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The Friedman rule under habit formation

Yoshimasa Aoki and Yasunobu Tomoda

Abstract.

This paper studies the Friedman rule on the optimal quantity of money in a money in the utility model with habit formation. If habits are formed only in consumption, or if habit formations in both consumption and real money holdings are symmetric, the Friedman rule is optimal. However, if habit forming is assumed in real money holdings, then the Freidman rule may not be optimal.

§1. Introduction

In the seminal paper "The Optimum Quantity of Money" (1969), Milton Friedman claimed that government should supply as large an amount of money as possible to the market. This follows since the amount of money supply is optimal if the marginal utility of real money holdings is equal to the marginal cost of money supply and the production cost of money is negligible. This optimal rule of money supply implies that the nominal interest rate is zero in market and that government should not tax the real money holdings of households.

Recently, a considerable number of studies have been made on the Friedman rule, including models with distortional taxes. In particular, Chari et al (1996) and Correia and Teles (1999) demonstrate that, in money in the utility models, the Friedman rule is optimal if the utility function is homothetic in consumption and real money holdings and separable in leisure[1].

In this study, we extend their models by introducing habit formation which arises from consumption and real balance holdings[2]. And we show that if habits are only formed in consumption, the Freidman rule is

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[1]Chari and Kehoe (1999) surveys the literature of this field.

[2]Habit formation is formalized by Ryder and Heal (1973).

optimal. If habits are symmetrically formed in both consumption and real balance holdings, the Friedman rule is still optimal. However, if habits are formed only in real money holdings, the Freidman rule is not necessarily optimal. We conclude that the optimality of the Friedman rule depends only on habits formed from real balance holdings with the exception of the symmetric habit parameter case.

This paper is organized as follows. In section 2, we set up the basic model with habit formation. In section 3, we show steady state solution and consider the optimal quantity of money. A summary is given section 4.

§2. The model

We consider an infinite horizon model in which a representative agent maximizes utility. Utility is a function of consumption, real balance holdings, and leisure. Consumption and real balance holdings include habit from past periods. The representative agent maximizes the discounted sum of utility over an infinite horizon:

$$\sum_{t=0}^{\infty} \beta^t U(X_t, M_t, l_t), \text{ with } U(\cdot) = \nu(X_t, M_t) + u(l_t) \tag{1}$$

where β represents the discount factor, l_t is leisure, and X_t and M_t are consumption and real balance holdings in relative amounts. $\nu(\cdot)$ and $u(\cdot)$ possess the following properties.

Assumption 1. $\nu(\cdot)$ is strictly concave, homogeneous of degree k (≥ 1), differentiable, and $\lim_{M_t \to \infty} \left(\frac{\partial U}{\partial M_t}\right) = 0$.

Assumption 2. $u(\cdot)$ is strictly concave and differentiable.

We also assume that X_t and M_t have the following forms.

$$X_t = \frac{x_t}{(h_t)^\eta}, \tag{2}$$

$$M_t = \frac{m_t}{(b_t)^\sigma}, \tag{3}$$

where $\eta \in [0, 1)$ and $\sigma \in [0.1)$. x_t and m_t represent the consumption good and real balance holdings in the current period, respectively. Let h_t and b_t represent the weighted average of past consumption as in the following: $h_t = \rho \sum_{t=0}^{\infty} (1-\rho)^j x_{t-1-j}$ and $b_t = \omega \sum_{t=0}^{\infty} (1-\omega)^j m_{t-1-j}$, where $\rho \in (0, 1)$ and $\omega \in (0.1)$. These equations are equivalent as

$$h_{t+1} = h_t + \rho \cdot (x_t - h_t), \tag{4}$$

$$b_{t+1} = b_t + \omega \cdot (m_t - b_t). \tag{5}$$

In period t, households trade money, bonds, and the consumption good in each market. The budget constraint of households is

$$(1-\tau_t)P_t(1-l_t) + n_t + (1+i_t)B_t \geq P_t x_t + n_{t+1} + B_{t+1}, t = 0, 1, \cdots \tag{6}$$

where τ_t, $(1-l_t)$, n_t, i_t, B_t, and P_t represent, respectively, the income tax, labor supply, nominal money holdings, the nominal interest rate, bond holdings from period t to $t+1$, and the price of the consumption good in units of money.

Next, we introduce the resource constraint. Households are endowed with one unit of time in each period. For simplicity, one unit of labor produces one unit of the consumption good. Thus, the resources constraint which the economy faces in period t is

$$1 - l_t \geq x_t + g, \tag{7}$$

where g is a given level of government expenditures that is constant over time.

§3. The Friedman rule

Under a given sequence $\{P_t, i_t, \tau_t\}_{t=0}^{\infty}$, the representative household maximizes the utility function (1) subject to budget constraints (6), the initial condition $n_{-1} = B_{-1} = 0$, and a no Ponzi games condition $\lim_{t\to\infty} I_t(n_t + B_t) \geq 0$, with $I_t = \frac{1}{\Pi_{s=0}^{t}(1+i_s)}$. The set of budget constraints can be written as a unique intertemporal budget constraint;

$$\sum_{t=0}^{\infty} I_t P_t (1-\tau_t)(1-l_t) \geq \sum_{t=0}^{\infty} I_t P_t x_t + \sum_{t=0}^{\infty} I_t i_t P_t m_t. \tag{8}$$

Therefore, the household maximizes (1) subject to (8). The first order conditions of this problem, for $t \geq 0$, are

$$\beta^t\left(U_{X_t}\cdot(h_{t-1})^{-\eta}+\sum_{T=t+1}^{\infty}\beta^{T-t}U_{X_t}\frac{\partial X_T}{\partial x_t}\right)=\lambda I_tP_t, \tag{9}$$

$$\beta^t\left(U_{M_t}\cdot(b_{t-1})^{-\sigma}+\sum_{T=t+1}^{\infty}\beta^{T-t}U_{M_t}\frac{\partial M_T}{\partial m_t}\right)=\lambda I_ti_tP_t, \tag{10}$$

$$\beta^tU_{l_t}=\lambda I_tP_t(1-\tau_t), \tag{11}$$

where λ is the Lagrange multiplier of the intertemporal budget constraint. Terms $U_{X_t}\cdot(h_{t-1})^{-\eta}$ and $U_{M_t}\cdot(b_{t-1})^{-\sigma}$ in (9) and (10) represent the current marginal utilities, which are positive terms. On the other hand, terms of $\sum_{T=t+1}^{\infty}\beta^{T-t}U_{X_t}\frac{\partial X_T}{\partial x_t}$ and $\sum_{T=t+1}^{\infty}\beta^{T-t}U_{M_t}\frac{\partial M_T}{\partial m_t}$ represent the marginal effects of habit formation. Because consumption and real balance holdings in period t reduce the utilities of future periods, these terms are negative. We assume that the sums of the left hand terms of (9) and (10) are non-negative. To simplify notation, we define $U_{x_t}=U_{X_t}\cdot(h_{t-1})^{-\eta}$, $U_{m_t}=U_{M_t}\cdot(b_{t-1})^{-\sigma}$, $V_{x_t}=\sum_{T=t+1}^{\infty}\beta^{T-t}U_{X_t}\frac{\partial X_T}{\partial x_t}$, and $V_{m_t}=\sum_{T=t+1}^{\infty}\beta^{T-t}U_{M_t}\frac{\partial M_T}{\partial m_t}$.

From (9), (10), and (11), we can derive the following conditions:

$$\left(U_{x_t}+V_{x_t}\right)\cdot i_t=\left(U_{m_t}+V_{m_t}\right), \tag{12}$$

$$\left(U_{m_t}+V_{m_t}\right)(1-\tau_t)=U_{l_t}, \tag{13}$$

From (12), the Friedman rule $i_t=0$ is equivalent to $(U_{m_t}+V_{m_t})=0$. Moreover, from (9), we have following difference equation:

$$U_{x_t}+V_{x_t}=\frac{\beta(1+i_{t+1})P_t}{P_{t+1}}\left(U_{x_{t+1}}+V_{x_{t+1}}\right). \tag{14}$$

Since consumption causes a habitual effect represented by V_{x_t} and $V_{x_{t+1}}$, (14) includes terms for not only period t and *t+1* but also all periods after period t.

A benevolent government chooses a policy assuming that households' maximize their utilities. The first order condition (12), (13), and (14) and the budget constraint (8) represent the sequence of households' behaviors on markets. From these equations, the implementability constraint of this problem is

$$\sum_{t=0}^{\infty} \beta^t \left[(U_{x_t} + V_{x_t})\, x_t + (U_{m_t} + V_{m_t}) m_t - U_{l_t}(1 - l_t) \right] \leq 0. \tag{15}$$

Government chooses a $\{c_t, m_t, l_t\}_{t=0}^{\infty}$ that maximizing (1) subject to the implementability constraints (15) and the resource constraints (7). First order conditions are

$$(U_{x_t}+V_{x_t})-\mu[(U_{x_t}+V_{x_t})+(U_{x_tx_t}+V_{x_tx_t})x_t+(U_{x_tm_t}+V_{x_tm_t})m_t]-\Psi_t = 0, \tag{16}$$

$$(U_{m_t}+V_{m_t})-\mu[(U_{m_t}+V_{m_t})+(U_{x_tm_t}+V_{x_tm_t})x_t+(U_{m_tm_t}+V_{m_tm_t})m_t] = 0, \tag{17}$$

$$U_{l_t} - \mu[U_{l_t} - U_{l_tl_t}(1 - l_t)] - \Psi_t = 0, \tag{18}$$

where $\Psi_t = \frac{\psi_t}{\beta^t}$, and μ and ψ_t are multipliers of the implementability constraint and resource constraints. From the first order conditions (16), (17), and (18), and constraints (7) and (15), five variables, i.e. x_t, m_t, l_t, μ, and Ψ_t, are determined. Since these conditions are independent of t, the solution is stationary in all variables. For the remainder of this section, we omit time subscript t. Now, we check whether $i = 0$ is the optimal policy in the steady state. First, we have the following proposition.

Proposition 1. *The Friedman rule, $i = 0$, is the optimal policy iff the following equation is satisfied.*

$$(U_{xm} + V_{xm})\, x + (U_{mm} + V_{mm})\, m = 0. \tag{19}$$

Moreover, when $\sigma\omega\beta \neq 1 - \beta(1 - \omega)$, $i = 0$ iff $U_M m^{-\sigma} = 0$.

Proof. From (12), $i = 0$ is equivalent to $(U_m + V_m) = 0$ since $(U_x + V_x) \neq \infty$. Substituting $(U_m + V_m)$ into (17), we have (19) since we consider an interior solution, i.e. $\mu \neq 0$. Moreover, since, in the steady state, $x = h$ and $m = b$ from (4) and (5), we have $U_m + V_m = U_M m^{-\sigma}\left(1 - \frac{\sigma\omega\beta}{1-\beta(1-\omega)}\right)$. Therefore, if $\sigma\omega\beta \neq 1 - \beta(1 - \omega)$, $U_m + V_m = 0 \Longleftrightarrow U_M m^{-\sigma} = 0$. Q.E.D.

From the definitions of U_j and V_j ($j = x_t, m_t$), we calculate the terms in equation (19). Since $x = h$ and $m = b$ in the steady state, from

(4) and (5) we have $X = x^{1-\eta}$, and $M = m^{1-\sigma}$.Thus, (19) is rewritten as

$$(20) \qquad A \cdot U_{XM} X m^{-\sigma} + B \cdot U_{MM} M m^{-\sigma} + C \cdot U_M m^{-\sigma} = 0,$$

where $A = 1 + \frac{\beta\eta\sigma\rho\omega}{1-\beta(1-\rho)(1-\omega)}$, $B = 1 + \frac{\beta\sigma^2\omega^2}{1-\beta(1-\omega)^2}$, and $C = \frac{\beta\sigma(\sigma+1)\omega^2}{1-\beta(1-\omega)^2}$. When $i = 0$, equation (20) is not always established. Therefore, $i = 0$ is not always the solution of this problem in the steady state. We consider several parameter cases to obtain a better description of the optimal money supply.

Case 1. *(Chari et al. 1996)*
If $\eta = \sigma = 0$ (households do not depend on habit formation), the Friedman rule is optimal.

Proof. Substituting $\eta = \sigma = 0$ into (20), because of $A = B = 1$ and $C = 0$, we have $U_{XM}X + U_{MM}M = 0$. Since the utility function is homogeneous of degree k, this equation is rewritten as $(k-1)U_M = 0$ using Euler's formula. This result stems from assumption 1. Q.E.D.

This case is identical to Chari et al. (1996). Thus, if preferences do not have habit formation, the Freidman rule is the optimal policy when preferences are separable and homogeneous of degree k.

Case 2. *If $\eta = \sigma$ and $\rho = \omega$ (households have symmetric habit parameters), the Friedman rule is optimal.*

Chari et al. (1996) proved the optimality of Freidman rule under homothetic preferences. If there is symmetric habit formation in both consumption and real balance holdings, preference is homothetic for x and m as well as X and M. Thus, Case 2 above is a special case of Chari et al. (1996).

Case 3. *If $\eta \neq 0$ and $\sigma = 0$ (i.e. only consumption is habit forming), the Friedman rule is optimal.*

Faria (2001) obtained a similar result. However, there is a big difference between Case 3 and Faria's result. In Faria's case, there is no constraint on the government's supply of money. On the other hand, we solved the problem in which government has to impose a distortional tax and/or an inflation tax since government needs financial sources for expenditure g in (8). Therefore, we demonstrate the optimality of the Friedman rule under a different setting.

Case 4. *Suppose $\eta = 0$ and $\sigma \neq 0$ (i.e. only real money holdings are habit forming). If $\sigma > \frac{1}{2}$, the Friedman rule is the optimal policy.*

Proof. Since, by substituting $\eta = 0$ into (20), we obtain $A = 1$, (20) is rewritten as $U_{XM}Xm^{-\sigma} + U_{MM}Mm^{-\sigma}B + U_M m^{-\sigma}C = 0$. The first and third terms in the left hand side of this equation are $\lim_{m\to\infty}(U_{XM}Xm^{-\sigma}) = \lim_{m\to\infty}(U_M m^{-\sigma}C) = 0$. However, since $M = m^{1-\sigma}$, the second term in the left hand side is rewritten as $U_{MM}m^{1-2\sigma}B$. Therefore, if $1 - 2\sigma < 0$, then $\lim_{m\to\infty}(U_{MM}m^{1-2\sigma}B) = 0$. Q.E.D.

This result contrasts with that of Case 3. In Case 3, if habits are only formed from consumption, the Friedman rule is independent of consumption based habit formation. However, Case 4 claims that, if habits are only formed from real balance holdings, the optimality of the Friedman rule depends on habits formed from real balance holdings.

From the proof of Case 4, we can extend Case 4 for cases where $\eta \neq 0$. Then, we obtain a more general proposition than the above cases.

Proposition 2. *If $\sigma = 0$, $\sigma > \frac{1}{2}$, or $\eta = \sigma$ and $\rho = \omega$, the Friedman rule is the optimal policy.*

We omit the proof because it is trivial given the above cases. In our study, we analyzed a model with habit formation arising from consumption and real money holdings. We introduced four habitual parameters, i.e. η, σ, ρ, and ω. However, we conclude that the optimality of the Friedman rule in habit forming models depends only on the habitual parameter of money holdings σ, with the exception of the symmetric parameter case, i.e. $\eta = \sigma$ and $\rho = \omega$.

§4. Summary

We studied the optimality of the Friedman rule in a model with habit formation in real balance holdings as well as consumption. If the functions of habit formation are symmetric, the Friedman rule is optimal. If habit formation is assumed only in consumption, the Friedman rule is again optimal. Moreover, if the habit parameter of real balance holdings σ is greater than one-half, the Friedman rule is optimal regardless of the habit parameter of consumption. We introduced four habitual parameters. However, we conclude that the optimality of the Friedman rule in our model only depend on the habit parameter of money holdings σ with the expection of the symmetric parameter case.

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Yoshimasa Aoki
Department of Economics
Sapporo Gakuin University

Yasunobu Tomoda
Institute of Economic Research
Kyoto University, JSPS Research Fellow

E-mail address: tomoda@kier.kyoto-u.ac.jp