (2.5), (2.6) and (2.7), then we have (2.8).

For the spherically symmetric solution u of $(\mathrm{KS})_m$, we can pinpoint the location of blow-up points. Indeed, it holds

Corollary 2.6. ([23], **Blow-up points for spherically symmetric solution**) *Let the Assumption hold. Suppose that* (u, v) *is the weak solution of* $(KS)_m$ *on* $[0, T)$ *with the additional properties (1.3)–(1.4) with* $T = T_0$. *If* u *is a spherically symmetric with the property (2.8) in Theorem 2.5, then it holds that* $S_u = \phi$, *or* $S_u = \{0\}$.

Remark 4. It seems to be an interesting question whether the solution (u, v) is spherically symmetric for such an intimal data as $u_0(x) = u_0(|x|)$.

As we have seen in Theorem 2.5, the continuity of the weak solution in $L^1(\mathbb{R}^N)$ plays an important role for the estimate of the size of blow-up set S_u. If we impose strong continuity in $L^1(\mathbb{R}^N)$ on $u(t)$ as $t \to T-0$, then u can be continued beyond $t = T$. Indeed, we have the following extension criterion.

Theorem 2.7. ([23], **Extension criterion**) *Let the Assumption hold. Suppose that* (u, v) *is an arbitrary weak solution of* $(KS)_m$ *on* $[0, T)$ *in Definition 1 with the additional properties (1.3)–(1.4) with* $T = T_0$. *If it holds that*

$$u \in C([0,T]; L^1(\mathbb{R}^N)), \tag{2.10}$$

then there exists $T' > T$ *such that* (u, v) *is a weak solution of* $(KS)_m$ *on* $[0, T')$.

Remark 5. It seems to be an interesting question that under what class of the initial data u_0, one can construct the weak solution satisfying (2.9) or (2.10). On the other hand, in [26], we have succeeded to construct the weak solution having the property (2.8). Such a delicate difference is seen only in the L^1-space since C_0^∞ is not dense in L^∞, the dual space of L^1.

In contrast with (2.1), for weak solutions u on $[0,T)$ with $^\sharp S_u < \infty$, we may take a larger constant $\alpha_{N,m}$ as in (2.11) below which guarantees the ε-regularity theorem.

Theorem 2.8. ([23], **ε-regularity theorem**) *Let the Assumption hold. Suppose that (u,v) is an arbitrary weak solution of (KS)$_m$ on $[0,T)$ in Definition 1 with the additional properties (1.3)–(1.4) with $T = T_0$. Suppose that $^\sharp S_u < \infty$. If u satisfies*

$$(2.11)\quad \sup_{0<t<T} \int_{B(x_0,\rho_0)} u(x,t)\, dx \;<\; \left(\frac{m\pi N^3}{N-1}\right)^{\frac{N}{2}} \frac{\Gamma(N/2)}{\Gamma(N)} \;=:\; \alpha_{N,m}$$

for some $x_0 \in \mathbb{R}^N$ and $\rho_0 > 0$, then it holds that

$$(2.12)\qquad \sup_{(x,t)\in B(x_0,\frac{\rho_0}{2})\times(0,T)} u(x,t) \;<\; C,$$

where $C = C(N, m, \gamma, \|u_0\|_{L^1\cap L^\infty}, T, \rho_0)$ is a constant independent of x_0.

In particular, for $\{x_1, x_2, \cdots, x_k\} =: S_u$ $(k \le k_0 - 1)$, we have

$$\limsup_{t\to T} \int_{B(x_i,\rho)} u(x,t)\, dx \;\ge\; \alpha_{N,m}, \quad i = 1, 2, \cdots, k$$

for all $\rho > 0$.

Remark 6. The balance of strength m of diffusion and the effect q of non-linearity plays an important role for existence of global solutions to (KS)$_m$. Indeed,

(i) For the case of $2 \le q < m + \frac{2}{N}$, (KS)$_m$ is globally solvable without any restriction on the size of the initial data u_0;
(ii) For the case of $q \ge m + \frac{2}{N}$, (KS)$_m$ is globally solvable for the small initial data u_0 in $L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)$. As for the large initial data, the solution of (KS)$_m$ with $q \ge m + \frac{2}{N}$ may have some singularities in a finite time even if the initial data is smooth. (See [18]–[21].)

From this point of view, in [24] we treated more general cases of $q \ge m + \frac{2}{N}$ and proved the corresponding ε-regularity theorem to the critical case of $q = m + \frac{2}{N}$. Indeed, we showed that if the solution u of (KS)$_m$ satisfies that

$$(2.13)\qquad \sup_{0<t<T} \int_{B(x_0,2\rho)} u^{\frac{N(q-m)}{2}}(x,t)\, dx \;<\; \varepsilon_0$$

for some $x_0 \in \mathbb{R}^N$ and $\rho > 0$, then it holds that

$$\sup_{(x,t)\in B(x_0,\rho)\times(0,T)} u(x,t) \; < \; C,$$

where C depends only on N, m, q, γ, ρ, $\|u_0\|_{L^1(\mathbb{R}^N)}$ and $\|u_0\|_{L^\infty(\mathbb{R}^N)}$ but not on x_0. In our generalized case, the space $L^\infty(0,\infty; L^{\frac{N(q-m)}{2}}(\mathbb{R}^N))$ is also a scaling invariant class associated with $(\mathrm{KS})_m$.

§3. Proof of Theorem 2.1 and Corollary 2.2

In what follows, we abbreviate simply as

$$\|\cdot\|_r = \|\cdot\|_{L^r(\mathbb{R}^N)}, \qquad 1 < r < \infty$$

and C denotes the constant which may change from line to line. In particular, $C = C(*, \cdots, *)$ denotes a constant depending only on the variables appearing in the parenthesis.

We give the sketch of the proof for our ε-regularity theorem. See [22], [23] for the complete proof.

First of all, we derive local bounds in L^r of u for all $1 < r < \infty$.

Lemma 3.1. *Let the Assumption hold. For every $1 \le r < \infty$, there is a positive constant ε_0 depending only on N, m and r such that if (u, v) is a weak solution of $(KS)_m$ on $[0,T)$ with (1.3)–(1.4) with $T = T_0$ and if u satisfies*

$$\text{(3.1)} \qquad \sup_{0<t<T} \int_{B(x_0,\rho_0+\delta)} u(x,t)\, dx \; < \; \varepsilon_0$$

for some $x_0 \in \mathbb{R}^N$, $\rho_0 > 0$ and $\delta > 0$, then it holds that

$$\int_{B(x_0,\rho_0)} u^r(x,t)\, dx \; \le \; C_r(T+1) \quad \text{for all } 0 < t < T,$$

where $C_r = C_r(r, N, m, \gamma, \delta, \|u_0\|_1, \|u_0\|_\infty)$.

We may put $x_0 = 0$ without loss of generality. Once the L^r-bound is established for all $1 \le r < \infty$ in Lemma 3.1, it follows from the representation $v = (-\Delta + \gamma)^{-1} u$ that

$$\text{(3.2)} \qquad \sup_{0<t<T} \|v(t)\|_{L^\infty(B(0,\rho_0+\delta))} \; \le \; C,$$

and

$$\sup_{0<t<T} \|\nabla v(t)\|_{L^\infty(B(0,\rho_0+\delta))} \leq C, \tag{3.3}$$

where $0 < \delta < \frac{\rho_0}{3}$ and $C = C(N, m, \gamma, \rho_0, \|u_0\|_1, \|u_0\|_\infty, T)$. It should be noted that the constant C in (3.3) can be taken independently of δ. Indeed, we shall show (3.3) according to the similar argument in [12]. From the assumption (2.1) in Theorem 2.1, it follows that

$$\sup_{0<t<T} \int_{B(0,\frac{5}{3}\rho_0+\delta)} u(x,t)\, dx \leq \varepsilon_0,$$

where $0 < \delta < \frac{\rho_0}{3}$. Hence we obtain from Lemma 3.1 with $r = N+1$ and (ρ_0, δ) replaced by $(\frac{4}{3}\rho_0 + \delta, \frac{1}{3}\rho_0)$ that

$$\sup_{0<t<T} \|u(\cdot,t)\chi_{B(0,\frac{4}{3}\rho_0+\delta)}\|_{N+1} \leq C_0,$$

where $C_0 = C_0(N, m, \gamma, \rho_0, \|u_0\|_1, \|u_0\|_\infty, T)$. We here consider

$$-\Delta v_1 + \gamma v_1 = u\chi_{B(0,\frac{4}{3}\rho_0+\delta)} \quad \text{in } \mathbb{R}^N. \tag{3.4}$$

Then, the function v_1 given by

$$v_1(x,t) = \int_{\mathbb{R}^N} G(x-y) u\chi_{B(0,\frac{4}{3}\rho_0+\delta)}(y,t)\, dy$$

is the strong solution of (3.4), where $G(x)$ is the kernel of the Bessel potential. Since $G \in L^{\frac{N}{N-1}}(\mathbb{R}^N)$ and $\nabla G \in L^{\frac{N+1}{N}}(\mathbb{R}^N)$, we see that

$$\begin{aligned} &\sup_{0<t<T} \|v_1(t)\|_\infty \\ &\leq \|G\|_{\frac{N}{N-1}} \cdot \sup_{0<t<T} \|u\chi_{B(0,\frac{4}{3}\rho_0+\delta)}(t)\|_N \leq C, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\sup_{0<t<T} \|\nabla v_1(t)\|_\infty \\ &\leq \|\nabla G\|_{\frac{N+1}{N}} \cdot \sup_{0<t<T} \|u\chi_{B(0,\frac{4}{3}\rho_0+\delta)}(t)\|_{N+1} \leq C, \end{aligned} \tag{3.6}$$

where $C = C(N, m, \gamma, \rho_0, \|u_0\|_1, \|u_0\|_\infty, T)$.

Next, we consider

$$-\Delta v_2 + \gamma v_2 = u - u\chi_{B(0,\frac{4}{3}\rho_0+\delta)} \quad \text{in } \mathbb{R}^N. \tag{3.7}$$

Then, the function v_2 given by

$$(3.8)\qquad v_2(x,t) = \int_{\mathbb{R}^N} G(x-y)\cdot(u - u\chi_{B(0,\frac{4}{3}\rho_0+\delta)})(y,t)\,dy$$

is the strong solution of (3.7). Since $G(x)$ satisfies the estimates $|G(x)| \le C|x|^{2-N}$ and $|\nabla G(x)| \le C|x|^{1-N}$ for all $x \in \mathbb{R}^N$, we have

$$\begin{aligned}
&\|v_2(t)\|_{L^\infty(B(0,\rho_0+\delta))} \\
&= \sup_{x\in B(0,\rho_0+\delta)} \Big| \int_{\mathbb{R}^N\setminus B(0,\frac{4}{3}\rho_0+\delta)} G(x-y)\times \\
&\qquad (u - u\chi_{B(0,\frac{4}{3}\rho_0+\delta)})(y,t)\,dy\Big| \\
(3.9)\quad &\le C|\frac{\rho_0}{3}|^{2-N}\cdot\|u_0\|_1 \le C \qquad \text{for all } 0<t<T, \\
&\|\nabla v_2(t)\|_{L^\infty(B(0,\rho_0+\delta))} \\
&= \sup_{x\in B(0,\rho_0+\delta)} \Big| \int_{\mathbb{R}^N\setminus B(0,\frac{4}{3}\rho_0+\delta)} \nabla G(x-y)\times \\
&\qquad (u - u\chi_{B(0,\frac{4}{3}\rho_0+\delta)})(y,t)\,dy\Big| \\
(3.10)\quad &\le C|\frac{\rho_0}{3}|^{1-N}\cdot\|u_0\|_1 \le C \qquad \text{for all } 0<t<T,
\end{aligned}$$

where $C = C(N,\gamma,\rho_0,\|u_0\|_1)$.

By (3.4) and (3.7), obviously, $v := v_1 + v_2$ gives the unique strong solution of the equation:

$$-\Delta v + \gamma v = u \quad \text{in } \mathbb{R}^N.$$

Thus from (3.5), (3.6), (3.9), (3.10), we obtain (3.3).

Let us introduce a cut-off function η with several properties.

Lemma 3.2. *Let $\rho_0 > 0$ and $\delta > 0$ as in (3.3). Let $\eta(x) = \eta(|x|)$ be as*

$$\eta(x) := \begin{cases} 1 & \text{for } 0 \le |x| < \rho_0, \\ \exp(1 - \frac{\delta}{\rho_0+\delta-|x|}) & \text{for } \rho_0 \le |x| < \rho_0+\delta, \\ 0 & \text{for } |x| \ge \rho_0+\delta. \end{cases}$$

Then, it holds that

$$\begin{aligned}
|\nabla\eta(x)| &\le \frac{c}{a^2\delta}\cdot\eta(x)^{1-a}, \\
|\Delta\eta(x)| &\le \frac{c}{a^4\delta^2}\cdot\eta(x)^{1-a},
\end{aligned}$$

for all $x \in \mathbb{R}^N$ and all $0 < a < 1$, where c is an absolute positive constant.

In what follows, we take ρ_0 and δ so that (3.3) holds. We now proceed to give the proof of Theorem 2.1. For every weak solution with (2.1), it holds that

$$\frac{1}{r}\int_{\mathbf{R}^N} u^r(x,t)\eta(x)\,dx = \int_0^t (I_1 + I_2)(s)\,ds + \frac{1}{r}\int_{\mathbf{R}^N} (u_0^r\eta)(x)\,dx, \tag{3.11}$$

where I_1 and I_2 are defined by

$$I_1 := -\int_{\mathbf{R}^N} \nabla u^m \cdot \nabla\big(u^{r-1}\cdot\eta\big)\,dx,$$

and

$$I_2 := \int_{\mathbf{R}^N} u^{q-1}\nabla v \cdot \nabla\big(u^{r-1}\cdot\eta\big)\,dx$$

and where η is the cut-off function which is centered at x_0 and determined by ρ_0 and δ as in Lemma 3.2.

Applying a variant of the Sobolev inequality together with the Young inequality, we may take r_* depending only on N, m such that

$$\begin{aligned} I_1 \le\; & -\frac{2m(r-1)}{(r+m-1)^2}\int_{\mathbf{R}^N} |\nabla u^{\frac{r+m-1}{2}}|^2\eta\,dx \\ & + C\Big(r+\frac{1}{a^2\delta}\Big)^C \|u\|^{r+m-1}_{L^{\frac{r}{4}}(B(x_0,\rho_0+\delta))} + 1, \quad 0 < a \le \frac{1}{3(N+1)} \end{aligned} \tag{3.12}$$

for all $r_* < r < \infty$, where $C = C(N, m, \gamma, \|u_0\|_1, \|u_0\|_\infty)$.

Furthermore, from (3.3) and the Young inequality, we obtain that

$$\begin{aligned} I_2 \le\; & \frac{3m(r-1)}{2(r+m-1)^2}\int_{\mathbf{R}^N} |\nabla u^{\frac{r+m-1}{2}}|^2\eta\,dx \\ & + C\Big(r+\frac{1}{a^2\delta}\Big)^C \Big(\|u\|^{r+2q-m-3}_{L^{\frac{r}{4}}(B(x_0,\rho_0+\delta))} + \|u\|^{r+q-2}_{L^{\frac{r}{4}}(B(x_0,\rho_0+\delta))} + 1\Big), \end{aligned} \tag{3.13}$$

for $0 < t < T$, for all $0 < a \le \frac{1}{3(N+1)}$ and for all $r_* < r < \infty$. See [22, Sections 3 and 4] for proof of (3.12) and (3.13).

From (3.11)–(3.13), it follows that

$$\begin{aligned}
&\int_{\mathbf{R}^N} u^r(x,t)\eta(x)\,dx \\
&\le -\frac{mr(r-1)}{2(r+m-1)^2}\int_0^t\int_{\mathbf{R}^N}|\nabla u^{\frac{r+m-1}{2}}|^2\eta\,dxds \\
&\quad + C\Big(r+\frac{1}{a^2\delta}\Big)^C \times \int_0^t \Big(\|u\|^{r+m-1}_{L^{\frac{r}{4}}(B(x_0,\rho_0+\delta))} + \\
&\qquad \|u\|^{r+2q-m-3}_{L^{\frac{r}{4}}(B(x_0,\rho_0+\delta))} + \|u\|^{r+q-2}_{L^{\frac{r}{4}}(B(x_0,\rho_0+\delta))}\Big)\,ds \\
&\quad +TC\Big(r+\frac{1}{a^2\delta}\Big)^C + \int_{\mathbf{R}^N}(u_0^r\eta)(x)\,dx, \qquad 0<a<\frac{1}{3(N+1)}, \qquad (3.14)
\end{aligned}$$

where $C = C(N,m,\gamma,\rho_0,\|u_0\|_1,\|u_0\|_\infty,T)$.

Since

$$r+m-1 \;>\; r+q-2 \;>\; r+2q-m-3$$

implied by $m-q+1 = 1-\frac{2}{N} > 0$ and since we may take a as an arbitrary number in $(0,\frac{1}{3(N+1)}]$, by setting $\delta = \dfrac{\rho_0}{r}$ in (3.14), we have

$$\begin{aligned}
&\sup_{0<t<T}\|u(t)\|_{L^r(B(x_0,\rho_0))} \\
&\le (Cr^C)^{\frac{1}{r}}\cdot\max\{\sup_{0<t<T}\|u\|^{\frac{r+m-1}{r}}_{L^{\frac{r}{4}}(B(x_0,\rho_0+\frac{\rho_0}{r}))},\ \|u_0\|_r,\ T+1\} \qquad (3.15)
\end{aligned}$$

for all $r_* < r < \infty$. Now we take p_0 such as $4^{p_0} > r_*$ and define α_p as

$$\alpha_p := \max\Big\{\sup_{0<t<T}\|u\|_{L^{4^p}(B(x_0,\rho_0-\sum_{i=1}^p\frac{\rho_0}{4^i}))}, \|u_0\|_1, \|u_0\|_\infty,\ T+1\Big\},$$

for $p > p_0$. Taking $r = 4^p$ in (3.15), we have

$$\begin{aligned}
\alpha_p &\le C^{1/4^p}4^{Cp/4^p} \\
&\quad \times\max\Big\{\sup_{0<t<T}\|u\|_{L^{4^p}(B(x_0,\rho_0-\sum_{i=1}^{p-1}\frac{\rho_0}{4^i}))}, \|u_0\|_1, \|u_0\|_\infty,\ T+1\Big\}^{1+\frac{m-1}{4^p}} \\
&= C^{1/4^p}4^{Cp/4^p}\cdot\alpha_{p-1}^{1+\frac{m-1}{4^p}} \\
&\le C\cdot\alpha_{p_0-1}^c \qquad \text{for all } p_0 < p < \infty,
\end{aligned}$$

which yields

$$\begin{aligned}
&\sup_{0<t<T} \|u\|_{L^{4^p}(B(x_0,\rho_0-\sum_{i=1}^{p}\frac{\rho_0}{4^i})}\\
&\leq \ C\cdot \alpha_{p_0-1}^c\\
(3.16) &= C\max\left\{\sup_{0<t<T}\|u\|_{L^{4^{p_0}}(B(x_0,\rho_0-\sum_{i=1}^{p_0}\frac{\rho_0}{4^i})}, \|u_0\|_1, \|u_0\|_\infty, T+1\right\}^c.
\end{aligned}$$

See [Proof of Lemma 5.1, 27] for detail. Under the hypothesis of (2.1), the assumption (3.1) in Lemma 3.1 is fulfilled with $\delta = \frac{\rho_0}{4^{p_0}}$, which makes it possible to take $r = 4^{p_0}$ with the estimate

$$\sup_{0<t<T} \|u\|_{L^{4^{p_0}}(B(x_0,\rho_0))} \ \leq \ C,$$

where $C = C(N, m, \gamma, p_0, \rho_0, \|u_0\|_1, \|u_0\|_\infty, T)$. Since $\sum_{i=1}^{p}\frac{\rho_0}{4^i} < \frac{\rho_0}{3}$ for all $1 < p < \infty$, by letting $p \to \infty$ in (3.16), we see that $u \in L^\infty(0,T; L^\infty(B(x_0, \frac{2\rho_0}{3})))$ with

$$\sup_{0<t<T} \|u(t)\|_{L^\infty(B(x_0,\frac{2\rho_0}{3}))} \ \leq \ C(T+1),$$

where $C = C(N, m, \gamma, p_0, , \rho_0, \|u_0\|_1, \|u_0\|_\infty, T)$. Thus we complete the proof of Theorem 2.1.

Obviously, Corollary 2.2 is an immediate consequence of Theorem 2.1.

§4. Proof of Theorems 2.3 and 2.4

In our quasi-linear case *i.e.,* $m > 1$, we do not have any information on the time derivative of u in the classical sense. Hence we need to treat the weak solution but not the classical solution, which is an essential difference between the semi-linear and quasi-linear cases. Without the regularity on $\partial_t u$ in the classical sense, assuming some additional integrability conditions such as (i)–(iii) in Theorem 2.3, we can show that our weak solution $u(\cdot, t)$ becomes weakly continuous in $L^1_{loc}(\mathbb{R}^N)$ on $[0, T]$ in the following lemma. See [23, Section 5] for the proof.

Lemma 4.1. *Let the Assumption hold. Suppose that (u, v) is the weak solution of $(KS)_m$ on $[0, T)$ with the additional properties (1.3)–(1.4).*

(1) Suppose that $x_0 \in S_u$ has the property that

$$S_u \cap \{x \in \mathbb{R}^N; d < |x - x_0| < 2d\} \ = \ \phi$$

for some $d > 0$. Then, for the cut-off function $\eta \in C_0^\infty(B_{2d}(x_0))$ centered at x_0 with $\rho_0 = d$ and $\delta = \frac{d}{2}$ as in Lemma 3.2, it holds that $\int_{\mathbf{R}^N} u(x,t)\eta(x)\,dx$ is continuous on $[0,T]$.

(2) If u satisfies one of three conditions (i), (ii) and (iii) in Theorem 2.3, then, it holds that $\int_{\mathbf{R}^N} u(x,t)\psi(x)\,dx$ is continuous on $[0,T]$ for each $\psi \in C_0^\infty(\mathbb{R}^N)$.

Once we establish Lemma 4.1, we can prove Theorem 2.3 by the similar argument to that in [12, Theorem 3] as follows.

Proof of Theorem 2.3 (1). In both cases (i) and (ii). we may prove that S_u never has more than $k_0 - 1$ isolated points, or more generally, more than $k_0 - 1$ isolated cluster points. We shall show by contradiction. Assume that $\{x_1, x_2, \cdots, x_{k_0}\}$ are k_0 isolated points of S_u. Then, there exists $d > 0$ such that $S_u \cap \{x \in \mathbb{R}^N; d < |x - x_i| < 2d\} = \phi$ for all $i = 1, 2, \cdots, k_0$ and

$$(4.1) B(x_i, 2d) \cap B(x_j, 2d) = \phi \quad \text{for all } i, j = 1, 2, \cdots, k_0 \text{ with } i \neq j.$$

By Lemma 4.1 (1), we see that the function $\int_{\mathbf{R}^N} u(x,t)\eta_i(x)\,dx$ is continuous on $[0,T]$, where $\eta_i \in C_0^\infty(B_{2d}(x_i))$ is the cut-off function centered at x_i with $\rho_0 = d$ and $\delta = \frac{d}{2}$ as in Lemma 3.2 for $i = 1, 2, \cdots, k_0$. Since $x_i \in S_u$, it follows from Corollary 2.2 that

$$(4.2) \quad \limsup_{t \to T} \int_{B(x_i,d)} u(x,t)\,dx \;>\; \varepsilon_0 \qquad \text{for all } i = 1, 2, \cdots k_0.$$

Then, we have by (4.2) and Lemma 4.1 that

$$\begin{aligned} k_0\varepsilon_0 &< \sum_{i=1}^{k_0} \limsup_{t \to T} \int_{B(x_i,d)} u(x,t)\,dx \\ &\leq \sum_{i=1}^{k_0} \limsup_{t \to T} \int_{B(x_i,\frac{3d}{2})} u(x,t)\eta_i(x)\,dx \\ &= \sum_{i=1}^{k_0} \liminf_{t \to T} \int_{B(x_i,\frac{3d}{2})} u(x,t)\eta_i(x)\,dx, \end{aligned} \tag{4.3}$$

where η_i is the cut-off function as in Lemma 3.2, which is centered at x_i and with $\rho_0 = d$ and $\delta = \frac{d}{2}$. On the other hand, for arbitrary $\varepsilon > 0$, there exists $\mu_i = \mu_i(\varepsilon)$ such that, for all $T - \mu_i < s < T$,

$$
\begin{aligned}
&\liminf_{\tau\to T} \int_{B(x_i,\frac{3d}{2})} u(x,\tau)\eta_i(x)\,dx - \varepsilon \\
&\le \|u(s)\eta_i\|_{L^1(B(x_i,\frac{3d}{2}))} \le \|u(s)\|_{L^1(B(x_i,\frac{3d}{2}))}.
\end{aligned} \tag{4.4}
$$

Now let us define $\mu := \min_{1\le i\le k_0} \mu_i$. Since $\|u(s)\|_1 = \|u_0\|_1$ for all $0 \le s \le T$, it follows from (4.1) and (4.4) that

$$
\begin{aligned}
&\sum_{i=1}^{k_0}\Big(\liminf_{t\to T} \int_{B(x_i,\frac{3d}{2})} u(x,t)\eta_i(x)\,dx - \varepsilon\Big) \\
&\le \sum_{i=1}^{k_0} \|u(T-\frac{\mu}{2})\|_{L^1(B(x_i,\frac{3d}{2}))} \le \|u(T-\frac{\mu}{2})\|_1 = \|u_0\|_1.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrarily taken, we see that

$$
\sum_{i=1}^{k_0} \liminf_{t\to T} \int_{B(x_i,\frac{3d}{2})} u(x,t)\eta_i(x)\,dx \le \|u_0\|_1. \tag{4.5}
$$

Combining (4.3) with (4.5), we have by (2.4) that

$$
\begin{aligned}
k_0\varepsilon_0 &< \sum_{i=1}^{k_0} \liminf_{t\to T} \int_{B(x_i,\frac{3d}{2})} u(x,t)\eta_i(x)\,dx \le \|u_0\|_1 \\
&< k_0\varepsilon_0,
\end{aligned} \tag{4.6}
$$

which causes a contradiction.

Proof of Theorem 2.3 (2). Assume that u satisfies one of three conditions (i), (ii) and (iii). Suppose that $^\sharp S_u \ge k_0$. Then, we can select k_0 points $x_1, x_2, \cdots, x_{k_0}$ in S_u so that (4.1) holds for some $d > 0$. By Lemma 4.1 (2), it holds that $\int_{\mathbf{R}^N} u(x,t)\eta_i(x)\,dx$ is continuous on $[0,T]$, where $\eta_i \in C_0^\infty(B_{2d}(x_i))$ is the same cut-off function as in (1). Now it is easy to see that a similar argument as above yields a contradiction. This completes the proof of Theorem 2.3.

§5. Proof of Theorem 2.4

Let us define $M_{i,r}$, $1 \le i \le k$ by

$$
M_{i,r} := \lim_{t\to T} \int_{B(x_i,r)} u(x,t)\eta_i(x)\,dx \qquad \text{for } r > 0, \tag{5.1}
$$

where η_i is the same cut-off function as in Lemma 3.2 such that supp $\eta_i \subset B(x_i, r)$ with $\rho_0 = \frac{r}{2}$ and $\delta = \frac{r}{2}$. It should be noted that the limit in (5.1) exists on account of Lemma 4.1. Since $M_{i,r}$ is monotone decreasing in r and bounded from below by ε_0 for all $i = 1, 2, \cdots, k$, there exists the limit of $M_{i,r}$ as $r \to 0$, *i.e.*, that

$$(5.2) \qquad M_i \ := \ \lim_{r \to 0} M_{i,r} \ < \ \infty \qquad \text{for all } i = 1, 2, \cdots, k.$$

We determine the regular part $f(x)$ of $u(x,t)$ as $t \to T$ in the following lemma without the regularity of $\partial_t u$ in the classical sense.

Lemma 5.1. *Let all assumptions in Theorem 2.4 hold. Then, there exist a function $f \in L^1(\mathbb{R}^N)$ and a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to T$ as $n \to \infty$ such that*

$$f(x) \ = \ \lim_{n \to \infty} u(x, t_n) \qquad a.a. \ x \in \mathbb{R}^N.$$

To establish Lemma 5.1, we deal with u^m instead of u itself, and show that

$$\partial_t u^m \in L^2\big(0, T; H^1(\Omega_r)^*\big), \quad \nabla u^m \in L^2(0, T; L^2(\Omega_r)),$$

where $\Omega_r := \mathbb{R}^N \backslash \bigcup_{i=1}^k B(x_i, r)$. Hence by the well-known interpolation argument, (see Lions–Magenus [9]), we conclude that

$$u^m \in C([0, T]; L^2(\Omega_r)).$$

This continuity of $u^m(\cdot, t)$ at T together with the L^1-conservation law yields Lemma 5.1. This process exhibits a remarkable difference between ours and the 2-D semi-linear case $(\mathrm{KS})_1$, because higher regularity as $u \in C^{2,1}(\mathbb{R}^2 \backslash \bigcup_{i=1}^k B(x_i, r) \times [0, T])$ can be obtained from the standard argument in the latter case.

Using Lemma 5.1, we shall now show that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} u(x, t_n) \psi(x) \, dx \ = \ \sum_{i=1}^k M_i \psi(x_i) \ + \ \int_{\mathbb{R}^N} f(x) \psi(x) \, dx$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Let us take the cut-off functions $\eta_i(x)$, $i = 1, \cdots, k$ as in (5.1). Since $1 - \eta_i(x) = 0$ for all $x \in B(x_i, \frac{r}{2})$, we have by a direct calculation that

$$
\begin{aligned}
&\int_{\mathbf{R}^N} u(x,t)\psi(x)\,dx \;-\; \sum_{i=1}^{k} M_i\psi(x_i) \;-\; \int_{\mathbf{R}^N} f(x)\psi(x)\,dx \\
&\quad= \int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{k} B(x_i,r)} \big(u(x,t)-f(x)\big)\psi(x)\,dx \\
&\qquad - \sum_{i=1}^{k}\int_{B(x_i,r)} f(x)\psi(x)\,dx \\
&\qquad + \sum_{i=1}^{k}\int_{B(x_i,r)} u(x,t)\eta_i(x)\,dx\cdot\psi(x_i) \;-\; \sum_{i=1}^{k} M_i\psi(x_i) \\
&\qquad - \sum_{i=1}^{k}\int_{B(x_i,r)} u(x,t)\eta_i(x)\,dx\cdot\psi(x_i) \\
&\qquad + \sum_{i=1}^{k}\int_{B(x_i,r)} u(x,t)\psi(x)\,dx \\
&\quad= \int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{k} B(x_i,r)} \big(u(x,t)-f(x)\big)\psi(x)\,dx \\
&\qquad - \sum_{i=1}^{k}\int_{B(x_i,r)} f(x)\psi(x)\,dx \\
&\qquad + \sum_{i=1}^{k}\Big(\int_{B(x_i,r)} u(x,t)\eta_i(x)\,dx \;-\; M_i\Big)\psi(x_i) \\
&\qquad + \sum_{i=1}^{k}\int_{B(x_i,r)\setminus B(x_i,\frac{r}{2})} \big(u(x,t)-f(x)\big)\psi(x)\cdot\big(1-\eta_i(x)\big)\,dx \\
&\qquad + \sum_{i=1}^{k}\int_{B(x_i,r)\setminus B(x_i,\frac{r}{2})} f(x)\psi(x)\cdot\big(1-\eta_i(x)\big)\,dx \\
&\qquad + \sum_{i=1}^{k}\int_{B(x_i,r)} u(x,t)\eta_i(x)\cdot\big(\psi(x)-\psi(x_i)\big)\,dx. \qquad (5.3)
\end{aligned}
$$

We have by the definition of the function f that

$$
\Big|\int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{k} B(x_i,r)} \big(u(x,t_n)-f(x)\big)\psi(x)\,dx\Big| \underset{n\to\infty}{\longrightarrow} 0,
$$

$$
\sum_{i=1}^{k}\Big|\int_{B(x_i,r)\setminus B(x_i,\frac{r}{2})} \big(u(x,t_n)-f(x)\big)\psi(x)\cdot\big(1-\eta_i(x)\big)\,dx\Big| \underset{n\to\infty}{\longrightarrow} 0.
$$

Substituting $t = t_n$ in (5.3) and then letting $n \to \infty$, we obtain from (5.1) that

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_{\mathbf{R}^N} u(x,t_n)\psi(x)\,dx - \sum_{i=1}^k M_i\psi(x_i) - \int_{\mathbf{R}^N} f(x)\psi(x)\,dx\right| \\
&\le \sum_{i=1}^k \int_{B(x_i,r)} f(x)\,dx\cdot\max_{x\in\mathbf{R}^N}|\psi(x)| + \sum_{i=1}^k |M_{i,r}-M_i||\psi(x_i)| \\
&\quad + \sum_{i=1}^k \int_{B(x_i,r)} f(x)\,dx\cdot\max_{x\in\mathbf{R}^N}|\psi(x)| \\
&\quad + \sum_{i=1}^k \|u_0\|_1\cdot\max_{x\in B(x_i,r)}|\psi(x)-\psi(x_i)| \\
(5.4) &=: F(r).
\end{aligned}$$

Since $\psi \in C_0^\infty(\mathbb{R}^N)$ and $f \in L^1(\mathbb{R}^N)$, we have, by (5.2), that

$$\lim_{r\to 0} F(r) = 0.$$

Since the left-hand side of (5.4) is independent of r, we conclude that

$$\lim_{n\to\infty}\left|\int_{\mathbf{R}^N} u(x,t_n)\psi(x)\,dx - \sum_{i=1}^k M_i\psi(x_i) - \int_{\mathbf{R}^N} f(x)\psi(x)dx\right| = 0,$$

which completes the proof of Theorem 2.4.

We refer to [23, Sections 4–7] for the proof of Theorem 2.5, of Corollary 2.6, and of Theorems 2.7 and 2.8.

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Department of Mathematics
Tsuda College
2-1-1, Tsuda-chou, Kodaira-shi
Tokyo, 187-8577
Japan

E-mail address: `sugiyama@tsuda.ac.jp`