Advanced Studies in Pure Mathematics 54, 2009
Algebraic Analysis and Around
pp. 167–186

Local geometric Langlands correspondence: the spherical case

Edward Frenkel[1] **and Dennis Gaitsgory**[2]

To Masaki Kashiwara on his 60th birthday

Abstract.

A module over an affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is called spherical if the action of the Lie subalgebra $\mathfrak{g}[[t]]$ on it integrates to an algebraic action of the corresponding group $G[[t]]$. Consider the category of spherical $\widehat{\mathfrak{g}}$-modules of critical level. In this paper we prove that this category is equivalent to the category of quasi-coherent sheaves on the ind-scheme of opers on the punctured disc which are unramified as local systems. This result is a categorical version of the well-known description of spherical vectors in representations of groups over local non-archimedian fields. It may be viewed as a special case of the local geometric Langlands correspondence proposed in [FG2].

§1. Introduction

A general framework for the local geometric Langlands correspondence was proposed in our earlier work [FG2] (see also [FG3]–[FG5] and [F2]). According to our proposal, to each "local Langlands parameter" σ, which is a $\check{G}$-local system on the punctured disc $\mathcal{D}^\times = \operatorname{Spec} \mathbb{C}((t))$ (or equivalently, a $\check{G}$-bundle with a connection on $\mathcal{D}^\times$), there should correspond a category $\mathcal{C}_\sigma$ equipped with an action of the formal loop group $G((t))$. Even more ambitiously, we expect that there exists a category $\mathcal{C}_{\mathrm{univ}}$ fibered over the stack $\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$ of $\check{G}$-local systems, equipped with a fiberwise action of the ind-group $G((t))$, whose fiber category at σ is $\mathcal{C}_\sigma$. Moreover, we expect $\mathcal{C}_{\mathrm{univ}}$ to be the universal category equipped with an action of $G((t))$. In other words, we expect that $\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$ is the universal parameter space for the categorical representations of $G((t))$.

Received December 2, 2007.
[1]Supported by DARPA and AFOSR through the grant FA9550-07-1-0543.
[2]Supported by NSF grant 0600903.

The ultimate form of the local Langlands correspondence for loop groups should be, roughly, the following statement:

(1.1) categories fibering over $\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$ $\iff$ categories equipped with action of $G((t))$

We should point out, however, that neither the notion of category fibered over a non-algebraic stack such as $\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$, nor the unversal property alluded to above are easy to formulate. So for now (1.1) should be understood heuristically, as a guiding principle.

As we explained in [FG2], the local geometric Langlands correspondence should be viewed as a categorification of the local Langlands correspondence for the group $G(F)$, where F is a local non-archimedian field. This means that the categories $\mathcal{C}_\sigma$, equipped with an action of $G((t))$, that we wish to attach to the Langlands parameters $\sigma \in \mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$ should be viewed as categorifications of smooth representations of $G(F)$ in the sense that we expect the Grothendieck groups of the categories $\mathcal{C}_\sigma$ to "look like" irreducible smooth representations of $G(F)$.

1.1. The spherical part

In the study of representations π of $G(F)$, a standard tool is to consider the subspaces π^K of vectors fixed by open compact subgroups K of $G(F)$.

This procedure has a categorical counterpart. Let K be a group-scheme contained in $G[[t]]$ and containing the Nth congruence subgroup K_N for some N (i.e., the subgroup of $G[[t]]$ consisting of elements congruent to 1 modulo $t^N\mathbb{C}[[t]]$). For example, K can be $G[[t]]$ itself, or the Iwahori subgroup I.

Given a category $\mathcal{C}$, acted on by $G((t))$, we can consider the corresponding K-equivariant category $\mathcal{C}^K$. Via (1.1), any such $\mathcal{C}^K$ is also a category fibered over $\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$.

This procedure applies in particular to $\mathcal{C}_{\mathrm{univ}}$. Although at present we do not know how to construct the entire category $\mathcal{C}_{\mathrm{univ}}$, we do have a guess what $\mathcal{C}^K_{\mathrm{univ}}$ is for some choices of K.

In this paper we specialize to the simplest case $K = G[[t]]$. (Another case, which can be explicitly analyzed is that of $K = I$, discussed in [FG2].) Based on the analogy with the classical local Langlands correspondence for spherical representations, we propose:

$$\mathcal{C}_{\mathrm{univ}}^{G[[t]]} \simeq \mathrm{Rep}(\check{G}). \tag{1.2}$$

Here $\mathrm{Rep}(\check{G})$ is the category of (algebraic) representations of $\check{G}$, which can be also thought as the category of quasi-coherent sheaves on the stack $\mathrm{pt}/\check{G}$. The structure of category fibered over $\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$ comes from the maps of stacks

$$\mathrm{pt}/\check{G} \simeq \mathrm{Loc}_{\check{G}}^{\mathrm{unr}} \to \mathrm{Loc}_{\check{G}}(\mathcal{D}^\times) \tag{1.3}$$

corresponding to the inclusion of the stack $\mathrm{Loc}_{\check{G}}^{\mathrm{unr}}$ of unramified local systems (or, equivalently, local systems on the unpunctured disc $\mathcal{D}$) into the stack $\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$ of all local systems.

1.2. Representations of critical level

In [FG2] we have considered a specific example of a category equipped with an action of $G(\!(t)\!)$; namely, the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod of modules over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ of critical level. It carries a canonical action of the ind-group $G(\!(t)\!)$ via its adjoint action on $\widehat{\mathfrak{g}}_{\mathrm{crit}}$.

What should be the relationship between $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod and the conjectural universal category $\mathcal{C}_{\mathrm{univ}}$?

We note that the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod naturally fibers over the ind-scheme $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ of $\check{G}$-opers on $\mathcal{D}^\times$ introduced in [BD]. This is because, according to [FF, F1], the center $\mathfrak{Z}_{\mathfrak{g}}$ of the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod is isomorphic to the algebra of functions on $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$.

The idea of [FG2] is that the latter fibration is a "base change" of $\mathcal{C}_{\mathrm{univ}}$, that is, there is a Cartesian diagram

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod} & \longrightarrow & \mathcal{C}_{\mathrm{univ}} \\ \downarrow & & \downarrow \\ \mathrm{Op}_{\check{G}}(\mathcal{D}^\times) & \xrightarrow{\alpha} & \mathrm{Loc}_{\check{G}}(\mathcal{D}^\times) \end{array} \tag{1.4}$$

which commutes with the action of $G(\!(t)\!)$ along the fibers of the two vertical maps. In other words,

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod} \simeq \mathcal{C}_{\mathrm{univ}} \underset{\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{G}}(\mathcal{D}^\times). \tag{1.5}$$

Given a $\check{\mathfrak{g}}$-oper χ, let us consider it as a point of $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})$, i.e., a character of $\mathfrak{Z}_{\mathfrak{g}}$. Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}_\chi$ be the full subcategory $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}_\chi$ of

$\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod whose objects are $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules, on which the $\mathfrak{Z}_{\mathfrak{g}}$ acts according to this character. This is the fiber category of the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod over $\chi \in \mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$.

Let $\sigma = \alpha(\chi) \in \mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$. By (1.5), we have:

$$\mathcal{C}_\sigma \simeq \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_\chi\,. \tag{1.6}$$

As was mentioned above, at the moment we do not have an independent definition of $\mathcal{C}_{\mathrm{univ}}$, and therefore we cannot make the equivalences (1.5) and (1.6) precise. But we use it as our guiding principle. This leads us to a number of interesting corollaries, some of which have been discussed in [FG2]–[FG5].

For example, if χ, χ' are two $\check{G}$-opers, such that the corresponding local systems $\alpha(\chi)$ and $\alpha(\chi')$ are isomorphic, for every choice of an isomorphism we are supposed to have an equivalence of categories:

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_\chi \simeq \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\chi'} \tag{1.7}$$

intertwining the actions of $G(\!(t)\!)$. This is a highly non-trivial conjecture about representations of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$.

1.3. Harish-Chandra categories

Let us return to the discussion of the category of K-equivariant objects in the context of $\mathcal{C} = \widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod. The corresponding category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^K$ identifies with the category of $(\widehat{\mathfrak{g}}_{\mathrm{crit}}, K)$ Harish-Chandra modules. When K is connected, this is a full abelian subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod, consisting of modules, on which the action of the Lie algebra $\mathrm{Lie}(K) \subset \widehat{\mathfrak{g}}_{\mathrm{crit}}$ is integrable, i.e., comes from an algebraic action of K.

Now specialize to the case $K = G[[t]]$. We call objects of the corresponding category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{G[[t]]}$ of $G[[t]]$-equivariant $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules *spherical*. Combining eqns. (1.2), (1.3) and (1.5), we arrive at the following equivalence:

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{G[[t]]} \simeq \mathrm{QCoh}\left(\mathrm{Loc}^{\mathrm{unr}}_{\check{G}} \underset{\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\right). \tag{1.8}$$

Here we should remark that although the stack $\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$ is a problematic object to work with, the fiber product

$$\mathrm{Loc}^{\mathrm{unr}}_{\check{G}} \underset{\mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$$

appearing on the right-hand side of (1.8) is a well-defined (non-reduced) ind-subscheme of $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$. This is the moduli ind-scheme of opers that are unramified as local systems. We denote this ind-scheme by $\mathrm{Op}^{\mathrm{unr}}_{\check{G}}$. It is a disjoint union of ind-schemes $\mathrm{Op}^{\mathrm{unr},\lambda}_{\check{G}}$, λ being a dominant weight, where the reduced scheme corresponding to each $\mathrm{Op}^{\mathrm{unr},\lambda}_{\check{G}}$ is the scheme $\mathrm{Op}^{\mathrm{reg},\lambda}_{\check{G}}$ of λ-regular opers introduced in [FG2].

Thus, the heuristic guess given by (1.8) leads to the following precise statement, which is the main result of this paper:

Main Theorem. *The category* $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}$ *of spherical* $\widehat{\mathfrak{g}}_{\mathrm{crit}}$*-modules is equivalent to the category of quasi-coherent sheaves on the ind-scheme* $\mathrm{Op}^{\mathrm{unr}}_{\check{G}}$ *of* $\check{G}$*-opers on* $\mathcal{D}^\times$ *unramified as local systems.*

Moreover, we show that a functor from the former category to the latter one is an analogue of the Whittaker functor.

1.4. Some corollaries

Let χ be a $\mathbb{C}$-point of $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$, and let us consider the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}_{\chi}$. As an abelian category, this is a full subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi}$, consisting of $G[[t]]$-integrable modules.

One can show (see [FG3], Corollary 1.11) that this category is 0 unless $\chi \in \mathrm{Op}^{\mathrm{unr}}_{\check{G}}$. In the latter case, from the Main Theorem we obtain that the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}_{\chi}$ is equivalent to the category of vector spaces. This result is the first test for our prediction that $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi}$, as a category equipped with a $G((t))$-action, depends only on $\alpha(\chi)$, as expected in (1.7). In addition, this equivalence is in agreement with a classical fact that the space of spherical vectors in an irreducible representation of $G(F)$ is either zero or one-dimensional.

As another corollary of the Main Theorem, we obtain the following description of the algebra of self-Exts of the Weyl modules $\mathbb{V}_\lambda$ in the derived category $D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]})$ of $(\widehat{\mathfrak{g}}_{\mathrm{crit}}, G[[t]])$ Harish-Chandra modules:

$$\mathrm{Ext}^\bullet_{D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]})}(\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}, \mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}) \simeq \Lambda^{\bullet}_{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}(\mathcal{N}^\lambda_{\mathrm{reg/\,unr}}),$$

where $\mathcal{N}^\lambda_{\mathrm{reg/\,unr}}$ is the normal bundle of $\mathrm{Op}^{\mathrm{reg},\lambda}_{\check{G}}$ in $\mathrm{Op}^{\mathrm{unr},\lambda}_{\check{G}}$. (In the above formula we identify the algebra of functions on $\mathrm{Op}^{\mathrm{reg},\lambda}_{\check{G}}$ with the corresponding quotient of $\mathfrak{Z}_{\mathfrak{g}}$, denoted $\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}$.) For $\lambda = 0$ this isomorphism was previously established in [FT] by other methods.

1.5. Structure of the proof

The proof of the Main Theorem is quite simple. The main idea is that the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod$^{G[[t]]}$ has a universal object, denoted $\mathcal{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$, which is the vacuum module of the chiral algebra of differential operators on G. The module $\mathcal{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$ is in fact a $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-bimodule, and for any other object $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod$^{G[[t]]}$ we have

$$\mathcal{M} \simeq \mathcal{D}^{\mathrm{ch}}_{G,\mathrm{crit}} \overset{\infty/2}{\underset{\mathfrak{g}((t))}{\otimes}} \mathcal{M}$$

(here $\overset{\infty/2}{\underset{\mathfrak{g}((t))}{\otimes}}$ stands for the semi-infinite Tor functor).

Therefore, in order to define functors and check isomorphisms on $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod$^{G[[t]]}$, it is enough to do so just for the module $\mathcal{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$. Thus, in Sect. 2 we prove a theorem that describes the structure of $\mathcal{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$ as a bi-module over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod$^{G[[t]]}$, and in Sect. 3 we derive our Main Theorem from this structure theorem.

§2. Chiral differential operators on G at the critical level

In this section we describe the structure of the chiral algebra of differential operators (CADO) on a simple connected simply-connected algebraic group G over $\mathbb{C}$ at the critical level, viewed as a bimodule over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$.

2.1. Notation

We will follow the notation of [FG2]. In particular, $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ is the critical central extension of the formal loop algebra $\mathfrak{g}((t))$, $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod is the category of discrete modules over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$, $\mathfrak{Z}_{\mathfrak{g}}$ is the center of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod (or, equivalently, of the completion of the enveloping algebra of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$). This is a topological commutative algebra. According to a theorem of [FF, F1], the corresponding ind-scheme $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})$ is canonically isomorphic to the moduli space $\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})$ of $\check{G}$-opers on the formal punctured disc, where $\check{G}$ is the Langlands dual group to G (of adjoint type). For the definition of $\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})$, see [BD].

For $\lambda \in \Lambda^+$, we let $\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}$ denote the quotient of $\mathfrak{Z}_{\mathfrak{g}}$ corresponding to the sub-scheme $\mathrm{Op}^{\mathrm{reg},\lambda}_{\check{G}} \subset \mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})$ introduced in [FG2], Section 2.9. Let

$$\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} = \mathrm{Ind}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}_{\mathfrak{g}[[t]]\oplus\mathbf{C1}}(V^{\lambda}) := U(\widehat{\mathfrak{g}}_{\mathrm{crit}}) \underset{U(\mathfrak{g}[[t]]\oplus\mathbf{C1})}{\otimes} V^{\lambda}$$

be the Weyl module with dominant integral highest weight $\lambda \in \Lambda^+$. According to [FG6], Theorem 1, the action of $\mathfrak{Z}_\mathfrak{g}$ on $\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}$ factors as follows:

$$\mathfrak{Z}_\mathfrak{g} \twoheadrightarrow \mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda} \simeq \mathrm{End}(\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}).$$

Furthermore, $\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}$ is flat (and in fact, free) as a $\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda}$-module.

Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}^{G[[t]]}$ be the full abelian subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod. In this paper we will work with the "naive" derived category $D(\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}^{G[[t]]})$. However, by generalizing the argument of [FG2], Sect. 20.16, one can identify $D(\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}^{G[[t]]})$ with the $G[[t]]$-equivariant derived category corresponding to $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod, as introduced in *loc. cit.*, Sect. 20.8.

In particular, for $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}^{G[[t]]}$ we have:

$$\mathrm{Ext}^i_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–}\mathrm{mod}^{G[[t]]}}(\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}, \mathcal{M}) \simeq \mathrm{Ext}^i_{G[[t]]}(V^\lambda, \mathcal{M}).$$

2.2. Unramified opers

Let $\mathrm{Op}^{\mathrm{unr}}_{\check{G}} \subset \mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ be the ind-subscheme of opers that are unramified as local systems. For any $\mathbb{C}$-algebra A, the set of A-points of $\mathrm{Op}^{\mathrm{unr}}_{\check{G}}$ is by definition the set of opers on $\mathrm{Spec}\, A((t))$, which are isomorphic, as local systems, to the trivial local system. We have:

$$\mathrm{Op}^{\mathrm{unr}}_{\check{G}} \simeq \bigcup_{\lambda\in\Lambda^+} \mathrm{Op}^{\mathrm{unr},\lambda}_{\check{G}}, \tag{2.1}$$

where $\mathrm{Op}^{\mathrm{unr},\lambda}_{\check{G}}$ are pairwise disjoint formal sub-schemes of $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ with

$$(\mathrm{Op}^{\mathrm{unr},\lambda}_{\check{G}})_{\mathrm{red}} \simeq \mathrm{Op}^{\mathrm{reg},\lambda}_{\check{G}}.$$

We will also use the notation $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr}}_\mathfrak{g})$, $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr},\lambda}_\mathfrak{g})$ for these ind-schemes. Let $\iota^\lambda_{\mathrm{reg/\,unr}}$ denote the closed embedding $\mathrm{Spec}(\mathfrak{z}^{\mathrm{reg},\lambda}_\mathfrak{g}) \hookrightarrow \mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr},\lambda}_\mathfrak{g})$; let $\mathbf{I}^\lambda$ denote the (closed) ideal of $\mathrm{Spec}(\mathfrak{z}^{\mathrm{reg},\lambda}_\mathfrak{g})$ in $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr},\lambda}_\mathfrak{g})$; let $\mathcal{N}^\lambda_{\mathrm{reg/\,unr}}$ be the normal to $\mathrm{Spec}(\mathfrak{z}^{\mathrm{reg},\lambda}_\mathfrak{g})$ in $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr},\lambda}_\mathfrak{g})$. It follows from [FG2], Section 4.6, that its sheaf of sections is a locally free $\mathfrak{z}^{\mathrm{reg},\lambda}_\mathfrak{g}$-module (in other words, $\mathcal{N}^\lambda_{\mathrm{reg/\,unr}}$ is s vector bundle over $\mathrm{Spec}(\mathfrak{z}^{\mathrm{reg},\lambda}_\mathfrak{g})$). Moreover, $\mathfrak{Z}_\mathfrak{g}$ carries a Poisson structure, which identifies $\mathcal{N}^\lambda_{\mathrm{reg/\,unr}}$ with $\Omega^1(\mathfrak{z}^{\mathrm{reg},\lambda}_\mathfrak{g})$.

The following fact was established in [FG3], Corollary 1.11 (note that $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr}}_\mathfrak{g})$ was denoted by $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{m.f.}}_\mathfrak{g})$ in [FG3]).

Theorem 2.3. *The support in* $\mathrm{Spec}(\mathfrak{Z}_\mathfrak{g})$ *of every* $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}^{G[[t]]}$ *is contained in* $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr}}_\mathfrak{g})$.

Thus, every $G[[t]]$-integrable $\widehat{\mathfrak{g}}$-module $\mathcal{M}$ splits as a direct sum $\bigoplus_{\lambda} \mathcal{M}^{\lambda}$, where $\mathcal{M}^{\lambda}$ is supported at $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{unr},\lambda})$ and has an increasing filtration whose sub-quotients are quotient modules of $\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}$.

2.4. Chiral differential operators

Let X be a smooth algebraic curve. We will work with Lie-* algebras and chiral algebras on X and with modules over them supported at a fixed point $x \in X$ (see [CHA] for the definitions). In this paper all chiral algebras will come from vertex algebras, and we will tacitly identify a chiral algebra with its vacuum module, i.e., its fiber at any point of a curve equipped with a coordinate. In fact, everything may be rephrased in terms of the corresponding vertex (Lie) algebras and modules over them, but we will use the formalism of chiral algebras for the sake of consistency with [FG1]–[FG3].

Recall from [AG, GMS] that for any level κ (i.e., an invariant bilinear form on $\mathfrak{g}$) we have the chiral algebra of differential operators (CADO), denoted by $\mathfrak{D}^{\mathrm{ch}}_{G,\kappa}$. It comes equipped with two mutually commuting embeddings

$$\mathcal{A}_{\mathfrak{g},\kappa} \xrightarrow{\mathfrak{l}_{\mathfrak{g}}} \mathfrak{D}^{\mathrm{ch}}(G)_{\kappa} \xleftarrow{\mathfrak{r}_{\mathfrak{g}}} \mathcal{A}_{\mathfrak{g},\kappa'}, \tag{2.2}$$

where

$$\kappa' = -\kappa + 2\kappa_{\mathrm{crit}}.$$

Recall that the fiber of $\mathcal{A}_{\mathfrak{g},\kappa}$ at x is the vacuum Weyl module $\mathbb{V}_{\mathfrak{g},\kappa}$ of level κ, and the fiber of $\mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$ at x with

$$\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa}}_{\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbf{1}}(\mathcal{O}_{G[[t]]}) := U(\widehat{\mathfrak{g}}_{\kappa}) \underset{U(\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbf{1})}{\otimes} \mathcal{O}_{G[[t]]}.$$

Here t is a formal coordinate at x and $\mathcal{O}_{G[[t]]}$ is the algebra of functions on the group $G[[t]]$, on which $\mathfrak{g}[[t]]$ acts trivially and $\mathbf{1}$ acts as the identity. This is a module over $\widehat{\mathfrak{g}}_{\kappa} \oplus \widehat{\mathfrak{g}}_{\kappa'}$. The action of $\widehat{\mathfrak{g}}_{\kappa}$ on $\mathfrak{D}^{\mathrm{ch}}(G)_{\kappa,x}$ corresponding to the left arrow in (2.2) is the natural action on this induced module, and the action corresponding to the right arrow in (2.2) was constructed in [AG, GMS]. We will refer to the two actions as the "left" and the "right" actions, respectively.

2.5. CADO at the critical level

We now specialize to the critical level $\kappa = \kappa_{\mathrm{crit}}$. Then $\kappa' = \kappa_{\mathrm{crit}}$, and so both left and right actions of $\widehat{\mathfrak{g}}$ correspond to the critical level. We will describe the structure of $\mathfrak{D}^{\mathrm{ch}}(G)_{\mathrm{crit},x}$ as a $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-bimodule. From

now on, when there is no confusion, we will skip the subscript x when describing the fiber of the chiral algebra at x.

Let $\mathfrak{z}_{\mathfrak{g}}$ denote the center of $\mathcal{A}_{\mathfrak{g},\mathrm{crit}}$ (note that $\mathfrak{z}_{\mathfrak{g}}$ identifies with $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},0}$). The following has been established in [FG1]:

Lemma 2.6. *The two embeddings*

$$\mathfrak{l},\mathfrak{r}:\mathfrak{z}_{\mathfrak{g}} \rightrightarrows \mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$$

differ by the automorphism of $\mathfrak{z}_{\mathfrak{g}}$*, induced by Cartan involution* τ *of* $\mathfrak{g}$.

As a bimodule over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$, $\mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$ is $G[[t]]$-integrable with respect to both actions. By Theorem 2.3, its support over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})$ is contained in $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr}}_{\mathfrak{g}})$ (note that by Lemma 2.6, the two actions of $\mathfrak{Z}_{\mathfrak{g}}$ on $\mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$ differ by τ). Hence, we have a direct sum decomposition of $\mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$ as a $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-bimodule:

$$\mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}} \simeq \bigoplus_{\lambda\in\Lambda^+} \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}$$

where $\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}$ is the summand supported at $\mathrm{Op}^{\mathrm{unr},\lambda}_{\check{G}} = \mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr},\lambda}_{\mathfrak{g}})$ (see formula (2.1)).

Recall that $\mathcal{O}_{G[[t]]}$ denotes the algebra of functions on $G[[t]]$. It has a natural structure of commutative chiral algebra, and as such it is a chiral subalgebra of $\mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$. The map

$$\mathcal{O}_{G[[t]]} \to \mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$$

respects the bimodule structure with respect to $\mathfrak{g}[[t]] \subset \widehat{\mathfrak{g}}_{\mathrm{crit}}$.

For $\lambda \in \Lambda^+$ we have a natural map

$$V^{\lambda} \otimes V^{\tau(\lambda)} \to \mathcal{O}_G \hookrightarrow \mathcal{O}_{G[[t]]},$$

compatible with the action of $\mathfrak{g}[[t]] \oplus \mathfrak{g}[[t]]$. Inducing, we obtain a map of bimodules over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$:

$$\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}} \to \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}.$$

From Lemma 2.6 we obtain:

Lemma 2.7. *The above map factors through a map*

$$\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}} \to \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}. \tag{2.3}$$

2.8. A description of the CADO

Recall that $\mathbf{I}^\lambda$ denotes the ideal of $\mathrm{Spec}(\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda})$ in $\mathrm{Spec}(\mathfrak{Z}_\mathfrak{g}^{\mathrm{unr},\lambda})$. Consider the canonical increasing filtration on $\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}$ numbered by $i = 0, 1, \ldots$ with $F^i(\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}})$ being the $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ sub-bimodule, annihilated by the $i+1$-st power of the ideal $\mathbf{I}^\lambda$. By construction, the image of the map (2.3) belongs to $F^0(\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}})$.

We are now ready to formulate the main result of this section:

Theorem 2.9.

(1) *The map* (2.3) *defines an isomorphism*

$$\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}} \simeq F^0(\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}).$$

(2) *The canonical maps*

$$(\mathbf{I}^\lambda)^n/(\mathbf{I}^\lambda)^{n+1} \underset{\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda}}{\otimes} \mathrm{gr}^0(\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}) \to \mathrm{gr}^n(\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}})$$

give rise to isomorphisms

$$\mathrm{gr}^n(\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}) \simeq \mathrm{gr}^0(\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}) \underset{\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda}}{\otimes} \mathrm{Sym}^n_{\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda}}(\mathcal{N}^\lambda_{\mathrm{reg/\,unr}}) \tag{2.4}$$

of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$*-bimodules, where* $\mathcal{N}^\lambda_{\mathrm{reg/\,unr}}$ *is the normal bundle to* $\mathrm{Op}^{\mathrm{reg},\lambda}_{\check{G}}$ *in* $\mathrm{Op}^{\mathrm{unr},\lambda}_{\check{G}}$.

The above theorem should be contrasted with the following:

Lemma 2.10. *For a generic* κ *(i.e., such that* κ/κ_c *is not a rational number) we have an isomorphism*

$$\mathfrak{D}^{\mathrm{ch}}_{G,\kappa} \simeq \bigoplus_{\lambda\in\Lambda^+} \mathbb{V}^\lambda_{\mathfrak{g},\kappa} \otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa'}, \tag{2.5}$$

of $\widehat{\mathfrak{g}}_\kappa \oplus \widehat{\mathfrak{g}}_{\kappa'}$ *modules.*

Proof. For any level κ we have a canonical non-zero homomorphism of $\widehat{\mathfrak{g}}_\kappa \oplus \widehat{\mathfrak{g}}_{\kappa'}$ modules

$$\mathbb{V}^\lambda_{\mathfrak{g},\kappa} \otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa'} \to \mathfrak{D}^{\mathrm{ch}}_{G,\kappa}. \tag{2.6}$$

If κ satisfies the conditions of the lemma, then both $\mathbb{V}^\lambda_{\mathfrak{g},\kappa}$ and $\mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa'}$ are irreducible modules. Therefore the above maps are injective. The assertion of the lemma then follows from the obvious fact that the characters of the two sides of (2.5) are equal to each other. Q.E.D.

For special values of κ, when $\kappa/\kappa_c \in \mathbb{Q}$, the modules $\mathbb{V}^{\lambda}_{\mathfrak{g},\kappa}$ and $\mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa'}$ may become reducible, and so the structure of $\mathfrak{D}^{\mathrm{ch}}_{G,\kappa}$ may become more complicated. Theorem 2.9 describes what happens at the critical level $\kappa = \kappa_{\mathrm{crit}}$. In this case the image of the homomorphism (2.6) is equal to $\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}^{\lambda}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}$. In other words, we observe the collapse of the degrees of freedom corresponding to $\mathfrak{z}^{\lambda}$. But these degrees of freedom are restored by the second factor in (2.4).

2.11. Proof of part (2)

We shall first prove part (2) of Theorem 2.9. Recall the chiral algebroid $\mathcal{A}^{\mathrm{ren},\tau}_{\mathfrak{g},\mathrm{crit}}$ of [FG1], Section 4, whose chiral enveloping algebra is $\mathfrak{D}^{\mathrm{ch},0}_{G,\mathrm{crit}}$, by Lemma 9.7 of *loc. cit.* (note that the assertion of Theorem 2.9 for $\lambda = 0$ follows in fact from this isomorphism and Lemma 7.4 of [FG1]).

A version of Kashiwara's theorem proved in Section 7 of *loc. cit.* implies the following:

Proposition 2.12. *Let $\mathcal{M}$ be a chiral $\mathcal{A}^{\mathrm{ren},\tau}_{\mathfrak{g},\mathrm{crit}}$-module, whose support over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})$ is contained in $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr},\lambda}_{\mathfrak{g}})$. Let $F^i(\mathcal{M})$, $i = 1, 2, \ldots$ be the canonical increasing filtration on $\mathcal{M}$ by the powers of $\mathbf{I}^{\lambda}$. Then*

(a) *$R^i(\iota^{\lambda})^!(\mathcal{M}) = 0$ for $i > 0$ and $R^0(\iota^{\lambda})^!(\mathcal{M}) \simeq F^0(\mathcal{M})$.*

(b) *The canonical maps*

$$(\mathbf{I}^{\lambda})^n/(\mathbf{I}^{\lambda})^{n+1} \underset{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}{\otimes} \mathrm{gr}^0(\mathcal{M}) \to \mathrm{gr}^n(\mathcal{M})$$

give rise to isomorphisms

$$\mathrm{gr}^n(\mathcal{M}) \simeq \mathrm{gr}^0(\mathcal{M}) \underset{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}{\otimes} \mathrm{Sym}^n_{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}(\mathcal{N}^{\lambda}_{\mathrm{reg/\,unr}}).$$

We apply this proposition to $\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}$, which is a chiral module over $\mathfrak{D}^{\mathrm{ch},0}_{G,\mathrm{crit}}$, and hence over $\mathcal{A}^{\mathrm{ren},\tau}_{\mathfrak{g},\mathrm{crit}}$, and the assertion of point (2) of Theorem 2.9 follows.

2.13. Proof of part (1)

To prove part (1) of Theorem 2.9, let us first show that the map (2.3) is injective. Indeed, let K denote its kernel; this is a bimodule over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$, supported at $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr},\lambda}_{\mathfrak{g}})$ and $G[[t]]$-integrable with respect to both actions. Hence, if $K \neq 0$, there exists a non-zero map of $\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \to K$ of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules, with respect to the left action.

However, we claim that the map

$$\text{(2.7)}\quad \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}) \to \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}})$$

is injective, and in fact an isomorphism. This would lead to a contradiction, implying that $K = 0$.

To show that (2.7) is an isomorphism, consider the composition

$$\text{(2.8)}\quad \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}} \to \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}) \to \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}).$$

We claim that the first arrow in (2.8) is an isomorphism. Indeed, since $\mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}$ is flat over $\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}$, we have

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}) \simeq \mathrm{End}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}) \underset{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}},$$

and by the main result of [FG6], the natural map

$$\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}} \to \mathrm{End}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}})$$

is an isomorphism.

Now we claim that the composition in (2.8) is an isomorphism. The latter is equivalent to point (a) of the following assertion, established in [AG]:

Lemma 2.14.

(a)

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}) \simeq \mathrm{Hom}_{G[[t]]}(V^{\lambda}, \mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}) \simeq \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}.$$

(b) *For $i > 0$,*

$$\mathrm{Ext}^{i}_{G[[t]]}(V^{\lambda}, \mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}) = 0.$$

2.15. Computation of characters

We are now ready to finish the proof of Theorem 2.9. Using the coordinate on the formal disc, we will view $\mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$ as acted on by $G \times G \times \mathbb{G}_m$, where the latter acts by loop rotations. It is easy to see that the isotypic components for the above action are finite-dimensional.

Using point (2) of Theorem 2.9 and the above injectivity result, we obtain that the theorem would follow once we show that for each

μ_1, μ_2, d,

$$\dim\Big(\mathrm{Hom}_{G\times G\times \mathbb{G}_m}(V^{\mu_1}\otimes V^{\mu_2}\otimes \mathbb{C}^d, \mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}})\Big) = \sum_\lambda \dim\Big(\mathrm{Hom}_{G\times G\times \mathbb{G}_m}\Big(V^{\mu_1}\otimes V^{\mu_2}\otimes \mathbb{C}^d, (\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}) \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}{\otimes} \mathrm{Sym}_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}(\mathcal{N}^\lambda_{\mathrm{reg/\,unr}})\Big)\Big).$$

Since $\mathcal{N}^\lambda_{\mathrm{reg/\,unr}} \simeq \Omega^1(\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda})$, and since each $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}$ is isomorphic to a polynomial algebra, the multiplicities of the irreducibles in the $G\times G\times \mathbb{G}_m$-modules

$$(\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}{\otimes} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}) \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}{\otimes} \mathrm{Sym}_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}(\mathcal{N}^\lambda_{\mathrm{reg/\,unr}}) \text{ and } \mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}\otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}$$

are the same.

Hence, it suffices to show that for each μ_1, μ_2, d,

$$\dim\left(\mathrm{Hom}_{G\times G\times \mathbb{G}_m}(V^{\mu_1}\otimes V^{\mu_2}\otimes \mathbb{C}^d, \mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}})\right) = \sum_\lambda \dim\left(\mathrm{Hom}_{G\times G\times \mathbb{G}_m}\left(V^{\mu_1}\otimes V^{\mu_2}\otimes \mathbb{C}^d, \mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}\otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}\right)\right).$$

However, the one-parameter families of $G\times G\times \mathbb{G}_m$-modules given by $\mathbb{V}^\lambda_{\mathfrak{g},\kappa_\hbar+\kappa_{\mathrm{crit}}}\otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},-\kappa_\hbar+\kappa_{\mathrm{crit}}}$ and $\mathfrak{D}^{\mathrm{ch}}_{G,\hbar}$, where $\kappa_\hbar = \hbar\kappa_0$ for some non-zero invariant inner product κ_0, are $\hbar$-flat. Hence, it is sufficient to check the equality

$$\dim\left(\mathrm{Hom}_{G\times G\times \mathbb{G}_m}(V^{\mu_1}\otimes V^{\mu_2}\otimes \mathbb{C}^d, \mathfrak{D}^{\mathrm{ch}}_{G,\hbar})\right) = = \sum_\lambda \dim\left(\mathrm{Hom}_{G\times G\times \mathbb{G}_m}\left(V^{\mu_1}\otimes V^{\mu_2}\otimes \mathbb{C}^d, \mathbb{V}^\lambda_{\mathfrak{g},\kappa_\hbar+\kappa_{\mathrm{crit}}}\otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},-\kappa_\hbar+\kappa_{\mathrm{crit}}}\right)\right)$$

for a generic $\hbar$. The latter equality indeed holds, since for $\hbar$ irrational we have an isomorphism of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-bimodules:

$$\mathfrak{D}^{\mathrm{ch}}_{G,\hbar} \simeq \bigoplus_\lambda \mathbb{V}^\lambda_{\mathfrak{g},\kappa_\hbar+\kappa_{\mathrm{crit}}}\otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},-\kappa_\hbar+\kappa_{\mathrm{crit}}}$$

by Lemma 2.5.

§3. The category of spherical modules

In this section we use the results on the CADO obtained in the previous section to prove the Main Theorem stated in the Introduction.

3.1. Semi-infinite cohomology functor

Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ its nilpotent radical. Choose a non-degenerate character $\psi : \mathfrak{n} \to \mathbb{C}$. Define the character

$$\chi_0 : \mathfrak{n}((t)) \to \mathbb{C} \tag{3.1}$$

by the formula

$$\chi_0(x \otimes t^n) = \begin{cases} \psi(x), & \text{if } n = -1, \\ 0, & \text{otherwise}. \end{cases}$$

We have the functors of semi-infinite cohomology (the quantum Drinfeld–Sokolov reduction) from the category of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules to the category of graded vector spaces,

$$\mathcal{M} \mapsto H^{\frac{\infty}{2}+i}(\mathfrak{n}, \mathfrak{n}[[t]], \mathcal{M} \otimes \chi_0), \tag{3.2}$$

introduced in [FF, FKW] (see also [FB], Ch. 15, and [FG2], Sect. 18; we follow the notation of the latter).

More generally, for a complex $\mathcal{M}^\bullet$ of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules (or of $\mathfrak{n}((t))$-modules), the corresponding semi-infinite Chevalley complex

$$\mathbf{C}^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathcal{M}^\bullet \otimes \chi_0)$$

gives rise to a well-defined triangulated functor

$$D^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}) \to D(\mathrm{Vect}).$$

This is an analogue of the Whittaker functor in representation theory of reductive groups over local fields.

Since $\mathfrak{Z}_\mathfrak{g}$ maps to the center of the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-mod, the above functor naturally lifts to a functor

$$D^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}) \to D(\mathfrak{Z}_\mathfrak{g}\text{-mod}).$$

By Theorem 2.3, the composed functor

$$D^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}) \to D^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}) \to D(\mathfrak{Z}_\mathfrak{g}\text{-mod}),$$

factors through a functor

$$\Psi : D^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}) \to D\left(\mathrm{QCoh}(\mathrm{Spec}(\mathfrak{Z}_\mathfrak{g}^{\mathrm{unr}}))\right).$$

The main result of this paper is the following:

Theorem 3.2. *The functor Ψ is exact (with respect to the natural t-structures) and defines an equivalence of abelian categories*

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Spec}(\mathfrak{Z}_\mathfrak{g}^{\mathrm{unr}})). \tag{3.3}$$

3.3. Strategy of the proof

We will derive Theorem 3.2 from the following two statements.

Proposition 3.4. *There exists an isomorphism of algebras*

$$R^\bullet \operatorname{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}}(\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}, \mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}) \simeq R^\bullet \operatorname{Hom}_{\mathrm{QCoh}(\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr}}_{\mathfrak{g}}))}(\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}, \mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}). \tag{3.4}$$

Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg},\lambda}$ be the full subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}$, consisting of modules, whose support over $\mathfrak{Z}_{\mathfrak{g}}$ is contained in $\mathrm{Spec}(\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}})$. Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}_{\mathrm{reg},\lambda}$ denote the intersection

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg},\lambda} \cap \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}.$$

Proposition 3.5. *The functors* Ψ *and* $\mathcal{L} \mapsto \mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}{\otimes} \mathcal{L}$ *define mutually quasi-inverse equivalences*

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}_{\mathrm{reg},\lambda} \leftrightarrows \mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}\text{-mod}.$$

For $\lambda = 0$ this was proved in [FG1] (see also Conjecture 10.3.12 of [F2]).

We remark that, conversely, both of these propositions follow from Theorem 3.2 and the isomorphism

$$\Psi(\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}) \simeq \mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}} \tag{3.5}$$

established in [FG6].

Note that $\mathrm{Spec}(\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}) \to \mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr}}_{\mathfrak{g}})$ is a regular embedding. Therefore we have

$$R^\bullet \operatorname{Hom}_{\mathrm{QCoh}(\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr}}_{\mathfrak{g}}))}(\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}, \mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}) \simeq \Lambda^\bullet_{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}(\mathcal{N}^\lambda_{\mathrm{reg}/\,\mathrm{unr}}).$$

Combining this with Proposition 3.4, we obtain:

Corollary 3.6.

$$R^\bullet \operatorname{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}}(\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}, \mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}) \simeq \Lambda^\bullet_{\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}}(\mathcal{N}^\lambda_{\mathrm{reg}/\,\mathrm{unr}}). \tag{3.6}$$

For $\lambda = 0$ the isomorphism (3.6) was established in [FT] by other methods. The above proof is independent of [FT] and therefore provides an alternative argument.

Let χ be a $\mathbb{C}$-point of $\mathrm{Op}^{\mathrm{unr}}_{\check{G}}$, that is, a λ-regular oper in $\mathrm{Op}^{\mathrm{reg},\lambda}_{\check{G}}$ for some $\lambda \in \Lambda^+$. We denote by $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi$ the category of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules

on which the center $\mathfrak{Z}_{\mathfrak{g}}$ acts according to the character associated to χ. Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi}^{G[[t]]}$ be the corresponding $G[[t]]$-equivariant category. This category contains the quotient $\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}(\chi)$ of the Weyl module $\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}$ by the central character χ. Theorem 3.2 then has the following corollary (see Conjecture 10.3.11 of [F2]):

Corollary 3.7. *For any $\chi \in \mathrm{Op}^{\mathrm{reg},\lambda}_{\check{G}}, \lambda \in \Lambda^+$, the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi}^{G[[t]]}$ is equivalent to the category of vector spaces: its unique, up to isomorphism, irreducible object is $\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}(\chi)$ and any other object is isomorphic to a direct sum of copies of $\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}(\chi)$. This equivalence is given by the functor Ψ.*

This provides a non-trivial test of our conjecture, described in the Introduction (see formula (1.7)), that the categories $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi}^{K}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi'}^{K}$ are equivalent whenever the local systems underlying χ and χ' are isomorphic to each other.

3.8. Computation of Ψ

The first step is to compute the functor Ψ on the objects $\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}$. Since the functor Ψ commutes with direct limits, from Theorem 2.9 and (3.5) we obtain that $\mathcal{B}^{\lambda}_{G} := \Psi(\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}})$ is acyclic off cohomological degree 0 (here we view $\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}$ as an object of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}$ via the left action $\mathfrak{l}$).

Proposition 3.9. *The functor Ψ defines an isomorphism*

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}) \to \mathrm{Hom}_{\mathfrak{Z}\text{-mod}}(\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}, \mathcal{B}^{\lambda}_{G})$$

(here we consider the left action $\mathfrak{l}$ of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ on $\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}$). Furthermore, the higher $R^i\,\mathrm{Hom}$'s,

$$R^i\,\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}})$$

$$\text{and} \quad R^i\,\mathrm{Hom}_{D(\mathrm{QCoh}(\mathrm{Spec}(\mathfrak{Z}^{\mathrm{unr}}_{\mathfrak{g}})))}(\mathfrak{z}^{\mathrm{reg},\lambda}_{\mathfrak{g}}, \mathcal{B}^{\lambda}_{G}),$$

vanish.

Proof. From Lemma 2.14, we know that

$$R^i\,\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}) = 0$$

for $i > 0$ and

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}) \simeq \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}}.$$

By Theorem 2.9(2) and formula (3.5), $\mathcal{B}_G^\lambda$ has a filtration with the associated graded quotients given by

$$\mathrm{gr}^n(\mathcal{B}_G^\lambda) \simeq \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda}}{\otimes} \mathrm{Sym}^n_{\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda}}(\mathcal{N}^\lambda_{\mathrm{reg}/\,\mathrm{unr}}).$$

Moreover, it follows from the definition of the filtration on $\mathfrak{D}^{\mathrm{ch}}_{G,\mathrm{crit}}$ that this filtration is the canonical one, given by the powers of annihilation by $\mathbf{I}^\lambda$. This implies that $R^i(\iota^\lambda)^!(\check{\mathcal{B}}_G^\lambda) = 0$ for $i > 0$ and that the natural map

$$\mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\mathrm{crit}} \to R^0(\iota^\lambda)^!(\check{\mathcal{B}}_G^\lambda)$$

is an isomorphism.

Q.E.D.

3.10. Proof of Proposition 3.4

Consider the relative Chevalley complex

$$\mathbf{C}^\bullet(\mathfrak{g}[[t]]; \mathfrak{g}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}} \otimes V^\lambda)$$

taken with respect to the *right* action of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ on $\mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}}$, as a complex of objects of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}$. By Lemma 2.14, it is quasi-isomorphic to $\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}$ itself. We need to show that the functor Ψ induces isomorphisms

$$R^i \operatorname{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}} \left(\mathbb{V}^\lambda_{\mathfrak{g},\mathrm{crit}}, \mathbf{C}^\bullet(\mathfrak{g}[[t]]; \mathfrak{g}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}} \otimes V^\lambda)\right) \to$$
$$\to R^i \operatorname{Hom}_{D(\mathrm{QCoh}(\mathrm{Spec}(\mathfrak{z}_\mathfrak{g}^{\mathrm{unr}})))} \left(\mathfrak{z}_\mathfrak{g}^{\mathrm{reg},\lambda}, \Psi\left(\mathbf{C}^\bullet(\mathfrak{g}[[t]]; \mathfrak{g}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}} \otimes V^\lambda)\right)\right).$$

Taking into account Proposition 3.9, it remains to show that the natural map

$$\Psi\left(\mathbf{C}^\bullet(\mathfrak{g}[[t]]; \mathfrak{g}, \mathfrak{D}^{\mathrm{ch},\lambda}_{G,\mathrm{crit}} \otimes V^\lambda)\right) \to \mathbf{C}^\bullet(\mathfrak{g}[[t]]; \mathfrak{g}, \mathcal{B}^\lambda_G \otimes V^\lambda) \tag{3.7}$$

is an isomorphism, i.e., that the corresponding spectral sequences converges.

The latter is established as follows: we endow the bi-complex in the LHS of (3.7) with an additional $\mathbb{Z}$-grading, as in [FG6], Section 4 (see also [FG2], Section 18.11). We obtain that in each graded degree, the corresponding bi-complex is concentrated in a shift of a positive quadrant, hence the convergence.

3.11. Proof of Proposition 3.5

We are now ready to derive Proposition 3.5. The fact that functor

$$\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}\text{-mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}_{\mathrm{reg},\lambda},$$

given by

$$\mathcal{L} \mapsto \mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}{\otimes} \mathcal{L},$$

is an equivalence, follows from Proposition 3.4 by repeating verbatim the argument in [FG1], Section 8.

It remains to show that $\Psi(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}{\otimes} \mathcal{L})$ is acyclic away from the cohomological degree 0, and that the 0-th cohomology is isomorphic to $\mathcal{L}$.

Since the functors appearing above commute with direct limits, we can assume that $\mathcal{L}$ is finitely presented. Since $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}$ is isomorphic to a polynomial algebra, we can further assume that $\mathcal{L}$ admits a finite resolution by free $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}$-modules. This reduces the assertion to the formula (3.5).

3.12. Exactness

We are now ready to show that the functor Ψ is exact, i.e., that for $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}$, the object $\Psi(\mathcal{M})$ is acyclic away from the cohomological degree 0.

Indeed, since Ψ commutes with direct limits, we can assume that $\mathcal{M}$ is supported at the k-th infinitesimal neighborhood of $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda})$ inside $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{unr},\lambda})$. By (finite) devissage, that is, by representing $\mathcal{M}$ as a k-iterated successive extension of modules supported at $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda})$, we may further assume that $\mathcal{M}$ belongs to $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}_{\mathrm{reg},\lambda}$. In the latter case, the assertion follows from Proposition 3.5.

3.13. Completion of the proof of Theorem 3.2

Let us now show that the functor Ψ induces isomorphisms

$$\begin{aligned} &R^i \operatorname{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}}\left(\mathbb{V}^{\lambda}_{\mathfrak{g},\mathrm{crit}}, \mathcal{M}\right) \\ &\quad \to R^i \operatorname{Hom}_{D(\mathrm{QCoh}(\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{unr}})))}\left(\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}, \Psi(\mathcal{M})\right) \end{aligned} \tag{3.8}$$

for any i and $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{G[[t]]}$.

Both sides commute with direct limits in $\mathcal{M}$, so we can again assume that $\mathcal{M}$ is supported at the k-th infinitesimal neighborhood of

$\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda})$ inside $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{unr},\lambda})$, and further that it is on object of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ $-\mathrm{mod}_{\mathrm{reg},\lambda}^{G[[t]]}$, i.e.,

$$\mathcal{M} \simeq \mathbb{V}_{\mathfrak{g},\mathrm{crit}}^{\lambda} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}{\otimes} \mathcal{L}$$

for some $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}$-module $\mathcal{L}$. Using commutation with direct limits again, we can assume that $\mathcal{L}$ is finitely presented, and hence admits a finite resolution by free $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}$-modules. In the latter case, the isomorphism of (3.8) follows from Proposition 3.4.

By the same devissage procedure we conclude that the functor Ψ induces isomorphisms

$$R^i \operatorname{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}-\mathrm{mod}^{G[[t]]}}(\mathcal{M}_1, \mathcal{M}_2) \to R^i \operatorname{Hom}_{D(\mathrm{QCoh}(\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{unr}})))}(\Psi(\mathcal{M}_1), \Psi(\mathcal{M}_2))$$

for any i and $\mathcal{M}_1, \mathcal{M}_2 \in \widehat{\mathfrak{g}}_{\mathrm{crit}}-\mathrm{mod}^{G[[t]]}$.

Finally, it remains to see that Ψ is essentially surjective. Again, by commutation with direct limits, it is sufficient to see that any $\mathcal{L} \in \mathrm{QCoh}(\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{unr}}))$ supported at the k-th infinitesimal neighborhood of $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda})$ lies in the image of Ψ.

Since Ψ induces an isomorphism on the level of Ext^1, by induction, we can assume that $k = 0$, i.e., $\mathcal{L} \in \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}-\mathrm{mod}$. In the latter case, the assertion follows from Proposition 3.5.

References

[AG] S. Arkhipov and D. Gaitsgory, Differential operators on the loop group via chiral algebras, IMRN 2002, no. 4, 165–210.

[BD] A. Beilinson and V. Drinfeld, Quantization of Hitchin's integrable system and Hecke eigensheaves, available at http://www.math.uchicago.edu/~arinkin/langlands/

[CHA] A. Beilinson and V. Drinfeld, Chiral algebras, Amer. Math. Soc. Colloq. Publ., **51**, Amer. Math. Soc., Providence, RI, 2004.

[FF] B. Feigin and E. Frenkel, Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras, In: Infinite Analysis, (eds. A. Tsuchiya, T. Eguchi and M. Jimbo), Adv. Ser. in Math. Phys., **16**, World Scientific, Singapore, 1992, pp. 197–215.

[F1] E. Frenkel, Wakimoto modules, opers and the center at the critical level, Adv. Math., **195** (2005), 297–404.

[F2] E. Frenkel, Langlands Correspondence for Loop Groups, Cambridge Stud. Adv. Math., **103**, Cambridge Univ. Press, 2007.

[FB] E. Frenkel and D. Ben-Zvi, Vertex Algebras and Algebraic Curves, Second ed., Math. Surveys Monogr., **88**, Amer. Math. Soc., 2004.

[FG1] E. Frenkel and D. Gaitsgory, D-modules on the affine Grassmannian and representations of affine Kac–Moody algebras, Duke Math. J., **125** (2004), 279–327.

[FG2] E. Frenkel and D. Gaitsgory, Local geometric Langlands correspondence and affine Kac–Moody algebras, In: Algebraic Geometry and Number Theory, Progr. Math., **253**, Birkhäuser Boston, 2006, pp. 69–260.

[FG3] E. Frenkel and D. Gaitsgory, Fusion and convolution: applications to affine Kac–Moody algebras at the critical level, Pure Appl. Math. Q., **2** (2006), 1255–1312.

[FG4] E. Frenkel and D. Gaitsgory, Localization of $\widehat{\mathfrak{g}}$-modules on the affine Grassmannian, preprint, math.RT/0512562.

[FG5] E. Frenkel and D. Gaitsgory, Geometric realizations of Wakimoto modules at the critical level, preprint, math.RT/0603524, to appear in Duke Math. J.

[FG6] E. Frenkel and D. Gaitsgory, Weyl modules and opers without monodromy, preprint, arXiv:0706.3725.

[FKW] E. Frenkel, V. Kac and M. Wakimoto, Characters and fusion rules for $\mathcal{W}$–algebras via quantized Drinfeld-Sokolov reduction, Comm. Math. Phys., **147** (1992), 295–328.

[FT] E. Frenkel and C. Teleman, Self-extensions of Verma modules and differential forms on opers, Compos. Math., **142** (2006), 477–500.

[Ga] D. Gaitsgory, The notion of category over an algebraic stack, preprint, math.AG/0507192.

[GMS] V. Gorbounov, F. Malikov and V. Schechtman, On chiral differential operators over homogeneous spaces, Int. J. Math. Math. Sci., **26** (2001), 83–106.

Edward Frenkel
Department of Mathematics
University of California
Berkeley, CA 94720
USA

E-mail address: frenkel@math.berkeley.edu

Dennis Gaitsgory
Department of Mathematics
Harvard University
Cambridge, MA 02138
USA

E-mail address: gaitsgde@math.harvard.edu