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Horospherical geometry in the hyperbolic space

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Abstract.

This is a survey article on the recent results of the "horospherical geometry" in the hyperbolic space. Detailed arguments for the results have been appeared or will appear in several different articles.

§1. Introduction: Elementary horocyclic geometry

Recently we discovered a new geometry on submanifolds in the hyperbolic n-space which is called *horospherical geometry* ([3, 8, 9, 10, 11, 12, 13, 14]). **What is Horospherical geometry?** Here we describe the basic idea of this geometry in the hyperbolic plane as the corresponding elementary geometry. We consider the Poincaré disk model D^2 of the hyperbolic plane which is an open unit disk in the (x, y) plane with Riemannian metric : $ds^2 = 4(dx^2 + dy^2)/(1 - x^2 - y^2)^2$. Therefore it is conformally equivalent to Euclidean plane, so that a circle in the Poincaré disk is also a circle in Euclidean plane. If we adopt geodesics as lines in the Poincaré disk, we have a model of Hyperbolic geometry (the non-Euclidean geometry of Gauss–Bolyai–Lobachevski). However, we have another kind of curves in the Poincaré disk which have an analogous property of lines in Euclidean plane. A *horocycle* is a Euclidean circle which is tangent to the ideal boundary (cf., Fig. 1).
We remark that a line in Euclidean plane can be considered as a limit of circles when the radii tend to infinity. A horocycle is also a curve as a limit of circles when the radii tend to infinity in the Poincaré disk (cf., Fig. 2). Therefore, horocycles are also an analogous notion of lines. If we adopt horocycles as lines, what kind of geometry we obtain? We say that two horocycles are *parallel* if they have the comon tangent point

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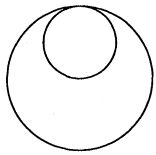


Fig. 1. Horocycle

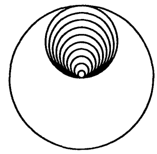


Fig. 2. The limit of circles

at the ideal boundary. Under this definition, the axiom of parallel is satisfied (cf., Fig. 3). However, for any two points in the disk, there are always two horocycles through the points, so that the axiom 1 of the Euclidean Geometry is not satisfied (cf., Fig. 4). We call this geometry a *horocyclic geometry.* Therefore, the horocyclic geometry is also a non-Euclidean geometry.

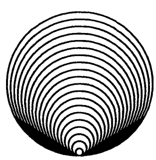


Fig. 3. The axiom of parallel

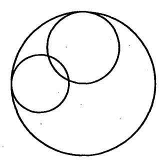


Fig. 4. The axiom 1

It might be said that horocycles have both the properties of lines and circles in Euclidean plane. We define the normal angle between two horocycles as follows: For a horocycle, we have a unit vector on Euclidean plane directed to the tangent points of the horocycle. We define that a *normal angle* between two horocycles is the Euclidean angle between corresponding two unit vectors (cf., Fig. 5, Fig. 6). It is clear that two horocycles are parallel if and only if the normal angle is zero. However, two horocycles are not parallel even if the normal angle is π.

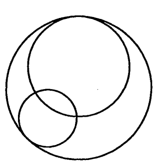


Fig. 5. Two horocycles

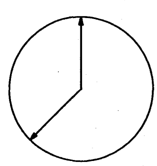


Fig. 6. Two unit horonormal vectors

We now consider three horocycles in the disk (cf., Fig. 7, Fig. 8). In this case, there are four horo-triangles in the disk. For the simplicity, we consider a horo-convex triangle. We say that a triangle is *horo-convex* if the horo-normal unit vector is directed to the inside of the triangle. If we have three horocycles sufficiently large radii parallel to given horocycles, there exists a horo-convex horo-triangle.

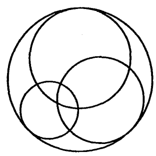


Fig. 7. Horo-triangles

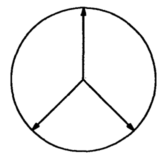


Fig. 8. Three unit horo-normal vectors

We can show the following theorem by observing Fig. 7 and Fig. 8:

Theorem 1. *The total sum of horo-normal angles of a horo-convex horo-triangle is* 2π.

If we consider the orientation of the horo-triangle, we have the similar theorem for non-horo-convex horo-triangles (under some careful considerations). Moreover, we can show that the total sum of the horo-normal angles of an oriented pieswise horo-cyclic curve is $2\pi \times$ the winding number, so that it is a topological invariant. This suggests us a kind of the Gauss–Bonnet type theorem holds if we define a suitable curvature of a surface in the hyperbolic space. By definition, the horo-normal angle is not a hyperbolic invariant. Nevertheless, the property that two horocycles are parallel (i.e., the angle is zero) is a hyperbolic invariant which corresponds to the flatness of the "horospherical curvature" in the hyperbolic space.

All maps considered here are of class C^∞ unless otherwise stated.

§2. Differential geometry in the hyperbolic space

We outline in this section the differential geometry of curves and surfaces in the hyperbolic 3-space which are developed in the previous papers [8, 9]. We adopt the Lorentzian model of the hyperbolic 3-space. Let $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) \mid x_i \in \mathbb{R}\ (i = 0, 1, 2, 3)\}$ be a 4-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$, $\boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i$. We call $(\mathbb{R}^4, \langle , \rangle)$ *Minkowski space.* We write $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. For a non-zero vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number c, we define *the hyperplane with pseudo normal* $\boldsymbol{v}$ by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}$. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We also define the *hyperbolic* 3*-space* by $H^3_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 \geq 1\}$ and *de Sitter* 3*-space* by $S^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$. For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is the canonical basis of $\mathbb{R}^4_1$ and $\boldsymbol{x}_i = (x_0^i, x_1^i, x_2^i, x_3^i)$. We can easily show that $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}\ \boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3)$, so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, 2, 3)$.

We also define a set $LC^*_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid x_0 > 0, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}$, which is called the *future lightcone* at the origin. We have three kinds of surfaces in $H^3_+(-1)$ which are given by intersections of $H^3_+(-1)$ with hyperplanes in $\mathbb{R}^4_1$. A surface $H^3_+(-1) \cap HP(\boldsymbol{v}, c)$ is called a *sphere*, a *equidistant surface* or a *horosphere* if $HP(\boldsymbol{v}, c)$ is spacelike, timelike or lightlike respectively. The equidistant surface is called a *hyperbolic plane* if $c = 0$. Especially we write a horosphere as $HS^2(\boldsymbol{v}, c) = H^3_+(-1) \cap HP(\boldsymbol{v}, c)$. If we consider a lightlike vector $\boldsymbol{v}_0 = -\boldsymbol{v}/c$, we have $HS^2(\boldsymbol{v}, c) = HS^2(\boldsymbol{v}_0, -1)$.

We construct the extrinsic differential geometry on curves in $H^3_+(-1)$ (cf., [9]). Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed curve. We have the tangent vector $\boldsymbol{t}(s) = \gamma'(s)$ with $\|\boldsymbol{t}(s)\| = 1$. In the case when $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, we have a unit vector $\boldsymbol{n}(s) = (\boldsymbol{t}'(s) - \gamma(s))/(\|\boldsymbol{t}'(s) - \gamma(s)\|)$. Moreover, define $\boldsymbol{e}(s) = \gamma(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, then we have a pseudo orthonormal frame $\{\gamma(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4_1$ along γ. By standard arguments, we have the following *Frenet–Serre type formulae*:

$$\begin{cases} \gamma'(s) &= \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) &= \kappa_h(s)\boldsymbol{n}(s) + \gamma(s) \\ \boldsymbol{n}'(s) &= -\kappa_h(s)\boldsymbol{t}(s) + \tau_h(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) &= -\tau_h(s)\boldsymbol{n}(s), \end{cases}$$

where $\kappa_h(s) = \|\boldsymbol{t}'(s) - \gamma(s)\|$ and

$$\tau_h(s) = \frac{-(\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s)))}{(\kappa_h(s))^2}.$$

We can easily show that the condition $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$ is equivalent to the condition $\kappa_h(s) \neq 0$. We say that γ is a *horocycle* if $\kappa_h(s) \equiv 1$ and $\tau_h(s) \equiv 0$.

On the other hand, we give a review on the explicit differential geometry on surfaces in $H^3_+(-1)$ (cf., [8]). Let $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ be a regular surface, where $U \subset \mathbb{R}^2$ is an open subset. We denote that $M = \boldsymbol{x}(U)$ and identify M with U through the embedding $\boldsymbol{x}$. Define a vector

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)\|},$$

then we have $\langle \boldsymbol{e}, \boldsymbol{x}_{u_i}\rangle \equiv \langle \boldsymbol{e}, \boldsymbol{x}\rangle \equiv 0, \langle \boldsymbol{e}, \boldsymbol{e}\rangle \equiv 1$, where $\boldsymbol{x}_{u_i} = \partial \boldsymbol{x}/\partial u_i$. Therefore we have a mapping $\mathbb{E} : U \longrightarrow S^3_1$ by $\mathbb{E}(u) = \boldsymbol{e}(u)$ which is called the *de Sitter Gauss image* of $\boldsymbol{x}$. Since $\boldsymbol{x}(u) \in H^3_+(-1)$, $\boldsymbol{e}(u) \in S^3_1$ and $\langle \boldsymbol{x}(u), \boldsymbol{e}(u)\rangle = 0$, we have $\boldsymbol{x}(u) \pm \boldsymbol{e}(u) \in LC^*_+$. We define a map

$$\mathbb{L}^{\pm} : U \longrightarrow LC^*_+ \ ; \ \mathbb{L}^{\pm}(u) = \boldsymbol{x}(u) \pm \boldsymbol{e}(u),$$

which is called the *lightcone* (*or, hyperbolic*) *Gauss image* of $\boldsymbol{x}$. We have shown that $D_v \mathbb{L}^{\pm} \in T_pM$ for any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, where D_v denotes the *covariant derivative* with respect to the tangent vector $\boldsymbol{v}$. We also showed that the surface $M = \boldsymbol{x}(U)$ is a part of a horosphere if and only if the lightcone Gauss image $\mathbb{L}^{\pm}$ is constant. Under the identification of U and M, the derivative $d\boldsymbol{x}(u_0)$ can be identified with the identity mapping 1_{T_pM} on the tangent space T_pM, where $p = \boldsymbol{x}(u_0)$. This means that $d\mathbb{L}^{\pm}(u_0) = 1_{T_pM} \pm d\mathbb{E}(u_0)$. We call the linear transformation $S^{\pm}_p = -d\mathbb{L}^{\pm}(u_0) : T_pM \longrightarrow T_pM$ the *hyperbolic shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We also call $A_p = -d\mathbb{E}(u_0) : T_pM \longrightarrow T_pM$ the *shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. In order to distinguish from the hyperbolic shape operator, we also call A_p the *de Sitter shape operator.* We denote the eigenvalues of $S^{\pm}_p$ by $\bar{\kappa}^{\pm}_i(p)$ $(i = 1, 2)$ and the eigenvalues of A_p by $\kappa_i(p)$. By the relation $S^{\pm}_p = -1_{T_pM} \pm A_p$, $S^{\pm}_p$ and A_p have same eigenvectors and relations $\bar{\kappa}^{\pm}_i(p) = -1 \pm \kappa_i(p)$. We call $\bar{\kappa}^{\pm}_i(p)$ *hyperbolic principal curvatures* and $\kappa_i(p)$ *principal curvatures* (or, *de Sitter principal curvatures* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. The *hyperbolic Gauss curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is $K^{\pm}_h(p) = \det S^{\pm}_p = \bar{\kappa}^{\pm}_1(p)\bar{\kappa}^{\pm}_2(p)$. The *hyperbolic mean curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is $H^{\pm}_h(p) = \mathrm{Trace} S^{\pm}_p/2 = (\bar{\kappa}^{\pm}_1(p) + \bar{\kappa}^{\pm}_2(p))/2$. The *extrinsic Gauss–Kronecker curvature* is $K_e(p) = \det A_p = \kappa_1(p)\kappa_2(p)$ and the *mean curvature* is $H(p) = \mathrm{Trace} A_p/2 = (\kappa_1(p) + \kappa_2(p))/2$. We clearly have that $H^{\pm}_h(p) = \pm H(p) - 1$. We say that a point $u_0 \in U$ or $p = \boldsymbol{x}(u_0)$ is an *umbilical point* if $\kappa_1(p) = \kappa_2(p)$, which is equivalent to the condition $\bar{\kappa}^{\pm}_1(p) = \bar{\kappa}^{\pm}_2(p)$. We say that $M = \boldsymbol{x}(U)$ is *totally umbilical* if all points on M are umbilical. The following classification theorem of totally umbilical surfaces is well-known (cf., [10]):

Proposition 2. *Suppose that $M = \boldsymbol{x}(U)$ is totally umbilical. Then $\kappa(p)$ is a constant κ. Under this condition, we have the following classification:*

1) *Suppose that* $\kappa^2 \neq 1$.
   a) *If* $\kappa \neq 0$ *and* $\kappa^2 < 1$, *then* M *is a part of an equidistant surface.*
   b) *If* $\kappa \neq 0$ *and* $\kappa^2 > 1$, *then* M *is a part of a sphere.*
   c) *If* $\kappa = 0$, *then* M *is a part of a hyperbolic plane.*
2) *If* $\kappa^2 = 1$, *then* M *is a part of a horosphere.*

By definition, $\kappa^2 = 1$ if and only if $\bar{\kappa}^{\pm} = 0$. Therefore, a horosphere is a totally umbilical surface with $\bar{\kappa}^{\pm} = 0$. We have the hyperbolic (respectively, de Sitter) version of the Weingarten formula. Since $\boldsymbol{x}_{u_i}$ $(i = 1,2)$ are spacelike vectors, we have the *first fundamental form* given by $ds^2 = \sum_{i=1}^{2} g_{ij}du_i du_j$ on $M = \boldsymbol{x}(U)$, where $g_{ij}(u) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$ and the *hyperbolic* (respectively, *de Sitter*) *second fundamental invariant* defined by $\bar{h}^{\pm}_{ij}(u) = \langle -\mathbb{L}^{\pm}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$ (respectively, $h_{ij}(u) = -\langle \mathbb{E}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$) for any $u \in U$. They satisfy the relation $\bar{h}^{\pm}_{ij}(u) = -g_{ij}(u) \pm h_{ij}(u)$. In [8, 14] it was shown the following formulae:

$$\mathbb{L}^{\pm}_{u_i} = -\sum_{j=1}^{2} \bar{h}^{\pm j}_{i} \boldsymbol{x}_{u_j} \quad \text{(The hyperbolic Weingarten formula)},$$

$$\mathbb{E}_{u_i} = -\sum_{j=1}^{2} h^{j}_{i} \boldsymbol{x}_{u_j} \quad \text{(The de Sitter Weingarten formula)},$$

where $\left(\bar{h}^{\pm j}_{i}\right) = \left(\bar{h}^{\pm}_{ik}\right)\left(g^{kj}\right)$, $\left(h^{j}_{i}\right) = \left(h_{ik}\right)\left(g^{kj}\right)$ and $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$. It follows that we have an explicit expression of the hyperbolic (respectively, extrinsic) Gauss–Kronecker curvature in terms of the first fundamental invariant and the hyperbolic (respectively, de Sitter) second fundamental invariant:

$$K^{\pm}_{h} = \det\left(\bar{h}^{\pm}_{ij}\right)/\det\left(g_{\alpha\beta}\right), \;\; K_e = \det\left(h_{ij}\right)/\det\left(g_{\alpha\beta}\right).$$

Since $H^3_+(-1)$ is a Riemannian manifold, we have the sectional curvaure of M. We denote it K_I which is called the *intrinsic Gauss curvature.* It is well-known the relation $K_e = K_I + 1$. Since $\bar{\kappa}^{\pm}_{i} = -1 \pm \kappa_i$, we deduce the above formula as follows: $K^{\pm}_{h} = 1 \mp 2H + K_e = 2 \mp 2H + K_I$. Therefore, $K^{\pm}_{h}$ is an extrinsic hyperbolic invariant of M.

§3. The horospherical geometry in the hyperbolic space

In this section we consider the notion of hyperbolic Gauss maps introduced by Bryant [2], Epstein [5] and Kobayashi[16] as follows: If

$\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\tilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \in S^2_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in LC^*_+ \mid x_0 = 1\ \}.$$

We call S^2_+ *the lightcone sphere.* We define a map $\widetilde{\mathbb{L}}^\pm : U \longrightarrow S^2_+$ by $\widetilde{\mathbb{L}}^\pm(u) = \widetilde{\mathbb{L}^\pm(u)}$ and call it the *hyperbolic Gauss map* of $M = \boldsymbol{x}(U)$. Let T_pM be the tangent space of M at p and N_pM be the pseudo-normal space of T_pM in $T_p\mathbb{R}^4_1$. We have the decomposition $T_p\mathbb{R}^4_1 = T_pM \oplus N_pM$, so that we have the Whitney sum $T\mathbb{R}^4_1 = TM \oplus NM$. Therefore we have the canonical projection $\Pi : T\mathbb{R}^4_1 \longrightarrow TM$. It follows that we have a linear transformation $\Pi_p \circ d\widetilde{\mathbb{L}}^\pm(u) : T_pM \longrightarrow T_pM$ for $p = \boldsymbol{x}(u_0)$ by the identification of U and $\boldsymbol{x}(U) = M$ via $\boldsymbol{x}$. In [14] we have shown the following horosphercal Weingarten formula:

$$\Pi_p \circ \widetilde{\mathbb{L}}^\pm_{u_i} = -\sum_{j=1}^{2} \frac{1}{\ell^\pm_0(u)} \bar{h}^{\pm j}_{\ i} \boldsymbol{x}_{u_j},$$

where $\mathbb{L}^\pm(u) = (\ell^\pm_0(u), \ell^\pm_1(u), \ell^\pm_2(u), \ell^\pm_3(u))$. We call the linear transformation $\widetilde{S}^\pm_p = -\Pi_p \circ d\widetilde{\mathbb{L}}^\pm$ the *horospherical shape operator* of $M = \boldsymbol{x}(U)$. We also define the *horospherical principal curvature* $\widetilde{\kappa}^\pm_i(p)$ $(i = 1, 2)$ as eigenvalues of $\widetilde{S}^\pm_p$. By the above formula, we have $\widetilde{\kappa}^\pm_i(p) = (1/\ell^\pm_0(p))\bar{\kappa}^\pm_i(p)$. The *horospherical Gauss–Kronecker curvature* of $M = \boldsymbol{x}(U)$ is $\widetilde{K}^\pm_h(p) = \det \widetilde{S}^\pm_p = \widetilde{\kappa}^\pm_1(p)\widetilde{\kappa}^\pm_2(p)$, so that we have the following relation between the horospherical Gauss–Kronecker curvature and the hyperbolic Gauss–Kronecker curvature:

$$\widetilde{K}^\pm_h(p) = \left(\frac{1}{\ell^\pm_0(u_0)}\right)^2 K^\pm_h(p).$$

We say that a point $u_0 \in U$ or $p = \boldsymbol{x}(u_0)$ is a *horo-umbilical point* if $\widetilde{S}^\pm_p = \widetilde{\kappa}^\pm(p)1_{T_pM}$. It follows from the horospherical Weingarten formula that p is a horo-umbilical point if and only if it is an umbilical point. We say that $M = \boldsymbol{x}(U)$ is *totally horo-umbilical* if all points on M are horo-umbilical as usual.

We remark that $\widetilde{\kappa}^\pm(p)$ is not invariant under hyperbolic motions but it is an $SO(3)$-invariant. However, we can make sense a point with vanishing horospherical principal curvature as a notion of the hyperbolic differential geometry[14].

Proposition 3. *For a point $p = \boldsymbol{x}(u)$, $\widetilde{\kappa}^\pm_i(p)$ is invariant under hyperbolic motions if and only if $\widetilde{\kappa}^\pm_i(p) = 0$.*

Corollary 4. *If $M = \boldsymbol{x}(U)$ is totally horo-umbilical and $\widetilde{\kappa}^{\pm}(p) = (1/\ell_0^{\pm}(u))\bar{\kappa}^{\pm}$ is a hyperbolic invariant, then M is a part of a horosphere (i.e., $\widetilde{\kappa}^{\pm} \equiv 0$).*

We now show that the notion of horospherical curvatures is independent of the choice of the model of the hyperbolic space. For the purpose, we introduce a smooth function on the unit tangent sphere bundle of the hyperbolic space which plays the principal role of the horospherical geometry. Let $SO_0(3,1)$ be the identity component of the matrix group

$$SO(3,1) = \{g \in SL(4,\mathbb{R}) \mid gI_{3,1}{}^t g = I_{3,1}\},$$

where

$$I_{3,1} = \left(\begin{array}{c|c} -1 & \mathbf{0} \\ \hline {}^t\mathbf{0} & I_3 \end{array}\right) \in GL(4,\mathbb{R}).$$

It is well-known that $SO_0(3,1)$ transitively acts on $H^3_+(-1)$ and the isotropic group at $p = (1,0,0,0)$ is $SO(3)$ which is naturally embedded in $SO_0(3,1)$. Moreover the action induces isometries on $H^3_+(-1)$.

On the other hand, we consider a submanifold $\Delta = \{(\boldsymbol{v},\boldsymbol{w}) \mid \langle \boldsymbol{v},\boldsymbol{w}\rangle = 0\ \}$ of $H^3_+(-1) \times S^3_1$ and the canonical projection $\bar{\pi} : \Delta \longrightarrow H^3_+(-1)$. Let $\pi : S(TH^3_+(-1)) \longrightarrow H^3_+(-1)$ be the unit tangent sphere bundle over $H^3_+(-1)$. For any $\boldsymbol{v} \in H^3_+(-1)$, we have the coordinates (v_1, v_2, v_3) of $H^3_+(-1)$ such that $\boldsymbol{v} = (\sqrt{v_1^2 + v_2^2 + v_3^2 + 1}, v_1, v_2, v_3)$. We can represent the tangent vector $\boldsymbol{w} = \sum_{i=1}^3 w_i \partial/\partial v_i \in T_v H^3_+(-1)$ by $\boldsymbol{w} = \left((\sum_{i=1}^3 w_i v_i)/v_0, w_1, w_2, w_3\right)$ as a vector in Minkowski 4-space. Then $\langle \boldsymbol{w},\boldsymbol{v}\rangle = (-(1/v_0)\sum_{i=1}^3 w_i v_i)v_0 + \sum_{i=1}^3 w_i v_i = 0$. It follows that $\boldsymbol{w} \in S(T_v H^3_+(-1))$ if and only if $\langle \boldsymbol{w},\boldsymbol{w}\rangle = 1$ and $\langle \boldsymbol{v},\boldsymbol{w}\rangle = 0$. These conditions are equivalent to the condition $(\boldsymbol{v},\boldsymbol{w}) \in \Delta$. This means that we can canonically identify $\pi : S(TH^3_+(-1)) \longrightarrow H^3_+(-1)$ with $\bar{\pi} : \Delta \longrightarrow H^3_+(-1)$. Moreover, the linear action of $SO_0(3,1)$ on $\mathbb{R}^4_1$ induces the canonical action on Δ (i.e., $g(\boldsymbol{v},\boldsymbol{w}) = (g\boldsymbol{v}, g\boldsymbol{w})$ for any $g \in SO_0(3,1)$). For any $(\boldsymbol{v},\boldsymbol{w}) \in \Delta$, the first component of $\boldsymbol{v} \pm \boldsymbol{w}$ is given by $v_0 \pm w_0 = \sqrt{v_1^2 + v_2^2 + v_3^2 + 1} \pm (\sum_{i=1}^3 v_i w_i)/\sqrt{v_1^2 + v_2^2 + v_3^2 + 1}$, so that it can be considered as a function on the unit tangent bundle $S(TH^3_+(-1))$. We now define a function $\mathcal{N}_h^{\pm} : \Delta \longrightarrow \mathbb{R}\ ;\ \mathcal{N}_h^{\pm}(\boldsymbol{v},\boldsymbol{w}) = 1/(v_0 \pm w_0)$. We call $\mathcal{N}_h^{\pm}$ a *horospherical normalization function* on $H^3_+(-1)$. Since $v_1^2 + v_2^2 + v_3^2 + 1$ and $\sum_{i=1}^3 v_i w_i$ are $SO(3)$-invariant functions, $\mathcal{N}_h^{\pm}$ is an $SO(3)$-invariant function. Therefore, $\mathcal{N}_h^{\pm}$ can be considered as a function on the unit tangent sphere bundle over the hyperbolic space $SO_0(3,1)/SO(3)$ which is independent of the choice of the model space. For any embedding $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$, we have the unit normal vector field $\mathbb{E} = \boldsymbol{e} : U \longrightarrow S^3_1$, so that $(\boldsymbol{x}(u), \boldsymbol{e}(u)) \in \Delta$ for any $u \in U$. It follows that $\widetilde{K}^{\pm}_h(u) = \mathcal{N}^{\pm}_h(\boldsymbol{x}(u), \boldsymbol{e}(u))^2 K^{\pm}_h(u)$. The right hand side of the above equality is independent of the choice of the model space.

§4. Total horospherical curvatures

We consider global properties of the horospherical Gauss–Kronecker curvature. Let M be a closed orientable 2-dimensional manifold and $f : M \longrightarrow H^3_+(-1)$ an immersion. Consider the unit normal $\mathbb{E}$ of $f(M)$ in $H^3_+(-1)$, then we define the lightcone Gauss image in the global $\mathbb{L}^{\pm} : M \longrightarrow LC^*_+$ by $\mathbb{L}^{\pm}(p) = f(p) \pm \mathbb{E}(p)$. The *global hyperbolic Gauss–Kronecker curvature function* $\mathcal{K}^{\pm}_h : M \longrightarrow \mathbb{R}$ is then defined in the usual way in terms of the global lightcone Gauss image $\mathbb{L}^{\pm}$. We also define the hyperbolic Gauss map in the global $\widetilde{\mathbb{L}}^{\pm} : M \longrightarrow S^2_+$ by $\widetilde{\mathbb{L}}^{\pm}(p) = \widetilde{\mathbb{L}^{\pm}(p)}$. We can define a *global horospherical Gauss–Kronecker curvature function* $\widetilde{\mathcal{K}}^{\pm}_h : M \longrightarrow \mathbb{R}$ which satisfies the relation $\widetilde{\mathcal{K}}^{\pm}_h(p) = \mathcal{N}^{\pm}_h(f(p), \mathbb{E}(p))^2 \mathcal{K}^{\pm}_h(p)$. In [14, 3] we have shown the following theorem:

Theorem 5. *If M is a closed orientable 2-dimensional surface in the hyperbolic 3-space, then*

$$\begin{aligned}
\int_M \widetilde{\mathcal{K}}^{\pm}_h d\mathfrak{a}_M &= 2\pi\chi(M) \text{ (the Gauss–Bonnet type formula[14])}, \\
\int_M |\widetilde{\mathcal{K}}^{\pm}_h| \, d\mathfrak{v}_M &\geqslant 2\pi(4 - \chi(M)) \text{ (the Chern–Lashof type inequality[3])},
\end{aligned}$$

where $\chi(M)$ is the Euler characteristic of M, $d\mathfrak{a}_M$ is the area form of M.

We also consider curves in $H^3_+(-1)$. Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed regular curve. The *horospherical Lipschitz–Killing curvature* of γ at s is

$$\widetilde{\kappa}_h(s) = \int_0^{2\pi} \mathcal{N}_h(\gamma(s), \cos\theta \boldsymbol{n}(s) + \sin\theta \boldsymbol{e}(s))^2 |(1 - \kappa_h(s))| d\theta.$$

We have the following theorem[3]:

Theorem 6. *Let $\gamma : S^1 \longrightarrow H^3_+(-1)$ be an embedding. Then we have the following inequality:*

$$\int_{S^1} \widetilde{\kappa}_h ds \geq 8\pi \quad \text{(the Fenchel type inequality)}.$$

Remark 1. We describe here why we say that the above theorem is a Fenchel type theorem. For an embedded curve $\gamma : S^1 \longrightarrow \mathbb{E}^3$ in Euclidean space, Fenchel's theorem asserts that

$$\int_{S^1} \kappa(s)ds \geq 2\pi.$$

In this case if we consider the canal surface $CM\gamma_r$ of γ with sufficiently small radius $r > 0$, we have a relation

$$4\int_{S^1} \kappa ds = \int_{CM\gamma_r} |K| d\mathfrak{v}_{CM\gamma_r}.$$

So the Fenchel's theorem follows from the Euclidean Chern–Lashof inequality. However for the horospherical curvature, we have no such a relation. We also remark that we have shown the Chern–Lashof type inequality for any codimensional submanifolds in a higher dimensional hyperbolic space[3].

We can also show the following theorem[3].

Theorem 7. *Let $\gamma : S^1 \longrightarrow H^3_+(-1)$ be an embedding. If γ is knotted, then*

$$\int_{S^1} \widetilde{\kappa}_h ds \geq 16\pi \quad \text{(the Fary–Milnor type inequality).}$$

We remark that the above inequalities for totally absolute horospherical cuarvatures are deeply related to the notion of horo-tight immersions in the hyperbolic space[4].

§5. Horospherical flat surfaces

In this section we investigate a special class of surfaces in the hyperbolic 3-space which are called *horospherical flat surfaces.* We say that a surface $M = \boldsymbol{x}(U)$ is *horospherical flat* (briefly, *horo-flat*) if $\widetilde{K}_h(p) = 0$ at any point $p \in M$. By a direct consequence of the relation in §3, $K_h(p) = 0$ if and only if $\widetilde{K}_h(p) = 0$, so that the horospherical flatness is a hyperbolic invariant. Moreover, there is an important class of surfaces called *linear Weingarten surfaces* which satisfy the relation $aK_I + b(2H - 2) = 0$ $((a, b) \neq (0, 0))$. In [6], the Weierstrass–Bryant type representation formula for such surfaces with $a + b \neq 0$ (called, a *linear Weingarten surface of Bryant type*) was shown. This class of surfaces contains flat surfaces (i.e., $a \neq 0, b = 0$) and CMC-1(constant mean curvature one) surfaces $(a = 0, b \neq 0)$. In the celebrated paper [2], Bryant showed the Weierstrass type representation formula for CMC-1 surfaces in the hyperbolic space. This is the reason why the class of the surface with $a+b \neq 0$ is called of Bryant type. By using such representation formula, there are a lot of results on such surfaces. We only refer [6, 17, 18, 20, 21] here. The horospherical flat surface is one of the linear Weingarten surfaces. It is, however, the exceptional case (a *linear Weingarten surface of non-Bryant type* : $a+b=0$). There are no Weierstarass–Bryant type representation formula for such surfaces so far as we know. Therefore the horospherical flat surfaces are also very important subjects in the hyperbolic geometry and we need a new approach for the study of such surfaces. If $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ is a surface without umbilical points, we may assume that both the u-curve and the v-curve are the lines of curvature for the coordinate system $(u,v) \in U$ such that the u-curve corresponds to the vanishing hyperbolic principal curvature. By the hyperbolic Weingarten formula, we have

$$\mathbb{L}_u(u,v) = \mathbf{0} \quad \mathbb{L}_v(u,v) = -\bar{\kappa}(u,v)\boldsymbol{x}_v(u,v),$$

where $\bar{\kappa}(u,v) \neq 0$. It follows that $\mathbb{L}(0,v) = \mathbb{L}(u,v)$. We define a function $F : H^3_+(-1) \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ by $F(\boldsymbol{X}, v) = \langle \mathbb{L}(0,v), \boldsymbol{X} \rangle + 1$, for sufficiently small $\varepsilon > 0$. For any fixed $v \in (-\varepsilon, \varepsilon)$, we have a horosphere $HS^2(\mathbb{L}(0,v), -1) = \{\boldsymbol{X} \in H^3_+(-1) \mid F(\boldsymbol{X}, v) = 0\}$, so that $F=0$ define a one-parameter family of horospheres. In [15] we have shown that the surface $M = \boldsymbol{x}(U)$ is the envelope of the family of horospheres defined by $F=0$.

On the other hand, we consider a surface $\widetilde{\boldsymbol{x}} : I \times J \longrightarrow H^3_+(-1)$ defined by

$$\widetilde{\boldsymbol{x}}(s,v) = \boldsymbol{x}(0,v) + s\frac{\boldsymbol{x}_u(0,v)}{\|\boldsymbol{x}_u(0,v)\|} + \frac{s^2}{2}\mathbb{L}(0,v),$$

where $I, J \subset \mathbb{R}$ are open intervals. We have also shown that the surface $\widetilde{M} = \widetilde{\boldsymbol{x}}(I \times J)$ is the envelope of the family of horospheres defined by $F=0$. It follows that a horo-flat surface can be reparametrized (at least locally) by $\widetilde{\boldsymbol{x}}(s,v)$. If we fix $v = v_0$, we denote that $\boldsymbol{a}_0 = \boldsymbol{x}(0,v_0)$, $\boldsymbol{a}_1 = \boldsymbol{x}_u(0,v_0)/\|\boldsymbol{x}_u(0,v_0)\|$, $\boldsymbol{a}_2 = \boldsymbol{e}(0,v_0)$. Then we have a curve

$$\gamma(s) = \boldsymbol{a}_0 + s\boldsymbol{a}_1 + \frac{s^2}{2}(\boldsymbol{a}_0 + \boldsymbol{a}_2).$$

We can show that $\gamma(s)$ is a horocycle. Moreover, any horocycle has the above parametrization. Therefore the horo-flat surface is given by the one-parameter family of horocycles. We say that a surface is a *horocyclic surface* if it is (at least locally) parametrized by one-parameter families of horocycles around any point. Eventually we have the following theorem[15]:

Theorem 8. *If $M \subset H^3_+(-1)$ is an umbilically free horo-flat surface, it is a horocyclic surface. Moreover, each horocycle is the line of curvatures with the vanishing hyperbolic principal curvature.*

It follows that our main subject is a class of horocyclic surfaces. Let $\gamma : I \longrightarrow H^3_+(-1)$ be a smooth map and $\boldsymbol{a}_i : I \longrightarrow S^3_1$ $(i = 1, 2)$ be smooth mappings from an open interval I with $\langle \gamma(t), \boldsymbol{a}_i(t)\rangle = \langle \boldsymbol{a}_1(t), \boldsymbol{a}_2(t)\rangle = 0$. We define a unit spacelike vector $\boldsymbol{a}_3(t) = \gamma(t) \wedge \boldsymbol{a}_1(t) \wedge \boldsymbol{a}_2(t)$, so that we have a pseudo-orthonormal frame $\{\gamma, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ of $\mathbb{R}^4_1$. We now define a mapping $F_{(\gamma, a_1, a_2)} : \mathbb{R} \times I \longrightarrow H^3_+(-1)$ by

$$F_{(\gamma, a_1, a_2)}(s, t) = \gamma(t) + s\boldsymbol{a}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}(t),$$

where $\boldsymbol{\ell}(t) = \gamma(t) + \boldsymbol{a}_2(t)$. We call $F_{(\gamma, a_1, a_2)}$ (or the image of it) a *horocyclic surface.* Each horocycle $F_{(\gamma, a_1, a_2)}(s, t_0)$ is called a *generating horocycle.* Since $\{\gamma, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is a pseudo-orthonormal frame of $\mathbb{R}^4_1$, we have

$$A(t) = \begin{pmatrix} \gamma(t) \\ \boldsymbol{a}_1(t) \\ \boldsymbol{a}_2(t) \\ \boldsymbol{a}_3(t) \end{pmatrix} \in SO_0(3,1), \text{ so that } C(t) = A'(t)A(t)^{-1} \in \mathfrak{so}(3,1),$$

where $\mathfrak{so}(3,1)$ is the Lie algebra of the Lorentzian group $SO_0(3,1)$. We remark that $C(t)$ has the form

$$C(t) = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ c_1(t) & 0 & c_4(t) & c_5(t) \\ c_2(t) & -c_4(t) & 0 & c_6(t) \\ c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix}.$$

Moreover, for any smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$, we apply the existence theorem on the linear systems of ordinary differential equations, so that there exists a unique curve $A : I \longrightarrow SO_0(3,1)$ such that $C(t) = A'(t)A(t)^{-1}$ with an initial data $A(t_0) \in SO_0(3,1)$. Therefore, a smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$ might be identified with a horocyclic surface in $H^3_+(-1)$. Let $C : I \longrightarrow \mathfrak{so}(3,1)$ be a smooth curve with $C(t) = A'(t)A(t)^{-1}$ and $B \in SO_0(3,1)$, then we have $C(t) = (A(t)B)'(A(t)B)^{-1}$. This means that the curve $C : I \longrightarrow \mathfrak{so}(3,1)$ is a Lorentzian invariant of $A(t)$, so that it is a hyperbolic invariant of the corresponding horocyclic surface. Let $C^\infty(I, \mathfrak{so}(3,1))$ be the space of smooth curves into $\mathfrak{so}(3,1)$ equipped with Whitney C^∞-topology. By the above arguments, we may regard $C^\infty(I, \mathfrak{so}(3,1))$ as the space of horocyclic surfaces, where I is an open interval or the unit circle.

On the other hand, we consider the singularities of horocyclic surfaces. By a straightforward calculation, (s,t) is a singular point of $F_{(\gamma,a_1,a_2)}(s,t)$ if and only if

$$c_2(t) + s(c_4(t) - c_1(t)) = 0, \quad \left(1 + \frac{s^2}{2}\right) c_3(t) + sc_5(t) + \frac{s^2}{2} c_6(t) = 0.$$

We have also shown in [15] that $F_{(\gamma,a_1,a_2)}(s,t)$ is horo-flat if and only if $c_2(t) = c_4(t) - c_1(t) = 0$. In this case each generating horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$ is a line of curvature. Therefore, the first equation for the singularities is automatically satisfied for a horo-flat horocyclic surface. In this case, the singular set is given by a family if quadratic equations $\sigma_C(s,t) = (c_3(t) + c_6(t))s^2 + 2C_5(t)s + 2c_3(t) = 0$.

We now consider the space of horo-flat horocyclic surfaces. Remember that $C^\infty(I, \mathfrak{so}(3,1))$ is the space of horocyclic surfaces. We consider a linear subspace of $\mathfrak{so}(3,1)$ defined by

$$\mathfrak{hf}(3,1) = \left\{ C \in \mathfrak{so}(3,1) \;\middle|\; c_2 = c_1 - c_4 = 0 \right\}.$$

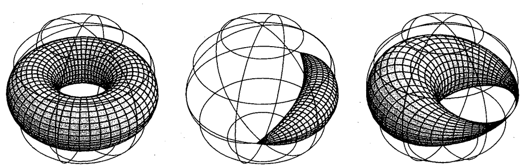
By the previous arguments, the *space of horo-flat horocyclic surfaces* is defined to be the space $C^\infty(I, \mathfrak{hf}(3,1))$ with Whitney C^∞-topology. We expect the analogous properties of developable surfaces in $\mathbb{R}^3$ which are ruled surfaces with vanishing Gaussian curvature. However the situation is quite different. In Euclidean space, complete non-singular developable surfaces are cylindrical surfaces [7]. There are various kinds of horo-flat horocyclic surfaces even if these are regular surfaces. We only give some interesting examples of regular horo-flat horocyclic surfaces and which suggest that the situation is quite different form the developable surfaces in Euclidean space. Suppose that $\gamma(t)$ is a unit speed curve with $\kappa_h(t) \neq 0$ and $\tau_h \equiv 0$ (i.e., a hyperbolic plane curve). Then we have the Frenet-type frame $\{\gamma(t), \boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{e}(t)\}$ with the constant binormal $\boldsymbol{e}(t) = \boldsymbol{e}$. We now define

$$F_{(\gamma,e,\pm n)}(s,t) = \gamma(t) + s\boldsymbol{e} + \frac{s^2}{2}(\gamma(t) \pm \boldsymbol{n}(t))$$

which is called a *binormal horocyclic surface of a hyperbolic plane curve.* By a straightforward calculation, the first fundamental form is given by $I_h = ds^2 + \left(1 + s^2(1 \mp \kappa_h(t))/2\right)^2 dt^2$. Here, $\boldsymbol{\ell}(t) = \gamma(t) \pm \boldsymbol{n}(t)$ is the lightlike normal vector field along the surface. Then we have

$$-\boldsymbol{\ell}'(t) == \frac{-2 \pm 2\kappa_h(t)}{2+s^2(1 \mp \kappa_h(t))} \frac{\partial F_{(\gamma,e,\pm n)}}{\partial t}(s,t)$$

It follow that the de Sitter principal curvatures are 1 and $1-(2 \mp 2\kappa_h(t))/(2+s^2(1\mp\kappa_h(t)))$. Since $\kappa_h(t) > 0$, $F_{(\gamma,e,-n)}$ is always umbilically free. We can draw the pictures of such surfaces in the Poincaré ball (cf., Fig. 9).



Horo-torus Banana Croissant

Fig. 9

However, $F_{(\gamma,e,n)}$ has umbilical points where $\kappa_h(t) = 1$. We can draw a horocylindrical surface which has umbilical points along the horocycle through $(0,0,0)$ in Fig. 10.

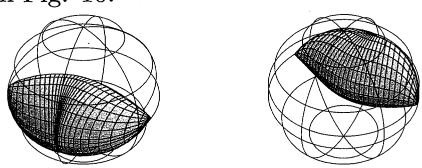


Fig. 10. Hips ($\kappa_h(0) = 1$ of γ, a_1 =constant)

This gives a concrete example of the surface with a constant principal curvature which is not umbilically free ([1], Example 2.1) which is a counter example of the hyperbolic version of the Shiohama–Takagi theorem[19, 23]. If $\kappa_h \equiv 1$ (i.e., $\gamma(t)$ is a horocycle), then $F_{(\gamma,e,n)}$ is totally umbilical (i.e., a horosphere).

§6. Singularities of horo-flat horocyclic surfaces

In this section we consider a horo-flat horocyclic suface $F_{(\gamma,a_1,a_2)}$ with singularities. Since the singularities satisfy the equation $\sigma_C(s,t) = 0$, $F_{(\gamma,a_1,a_2)}$ has at most two branches of singularities under the condition that $c_3(t) + c_4(t) \neq 0$. We suppose that one of the branches of the singularities is given by $\bar{\gamma}(t) = \gamma(t) + s(t)\boldsymbol{a}_1(t) + (s(t)^2/2)\boldsymbol{\ell}(t)$, where $s = s(t)$ is one of the real solutions of $\sigma_C(s,t)$ for any t. In this case we can reparametrize the horocyclic surface by $S = s - s(t), T = t$, $\bar{\boldsymbol{a}}_1(T) =$

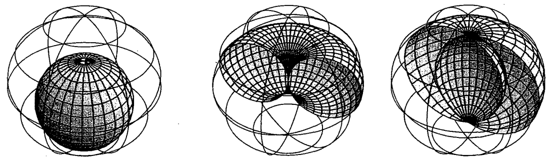
$\boldsymbol{a}_1(t) + s(t)\boldsymbol{\ell}(t)$ and $\bar{\boldsymbol{a}}_2(T) = \boldsymbol{\ell}(t) - \bar{\gamma}(t)$, then we have $F_{(\gamma,a_1,a_2)}(s,t) = F_{\bar{\gamma},\bar{a}_1,\bar{a}_2}(S,T)$. We can directly show that $c_2(t) = c_1(t) - c_4(t) = 0$ if and only if $\bar{c}_2(T) = \bar{c}_1(T) - \bar{c}_4(T) = 0$, so that one of the branch of the singularities is located on the curve $S = 0$. Therefore, we may always assume that one of the branch of singularities are located on $\gamma(t)$. In this case, such singularities satisfy the condition $c_3(t) = 0$. Moreover, another branch of the singularities is given by the equation $2c_5(t) + sc_6(t) = 0$. If $c_6(t) \neq 0$, we denote that $\gamma^\sharp(t) = \gamma(t) + s(t)\boldsymbol{a}_1(t) + (s(t)^2/2)\boldsymbol{\ell}(t)$, where $s(t) = -2c_5(t)/c_6(t)$. We remark that the conditon $c_6(t) \not\equiv 0$ is a generic condition for $C(t) \in C^\infty(I, \mathfrak{hl}(3,1))$.

A cone is one of the typical developable surfaces in Euclidean space. We have horo-flat horocyclic surfaces with analogous properties with cones, but the situation is complicated too. We call $F_{(\gamma,a_1,a_2)}$ a *generalized horo-cone* if $\gamma(t)$ is constant, $\boldsymbol{a}_1'(t) = c_5(t)\boldsymbol{a}_3(t)$ and $\boldsymbol{a}_2'(t) = c_6(t)\boldsymbol{a}_3(t)$. This condition is equivalent to the condition that $c_1(t) = c_2(t) = c_3(t) = c_4(t) = 0$. We say that a generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is a *horo-cone with a single vertex* if $c_1(t) = c_2(t) = c_3(t) = c_4(t) = c_5(t) = 0$ and $c_6(t) \neq 0$. In this case, both of $\gamma(t)$ and $\gamma^\sharp(t)$ are constant and $\gamma = \gamma^\sharp$. A generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is called a *horo-cone with two vertices* if both of $\gamma(t)$ and $\gamma^\sharp(t)$ are constant and $\gamma \neq \gamma^\sharp$. By the calculation of the derivative of $\gamma^\sharp(t)$, the above condition is equivalent to the condition that $c_1(t) = c_2(t) = c_3(t) = c_4(t) = 0$, $c_5(t) \neq 0$ and there exists a real number λ such that $c_5(t) = \lambda c_6(t)$. If the condition $c_1(t) = c_2(t) = c_3(t) = c_4(t) = c_6(t) = 0$, $c_5(t) \neq 0$ is satisfied, then $\boldsymbol{a}_2(t)$ is constant. It follows that the image of the generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is a part of a horosphere (i.e., we call it a *conical horosphere*). We simply call $F_{(\gamma,a_1,a_2)}$ a *horo-cone* if it is one of the above three cases. We can draw the pictures of horo-cones in the Poincaré ball (Fig. 11). We also have the notion of *semi-horo-cones* which belongs to the class of generalized horo-cones. However, we omit the detail. Finally, we say that $F_{(\gamma,a_1,a_2)}$ is a *horo-flat tangent horocyclic surface* if both of γ and $\gamma^\sharp$ are not constant or γ is not constant and $c_6(t) = 0$.

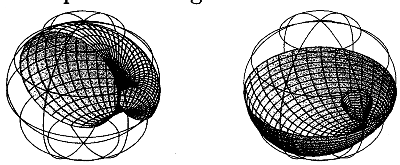
By the previous arguments, we also consider the linear subspace of $\mathfrak{so}(3,1)$ defined by

$$\mathfrak{hf}_\sigma(3,1) = \left\{ C \in \mathfrak{so}(3,1) \;\middle|\; c_2 = c_1 - c_4 = c_3 = 0 \right\}.$$

Therefore the *space of horo-flat singular horocyclic surfaces* can be regarded as the space $C^\infty(I, \mathfrak{hf}_\sigma(3,1))$ with Whitney C^∞-topology. In this terminology, one of the branches of the singularities of the horo-flat surface is always located on the image of γ. In this space the condition $c_5(t) = 0$ is a codimension one condition (in the sufficiently higher



Conical horosphere Single vertex Two vertices



Shifted single vertex Shifted two vertices
Fig. 11

order jet space $J^\ell(I, \mathfrak{hf}_\sigma(3,1))$. Therefore, we cannot generically avoid the points where $c_5(t) = 0$. Two branches of the singularities meet at such points. This fact suggests us the situation is also quite different from the singularities of general wavefront sets or tangent developables in Euclidean space. In[15] we have shown the following theorem:

Theorem 9. *Let $F_{(\gamma,a_1,a_2)}$ be a horo-flat tangent horocyclic surface with singularities along γ.*
(A) *Suppose that $c_5(t_0) \neq 0$ and $c_6(t_0) \neq 0$, then both the points $(0, t_0)$ and $(-s(t_0), t_0)$ are singularities, where $s(t) = 2c_5(t)/c_6(t)$. In this case we have the following:*
(1) *The point $(0, t_0)$ is the cuspidal edge if and only if $c_1(t_0) \neq 0$.*
(2) *The point $(0, t_0)$ is the swallowtail if and only if*

$$c_1(t_0) = 0 \text{ and } c_1'(t_0) \neq 0.$$

(3) *The point $(-s(t_0), t_0)$ is the cuspidal edge if and only if*

$$(c_1 - s')(t_0) \neq 0.$$

(4) *The point $(-s(t_0), t_0)$ is the swallowtail if and only if*

$$(c_1 - s')(t_0) = 0 \text{ and } (c_1 - s')'(t_0) \neq 0.$$

(B) *Suppose that $c_5(t_0) = 0$ and $c_6(t_0) \neq 0$, then $s(t_0) = 0$, so that $(0, t_0) = (-s(t_0), t_0)$ is a singular point. In this case, the point $(0, t_0)$ is the cuspidal beaks if and only if*

$$c_5'(t_0) \neq 0,\ c_1(t_0) \neq 0 \text{ and } (c_1 - s')(t_0) \neq 0.$$

(C) *Suppose that $c_5(t_0) \neq 0$ and $c_6(t_0) = 0$, then the point $(0, t_0)$ is the cuspidal cross cap if and only if*

$$c_1(t_0) \neq 0 \text{ and } c_6'(t_0) \neq 0.$$

In this case , $\gamma(t_0)$ is the only singular point on the generating horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$.

In the above theorem, the *cuspidal edge* is a germ of surface diffeomorphic to

$$CE = \{(x_1,x_2,x_3)|{x_1}^2 = {x_2}^3\},$$

the *swallowtail* is a germ of surface diffeomorphic to

$$SW = \{(x_1,x_2,x_3)|x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\},$$

the *cuspidal cross cap* is a germ of surface diffeomorphic to

$$CCR = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 = u, x_2 = uv^3, x_3 = v^2\}$$

and the *cuspidal beaks* is a germ of surface diffeomorphic to

$$CBK = \{(x_1,x_2,x_3)|x_1 = v, x_2 = -2u^3 + v^2u, x_3 = 3u^4 - v^2u^2\}.$$

These singularities are depicted in Fig. 12.

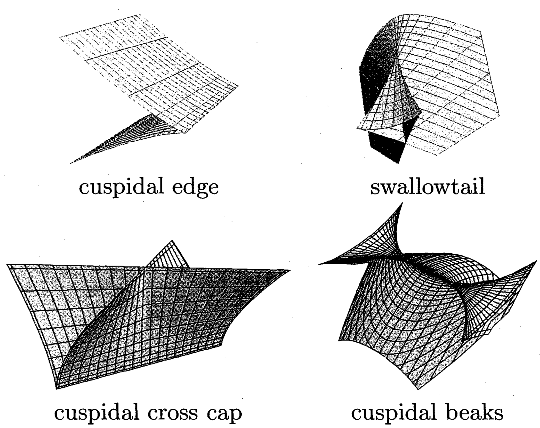


Fig. 12

By Thom's jet-transversality theorem, we can show that the conditions on $C(t)$ in Theorem 9 is generic in the space $C^\infty(I, \mathfrak{hf}_\sigma(3,1))$. This means that these conditions are generic (i.e., stable conditions) in the space of horo-flat tangent horocyclic surfaces. Moreover, we emphasize that the conditions on $C(t)$ are the exact conditions for the above singularities, so that we can easily recognize the singularities for given horo-flat horocyclic surfaces. We also remark that the cuspidal beaks appears as the center of one of the generic one-parameter bifurcations of wave front sets[22]. Usually it bifurcates into two swallowtails or two cuspidal edges. However, it never bifurcates under any small perturbations in the space of horo-flat horocyclic surfaces.

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