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On b-function, spectrum and multiplier ideals

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Dedicated to Professor Masaki Kashiwara

Abstract.

We survey some recent developments in the theory of b-function, spectrum, and multiplier ideals together with certain interesting relations among them including the case of arbitrary subvarieties.

§ Introduction

It has been known that there are certain interesting relations between b-function, spectrum and multiplier ideals. We give a survey on this topic. We first consider the case of hypersurfaces and then arbitrary subvarieties. We recall the definition of b-function, spectrum and multiplier ideals, and explain certain properties together with interesting relations among them. We also explain the cases of hyperplane arrangements and monomial ideals.

In Section 1 we recall the definition of b-function in the hypersurface case and explain some related topics including the V-filtration of Kashiwara and Malgrange. In Section 2 we recall the definition of spectrum in the hypersurface case and explain some known results mainly due to Steenbrink. In Section 3 we recall the definition of multiplier ideals in the general case and give an extension theorem generalizing Mustaţă's formula in the case of hyperplane arrangements. In Section 4 we explain certain relations among b-function, spectrum and multiplier ideals in the hypersurface case. In Section 5 we treat the case of hyperplane arrangements. In Section 6 we define the b-function in the general case and explain a relation with the multiplier ideals. In Section 7 we define the spectrum in the general case and explain a relation with the multiplier ideals. In Section 8 we treat the monomial ideal case.

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In this paper we use the following

Notation. $b_f(s) = b$-function of f, R_f = roots of $b_f(-s)$, $\alpha_f = \min R_f$, $m_{f,\alpha}$ = multiplicity of $\alpha \in R_f$. Similarly for $\widetilde{R}_f, \widetilde{m}_{f,\alpha}, \widetilde{\alpha}_f$ with $b_f(s)$ replaced by $\widetilde{b}_f(s) := b_f(s)/(s+1)$. $R_{f,x}, m_{f,x,\alpha}, \alpha_{f,x}$ are associated to the local b-function $b_{f,x}(s)$. $R'_{f,x} = \cup_{y \ne x} R_{f,y}$, $\alpha'_{f,x} = \min_{y \ne x}\{\alpha_{f,y}\}$.

§1. b-function of a hypersurface

In this section we recall the definition of b-function in the hypersurface case and explain some related topics including the V-filtration of Kashiwara and Malgrange.

1.1. Definition. Let X be a complex manifold or a smooth complex algebraic variety, and f be a holomorphic or algebraic function on X. Let $\mathcal{D}_X$ be the sheaf of linear differential operators on X. Set $\partial_i = \partial/\partial x_i$ for local coordinates $x_1, \dots, x_n$. Then

$$\mathcal{D}_X[s]f^s \subset \mathcal{O}_X[\tfrac{1}{f}][s]f^s \quad \text{with } \partial_i f^s = s(\partial_i f)f^{s-1}.$$

The b-function (i.e. the Bernstein–Sato polynomial) $b_f(s)$ is the monic polynomial of the smallest degree such that

$$b_f(s)f^s = P(x, \partial_x, s)f^{s+1} \quad \text{in } \mathcal{O}_X[\tfrac{1}{f}][s]f^s,$$

where $P(x, \partial_x, s) \in \mathcal{D}_X[s]$. Locally, this coincides with the minimal polynomial of the action of s on

$$\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}.$$

The latter definition is valid in a more general case.

We define $b_{f,x}(s)$ replacing $\mathcal{D}_X$ with $\mathcal{D}_{X,x}$.

1.2. Remark. The b-function or Bernstein–Sato polynomial for a hypersurface was introduced by Sato [41] and Bernstein [3], see also [4].

1.3. Observation. Let $i_f : X \to \widetilde{X} := X \times \mathbf{C}$ denote the graph embedding. Set

$$(1.3.1) \qquad \widetilde{M} = i_{f+}\mathcal{O}_X = \mathcal{D}_{\widetilde{X}}\delta(f-t) = \mathcal{O}_{X\times\mathbf{C}}\left[\tfrac{1}{f-t}\right]/\mathcal{O}_{X\times\mathbf{C}},$$

This is a free $\mathcal{O}_X[\partial_t]$-module of rank 1 with basis $\delta(f-t)$ which is identified with the class of $\frac{1}{f-t}$. Here i_{f+} denotes the direct image as a

$\mathcal{D}$-module, and t is the coordinate of $\mathbf{C}$. The action of ∂_i, t on $\delta(f-t)$ is given by

$$(1.3.2) \qquad \partial_i\delta(f-t) = -(\partial_i f)\partial_t\delta(f-t), \quad t\delta(f-t) = f\delta(f-t).$$

Then f^s is canonically identified with $\delta(f-t)$ by setting $s = -\partial_t t$, and there is a canonical isomorphism as $\mathcal{D}_X[s]$-modules

$$(1.3.3) \qquad \mathcal{D}_X[s]f^s = \mathcal{D}_X[s]\delta(f-t).$$

1.4. V-filtration. We say that V is a filtration of Kashiwara [25] and Malgrange [28] along f if V is exhaustive, separated, and satisfies the following conditions for any $\alpha \in \mathbf{Q}$:

(i) $V^\alpha\widetilde{M}$ is a coherent $\mathcal{D}_X[s]$-submodule of $\widetilde{M}$.

(ii) $tV^\alpha\widetilde{M} \subset V^{\alpha+1}\widetilde{M}$ and the equality holds for $\alpha \gg 0$.

(iii) $\partial_t V^\alpha\widetilde{M} \subset V^{\alpha-1}\widetilde{M}$.

(iv) $\partial_t t - \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha\widetilde{M}$.

(If V exists, it is unique.)

1.5. Relation with the b-function. Assume X is affine or Stein and relatively compact. Then the multiplicity of a root α of $b_f(s)$ is given by the degree of the minimal polynomial of $s - \alpha$ on

$$(1.5.1) \qquad \mathrm{Gr}_V^\alpha(\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}),$$

using the isomorphism (1.3.3) where $s = -\partial_t t$. Note that $V^\alpha\widetilde{M}$ for $\alpha \in \mathbf{Q}$ and $\mathcal{D}_X[s]f^{s+i}$ for $i \in \mathbf{N}$ are 'lattices' of $\widetilde{M}$, i.e.

$$(1.5.2) \qquad V^\alpha\widetilde{M} \subset \mathcal{D}_X[s]f^{s+i} \subset V^\beta\widetilde{M} \quad \text{for } \alpha \gg i \gg \beta,$$

and $V^\alpha\widetilde{M}$ is an analogue of the Deligne extension [11] with eigenvalues in $[\alpha, \alpha+1)$. This is quite similar to the case of differential equations of one variable with regular singularities. The existence of V is equivalent to the existence of $b_f(s)$ locally.

1.6. Theorem (Kashiwara [24], [25], Malgrange [28]). *The filtration V exists on $\widetilde{M} := i_{f+}M$ for any holonomic $\mathcal{D}_X$-module M (where V is indexed by $\mathbf{C}$).*

1.7. Remarks. (i) There are lots of ways to show this theorem. Indeed, it is essentially equivalent to the existence of the b-function in a generalized sense. In case M is regular, one way is to use a resolution of singularities and reduce to the case where the characteristic variety $\mathrm{CV}(M)$ has normal crossings.

(ii) A holonomic $\mathcal{D}$-module M is called quasi-unipotent if the local monodromies of the local systems $\mathcal{H}^j\mathrm{DR}(M)|_{S_i}$ are quasi-unipotent where $\{S_i\}$ is a suitable Whitney stratification. This condition is equivalent to the condition that the filtration V along f is indexed by $\mathbf{Q}$ for any locally defined function f. Indeed, the last condition is equivalent to the first condition since the last condition using V is stable by subquotients so that we can argue by induction on $\dim \mathrm{Supp}\, M$.

1.8. Relation with vanishing cycle functors. Let $\rho : X_t \to D$ be a 'good' retraction where $D = f^{-1}(0)$, and $X_t = f^{-1}(t)$ with $t \neq 0$ sufficiently near 0. This is obtained by using an embedded resolution of singularities of (X, D), since the existence of such a retraction is well known in the normal crossing case and it is enough to compose it with the blown-down. Then there are canonical isomorphisms

$$\psi_f \mathbf{C}_X = \mathbf{R}\rho_* \mathbf{C}_{X_t}, \quad \varphi_f \mathbf{C}_X = \psi_f \mathbf{C}_X / \mathbf{C}_D, \tag{1.8.1}$$

where $\psi_f \mathbf{C}_X, \varphi_f \mathbf{C}_X$ are nearby and vanishing cycle sheaves, see [13].

Let F_x denote the Milnor fiber around $x \in D$. Then we have

$$\begin{aligned} (\mathcal{H}^j \psi_f \mathbf{C}_X)_x &= H^j(F_x, \mathbf{C}), \\ (\mathcal{H}^j \varphi_f \mathbf{C}_X)_x &= \widetilde{H}^j(F_x, \mathbf{C}). \end{aligned} \tag{1.8.2}$$

For a $\mathcal{D}_X$-module M admitting the V-filtration on $\widetilde{M} = i_{f+}M$ indexed by $\mathbf{Q}$, we define $\mathcal{D}_X$-modules

$$\psi_f M = \textstyle\bigoplus_{0<\alpha\le 1} \mathrm{Gr}_V^\alpha \widetilde{M}, \quad \varphi_f M = \textstyle\bigoplus_{0\le\alpha<1} \mathrm{Gr}_V^\alpha \widetilde{M}. \tag{1.8.3}$$

1.9. Theorem (Kashiwara [25], Malgrange [28]). *For a quasi-unipotent regular holonomic $\mathcal{D}_X$-module M, we have the canonical isomorphisms*

$$\begin{aligned} \mathrm{DR}_X \psi_f(M) &= \psi_f \mathrm{DR}_X(M)[-1], \\ \mathrm{DR}_X \varphi_f(M) &= \varphi_f \mathrm{DR}_X(M)[-1], \end{aligned} \tag{1.9.1}$$

such that $\exp(-2\pi i \partial_t t)$ *on the left-hand side corresponds to the monodromy T on the right-hand side.*

1.10. Definition. Set

$R_f = \{\text{roots of } b_f(-s)\}$, $\alpha_f = \min R_f$, $m_{f,\alpha}$: the multiplicity of $\alpha \in R_f$.

(Similarly for $R_{f,x}$, etc. for $b_{f,x}(s)$.)

1.11. Theorem (Kashiwara [23]). *We have* $R_f \subset \mathbf{Q}_{>0}$.

(This is proved by using a resolution of singularities.)

1.12. Theorem (Kashiwara [25], Malgrange [28]). *We have*
(i) $e^{-2\pi i R_f} = \{$*the eigenvalues of* T *on* $H^j(F_x, \mathbf{C})$ *for any* $x \in D, j \in \mathbf{Z}\}$.
(ii) $m_{f,\alpha} \le \min\{i \mid N^i \psi_{f,\lambda}\mathbf{C}_X = 0\}$ *with* $\lambda = e^{-2\pi i\alpha}$.

Here $\psi_{f,\lambda} = \mathrm{Ker}(T_s - \lambda) \subset \psi_f$ in the abelian category of perverse sheaves [2], and $N = \log T_u$ with $T = T_s T_u$ the Jordan decomposition.

1.13. Remark. This is a corollary of the above Theorem (1.9) of Kashiwara and Malgrange, and is a generalization of a formula of Malgrange [27] in the isolated singularity case, see (4.6).

1.14. Microlocal *b*-function. Define $\widetilde{R}_f, \widetilde{m}_{f,\alpha}, \widetilde{\alpha}_f$ with $b_f(s)$ replaced by the *microlocal* (or reduced) *b*-function

$$\widetilde{b}_f(s) := b_f(s)/(s+1). \tag{1.14.1}$$

By [38], $\widetilde{b}_f(s)$ is the monic polynomial of the smallest degree such that

$$\widetilde{b}_f(s)\delta(f-t) = \widetilde{P}\partial_t^{-1}\delta(f-t), \tag{1.14.2}$$

where $\widetilde{P} \in \mathcal{D}_X[s, \partial_t^{-1}]$.

Put $n = \dim X$. Then

1.15. Theorem. *We have*

$$\widetilde{R}_f \subset [\widetilde{\alpha}_f, n - \widetilde{\alpha}_f], \quad \widetilde{m}_{f,\alpha} \le n - \widetilde{\alpha}_f - \alpha + 1.$$

(This follows from the filtered duality for φ_f, see loc. cit.)

1.16. Remark. If f is weighted-homogeneous with an isolated singularity at the origin, then we have by an unpublished result of Kashiwara (mentioned in the end of Introduction of [27])

$$\widetilde{R}_f = E_f, \quad \widetilde{m}_{f,\alpha} = 1 \text{ for } \alpha \in \widetilde{R}_f, \tag{1.16.1}$$

where E_f is the set of exponents, see (2.1.2) below. This assertion also follows from a result of Malgrange in loc. cit., see Th. (4.6) below.

If $f = \sum_i x_i^2$, then $\widetilde{\alpha}_f = n/2$ and (1.16.1) follows from the above Theorem (1.15).

§2. Spectrum of a hypersurface

In this section we recall the definition of spectrum in the hypersurface case and explain some known results mainly due to Steenbrink.

2.1. Spectrum. Let f be a function on a complex manifold or a smooth complex algebraic variety X of dimension n. Let F_x denote the Milnor fiber around $x \in D = f^{-1}(0)$. Following Steenbrink [45], [47] we define the *spectrum*

$$\begin{gathered}\mathrm{Sp}(f,x) = \mathrm{Sp}(D,x) = \textstyle\sum_{\alpha>0} n_{f,\alpha} t^{\alpha} \quad \text{where} \\ n_{f,\alpha} = \textstyle\sum_j (-1)^{j-n+1} \dim \mathrm{Gr}_F^p \widetilde{H}^j(F_x, \mathbf{C})_\lambda \quad \text{with} \\ p = [n-\alpha], \ \lambda = \exp(-2\pi i \alpha).\end{gathered} \tag{2.1.1}$$

Here F is the Hodge filtration ([12], [45]) on $\widetilde{H}^j(F_x, \mathbf{C})_\lambda := \mathrm{Ker}(T_s - \lambda)$ with $T = T_s T_u$ the Jordan decomposition. We define the *exponents* by

$$E_f = \{\alpha \in \mathbf{Q} \mid n_{f,\alpha} \neq 0\} \subset \mathbf{Q}_{>0}. \tag{2.1.2}$$

2.2. Isolated singularity case. In this case we have by [45] symmetry and positivity

$$n_{f,\alpha} = n_{f,n-\alpha} \geq 0. \tag{2.2.1}$$

Moreover, by Scherk–Steenbrink [43] and Varchenko [49], we have for f, g on X, Y

$$\mathrm{Sp}(f+g,(x,y)) = \mathrm{Sp}(f,x)\mathrm{Sp}(g,y), \tag{2.2.2}$$

where the product on the right-hand side is taken in $\mathbf{Q}[t^{1/e}]$ for some $e \in \mathbf{Z}_{>0}$. This can be extended to the non-isolated singularity case (unpublished).

2.3. Weighted homogeneous isolated singularity case. Assume f is weighted homogeneous with positive weights $w_1, \ldots, w_n$, i.e. $f = \sum_\nu c_\nu x^\nu$ with $c_\nu = 0$ for $\sum_i w_i \nu_i \neq 1$. Assume further $\mathrm{Sing}\, D = \{0\}$. Then we have by Steenbrink [44]

$$\mathrm{Sp}(f,x) = \textstyle\prod_i (t - t^{w_i})/(t^{w_i} - 1). \tag{2.3.1}$$

Indeed, he showed that the left-hand side is given by the Poincare polynomial of the graded vector space

$$\Omega_X^n / df \wedge \Omega_X^{n-1}, \tag{2.3.2}$$

and it is well known that the latter is calculated by using the morphism $(f_1, \ldots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ (where $f_i = \partial f / \partial x_i$).

2.4. Nondegenerate Newton boundary case. If $n = 2$ and f has nondegenerate Newton boundary ∂P_f such that $\mathbf{R}^2_{\geq 0} \setminus P_f$ is bounded, then by Steenbrink [45]

$$(2.4.1)\qquad \begin{aligned} &E_f \cap (0,1] = \bigcup_\sigma E_\sigma^{\leq 1} \quad \text{with} \\ &E_\sigma^{\leq 1} = \{L_\sigma(u) \mid u \in \mathbf{Z}^2_{>0} \cap (\{0\} \cup \sigma)^{\text{conv.hull}}\}, \end{aligned}$$

were L_σ is a linear function such that $L_\sigma^{-1}(1) \supset \sigma$. Here the symmetry of E_f with center 1 is used, see (2.2.1).

For $n > 2$, the filtration V on $\Omega_X^n / df \wedge \Omega_X^{n-1}$ is induced by the Newton filtration, and there is a combinatorial description by Steenbrink [45], see also [33], [51]. (Note that [33] was the origin of the theory of bifiltered strict complexes.)

2.5. Semicontinuity (Steenbrink [46]). For a deformation $\{f_\lambda\}_{\lambda \in \Delta}$ with isolated singularities the number of exponents in $(\alpha, \alpha+1]$ (counted with multiplicity) is upper-semicontinuous for any $\alpha \in \mathbf{R}$. This gives a necessary condition for adjacent relation of isolated hypersurface singularities, and implies a counterexample to some conjecture about the adjacent relation. (For a lower weight deformation of a weighted homogeneous polynomial, this is due to Varchenko [50].)

2.6. Steenbrink's conjecture [47]. If $\dim \operatorname{Sing} f = 1$, and g is generic with $dg \neq 0$, then we have for $r \gg 0$

$$(2.6.1)\qquad \begin{aligned} &\operatorname{Sp}(f + g^r, x) - \operatorname{Sp}(f, x) \\ &= \textstyle\sum_{k,j} t^{\alpha_{k,j} + (\beta_{k,j}/m_k r)} (1-t)/(1 - t^{1/m_k r}), \end{aligned}$$

where $m_k = \operatorname{mult}_x Z_k$ with Z_k the irreducible components of $(\operatorname{Sing} f)_{\text{red}}$, the $\alpha_{k,j}$ are the exponents (counted with multiplicities) at $y \in Z_k \setminus \{x\}$, and $\beta_{k,j}$ are rational numbers in $(0,1]$ such that $\exp(-2\pi i \beta_{k,j})$ are the eigenvalues of the monodromy along $Z_k \setminus \{x\}$ (compatible with $\alpha_{k,j}$), see [36] for a proof.

The formula (2.6.1) can be used for the calculation of $\operatorname{Sp}(f + g^r, x)$, see [47].

§3. Multiplier ideals and an extension theorem

In this section we recall the definition of multiplier ideals in the general case and give an extension theorem generalizing Mustaţă's formula in the case of hyperplane arrangements.

3.1. Definition. Let Z be a subvariety of a complex manifold or a smooth complex algebraic variety X. The multiplier ideal $\mathcal{J}(X, \alpha Z)$ for $\alpha \in \mathbf{Q}_{>0}$ is defined by

$$g \in \mathcal{J}(X, \alpha Z) \Leftrightarrow |g|^2/(\sum |f_i|^2)^\alpha \text{ is locally integrable,} \tag{3.1.1}$$

where $f_1, \ldots, f_r$ are local generators of the ideal of Z, or

$$\mathcal{J}(X, \alpha Z) = \rho_* \omega_{\widetilde{X}/X}(-\textstyle\sum_i [\alpha m_i] \widetilde{D}_i)), \tag{3.1.2}$$

where $\rho : (\widetilde{X}, \widetilde{D}) \to (X, Z)$ is an embedded resolution such that $\rho^{-1}\mathcal{I}_Z$ generates the ideal $\mathcal{I}_{\widetilde{D}}$ of $\widetilde{D} = \sum_i m_i \widetilde{D}_i$.

Define for any α (with $0 < \varepsilon \ll 1$)

$$\mathcal{G}(X, \alpha Z) = \mathcal{J}(X, (\alpha - \varepsilon)Z)/\mathcal{J}(X, \alpha Z). \tag{3.1.3}$$

We say that α is a jumping number of Z if and only if $\mathcal{G}(X, \alpha Z) \neq 0$. Set

$$\mathrm{JN}(Z) = \{\text{Jumping numbers of } Z\} \subset \mathbf{Q}_{>0}. \tag{3.1.4}$$

3.2. Extension of multiplier ideals. Assume $X = Y \times \mathbf{C}^r$ and $D = f^{-1}(0)$ with $\lambda^* f = f$ for $\lambda \in \mathbf{C}^*$, where the action of λ is defined by

$$\lambda \cdot (y, z_1, \ldots, z_r) = (y, \lambda^{w_1} z_1, \ldots, \lambda^{w_1} z_r) \in Y \times \mathbf{C}^r,$$

with $w_i > 0$. For $y \in Y = Y \times \{0\} \subset X$, let

$$G^{>\alpha}\mathcal{O}_{X,y} = \{g \in \mathcal{O}_{X,y} \mid v(g) > \alpha\} \text{ with}$$
$$v(\textstyle\sum a_\nu z^\nu) = \min\{\sum_i w_i(\nu_i + 1) \mid a_\nu \neq 0\}.$$

Let $X' = X \setminus (Y \times \{0\})$, $D' = D \cap X'$ with the inclusion $j : X' \to X$. Then

3.3. Theorem [39]. *We have*

$$\mathcal{J}(X, \alpha D)_y = (j_* \mathcal{J}(X', \alpha D'))_y \cap G^{>\alpha}\mathcal{O}_{X,y}.$$

This implies the following generalization of Mustaţă's formula [29] in the case of hyperplane arrangements (see (5.17) below).

3.4. Corollary. *Assume D is the affine cone of a divisor Z on $\mathbf{P}^{n-1}$. Let $d = \deg Z = \deg f$. Then*

$$\mathcal{J}(X, \alpha D) = I_0^k \textit{ with } k = [d\alpha] - n + 1 \textit{ if } \alpha < \alpha'_{f,0}, \tag{3.4.1}$$

where I_0 *is the ideal of* 0 *and* $\alpha'_{f,0} = \min_{x\neq 0}\{\alpha_{f,x}\}$.

3.5. Corollary. *With the above assumption*

$$\dim F^{n-1}H^{n-1}(F_0,\mathbf{C})_{\mathbf{e}(-k/d)} = \binom{k-1}{n-1} \text{ for } 0 < \tfrac{k}{d} < \alpha'_{f,0},$$

and the same holds with F *replaced by* P.

3.6. Corollary. *With the above assumption, we have*

$$\alpha_f = \min\left(\alpha'_{f,0}, \frac{n}{d}\right).$$

§4. Relations in the hypersurface case

In this section we explain certain relations among b-function, spectrum and multiplier ideals in the hypersurface case.

4.1. Theorem (Budur [7]) *Assume* $\mathrm{Sing}\, f = \{x\}$. *Then*

$$\begin{aligned} n_{f,\alpha} &= \dim \mathcal{G}(X,\alpha D)_x \quad (\alpha \in (0,1)), \\ \mathrm{JN}(D) \cap (0,1) &= E_f \cap (0,1). \end{aligned} \tag{4.1.1}$$

(This is generalized to the non-hypersurface case in Th. (7.4).)

4.2. Theorem (Budur, S. [10]). *Let* V *denote also the induced filtration on* $\mathcal{O}_X \subset \mathcal{O}_X[\partial_t]\delta(f-t)$. *If* α *is not a jumping number,*

$$\mathcal{J}(X,\alpha D) = V^{\alpha}\mathcal{O}_X. \tag{4.2.1}$$

For α *general we have for* $0 < \varepsilon \ll 1$

$$\mathcal{J}(X,\alpha D) = V^{\alpha+\varepsilon}\mathcal{O}_X, \quad V^{\alpha}\mathcal{O}_X = \mathcal{J}(X,(\alpha-\varepsilon)D). \tag{4.2.2}$$

Note that V is left-continuous and $\mathcal{J}(X,\alpha D)$ is right-continuous, i.e.

$$V^{\alpha}\mathcal{O}_X = V^{\alpha-\varepsilon}\mathcal{O}_X, \quad \mathcal{J}(X,\alpha D) = \mathcal{J}(X,(\alpha+\varepsilon)D). \tag{4.2.3}$$

The proof of (4.2) uses the theory of bifiltered direct images [34], [35] to reduce the assertion to the normal crossing case.

As a corollary we get another proof of the results of Ein, Lazarsfeld, Smith and Varolin [18], and of Lichtin, Yano and Kollár [26].

4.3. Corollary. (i) $\mathrm{JN}(D) \cap (0,1] \subset R_f$ (*see* [18]).
(ii) $\alpha_f = \min \mathrm{JN}(D)$ (*see* [26]).

Let $\alpha'_{f,x} = \min_{y\neq x}\{\alpha_{f,y}\}$. We have a partial converse of Cor. (4.3)(i) as follows.

4.4. Theorem. *If $\xi f = f$ for a vector field ξ, then*

$$R_f \cap (0, \alpha'_{f,x}) = \mathrm{JN}(D) \cap (0, \alpha'_{f,x}). \tag{4.4.1}$$

This does not hold without the assumption on ξ nor without restricting to $(0, \alpha'_{f,x})$.

4.5. Brieskorn lattice (isolated singularities case). The Brieskorn lattice [5] and its saturation are defined by

$$\begin{aligned} H''_f &= \Omega^n_{X,x}/df \wedge d\Omega^{n-2}_{X,x}, \\ \widetilde{H}''_f &= \textstyle\sum_{i\geq 0}(t\partial_t)^i H''_f \subset H''_f[t^{-1}]. \end{aligned} \tag{4.5.1}$$

These are finite $\mathbf{C}\{t\}$-modules with a regular singular connection. Here $\Omega^\bullet_X$ is analytic and $n = \dim X$. Note that the action of ∂_t^{-1} on H''_f is well-defined by $\partial_t^{-1}[\omega] = [df \wedge \xi]$ where $\xi \in \Omega^{n-1}_{X,x}$ such that $d\xi = \omega$ in $\Omega^n_{X,x}$.

4.6. Theorem (Malgrange [27]). *In the isolated singularity case, the reduced b-function $\widetilde{b}_f(s)$ coincides with the minimal polynomial of $-\partial_t t$ on $\widetilde{H}''_f/t\widetilde{H}''_f$.*

(The above formula of Kashiwara on b-function (1.16.1) can be proved by using this together with Brieskorn's calculation.)

4.7. Asymptotic Hodge structures (Varchenko [49] and Scherk–Steenbrink [43]). In the isolated singularity case, let $\mathcal{G}_f$ be the Gauss–Manin system $H''_f[\partial_t]$ (which is the localization of H''_f by the action of the microdifferential operator ∂_t^{-1}). Let V be the filtration of Kashiwara and Malgrange on $\mathcal{G}_f$. Set $n = \dim X$. Then

$$\begin{aligned} F^p H^{n-1}(F_x, \mathbf{C})_\lambda &= \mathrm{Gr}^\alpha_V H''_f \\ \text{for } p &= [n-\alpha], \lambda = e^{-2\pi i\alpha}, \end{aligned} \tag{4.7.1}$$

under the canonical isomorphism

$$H^{n-1}(F_x, \mathbf{C})_\lambda = \mathrm{Gr}^\alpha_V \mathcal{G}_f, \tag{4.7.2}$$

together with $\partial_t^i : \mathrm{Gr}^\alpha_V \mathcal{G}_f \xrightarrow{\sim} \mathrm{Gr}^{\alpha-i}_V \mathcal{G}_f$ for $i \in \mathbf{Z}$. Note that Varchenko's filtration is defined by using t^{-i} instead of ∂_t^i.

The formula (4.7.1) can be generalized to the non-isolated singularity case using mixed Hodge modules.

4.8. Reformulation of Malgrange's formula. Set

$$\widetilde{F}^p H^{n-1}(F_x, \mathbf{C})_\lambda = \mathrm{Gr}_V^\alpha \widetilde{H}_f'' \quad \text{for } p = [n-\alpha], \lambda = e^{-2\pi i\alpha}, \tag{4.8.1}$$

under the canonical isomorphism (4.7.2). Then

$$\widetilde{m}_{f,\alpha} = \deg(\min \operatorname{poly}(N \mid \mathrm{Gr}_{\widetilde{F}}^p H^{n-1}(F_x, \mathbf{C})_\lambda)), \tag{4.8.2}$$

where min poly means the minimal polynomial.

4.9. Remarks. (i) If f has an isolated singularity, then, as a corollary of the results of Malgrange [27], Varchenko [49], Scherk–Steenbrink [43] explained in (4.6–7), we have

$$\widetilde{R}_f \subset \textstyle\bigcup_{0 \le k < n} (E_f - k), \quad \widetilde{\alpha}_f = \min \widetilde{R}_f = \min E_f. \tag{4.9.1}$$

(ii) If f is weighted homogeneous with an isolated singularity, then by the result of Kashiwara explained in (1.16) we have

$$\widetilde{F} = F, \quad \widetilde{R}_f = E_f. \tag{4.9.2}$$

(iii) Let g be a weighted homogenous polynomial with an isolated singularity, and h be a monomial x^u with modified degree $\beta := \sum_i w_i u_i > 1$ where $w_1, \ldots, w_n$ are the weights associated to g, i.e. $\sum_i w_i x_i \partial g/\partial x_i = g$. Set $f = g + h$, and assume $h \notin (\partial g)$. Then $E_f \ne \widetilde{R}_f$. Indeed, we have $\partial_t[x^{u+v}dx] \in \widetilde{H}_f''$ for any monomial x^v since

$$\left(\textstyle\sum_i (v_i + 1) w_i\right) \partial_t^{-1}[x^v dx] - t[x^v dx] = (\beta - 1)[x^{u+v} dx].$$

We can apply this to a monomial x^v such that x^{u+v} generates the highest modified degree part of $\mathbf{C}[x]/(\partial g)$ which is 1-dimensional.

4.10. Example. Let $f = x^5 + y^4 + x^3y^2$. Then

$$E_f = \{\tfrac{i}{5} + \tfrac{j}{4} \mid 1 \le i \le 4,\ 1 \le j \le 3\},$$
$$\widetilde{R}_f = E_f \cup \{\tfrac{11}{20}\} \setminus \{\tfrac{31}{20}\}.$$

This is the simplest example such that $E_f \ne \widetilde{R}_f$.

4.11. Relation with rational singularities [37]. *Assume $D := f^{-1}(0)$ is reduced. Then D has rational singularities if and only if $\widetilde{\alpha}_f > 1$. Moreover,*

$$\omega_D/\rho_*\omega_{\widetilde{D}} \simeq F_{1-n}\varphi_f \mathcal{O}_X,$$

where $\rho : \widetilde{D} \to D$ is a resolution of singularities.

In the isolated singularities case, this was proved in [32] using the coincidence of $\widetilde{\alpha}_f$ and the minimal exponent.

4.12. Relation with the pole order filtration [37]. Let P be the pole order filtration on $\mathcal{O}_X(*D)$, i.e. $P_i = \mathcal{O}_X((i+1)D)$ if $i \geq 0$, and $P_i = 0$ if $i < 0$. Let F be the Hodge filtration on $\mathcal{O}_X(*D)$. Then we have $F_i \subset P_i$ in general, and

$$F_i = P_i \text{ on a neighborhood of } x \text{ if } i \leq \widetilde{\alpha}_{f,x} - 1.$$

4.13. Remark. In the case $X = \mathbf{P}^n$, replacing $\widetilde{\alpha}_{f,x}$ with $[(n-r)/d]$ where $r = \dim \operatorname{Sing} D$ and $d = \deg D$, the assertion was obtained by Deligne (unpublished).

§5. Hyperplane arrangement case

In this section we treat the case of hyperplane arrangements.

5.1. Let D be a central hyperplane arrangement in $X = \mathbf{C}^n$, i.e. D is an affine cone of a projective hyperplane arrangement $Z \subset \mathbf{P}^{n-1}$. Let f be the reduced equation of D with $d = \deg f > n$. Assume D is not the pull-back of $D' \subset \mathbf{C}^{n'}$ with $n' < n$. Then we have

5.2. Theorem. (i) $\max R_f < 2 - \frac{1}{d}$. (ii) $m_1 = n$.

For the proof of (i) we use a partial generalization of a solution of Aomoto's conjecture due to Esnault, Schechtman, Viehweg, Terao, Varchenko ([19], [42]) together with the following generalization of Malgrange's formula in (4.8).

5.3. Theorem (Generalization of Malgrange's formula) [39]. *There exists a pole order filtration P on $H^{n-1}(F_0, \mathbf{C})_\lambda$ satisfying the following property.*

If $(\alpha + \mathbf{N}) \cap R'_{f,0} = \emptyset$ with $R'_{f,0} = \cup_{x \neq 0} R_{f,x}$, then

$$\alpha \in R_f \iff \operatorname{Gr}_P^p H^{n-1}(F_0, \mathbf{C})_\lambda \neq 0, \tag{5.3.1}$$

where $p = [n - \alpha], \lambda = e^{-2\pi i \alpha}$.

Using this, the proof of (5.2)(i) is reduced to

$$P^i H^{n-1}(F_0, \mathbf{C})_\lambda = H^{n-1}(F_0, \mathbf{C})_\lambda, \tag{5.3.2}$$

for $i = n-1$ if $\lambda = 1$ or $e^{2\pi i/d}$, and $i = n-2$ otherwise.

5.4. Construction of the pole order filtration P. Let $U = \mathbf{P}^{n-1} \setminus Z$, and $F_0 = f^{-1}(1) \subset \mathbf{C}^n$. Then F_0 is canonically identified with a cyclic d-fold covering $\pi : \widetilde{U} \to U$ ramified over Z. Let $L^{(k)}$ be the local systems of rank 1 on U such that $\pi_*\mathbf{C} = \bigoplus_{0 \le i < d} L^{(k)}$ and T acts on $L^{(k)}$ by $e^{-2\pi i k/d}$. Then we have canonical isomorphisms

$$H^j(U, L^{(k)}) = H^j(F_0, \mathbf{C})_{\mathbf{e}(-k/d)}, \tag{5.4.1}$$

and P is induced by the pole order filtration on the meromorphic extension $\mathcal{L}^{(k)}$ (see [11]) of $L^{(k)} \otimes_{\mathbf{C}} \mathcal{O}_U$ over $\mathbf{P}^{n-1}$, see [16], [39], [40]. This is closely related to [1] and also the following.

5.5. Solution of Aomoto's conjecture ([19], [42]). Let Z_i be the irreducible components of Z $(1 \le i \le d)$. Let g_i be the defining equation of Z_i on $\mathbf{P}^{n-1} \setminus Z_d$ for $i < d$, and set

$$\omega = \textstyle\sum_{i<d} \alpha_i \omega_i \quad \text{with } \omega_i = dg_i/g_i,\ \alpha_i \in \mathbf{C}.$$

Let ∇ be the connection on $\mathcal{O}_U$ defined by

$$\nabla u = du + \omega \wedge u.$$

Set $\alpha_d = -\sum_{i<d} \alpha_i$. Then $H^\bullet_{\mathrm{DR}}(U, (\mathcal{O}_U, \nabla))$ is calculated by the complex

$$(\mathcal{A}^\bullet_\alpha, \omega\wedge) \quad \text{with} \quad \mathcal{A}^p_\alpha = \textstyle\sum \mathbf{C}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}, \tag{5.5.1}$$

if the following condition is satisfied:

$$\textstyle\sum_{Z_i \supset L} \alpha_i \notin \mathbf{N} \setminus \{0\} \text{ for any } \textit{dense} \text{ edge } L \subset Z, \tag{5.5.2}$$

see (5.7) below for dense edges.

For the proof of (5.2)(ii) we use

5.6. Proposition. *If* $\mathrm{Gr}^W_{2n-2} H^{n-1}(F_x, \mathbf{C})_\lambda \ne 0$, *then* $N^{n-1}\psi_{f,\lambda}\mathbf{C} \ne 0$.

(Indeed, by the definition of W, we have the isomorphism

$$N^{n-1} : \mathrm{Gr}^W_{2n-2}\psi_{f,\lambda}\mathbf{C} \xrightarrow{\sim} \mathrm{Gr}^W_0 \psi_{f,\lambda}\mathbf{C},$$

and the assumption of (5.6) implies $\mathrm{Gr}^W_{2n-2}\psi_{f,\lambda}\mathbf{C} \ne 0$.)

Note that Proposition (5.6) implies (5.2)(ii), since we have the non-vanishing of $\omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-1}}$ in

$$\mathrm{Gr}^W_{2n-2} H^{n-1}(\mathbf{P}^{n-1} \setminus Z, \mathbf{C}) = \mathrm{Gr}^W_{2n-2} H^{n-1}(F_x, \mathbf{C})_1.$$

5.7. Dense edges. Let $D = \cup_i D_i$ be the irreducible decomposition. Then $L = \cap_{i\in I} D_i$ for $I \neq \emptyset$ is called an edge of D. An edge L is called *dense* if $\{D_i/L \,|\, D_i \supset L\}$ is indecomposable. Here $\mathbf{C}^n \supset D$ is decomposable if $\mathbf{C}^n = \mathbf{C}^{n'} \times \mathbf{C}^{n''}$ such that D is the union of the pull-backs from $\mathbf{C}^{n'}, \mathbf{C}^{n''}$ where $n', n'' \neq 0$.

Let $m_L = \#\{D_i \,|\, D_i \supset L\}$, and for $\lambda \in \mathbf{C}$

$$\mathcal{DE}(D) = \{\text{dense edges of } D\},$$
$$\mathcal{DE}(D,\lambda) = \{L \in \mathcal{DE}(D) \,|\, \lambda^{m_L} = 1\}.$$

We say that L, L' are *strongly adjacent* if $L \subset L'$ or $L \supset L'$ or $L \cap L'$ is non-dense. Let

$$m(\lambda) = \max\{|S| \,|\, S \subset \mathcal{DE}(D,\lambda) \text{ such that any two edges belonging to } S \text{ are strongly adjacent}\}.$$

5.8. Theorem [40]. *We have* $m_{f,\alpha} \le m(\lambda)$ *with* $\lambda = e^{-2\pi i\alpha}$.

5.9. Corollary. *We have* $R_f \subset \bigcup_{L\in\mathcal{DE}(D)} \mathbf{Z}m_L^{-1}$.

5.10. Corollary. *Assume that* $\mathrm{GCD}(m_L, m_{L'}) = 1$ *for any strongly adjacent* $L, L' \in \mathcal{DE}(D)$. *Then* $m_{f,\alpha} = 1$ *for any* $\alpha \in R_f \setminus \mathbf{Z}$.

For Theorem (5.2) we use the canonical embedded resolution of singularities $\pi : (\widetilde{X}, \widetilde{D}) \to (\mathbf{P}^{n-1}, D)$, see [42]. This is obtained by blowing up along the proper transforms of the dense edges. Note that we have $\mathrm{mult}\, \widetilde{D}(\lambda)_{\mathrm{red}} \le m(\lambda)$, where $\widetilde{D}(\lambda)$ is the union of $\widetilde{D}_i$ such that $\lambda^{\widetilde{m}_i} = 1$ and $\widetilde{m}_i = \mathrm{mult}_{\widetilde{D}_i} \widetilde{D}$.

5.11. Generic case. If D is a generic central hyperplane arrangement, then

$$b_f(s) = (s+1)^{n-1} \prod_{j=n}^{2d-2} (s + \tfrac{j}{d}) \tag{5.11.1}$$

by U. Walther [53] (except for the multiplicity of -1) using a completely different method.

Note that Theorems (5.2) and (5.8) imply that the left-hand side divides the right-hand side of (5.11.1), and the equality follows using also (3.5).

5.12. Explicit calculation. Let $\alpha = k/d$, $\lambda = e^{-2\pi i\alpha}$ with $k \in \{1,\dots,d\}$. If $\alpha \ge \alpha'_{f,0} := \min_{x\neq 0}\{\alpha_{f,x}\}$, we assume there is $I \subset \{1,\dots,d-1\}$ with $|I| = k-1$, and also the condition of [42] (i.e. (5.5.2) above) is satisfied for

$$\alpha_i = 1 - \alpha \text{ if } i \in I \cup \{d\}, \text{ and } -\alpha \text{ otherwise.} \tag{5.12.1}$$

Let $V(I)$ be the subspace of $H^{n-1}\mathcal{A}_\alpha^\bullet$ generated by

$$\omega_{i_1}\wedge\cdots\wedge\omega_{i_{n-1}} \quad \text{for} \quad \{i_1,\dots,i_{n-1}\}\subset I.$$

5.13. Theorem. *With the above notation and assumptions, we have for $\alpha = k/d$ and $\lambda = e^{-2\pi i\alpha}$ with $k\in\{1,\dots,d\}$ the following.*
(a) *In case $k = d-1$ or d, we have $\alpha\in R_f$, $\alpha+1\notin R_f$.*
(b) *In case $\alpha < \alpha'_{f,0}$, we have $\alpha\in R_f$ if and only if $k\ge d$.*
(c) *In case $\binom{k-1}{n-1} < \dim H^{n-1}(F_0,\mathbf{C})_\lambda$, we have $\alpha+1\in R_f$.*
(d) *In case $\alpha<\alpha'_{f,0}$, $\alpha\notin R'_{f,0}+\mathbf{Z}$ and $\binom{k-1}{n-1}=\chi(U)$, we have $\alpha+1\notin R_f$.*
(e) *In case $\alpha\ge\alpha'_{f,0}$ and $V(I)\ne 0$, we have $\alpha\in R_f$.*
(f) *In case $\alpha\ge\alpha'_{f,0}$ and $V(I)=H^{n-1}\mathcal{A}_\alpha^\bullet$, we have $\alpha+1\notin R_f$.*

5.14. Theorem [40]. *Assume $n=3$, $\max\{\mathrm{mult}_z Z\mid z\in Z\}=3$, and $d\le 7$. Let ν_3 be the number of triple points of Z, and assume $\nu_3\ne 0$. Then*

$$b_f(s) = (s+1)\prod_{i=2}^4(s+\tfrac{i}{3})\prod_{j=3}^r(s+\tfrac{j}{d}), \tag{5.14.1}$$

with $r=2d-2$ or $2d-3$. We have $r=2d-2$ if $\nu_3<d-3$. The converse holds for $d<7$. In the case $d=7$, we have $r=2d-3$ if $\nu_3>4$. However, r can be both $2d-2$ and $2d-3$ if $\nu_3=4$.

5.15. Remarks. (i) We have $\nu_3<d-3$ if and only if we have

$$\chi(U)=\tfrac{(d-2)(d-3)}{2}-\nu_3>\tfrac{(d-3)(d-4)}{2}=\binom{d-3}{2}.$$

(ii) By (5.4.1) we have

$$\chi(U)=h^2(F_0,\mathbf{C})_\lambda-h^1(F_0,\mathbf{C})_\lambda \ \text{ if } \lambda^d=1 \text{ with } \lambda\ne 1.$$

(iii) Let ν'_i be the number of i-ple points of $Z':=Z\cap\mathbf{C}^2$. Then we have by [6]

$$b_0(U)=1,\quad b_1(U)=d-1,\quad b_2(U)=\nu'_2+2\nu'_3,$$

5.16. Example. Assume Z' is defined by $(x^2-y^2)(x^2-1)(y^2-1)=0$ in $\mathbf{C}^2$ with $d=7$. Then (5.14.1) holds with $r=11$, and $12/7\notin R_f$. In this case we have

$$b_1(U)=6,\ b_2(U)=9,\ \chi(U)=4,$$
$$h^2(F_0,\mathbf{C})_\lambda=4 \text{ if } \lambda^7=1 \text{ and } \lambda\ne 1.$$

Then $5/7\in R_f$ by (e) and $12/7\notin R_f$ by (f), where I^c corresponds to $(x+1)(y+1)=0$. Note that $5/7$ is not a jumping number.

5.17. Multiplier ideals of hyperplane arrangements. Let $m_L = \mathrm{mult}_L D$, $r = \mathrm{codim}_X L$, and $\mathcal{I}_L$ be the ideal of an edge $L \subset X$. Then by Mustaţă [29]

$$\mathcal{J}(X, \alpha D) = \textstyle\bigcap_L \mathcal{I}_L^{[\alpha m_L]+1-r_L}. \tag{5.17.1}$$

(This is generalized as in Cor. (3.4) above.)

As for the spectrum, it does not seem easy to give a combinatorial formula even for the generic case, see e.g. [39], 5.6.

§6. b-function of a subvariety

In this section we define the b-function in the general case and explain a relation with the multiplier ideals.

6.1. Let Z be a closed subvariety of a complex manifold or a smooth complex algebraic variety X. Let $f = (f_1, \dots, f_r)$ be generators of the ideal of Z. (We do not assume Z reduced nor irreducible.) Define the action of t_j on

$$\mathcal{O}_X\big[\tfrac{1}{f_1 \cdots f_r}\big][s_1, \dots, s_r]\textstyle\prod_i f_i^{s_i},$$

by $t_j(s_i) = s_i + 1$ if $i = j$, and $t_j(s_i) = s_i$ otherwise. Set

$$s_{i,j} = s_i t_i^{-1} t_j, \quad s = \textstyle\sum_i s_i.$$

Then $b_f(s)$ is the monic polynomial of the smallest degree such that

$$b_f(s)\textstyle\prod_i f_i^{s_i} = \sum_{k=1}^r P_k t_k \prod_i f_i^{s_i}, \tag{6.1.1}$$

where P_k belong to the ring generated by $\mathcal{D}_X$ and $s_{i,j}$.

Here we can replace $\prod_i f_i^{s_i}$ with $\prod_i \delta(t_i - f_i)$, using the direct image by the graph of $f : X \to \mathbf{C}^r$. Note that the existence of $b_f(s)$ follows from the theory of the V-filtration of Kashiwara and Malgrange. This b-function has appeared in work of Sabbah [31] and Gyoja [20] for the study of b-functions of several variables.

6.2. Theorem (Budur, Mustaţă, S. [8]). *Let $c = \mathrm{codim}_X Z$. Then $b_Z(s) := b_f(s - c)$ depends only on Z, i.e. it is independent of the choice of X, $f = (f_1, \dots, f_r)$, and also of r.*

6.3. Equivalent definition. The b-function $b_f(s)$ coincides with the monic polynomial of the smallest degree such that

$$b_f(s)\textstyle\prod_i f^{s_i} \in \sum_{|c|=1} \mathcal{D}_X[\mathrm{s}] \prod_{c_i<0} \binom{s_i}{-c_i} \prod_i f_i^{s_i+c_i}, \tag{6.3.1}$$

where $c = (c_1, \dots, c_r) \in \mathbf{Z}^r$ with $|c| := \sum_i c_i = 1$. Here $\mathcal{D}_X[\mathbf{s}] = \mathcal{D}_X[s_1, \cdots, s_r]$.

This is due to Mustaţă, and is used in the monomial ideal case, see (8.7) below. Note that the well-definedness does not hold without the term $\prod_{c_i<0} \binom{s_i}{-c_i}$.

We denote also by V the induced filtration under the inclusion

$$\mathcal{O}_X \subset i_{f+}\mathcal{O}_X = \mathcal{O}_X[\partial_1, \dots, \partial_r]\textstyle\prod_i \delta(t_i - f_i).$$

6.4. Theorem (Budur, Mustaţă, S. [8]). *For $\alpha \notin \mathrm{JN}(Z)$, we have*

$$\mathcal{J}(X, \alpha Z) = V^{\alpha}\mathcal{O}_X. \tag{6.4.1}$$

In general we have for any α (with $0 < \varepsilon \ll 1$)

$$\mathcal{J}(X, \alpha Z) = V^{\alpha+\varepsilon}\mathcal{O}_X, \quad V^{\alpha}\mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)Z). \tag{6.4.2}$$

6.5. Corollary (Budur, Mustaţă, S. [8]). *We have the inclusion*

$$\mathrm{JN}(Z) \cap [\alpha_f, \alpha_f + 1) \subset R_f. \tag{6.5.1}$$

6.6. Theorem (Budur, Mustaţă, S. [8]). *Assume Z is reduced and is a local complete intersection. Then Z has at most rational singularities if and only if $\alpha_f = r$ with multiplicity* 1.

§7. Spectrum of a subvariety

In this section we define the spectrum in the general case and explain a relation with the multiplier ideals.

7.1. Let Z be a closed subvariety of a complex manifold or a smooth complex algebraic variety X, and $\mathcal{I}_Z \subset \mathcal{O}_X$ be the ideal sheaf of Z. The Verdier specialization [52] is defined by

$$\mathrm{Sp}_Z \mathbf{Q}_X = \psi_t \, \mathbf{R}j_* \mathbf{Q}_{X \times \mathbf{C}^*}, \tag{7.1.1}$$

where

$$j : X \times \mathbf{C}^* \, (= \mathcal{S}pec_X \mathcal{O}_X[t, t^{-1}]) \to \mathcal{S}pec_X(\textstyle\bigoplus_{i \in \mathbf{Z}} \mathcal{I}_Z^{-i} \otimes t^i)$$

is the inclusion to the total space of the deformation to the normal cone

$$N_Z X = \mathcal{S}pec_Z(\textstyle\bigoplus_{i \in \mathbf{N}} \mathcal{I}_Z^i / \mathcal{I}_Z^{i+1}). \tag{7.1.2}$$

Let Λ be an irreducible component of the fiber $(N_Z X)_z$ over $z \in Z$, and $\xi \in \Lambda$ be a sufficiently general point of Λ with the inclusion $i_\xi : \{\xi\} \to N_Z X$. Set $c_\Lambda = \dim X - \dim \Lambda$. Define the non-reduced spectrum and the (reduced) spectrum

$$\widehat{\mathrm{Sp}}(Z,\Lambda) = \textstyle\sum_{\alpha>0} n_{\Lambda,\alpha} t^\alpha,$$
$$\mathrm{Sp}(Z,\Lambda) = \widehat{\mathrm{Sp}}(Z,\Lambda) - (-1)^{c_\Lambda} t^{c_\Lambda+1},$$

where

$$\begin{aligned} n_{\Lambda,\alpha} &= \textstyle\sum_j (-1)^j \dim \mathrm{Gr}_F^p H^{j+c_\Lambda}(i_\xi^* \mathrm{Sp}_Z \mathbf{C}_X)_\lambda \quad \text{with} \\ p &= [c_\Lambda + 1 - \alpha], \ \lambda = \exp(-2\pi i\alpha), \end{aligned} \tag{7.1.3}$$

If $(N_Z X)_x$ is irreducible (e.g. if Z is a complete intersection), set

$$\widehat{\mathrm{Sp}}(Z,x) = \widehat{\mathrm{Sp}}(Z,\Lambda), \text{ etc. for } \Lambda = (N_Z X)_x.$$

This generalizes the definition for hypersurfaces.

7.2. Remarks. (i) In general, we have

$$n_{\Lambda,\alpha} = 0 \ (\alpha \le 0), \quad n_{\Lambda,\beta} \ge 0 \ (\beta \in (0,1]).$$

In the isolated complete intersection singularity case, we have

$$\widetilde{n}_{x,\alpha} \ge 0 \quad \text{with} \quad \mathrm{Sp}(Z,x) = \textstyle\sum_\alpha \widetilde{n}_{x,\alpha} t^\alpha,$$

but symmetry and semicontinuity do not hold, see [17], [30], [48].

(ii) In the isolated complete intersection singularity case, our definition coincides with the one by Ebeling and Steenbrink [17] except for $n_{x,\alpha}$ with $\alpha \in \mathbf{Z}$. Indeed, they take generic 1-parameter smoothings

$$f : X' \to \mathbf{C} \ \text{ of } \ Z, \quad g : X'' \to \mathbf{C} \ \text{ of } \ X',$$

and consider $\varphi_f \psi_g \mathbf{Q}_{X''}[n]$ (where $n = \dim Z$) together with the exact sequence

$$0 \to \widetilde{H}^n(F_f, \mathbf{C}) \to \varphi_f \psi_g \mathbf{Q}_{X''}[n] \to H^{n+1}(F_g, \mathbf{C}) \to 0,$$

where $\psi_g \mathbf{Q}_{X''}|_{X'\setminus\{0\}} = \mathbf{Q}$, $(\psi_g \mathbf{Q}_{X''})_0 = \mathbf{R}\Gamma(F_g, \mathbf{C})$. The action of the monodromy on $H^{n+1}(F_g, \mathbf{C})$ is associated to the functor φ_f, and is the identity.

7.3. Let $\mathcal{I}_Z$ be the ideal sheaf of $Z \subset X$. For $z \in Z$ and $\beta \in (0,1] \cap \mathbf{Q}$, let

$$\begin{aligned} \mathcal{M}(\beta) &= \textstyle\bigoplus_{i\ge0} \mathcal{G}(X, (\beta+i)Z), \quad \bar{\mathcal{A}} = \bigoplus_{j\ge0} \mathcal{I}_Z^j/\mathcal{I}_Z^{j+1}, \\ \mathcal{M}(\beta, z) &= \mathcal{M}(\beta)/\mathfrak{m}_{Z,z}\mathcal{M}(\beta), \quad \bar{\mathcal{A}}(z) = \bar{\mathcal{A}}/\mathfrak{m}_{Z,z}\bar{\mathcal{A}}. \end{aligned} \tag{7.3.1}$$

Then $\mathcal{M}(\beta)$, $\mathcal{M}(\beta,z)$ are graded modules over $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}(z)$, because

$$(7.3.2)\qquad (\mathcal{I}_Z^j/\mathcal{I}_Z^{j+1})\,\mathcal{G}(X,(\beta+i)Z)\subset\mathcal{G}(X,(\beta+i+j)Z).$$

For $z\in Z$ and an irreducible component Λ of $(N_ZX)_z=\mathrm{Spec}\,\bar{\mathcal{A}}(z)$,

$$(7.3.3)\qquad \mu_{\Lambda,\beta}:=\dim_{\mathbf{C}(\Lambda)}\mathcal{M}(\beta,z)\otimes_{\bar{\mathcal{A}}(z)}\mathbf{C}(\Lambda),$$

where $\mathbf{C}(\Lambda)$ is the function field of Λ.

7.4. Theorem (Dimca, Maisonobe, S. [14]). *Let* $\beta\in(0,1]\cap\mathbf{Q}$.
(i) *We have* $0\le n_{\Lambda,\beta}\le\mu_{\Lambda,\beta}$ *(in particular,* $n_{\Lambda,\beta}=0$ *if* $z\notin\mathrm{Supp}\,\mathcal{M}(\beta)$*).*
(ii) *We have* $n_{\Lambda,\beta}=\mu_{\Lambda,\beta}$ *if* $\mathrm{Supp}_{\bar{\mathcal{A}}}\mathcal{M}(\beta)\subset(N_ZX)_z$ *on a neighborhood of the generic point of* Λ.

(For hypersurfaces, this is due to Budur [7].)

7.5. Corollary (DMS [14]). *If* $n_{\Lambda,\alpha}\ne0$ *with* $\alpha\in(0,1)$*, then there is a nonnegative integer* j_0 *such that* $\alpha+j\in\mathrm{JN}(Z)$ *for any* $j\ge j_0\in\mathbf{N}$.

7.6. Theorem (DMS [14]). *If* T *is a transversal slice to a stratum of a good Whitney stratification and* $r=\mathrm{codim}\,T$*, we have*

$$\widehat{\mathrm{Sp}}(Z,\Lambda)=(-t)^r\,\widehat{\mathrm{Sp}}(Z\cap T,\Lambda).$$

(For the constantness of the jumping numbers under a topologically trivial deformation of divisors, see [15].)

7.7. Remark. Let $E_{Z,\Lambda}=\{\alpha\,|\,n_{\Lambda,\alpha}\ne0\}$. Then

$$\textstyle\bigcup_\Lambda \exp(-2\pi iE_{Z,\Lambda})\subset\exp(-2\pi iR_{f,x}),$$

where Λ runs over the irreducible components of $(N_ZX)_x$. However, the equality does not always hold (e.g. if $f=x^2y$) unless we take the union over the irreducible components Λ of $(N_ZX)_y$ for any $y\in Z$ sufficiently near x.

8. Monomial ideal case

In this section we treat the monomial ideal case.

8.1. Multiplier ideals. Let $\mathfrak{a}\subset\mathbf{C}[x]:=\mathbf{C}[x_1,\dots,x_n]$ a monomial ideal. We have the associated semigroup defined by

$$\Gamma_\mathfrak{a}=\{u\in\mathbf{N}^n\mid x^u\in\mathfrak{a}\}.$$

Let $P_\mathfrak{a}$ be the convex hull of $\Gamma_\mathfrak{a}$ in $\mathbf{R}^n_{\ge0}$. Set $\mathbf{1}=(1,\dots,1)$, and

$$U(\alpha):=\{\nu\in\mathbf{N}^n\,|\,\nu+\mathbf{1}\in(\alpha+\varepsilon)P_\mathfrak{a}\ (0<\varepsilon\ll1)\}.$$

Let Z be the subvariety of $X = \mathbf{C}^n$ defined by $\mathfrak{a}$.

By Howald we have the following.

8.2. Theorem (*Multiplier ideals*) (Howald [21]). *We have*

$$\mathcal{J}(X, \alpha Z) = \textstyle\sum_{\nu \in U(\alpha)} \mathbf{C}\, x^{\nu}.$$

8.3. Corollary. *Set* $\phi(\nu) = L_\sigma(\nu)$ *if* $\nu \in \mathrm{Cone}(0,\sigma) := \bigcup_{\lambda \geq 0} \lambda\sigma$, *where* L_σ *is as in* 8.4 *below. Then*

$$\mathrm{JN}(Z) = \{\phi(\nu) \mid \nu \in \mathbf{Z}^n_{>0}\}.$$

8.4. Spectrum. For a maximal face σ of $P_\mathfrak{a}$, set

L_σ : the linear function such that $L_\sigma^{-1}(1) \supset \sigma$,

c_σ : the smallest positive integer such that $c_\sigma L_\sigma \in \mathbf{Z}[x]$,

$e_\sigma = |G'_\sigma / G_\sigma|$,

where $G'_\sigma = \mathbf{Z}^n \cap L_\sigma^{-1}(0)$ and G_σ is generated by $\nu - \nu'$ with $\nu, \nu' \in \Gamma_\mathfrak{a} \cap \sigma$.

8.5. Theorem (*Spectrum*) (Dimca, Maisonobe, S. [14]). *We have a one-to-one correspondence between the maximal compact faces* σ *of* $P_\mathfrak{a}$ *and the irreducible components* Λ *of the fiber* $(N_Z X)_0$, *and*

$$\widehat{\mathrm{Sp}}(Z, \Lambda) = \textstyle\sum_{i=1}^{c_\sigma} e_\sigma t^{i/c_\sigma}.$$

8.6. b-function. For a face σ of $P_\mathfrak{a}$, set

V_σ : the linear subspace generated by σ,

M_σ : the subsemigroup generated by $u - v$ with $u \in \Gamma_\mathfrak{a}$, $v \in \Gamma_\mathfrak{a} \cap \sigma$,

$M'_\sigma = v_0 + M_\sigma$ with $v_0 \in \Gamma_\mathfrak{a} \cap \sigma$ (independent of v_0),

$R_\sigma = \{L_\sigma(u) \mid u \in ((M_\sigma \setminus M'_\sigma) + \mathbf{1}) \cap V_\sigma\}$, where $\mathbf{1} = (1, \dots, 1)$,

$R_\mathfrak{a} = \{\text{roots of } b_\mathfrak{a}(-s)\}$ where $b_\mathfrak{a}(s) = b_Z(s)$.

8.7. Theorem (*b-function*) (Budur, Mustaţă, S. [9]). *We have* $R_\mathfrak{a} = \bigcup_\sigma R_\sigma$ *with* σ *not contained in any coordinate hyperplanes.*

8.8. Remark. It is possible that R_σ depends on the other σ'. Indeed, we have the following (see [9], Ex. 4.4).

(i) If $\mathfrak{a} = (xy^5, x^3y^2, x^5y)$, then $R_\mathfrak{a} = R_\sigma \cup R_{\sigma'}$ with

$$R_\sigma = \left\{\tfrac{5}{13}, \tfrac{i}{13}\,(7 \leq i \leq 17), \tfrac{19}{13}\right\}, \quad R_{\sigma'} = \left\{\tfrac{j}{7}\,(3 \leq j \leq 9)\right\}.$$

So $R_\sigma = \{\frac{3i+2j}{13}\,(1 \le i \le 3, 1 \le j \le 5)\}$ with $L_\sigma(i,j) = \frac{3i+2j}{13}$.

(As for $R_{\sigma'}$ there is a misprint in loc. cit. as remarked by a student of W. Veys.)

(ii) If $\mathfrak{a} = (xy^5, x^3y^2, x^4y)$, then $R_\mathfrak{a} = R_\sigma \cup R_{\sigma'}$ with

$$R_\sigma = \{\tfrac{i}{13}\,(5 \le i \le 17)\}, \quad R_{\sigma'} = \{\tfrac{j}{5}\,(2 \le j \le 6)\}.$$

So $R_\sigma \ne \{\frac{3i+2j}{13}\,(1 \le i \le 3, 1 \le j \le 5)\}$ with $\frac{19}{13}$ shifted to $\frac{6}{13}$.

8.9. Comparison. Let $D = f^{-1}(0)$ for $f = \sum c_\nu x^\nu \in \mathbf{C}[x]$ with non-degenerate Newton boundary $\partial P_f = \partial P_\mathfrak{a}$. Assume $Z_{\rm red} = \{0\}$ so that $\operatorname{Sing} D = \{0\}$. Then

$$(4.5.3) \qquad \begin{array}{ccc} \mathrm{JN}(D) \cap (0,1) & \overset{(1)}{=\!=} & \mathrm{JN}(Z) \cap (0,1) \\ {\scriptstyle (2)}\| & & \cap{\scriptstyle (3)} \\ E_f \cap (0,1) & \underset{(4)}{\subset} & \bigcup_\Lambda E_{Z,\Lambda} \cap (0,1) \end{array}$$

where $E_{Z,\Lambda} = \{\alpha \,|\, n_{\Lambda,\alpha} \ne 0\}$. Indeed, we have (1) by Howald [23], and (2) by Budur [7]. The composition of (1) and (2) is an equality by comparing the formulas of Howald [21] and Steenbrink [45] (see also [33], [51]). Finally we have (3) and (4) by [14]. (In general (3) (4) are not equality.)

8.10. Example. If $\mathfrak{a} = (x_1^{m_1}, \dots, x_n^{m_n})$, set

$$c_\sigma = \mathrm{LCM}(m_1, \dots, m_n), \quad e_\sigma = m_1 \cdots m_n / c_\sigma,$$
$$E = \{(a_1, \dots, a_n) \in \mathbf{N}^n \,|\, a_i \in [1, m_i]\}.$$

Then

$$\mathrm{JN}(Z) = \Big\{ \sum_{i=1}^n \frac{a_i}{m_i} \,\Big|\, a_i \in \mathbf{Z}_{>0} \Big\},$$

$$\widehat{\mathrm{Sp}}(Z,0) = \sum_{i=1}^{c_\sigma} e_\sigma\, t^{i/c_\sigma},$$

$$b_\mathfrak{a}(s) = \Big[\prod_{(a_1,\dots,a_n)\in E} \Big(s + \sum_{i=1}^n \frac{a_i}{m_i} \Big) \Big]_{\rm red}.$$

Here $[\prod_j (s+\beta_j)^{n_j}]_{\rm red} = \prod_j (s+\beta_j)$ if the β_j are mutually different and $n_j \in \mathbf{Z}_{>0}$.

This may be compared with the following.

8.11. Example. If $f = \sum_i x_i^{m_i}$ and $D = f^{-1}(0)$, set

$$\widetilde{E} = \{(a_1, \dots, a_n) \in \mathbf{N}^n \mid a_i \in [1, m_i - 1]\}.$$

Then

$$\mathrm{JN}(D) \cap (0,1] = \Big\{ \sum_{i=1}^n \frac{a_i}{m_i} \,\Big|\, a_i \in \mathbf{Z}_{>0} \Big\} \cap (0,1],$$

$$\text{with } \mathrm{JN}(D) = \big(\mathrm{JN}(D) \cap (0,1]\big) + \mathbf{N},$$

$$\mathrm{Sp}(D,0) = \prod_{i=1}^n (t - t^{1/m_i})/(t^{1/m_i} - 1),$$

$$\widetilde{b}_f(s) = \Big[\prod_{(a_1,\dots,a_n)\in\widetilde{E}} \Big(s + \sum_{i=1}^n \frac{a_i}{m_i} \Big) \Big]_{\mathrm{red}}.$$

Indeed, for the assertion on $\mathrm{JN}(D)$, we can apply [22] or [7] (i.e. Th. (4.1) above), see also Th. (4.4). The other assertions follow from (1.16) and (2.3). Note that the assertions hold for an isolated weighted homogeneous singularities with weights $w_1, \dots, w_n$ if we replace $1/m_i$ by w_i.

8.12. Remark. In the monomial ideal case, j_0 in Cor. (7.5) is bounded by $n-1$, and $\mathrm{JN}(Z)$ is stable by adding any positive integers, see [14]. Note that $j_0 = n-1$ if the m_i in (8.10) are mutually prime. In general it is unclear whether j_0 is always bounded by $n-1$.

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