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Instanton counting and the chiral ring relations in supersymmetric gauge theories

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Abstract.

We compute topological one-point functions of the chiral operator Tr φ^k in the maximally confining phase of $U(N)$ supersymmetric gauge theory. These chiral one-point functions are of particular interest from gauge/string theory correspondence, since they are related to the equivariant Gromov–Witten theory of $\mathbf{P}^1$. By considering the power sums of Jucys–Murphy elements in the class algebra of the symmetric group we can derive a combinatorial identity that leads the relations among chiral one-point functions. Using the operator formalism of free fermions, we also compute the vacuum expectation value of the loop operator $\langle \text{Tr } e^{it\varphi} \rangle$ which gives the generating function of the one-point functions.

§1. Introduction

Among the "web" of various dualities in string theory, one of the most intriguing ideas is *gravity/gauge theory correspondence.* The fact that gravity is described by closed string has been regarded as a miracle of string theory, while the boundaries of open string can incorporate the degrees of freedom of gauge theory. Hence, the gravity/gauge theory correspondence is a kind of the duality of open and closed strings. The discovery of D-branes has led a completely new insight for the correspondence, for example, involving the Chern–Simons theory [27, 7].

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Topological gauge theories and topological string theories often provide exactly solvable models of gauge theory and string theory. In mathematical physics exactly solvable models have played quite important roles in understanding interesting phenomena which are difficult to analyze in perturbation theory; for example, sine-Gordon model in two dimensional quantum field theory for soliton dynamics and boson/fermion correspondence and the Ising model in statistical physics for phase transition. Thus it is interesting to see if we can use topological gauge/string theories to test the idea of gravity/gauge theory correspondence.

Recently Nekrasov proposed a partition function $Z_{\rm Nek}(\epsilon, a_\ell, q)$ that encodes the information of the instanton counting in four dimensional gauge theory [23]. The leading part of the free energy $\log Z_{\rm Nek}(\epsilon, a_\ell, q)$ gives the Seiberg–Witten prepotential [24, 21]. In [23] integrations over the instanton moduli space, which are required for computing the nonperturbative partition function and correlation functions, are evaluated by the equivariant localization principle, where the toric action $(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$ on flat four dimensional space-time $(z_1, z_2) \in \mathbb{C}^2 \simeq \mathbb{R}^4$ is introduced. Physically this corresponds to a special gravitational background, called "Ω background" of Nekrasov, on the six dimensional space-time of a $\mathbb{C}^2$ fibration on two dimensional torus. In this paper we consider the case where the equivariant parameters (ϵ_1, ϵ_2) satisfy the self-duality $\hbar = \epsilon_1 = -\epsilon_2$. The fixed points of the toric action are isolated and labeled by the partitions or in other words the Young diagrams. Thus, we can compute the partition function and correlation functions by taking the summation of appropriate functions on the set of Young diagrams. They are Laurent polynomials in $\hbar$ and q, the parameter of instanton expansion. We can show that a five dimensional lift (or the K-theoretic version [22]) of Nekrasov's partition function $Z^{\rm 5D}_{\rm Nek}$ is nothing but topological string amplitudes (the generating function of the Gromov–Witten invariants) $Z^{(K_S)}_{\rm top\ str}$ on a local toric Calabi–Yau 3-fold K_S, where S is an appropriate toric surface [8, 9, 2, 3, 28]. This agreement of $Z^{\rm 5D}_{\rm Nek}$ and $Z^{(K_S)}_{\rm top\ str}$ is one of the examples of gauge/string theory correspondence in topological theory, which is expected from the idea of geometric engineering [13].

In this article, we will explore a similar example of gauge/string theory correspondence, which involves the chiral operators $\mathrm{Tr}\ \varphi^J$ in $\mathcal{N}=1$ $U(N)$ supersymmetric gauge theory, where φ is the (Higgs) scalar field in the adjoint representation. $\mathcal{N}=1$ theory is obtained by turning on a tree level superpotential $W(\Phi)$ that softly breaks $\mathcal{N}=2$ supersymmetry. We will compute one-point functions of $\mathrm{Tr}\ \varphi^J$ in the maximally confining phase, where effective low energy symmetry is reduced to the diagonal subgroup $U(1) \subset U(N)$. In [16] it is claimed that there is a correspondence between the chiral operators in supersymmetric gauge theory and the topological operators in the Gromov–Witten theory of the rational curve $\mathbf{P}^1$ developed by Okounkov and Pandharipande [25, 26]. More concretely, we expect the correspondence Tr $\varphi^J \Longleftrightarrow \tau_p(\omega)$, where $\tau_p(\omega)$ is the p-th gravitational descendant of the Kähler class ω of $\mathbf{P}^1$. We have checked that the one point functions we compute exactly agree to a particular sector of the equivariant Gromov–Witten invariants of $\mathbf{P}^1$.

We show that the chiral one-point functions $\langle \mathrm{Tr}\ \varphi^{2j} \rangle$ satisfy the relations

$$\sum_{j=1}^{r} c_j^r \hbar^{2(r-j)} \langle \mathrm{Tr}\ \varphi^{2j} \rangle = \frac{(2r)!}{(r!)^2} q^r , \tag{1}$$

where the coefficients c_j^r are defined by $\prod_{j=0}^{r-1}(x^2 - j^2) = \sum_{j=1}^{r} c_j^r x^{2j}$ or a specialization of the elementary symmetric functions $e_n(x)$; $c_j^r = (-1)^{r-j} e_{r-j}(1^2, 2^2, \cdots, (r-1)^2)$. The relation (1) is proved in [6]. Our proof is based on combinatorial identities, which we obtain by considering the power sums of Jucys–Murphy elements in the class algebra of the symmetric group [10, 18, 14]. We also compute the generating function of $\langle \mathrm{Tr}\ \varphi^{2j} \rangle$, which is related to the vacuum expactation value of the loop operator $\langle \mathrm{Tr}\ e^{it\varphi} \rangle$ by the Laplace transformation. The operator formalism of free fermions, which was also employed in [25, 26], gives a rather simple result;

$$\langle \mathrm{Tr}\ e^{it\varphi} \rangle = I_0 \left(4\sqrt{q}\ \frac{\sinh(it\hbar/2)}{\hbar} \right) \to I_0(2i\sqrt{q}t) , \quad (\hbar \to 0) , \tag{2}$$

with $I_0(x)$ being the modified Bessel function. By taking the Laplace transformation of (2), we find that the generating function $T(z)$ is given by

$$T(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} q^n \prod_{m=-n}^{n} \frac{1}{z - m\hbar} \to \frac{1}{\sqrt{z^4 - 4q}} , \quad (\hbar \to 0) , \tag{3}$$

which is consistent with the relations (1).

The paper is organized as follows. In section 2 we summarize basic facts on the chiral ring of supersymmetric gauge theory and give a characterization of the maximally confining phase. In section 3 we first explain what the one-point function $\langle \mathrm{Tr}\ \varphi^{2j} \rangle$ means mathematically. Then we outline a derivation of (1). The operator formalism is introduced in section 4. The computation of $\langle \mathrm{Tr}\ e^{it\varphi} \rangle$ and the generating function $T(z)$ is worked out in section 5.

§2. Chiral ring of supersymmetric gauge theory

In this section we review the generating functions of one-point functions of chiral ring generators, closely following [1]. Chiral operators $\mathcal{O}$ in supersymmetric field theories are, by definition, annihilated by the fermionic charges $\overline{Q}_{\dot\alpha}$ of one chirality; $[\overline{Q}_{\dot\alpha}, \mathcal{O}]_{\pm} = 0$, considered modulo $\overline{Q}_{\dot\alpha}$-exact operators; $\mathcal{O} \simeq \mathcal{O} + [\overline{Q}_{\dot\alpha}, \Lambda]_{\pm}$. The set of chiral operators forms a ring, called the chiral ring. From the supersymmetry algebra in four dimensions, $[Q_\alpha, \overline{Q}_{\dot\alpha}]_+ = \sigma^\mu_{\alpha\dot\alpha} P_\mu$, we can see that correlation functions of chiral operators are "topological" in the sense that they are independent of the positions of operators. Especially, topological one-point functions characterize the vacuum structure (phase) of the theory, namely they are the moduli of vacua.

Let us consider a four dimensional $\mathcal{N} = 1$ supersymmetric gauge theory with a single chiral multiplet Φ. This might be regarded as a (softly) broken $\mathcal{N} = 2$ theory by a tree level superpotential $W(\Phi)$. We have a vector multiplet (A_m, λ_α) and a chiral multiplet $\Phi = (\varphi, \psi_\alpha)$. They are all in the adjoint representation of $U(N)$. The operators $\mathrm{Tr}\ \varphi^k$ $(k \in \mathbb{Z}_+)$ are examples of gauge invariant chiral operators. One can show that the generators of the chiral ring are of the form $\mathrm{Tr}\ \varphi^k$, $\mathrm{Tr}\ \lambda_\alpha \varphi^k$ and $\mathrm{Tr}\ \lambda_\alpha \lambda^\alpha \varphi^k$. Classically they are subject to several relations of group theoretical origin, but we expect they are modified by instanton corrections. The modified relations are realized by those among topological correlation functions of chiral operators. In [1] the Ward identity of the generalized Konishi anomaly in supersymmetric gauge theory was formulated in terms of the generating functions of one-point functions:

$$
\begin{aligned}
T(z) &= \sum_{k=0}^{\infty} z^{-1-k} \langle \mathrm{Tr}\ \varphi^k \rangle = \left\langle \mathrm{Tr}\ \frac{1}{z-\varphi} \right\rangle , \\
w_\alpha(z) &= \sum_{k=0}^{\infty} z^{-1-k} \langle \mathrm{Tr}\ \lambda_\alpha \varphi^k \rangle = \left\langle \mathrm{Tr}\ \lambda_\alpha \frac{1}{z-\varphi} \right\rangle , \\
R(z) &= -\frac{1}{2} \sum_{k=0}^{\infty} z^{-1-k} \langle \mathrm{Tr}\ \lambda_\alpha \lambda^\alpha \varphi^k \rangle = -\frac{1}{2} \left\langle \mathrm{Tr}\ \lambda_\alpha \lambda^\alpha \frac{1}{z-\varphi} \right\rangle .
\end{aligned}
$$

When the fermionic expectation values vanish, $w_\alpha(z) = 0$, the generalized Konishi anomaly equations are

$$
R(z)^2 - W'(z) R(z) - \frac{1}{4} f(z) = 0 , \tag{4}
$$

$$
2R(z)T(z) - W'(z)T(z) - \frac{1}{4} c(z) = 0 , \tag{5}
$$

where $W(\Phi) = \sum_{r=0}^{k} \frac{g_r}{r+1} \mathrm{Tr}\ \Phi^{r+1}$ is a superpotential of degree $k+1$. $f(z)$ and $c(z)$ are polynomials of degree $k-1$ that depend on the superpotential. In deriving the above equations one uses the fact that the correlation functions of a product of chiral operators factorize. Solving the above equations, we obtain

$$\begin{aligned} R(z) &= \frac{1}{2}\left(W'(z) - \sqrt{W'(z)^2 + f(z)}\right) , \\ T(z) &= -\frac{c(z)}{4\sqrt{W'(z)^2 + f(z)}} . \end{aligned}$$

The point is that the equation (4) for $R(z)$ is exactly the same as the loop equation in the matrix model in large N limit, while it is $T(z)$ that can be directly related to the Seiberg–Witten theory. It is a crucial observation in [1] that these two generating functions $R(z)$ and $T(z)$, which are related to the large N matrix model and the Seiberg–Witten theory, are in the same multiplet under a (hidden) fermionic symmetry. Hence we may establish a connection of the Seiberg–Witten theory to the matrix model.

For $U(N)$ gauge theory the generating function $T(z)$ is classically given by

$$T(z) = \frac{P_N'(z)}{P_N(z)} , \qquad P_N(z) := \langle \det\ (z - \varphi) \rangle .$$

However, the Seiberg–Witten theory tells

$$T(z) = \frac{P_N'(z)}{\sqrt{P_N(z)^2 - 4\Lambda^{2N}}} ,$$

where $y^2 = P_N(z)^2 - 4\Lambda^{2N}$ is the Seiberg–Witten curve. The coefficients of the characteristic polynomial $P_N(z)$ are the moduli parameters of $\mathcal{N} = 2$ vacua. When the superpotential $W(\Phi) = \sum_{r=0}^{k} \frac{g_r}{r+1} \mathrm{Tr}\ \Phi^{r+1}$ is turned on, the vacua are on a codimension $N-k$ submanifold of the moduli space. This is described as the following factorization of the Seiberg–Witten curve

$$y^2 = P_N(z)^2 - 4\Lambda^{2N} = H_{N-k}^2(z) F_{2k}(z) , \tag{6}$$

which appears on the codimension $N-k$ submanifold of the moduli space of $\mathcal{N} = 2$ vacua. The factorization (6) implies that $N-k$ mutually non-intersecting one-cycles of the Seiberg–Witten curve collapse, which physically means the emergence of massless monopoles, or dyons which has both electric and magnetic charges. On such vacua $SU(N-k+1) \subset U(N)$ symmetry is confined due to massless monopoles and the residual symmetry is $U(1)^k$. $F_{2k}(z)$ is called reduced Seiberg–Witten curve. When the Seiberg–Witten curve factorizes as above, we can also factorize $P'_N(z) = H_{N-k}(z)R_{k-1}(z)$ and consequently

$$T(z) = \frac{P'_N(z)}{\sqrt{P_N(z)^2 - 4\Lambda^{2N}}} = \frac{R_{k-1}(z)}{\sqrt{F_{2k}(z)}} .$$

In particular, if the tree level superpotential is a mass perturbation $W(\Phi) = \frac{1}{2}m\mathrm{Tr}\ \Phi^2$, the vacua are those with $N-1$ massless monopoles and the reduced curve is $F_2(z) = m^2z^2 - \Lambda^2$. This is called maximally confining phase, where the remaining low energy effective symmetry is $U(1)$. In this phase, we have

$$T(z) = \frac{R_0(z)}{\sqrt{F_2(z)}} = \frac{N}{\sqrt{z^2 - \frac{\Lambda^2}{m^2}}} , \tag{7}$$

where $R_0 = Nm$. In the following we will compute $\langle \mathrm{Tr}\ \varphi^k \rangle$ by the instanton counting of $U(1)$ theory and derive the chiral ring relations among $\langle \mathrm{Tr}\ \varphi^k \rangle$. In particular, the relation implies the generating function $T(z)$ is given by (7), which means the theory is in the maximally confining phase.

§3. One-point functions in maximally confining phase

The computation of the one-point function $\langle \mathrm{Tr}\ \varphi^{2n} \rangle$ mathematically involves the Chern class $\mathrm{Tr}\ \mathcal{F}^{2n}$ of a universal sheaf $\mathcal{E}$ on $\mathbb{C}^2 \times \mathcal{M}_{N,k}$ [16, 5, 6], where $\mathcal{M}_{N,k}$ is the moduli space of (framed) instantons in $U(N)$ gauge theory with instanton number k. The moduli space $\mathcal{M}_{N,k}$ is defined, for example, by the ADHM construction. The ADHM construction tells us that when $N = 1$, $\mathcal{M}_{1,k}$, after the resolution of singularities, is isomorphic to the Hilbert scheme of points $(\mathbb{C}^2)^{[k]}$ on $\mathbb{C}^2$. Over the instanton moduli space $\mathcal{M}_{N,k}$ we have two vector bundles W and V of rank N and k, which naturally arise in the ADHM construction. The ADHM data are identified as $B_1, B_2 \in \mathrm{Hom}(V,V)$ and $J, I^\dagger \in \mathrm{Hom}(W,V)$. Roughly speaking, the vector bundle W comes from a local trivialization of the instanton at infinity[1], while the bundle V is the bundle of Dirac zero modes. The fiber of V is the space of normalizable solutions to the Dirac equation in the instanton background. The Riemann–Roch theorem tells that the number of Dirac

[1]The moduli space $\mathcal{M}_{N,k}$ is defined by the quotient by the gauge transformations that fix the "framing" at infinity.

zero modes is just the instanton number k. From vector bundles E_1 on $\mathbb{C}^2$ and E_2 on $\mathcal{M}_{N,k}$, we can construct an external tensor product bundle $E_1 \boxtimes E_2 := p_1^* E_1 \otimes p_2^* E_2$ on $\mathbb{C}^2 \times \mathcal{M}_{N,k}$, where p_i denotes the projection to the i-th component. Then as an element of the equivariant K-cohomology group the universal sheaf is isomorphic to the virtual vector bundle [20, 16];

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{C}^2} \boxtimes W \oplus (S^- - S^+) \boxtimes V ,$$

where $S^\pm$ are the positive and the negative spinor bundles on $\mathbb{C}^2$. Their equivariant characters are

$$\mathrm{Ch}(S^+)\ (t) = 1 + e^{it(\epsilon_1+\epsilon_2)} , \quad \mathrm{Ch}(S^-)\ (t) = e^{it\epsilon_1} + e^{it\epsilon_2} ,$$

where (ϵ_1, ϵ_2) are equivariant parameters of the toric action $(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$ on $\mathbb{C}^2$, which induces a toric action on $\mathcal{M}_{N,k}$. Note that we have another toric action on $\mathcal{M}_{N,k}$ that comes from the maximal abelian subgroup $U(1)^N$ of the gauge group $U(N)$. According to the work of Nakajima [19], the fixed point set of the $U(1)^2 \times U(1)^N$ action on $\mathcal{M}_{N,k}$ is in one-to-one correspondence with the set $\mathcal{P}_N(k)$ of N-tuples of Young diagrams whose total number of boxes is equal to k. At a fixed point labeled by N-tuples of Young diagrams Y_α the contribution to the Chern character of $\mathcal{E}$ is [20, 16],

$$\begin{aligned}&\mathrm{Ch}(\mathcal{E})_{\underline{Y}}(t)\\ &= \sum_{\alpha=1}^{N} e^{ita_\alpha} \left(1 - \left(1 - e^{it\epsilon_1}\right)\left(1 - e^{it\epsilon_2}\right) \sum_{(k,\ell)\in Y_\alpha} e^{it\epsilon_1(k-1)+it\epsilon_2(\ell-1)} \right) .\end{aligned}$$

Since we identify $\mathcal{F}$ as a curvature on the universal bundle $\mathcal{E}$, we have $\mathrm{Tr}\ \mathcal{F}^n = c_n(\mathcal{E})$, where the n-th Chern class $c_n(\mathcal{E})$ is defined by the expansion

$$\mathrm{Ch}(\mathcal{E})_{\underline{Y}}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} c_n(\mathcal{E})_{\underline{Y}} . \tag{8}$$

In the maximally confining phase introduced in section 2, the residual symmetry is the diagonal subgroup $U(1) \subset U(N)$. Hence it is sufficient to consider the $U(1)$ theory and we put $\varphi_{cl} = a = 0$ for simplicity. The fixed points are labeled by a single Young diagram Y and the Chern character is

$$\mathrm{Ch}(\mathcal{E})_Y = 1 - \left(1 - e^{it\epsilon_1}\right)\left(1 - e^{it\epsilon_2}\right) \sum_{(k,\ell)\in Y} e^{it\epsilon_1(k-1)+it\epsilon_2(\ell-1)} . \tag{9}$$

Putting $\epsilon_1 = -\epsilon_2 = \hbar$ and comparing (8) and (9), we find $\mathrm{Tr}\ \varphi_Y^{2n} = c_{2n}(\mathcal{E})_Y$ is given by

$$\begin{aligned}\mathrm{Tr}\ \varphi_Y^{2n} &= \hbar^{2n} \sum_{(k,\ell)\in Y} \left[(k-\ell+1)^{2n} + (k-\ell-1)^{2n} - 2(k-\ell)^{2n}\right] \\ &= \hbar^{2n} \sum_{m=0}^{n-1} 2\binom{2n}{2m} \sum_{\square\in Y} c(\square)^{2m} , \qquad (10)\end{aligned}$$

where $c(\square) := (\ell - k)$ is called the content at $\square = (k,\ell)$.

The Nekrasov's partition function of $U(1)$ gauge theory is

$$Z_{U(1)} = \sum_{k=0}^{\infty} \sum_{|Y|=k} \frac{1}{\prod_{\square\in Y}(\hbar h(\square))^2} q^k , \qquad (11)$$

where $h(\square)$ is the hook length at $\square$ and we have introduced a parameter q of the instanton expansion. It is a classical result in representation theory that

$$\prod_{\square\in Y} \frac{1}{h(\square)} = \frac{\dim\ S^Y}{k!} ,$$

where S^Y is the irreducible representation of the symmetric group labeled by the Young diagram Y. We obtain

$$\sum_{|Y|=k} \prod_{\square\in Y} h(\square)^{-2} = \frac{1}{k!} .$$

by the Plancherel formula $\sum_{|Y|=k} (\dim\ S^Y)^2 = k!$. Hence we find that the summation over the instanton number k in (11) is organized into a simple form;

$$Z_{U(1)} = \exp\left(\frac{q}{\hbar^2}\right) . \qquad (12)$$

The correlation functions of our interest are

$$\langle \mathrm{Tr}\ \varphi^{2n} \rangle = \frac{1}{Z_{U(1)}} \sum_{k=1}^{\infty} \sum_{|Y|=k} \frac{\mathrm{Tr}\ \varphi_Y^{2n}}{\hbar^{2k} \prod_{\square\in Y} h(\square)^2} q^k . \qquad (13)$$

Substituting the formula (10), we have

$$\langle \mathrm{Tr}\ \varphi^{2n} \rangle \exp\left(\frac{q}{\hbar^2}\right) = 2 \sum_{m=0}^{n-1} \binom{2n}{2m} \sum_{k=1}^{\infty} S_m(k) \hbar^{2(n-k)} q^k ,$$

where we have introduced

$$S_n(k) := \sum_{|Y|=k} \frac{\sum_{\square \in Y} c(\square)^{2n}}{\prod_{\square \in Y} h(\square)^2} . \tag{14}$$

Thus the computation of $\langle \mathrm{Tr}\ \varphi^{2n} \rangle$ is equivalent to giving a summation formula for $S_n(k)$ over Young diagrams. Looking at the Young diagrams with lower number of boxes explicitly, we find

$$\begin{aligned}
\langle \mathrm{Tr}\ \varphi^2 \rangle &= 2q , \\
\langle \mathrm{Tr}\ \varphi^4 \rangle &= 6q^2 + 2q\hbar^2 , \\
\langle \mathrm{Tr}\ \varphi^6 \rangle &= 20q^3 + 30q^2\hbar^2 + 2q\hbar^4 , \\
\langle \mathrm{Tr}\ \varphi^8 \rangle &= 70q^4 + 280q^3\hbar^2 + 126q^2\hbar^4 + 2q\hbar^6 .
\end{aligned}$$

In [6] we have proved the following formula;

$$\sum_{j=1}^{n} c_j^n S_j(k) = \frac{(2n)!}{((n+1)!)^2} \frac{1}{(k-n-1)!} , \tag{15}$$

which recursively determines $S_n(k)$. The coefficients c_j^n in (15) are defined by[2]

$$\mathcal{P}_{2n}(x) := x^{\overline{n}} \cdot x^{\underline{n}} = \prod_{j=0}^{n-1} (x^2 - j^2) = \sum_{j=1}^{n} c_j^n x^{2j} . \tag{16}$$

Note that c_j^n are given by a specialization of the elementary symmetric functions $e_r(x)$; $c_j^n = (-1)^{n-j} e_{n-j}(1^2, 2^2, \cdots, (n-1)^2)$. The formula (15) implies the following relation among topological one-point functions valid in the maximally confining phase;

$$\sum_{j=1}^{r} c_j^r \hbar^{2(r-j)} \langle \mathrm{Tr}\ \varphi^{2j} \rangle = \frac{(2r)!}{(r!)^2} q^r . \tag{17}$$

[2] The functions $x^{\overline{n}}$ and $x^{\underline{n}}$ are natural power functions in the calculus of difference.

To prove (17), we first plug the formula (10) into the definition (13) of $\langle \mathrm{Tr}\ \varphi^{2j} \rangle$ to obtain

$$\begin{aligned}
&\sum_{j=1}^{r} c_j^r \hbar^{2(r-j)} \langle \mathrm{Tr}\ \varphi^{2j} \rangle Z_{U(1)} \\
&= \hbar^{2r} \sum_{k=1}^{\infty} \sum_{|Y|=k} \left(\frac{q}{\hbar^2}\right)^k \sum_{j=1}^{r} c_j^r \frac{\sum_{\square \in Y} \left(c(\square)+1\right)^{2j} - 2\left(c(\square)\right)^{2j} + \left(c(\square)-1\right)^{2j}}{\prod_{\square \in Y} h(\square)^2} \\
&= \hbar^{2r} \sum_{k=1}^{\infty} \sum_{|Y|=k} \left(\frac{q}{\hbar^2}\right)^k \frac{\sum_{\square \in Y} \left[\mathcal{P}_{2r}(c(\square)+1) - 2\mathcal{P}_{2r}(c(\square)) + \mathcal{P}_{2r}(c(\square)-1)\right]}{\prod_{\square \in Y} h(\square)^2} \\
&= 2r(2r-1)\hbar^{2r} \sum_{k=1}^{\infty} \sum_{|Y|=k} \left(\frac{q}{\hbar^2}\right)^k \frac{\sum_{\square \in Y} \prod_{j=0}^{r-2} \left(c(\square)^2 - j^2\right)}{\prod_{\square \in Y} h(\square)^2} ,
\end{aligned}$$

where in the last line we have used the following relation satisfied by $\mathcal{P}_{2n}(x)$[3];

$$\Delta^2 \mathcal{P}_{2n}(x) := \mathcal{P}_{2n}(x+1) - 2\mathcal{P}_{2n}(x) + \mathcal{P}_{2n}(x-1) = 2n(2n-1)\mathcal{P}_{2n-2}(x) .$$

Finally we note that (15) means the following combinatorial formula;

$$\sum_{|Y|=k} \frac{\sum_{\square \in Y} \prod_{j=0}^{r-2} (c^2 - j^2)}{\prod_{\square \in Y} h(\square)^2} = \frac{(2(r-1))!}{(r!)^2} \frac{1}{(k-r)!} . \tag{18}$$

This allows us to factor out the partition function $Z_{U(1)}$ as follows;

$$\begin{aligned}
\sum_{j=1}^{r} c_j^r \hbar^{2(r-j)} \langle \mathrm{Tr}\ \varphi^{2j} \rangle Z_{U(1)} &= \frac{(2r)!}{(r!)^2} q^r \sum_{k=1}^{\infty} \frac{1}{(k-r)!} \left(\frac{q}{\hbar^2}\right)^{k-r} \\
&= \frac{(2r)!}{(r!)^2} q^r \exp\left(\frac{q}{\hbar^2}\right) .
\end{aligned}$$

Dividing both sides by $Z_{U(1)}$, we obtain (17).

§4. Computations in operator formalism

Due to the correspondence of Young diagrams (or Maya diagrams) and the fermion Fock states with neutral charge, the operator formalism

[3] This formula is a discrete version of $\frac{d^2}{dx^2} x^{2n} = 2n(2n-1)x^{2n-2}$.

is a very powerful tool for computing summations over the set of Young diagrams. Let us introduce a pair of charged free fermions

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r z^{-r-\frac{1}{2}} , \quad \psi^*(z) = \sum_{s\in\mathbb{Z}+\frac{1}{2}} \psi^*_s z^{-s-\frac{1}{2}} ,$$

with the anti-commutation relation

$$\{\psi_r, \psi^*_s\} = \delta_{r+s,0} , \quad r, s \in \mathbb{Z} + \frac{1}{2} .$$

The Fock vacuum $|0\rangle$ is defined by

$$\psi_r|0\rangle = \psi^*_s|0\rangle = 0 , \qquad r, s > 0 .$$

Using Young/Maya diagram correspondence, for each partition λ, we have a state $|\lambda\rangle$ in the charge zero sector of the fermion Fock space, which is given by

$$|\lambda\rangle = \prod_{i=1}^{\infty} \psi_{i-\lambda_i-\frac{1}{2}} \|0\rangle\rangle ,$$

with

$$\psi^*_s\|0\rangle\rangle = 0 , \qquad \forall s \in \mathbb{Z} + \frac{1}{2} .$$

Recall the standard bosonization rule;

$$J(z) =: \psi(z)\psi^*(z) := \sum_{n\in\mathbb{Z}} J_n z^{-n-1} , \quad J_n = \sum_{r\in\mathbb{Z}+\frac{1}{2}} : \psi_r \psi^*_{n-r} : , \tag{19}$$

$$J(z) = i\partial\phi(z), \qquad \psi(z) =: e^{i\phi(z)} :, \quad \psi^*(z) =: e^{-i\phi(z)} :,$$

where : : means the normal ordering. Now a crucial point is the following formula

$$\exp\left(\frac{J_{-1}}{\hbar}\right)|0\rangle = \sum_{k=0}^{\infty} \frac{1}{\hbar^k} \sum_{|\lambda|=k} \frac{1}{\prod_{\square\in\lambda} h(\square)} |\lambda\rangle , \tag{20}$$

which is eq.(5.29) of [24]. In the language of symmetric functions, the corresponding formula is given in [17].

What we want to compute is $S_n(k)$ defined by (14). Let us introduce the generating function of $S_n(k)$;

$$\mathrm{Ch}[k](z) := \sum_{|\lambda|=k} \frac{\sum_{\square\in\lambda} \exp(zc(\square))}{\prod_{\square\in\lambda} h(\square)^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} S_n(k) , \tag{21}$$

where we have used the fact that $c(\square)$ is odd under the transpose of the Young diagram. This generating function gives the Chern character of the tautological vector bundle over $(\mathbb{C}^2)^{[k]}$ considered in [15]. The sum in the numerator is

$$\begin{aligned}\sum_{\square\in\lambda} \exp\left(zc(\square)\right) &= \sum_{j=1}^{d(\lambda)}\sum_{i=1}^{\lambda_j} e^{z(i-j)} = \sum_{j=1}^{d(\lambda)} e^{-jz}\frac{e^z - e^{z(\lambda_j+1)}}{1-e^z} \\ &= \sum_{j=1}^{d(\lambda)} \frac{e^{z(\lambda_j-j+\frac{1}{2})} - e^{z(-j+\frac{1}{2})}}{e^{\frac{z}{2}} - e^{-\frac{z}{2}}} .\end{aligned}$$

Following Okounkov and Pandharipande [25, 26], we consider the operator

$$\mathcal{E}_n(z) := \sum_{r\in\mathbb{Z}+\frac{1}{2}} e^{z\left(r-\frac{n}{2}\right)} E_{n-r,r} , \tag{22}$$

where $E_{r,s} :=: \psi_r\psi_s^* :$ is the standard basis of $\mathfrak{gl}(\infty)$ acting on the fermion Fock space. We can see that $\mathcal{E}_n(z)$ satisfies the commutation relation[4]

$$[\mathcal{E}_n(z), \mathcal{E}_m(w)] = \mathrm{sh}(nw-mz)\ \mathcal{E}_{n+m}(z+w) + \delta_{n+m,0}\frac{\mathrm{sh}\, n(z+w)}{\mathrm{sh}\,(z+w)} , \tag{23}$$

where $\mathrm{sh}(z) := e^{\frac{z}{2}} - e^{-\frac{z}{2}}$. We have $\mathcal{E}_n(0) = J_n$ with J_n being the modes of the $U(1)$ current in (19). We also find a useful relation

$$\mathcal{E}_0(z)|\lambda\rangle = \sum_{i=1}^{\infty}\left(e^{z\left(\lambda_i-i+\frac{1}{2}\right)} - e^{z\left(\frac{1}{2}-i\right)}\right)|\lambda\rangle = \mathrm{sh}(z)\sum_{\square\in\lambda}\exp\left(zc(\square)\right)|\lambda\rangle .$$

The second term comes from $\mathcal{E}_0(z)|0\rangle\rangle = -(\mathrm{sh}(z))^{-1}|0\rangle\rangle$, which can be calculated directly from the definition of $|0\rangle\rangle$ or from the consistency $\mathcal{E}_0(z)|0\rangle = 0$. By the formula (20), the generating function of $S_n(k)$ is expressed in the operator formalism as follows;

$$(k!)^2\mathrm{sh}(z)\mathrm{Ch}[k](z) = \langle 0|J_1^k\mathcal{E}_0(z)J_{-1}^k|0\rangle .$$

[4]The infinite-dimensional Lie algebra with the commutation relation (23) appeared first in [4], where it was called area-preserving torus diffeomorphism algebra. The representation theory was initiated in [11] and the algebra was identified with $W_{1+\infty}$ algebra.

The right hand side can be computed by the commutation relation (23), which implies

$$J_1^k \mathcal{E}_0(z) = \sum_{\ell=0}^{k} \binom{k}{\ell} \mathrm{sh}^\ell(z) \mathcal{E}_\ell(z) J_1^{k-\ell} \, .$$

We also use

$$J_1^n J_{-1}^k |0\rangle = \frac{k!}{(k-n)!} J_{-1}^{k-n} |0\rangle \, , \quad (n \le k) \, ,$$

which is derived from

$$e^{zJ_1} e^{wJ_{-1}} |0\rangle = e^{[zJ_1, wJ_{-1}]} e^{wJ_{-1}} e^{zJ_1} |0\rangle = e^{zw} e^{wJ_{-1}} |0\rangle.$$

By these formulas, we obtain

$$\begin{aligned} \langle 0| J_1^k \mathcal{E}_0(z) J_{-1}^k |0\rangle &= \sum_{\ell=0}^{k} \binom{k}{\ell} \mathrm{sh}^\ell(z) \langle 0| \mathcal{E}_\ell(z) \frac{k!}{\ell!} J_{-1}^\ell |0\rangle \\ &= (k!)^2 \sum_{\ell=1}^{k} \frac{\mathrm{sh}(z)^{2\ell-1}}{(\ell!)^2 (k-\ell)!} \, . \end{aligned}$$

The contribution from $\ell = 0$ is simply zero because $\langle 0| \mathcal{E}_0(z) |0\rangle = 0$. Thus we find

$$\mathrm{Ch}[k](z) = \sum_{\ell=1}^{k} \frac{\mathrm{sh}(z)^{2\ell-2}}{(\ell!)^2 (k-\ell)!} \, . \tag{24}$$

We have computed the Taylor expansion of $\mathrm{Ch}[k](z)$ for each fixed k and found exact agreements with the results of the formula (15). Note that the formula (15) rather gives $S_n(k)$ as a function of k for each fixed n.

§5. Loop operator and the generating function

The vacuum expectation value of the Wilson loop operator $\langle \mathrm{Tr}\ e^{it\varphi} \rangle$ is related to the generating function of one-point functions simply by the Laplace transformation:

$$T(z) = \left\langle \mathrm{Tr}\ \frac{1}{z-\varphi} \right\rangle = \int_0^\infty dl e^{-lz} \langle \mathrm{Tr}\ e^{l\varphi} \rangle. \tag{25}$$

By (13) together with the formula (10), the Wilson loop operator can be expressed as

$$\langle \mathrm{Tr}\ e^{it\varphi} \rangle - 1 = \frac{1}{Z_{U(1)}} \sum_{k=1}^{\infty} \left[\frac{q}{\hbar^2} \right]^k \frac{\sum_{\square \in Y} (e^{z(c(\square)+1)} + e^{z(c(\square)-1)} - 2e^{z(c(\square))})}{\prod_{\square \in Y} (h(\square))^2}$$

where $z = it\hbar$. We can further put it into

$$\langle \mathrm{Tr}\ e^{it\varphi} \rangle - 1 = \frac{1}{Z_{U(1)}} \sum_{k=1}^{\infty} \left[\frac{q}{\hbar^2} \right]^k (\mathrm{sh}(z))^2 \mathrm{Ch}[k](z), \tag{26}$$

using the generating function $\mathrm{Ch}[k](z)$ defined in (21). The final expression (24) of $\mathrm{Ch}[k](z)$ derived in section 4 implies

$$\begin{aligned} \langle \mathrm{Tr}\ e^{it\varphi} \rangle - 1 &= \frac{1}{Z_{U(1)}} \sum_{k=1}^{\infty} \sum_{l=1}^{k} \left[\frac{q}{\hbar^2} \right]^k \frac{\mathrm{sh}(z)^{2l}}{(l!)^2 (k-l)!} \\ &= \frac{1}{Z_{U(1)}} \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \left[\frac{q}{\hbar^2} \right]^k \frac{\mathrm{sh}(z)^{2l}}{(l!)^2 (k-l)!}, \end{aligned}$$

where in the last equation we have exchanged the k summation and the l summation. Performing the k summation first,

$$\sum_{k=l}^{\infty} \frac{1}{(k-l)!} \left[\frac{q}{\hbar^2} \right]^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{q}{\hbar^2} \right]^{k+l} = Z_{U(1)} \left[\frac{q}{\hbar^2} \right]^l,$$

we find the Wilson loop operator is given as

$$\langle \mathrm{Tr}\ e^{it\varphi} \rangle = 1 + \sum_{l=1}^{\infty} \frac{1}{(l!)^2} \left[\frac{q}{\hbar^2} (\mathrm{sh}(z))^2 \right]^l = I_0\big(2\sqrt{q}\,\mathrm{sh}(it\hbar)/\hbar\big), \tag{27}$$

where $I_0(x)$ is the modified Bessel function. It is somewhat surprising that we can perform the instanton sum of the loop operator completely and obtain an exact result (27) in a closed form. We can also obtain an exact result on $T(z)$ from (27). The Laplace transformation (25) implies

$$\begin{aligned} T(z) &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{q}{\hbar^2} \right)^n \int_0^{\infty} dl e^{-lz} \mathrm{sh}^{2n}(l\hbar) \\ &= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{q}{\hbar^2} \right)^n \sum_{m=-n}^{n} \frac{(-1)^{n-m}}{(n-m)!(n+m)!} \frac{1}{z - m\hbar}. \end{aligned}$$

By computing the residue at $z = m\hbar$ $(-n \le m \le n)$, we find a partial fraction expansion

$$\prod_{m=-n}^{n} \frac{1}{z - m\hbar} = \hbar^{-2n} \sum_{m=-n}^{n} \frac{(-1)^{n-m}}{(n-m)!(n+m)!} \frac{1}{z - m\hbar},$$

which gives

$$
\begin{aligned}
(28) \qquad T(z) &= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} q^n \prod_{m=-n}^{n} \frac{1}{z - m\hbar} \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} q^n z^{-2n-1} \prod_{m=1}^{n} \sum_{k=0}^{\infty} \left(\frac{m\hbar}{z} \right)^{2k} .
\end{aligned}
$$

This is a remarkably simple answer to the $\hbar$ dependence of $T(z)$.

We can check that the final result (28) is consistent with the chiral ring relations (17) derived in section 3. Recall that the coefficients c_j^r in the relation (17) are defined by (16). We find

$$
c_r^r = 1 , \qquad c_{r-1}^r = -\sum_{j=0}^{r-1} j^2 = -\frac{1}{6} r(r-1)(2r-1) .
$$

Substituting these to the relation (17), we obtain

$$
\langle \mathrm{Tr}\ \varphi^{2r} \rangle = \frac{(2r)!}{(r!)^2} q^r + \frac{\hbar^2}{12} \frac{(2r)!}{(r-1)!(r-2)!} q^{r-1} + O(\hbar^4) .
$$

Hence

$$
T(z) = \sum_{n=0}^{\infty} z^{-2n-1} \frac{(2n)!}{(n!)^2} q^n + \sum_{n=2}^{\infty} z^{-2n-1} \frac{\hbar^2}{12} \frac{(2n)!}{(n-1)!(n-2)!} q^{n-1} + O(\hbar^4) ,
$$

which agrees to the $\hbar$ expansion of (28). Finally from the Taylor expansion;

$$
\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n , \qquad |x| < \frac{1}{4} ,
$$

and its derivatives, we have

$$
\begin{aligned}
T(z) &= \frac{1}{\sqrt{z^2 - 4q}} + \hbar^2 \frac{2q(q+z^2)}{(z^2 - 4q)^{\frac{7}{2}}} + O(\hbar^4) \\
&= \frac{1}{\sqrt{z^2 - 4q}} \left(1 + \hbar^2 \frac{2q(q+z^2)}{(z^2-4q)^3} + O(\hbar^4) \right) ,
\end{aligned}
$$

whose leading term agrees with the result reviewed in section 2. By similar computations of the two-point function $\langle \mathrm{Tr}\ e^{it\varphi}\ \mathrm{Tr}\ e^{is\varphi} \rangle$ of the loop operators, we can also obtain the other generating function $R(z)$ appeared in section 2. The result again exactly agrees to the prediction of the Konishi anomaly equation or the matrix models. See [12] for more details.

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