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On variational methods for second order discrete periodic problems

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Abstract.

We use some basic results from the theory of variational methods to prove the exisetnce and uniqueness of periodic solutions to second-order nonlinear discrete problems. Our method has no continuous counterpart since it is based on the finite dimension of the corresponding function space. Our main tools include matrix theory and the Fundamental theorem of calculus for Lebesgue Integral.

§1. Introduction

In recent years, the number of publications concerning difference equations and corresponding boundary value problems has sharply risen. Most of the existence and multiplicity results are based on the corresponding statements from differential equations or are even directly proven in the time scales settings, i.e. in a general scheme which encompasses both continuous and discrete problems (see e.g. Bohner, Peterson [6] for more details).

In this short note we show that some purely discrete approaches can yield results which can't be obtained in a traditional, continuous-like, way. Namely, we reformulate the second-order nonlinear periodic boundary value problem as a nonlinear equation in $\mathbb{R}^N$. Then we take advantage of the finite dimension of the considered problem and use some basic properties of positive (semi)definite matrices and Lebesgue integration. In contrast to traditional results, we are able to prove existence and uniqueness theorems for problems with non-continuous right-hand sides.

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Throughout this paper, we consider the second order nonlinear periodic problem:

(1)
$$\begin{cases} \Delta\left(p(k-1)\Delta x(k-1)\right)) + q(k)x(k) = g(k, x(k)), & k = 1, 2, \ldots, N, \\ x(0) = x(N), \quad p(0)\Delta x(0) = p(N)\Delta x(N), \end{cases}$$

where $p : \{0, 1, \ldots, N-1, N\} \to \mathbb{R}$ is such that $p(k) > 0$ and $q : \{1, \ldots, N-1, N\} \to \mathbb{R}$ satisfies either

$$(\mathrm{Q}^0) \qquad q(k) \equiv 0,$$

or

$$(\mathrm{Q}^+) \qquad q(k) \geq 0 \text{ for all } k, \quad q(k_1) > 0 \text{ for some } k_1 \in \{1, 2, \ldots, N\}.$$

By the solution to (1) we understand a vector

$$[x(0), x(1), \ldots, x(N), x(N+1)]^T,$$

such that the equation in (1) holds for every $k = 1, 2, \ldots, N$, and the boundary conditions there are satisfied as well.

Similar problems have been studied recently in different settings. Firstly, Atici, Cabada [3] considers the problem with $p \equiv 1$ and uses the method of lower and upper solutions to prove, for instance, that if g is continuous and nondecreasing in x and q satisfies (Q^+), then the problem (1) has a solution. Furthermore, if $q(k) > 0$ for all k and g is stricly increasing in x then the solution is unique. Bereanu and Mawhin [5] lists discrete results which correspond to Ambrosetti-Prodi and Landesman-Lazer problems for differential equations. Those results are proven for continuous right-hand sides as well.

The main contribution of this new technique consists in the facts that

(i) we prove the existence and uniqueness results also for non-continuous right-hand sides of (1), and
(ii) we prove the existence and uniqueness also in the case (Q^0).

§2. Matrix formulation

Taking into account the finite dimension of the space in which we seek a solution of (1), we rewrite now the periodic nonlinear problem (1) into a nonlinear equation in $\mathbb{R}^N$. First, let us consider the case (Q^0), i.e. the case without the term $q(k)x(k)$. Exploiting the difference relation

$\Delta x(k) = x(k+1) - x(k)$ and the boundary value problems in (1) we can reformulate (1) as

$$\overline{A}_{pq} x = G(x), \tag{2}$$

where $x = [x(1), x(2), \dots, x(N)]^T$, $\overline{A}_{pq}$ is an $N \times N$ matrix

$$\overline{A}_{pq} = \begin{bmatrix} p(0)+p(1) & -p(1) & & & -p(0) \\ -p(1) & p(1)+p(2) & -p(2) & & \\ & \ddots & \ddots & \ddots & \\ & & -p(N-2) & p(N-2)+p(N-1) & -p(N-1) \\ -p(0) & & & -p(N-1) & p(N-1)+p(0) \end{bmatrix},$$

and $G : \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear function defined by ($x(i)$ is abbreviated as x_i)

$$G(x) = \begin{bmatrix} g_1(x_1) \\ g_2(x_2) \\ g_3(x_3) \\ \vdots \\ g_{N-1}(x_{N-1}) \\ g_N(x_N) \end{bmatrix} := \begin{bmatrix} g(1, x_1) \\ g(2, x_2) \\ g(3, x_3) \\ \vdots \\ g(N-1, x_{N-1}) \\ g(N, x_N) \end{bmatrix}. \tag{3}$$

Furthermore, if we consider the general case with $q \geq 0$, then we can rewrite the problem (1) into a nonlinear equation in $\mathbb{R}^N$

$$A_{pq} x = G(x), \tag{4}$$

with $A_{pq} = \overline{A}_{pq} + Q$ where Q is a diagonal matrix with entries $q(1), q(2), \dots, q(N)$.

Obviously, if we find a solution $x_0 \in \mathbb{R}^N$ of the problem (4) then the vector

$$[x(N), x(1), x(2), \dots, x(N-1)x(N), x(1)]^T$$

is the desired solution of (1).

§3. Definiteness of matrices A_{pq}

In this section we show some basic properties of matrices A_{pq}. We follow the standard notation of Lütkepohl [14]. The interested reader can find more about matrices which are connected to discrete operators in Agarwal [1].

Lemma 1. *Let us suppose that* (Q^0) *holds, then the matrix* $A_{pq} = \overline{A}_{pq}$ *is positive semidefinite and* $\lambda = 0$ *is eigenvalue. If the condition* (Q^+) *is satisfied then the matrix* A_{pq} *is positive definite*

Proof. Firstly, let us notice that in all cases the matrices A_{pq} are symmetric. Therefore the matrix A_{pq} is positive (semi)definite if and only if its eigenvalues are positive (non-negative in semidefinite case) (e.g. [9, Theorem 9.12]).

Secondly, let us realize that symmetric matrix has only real eigenvalues (e.g. [9, Corollary 9.11]). Therefore, the Gerschgorin Circle Theorem (e.g. [9, Theorem 9.13]) yields that all eigenvalues of A_{pq} lies in the interval

$$\left[0, \max\left\{\max_{i=1,2,\dots,N-1} 2\left(p(i-1)+p(i)\right); 2\left(p(0)+p(N-1)\right)\right\}\right].$$

Let us consider first the case (Q^0), i.e. $A_{pq} = \overline{A}_{pq}$. All negative off-diagonal entries are counterbalanced by the occurence of their positive counterparts on the diagonal. Therefore, $\overline{A}_{pq}v = 0$ for $v = t\,[1,1,\dots,1]^T$ with $t \in \mathbb{R}$. This implies that $\lambda = 0$ is eigenvalue of $\overline{A}_{pq}$.

Furthermore, if (Q^+) holds then

$$x^T A_{pq} x = x^T(\overline{A}_{pq} + Q)x = x^T\overline{A}_{pq}x + x^T Q x \geq 0, \tag{5}$$

i.e. A_{pq} is at least positive semidefinite. But the equality in (5) could have occurred only if both terms $x^T\overline{A}_{pq}x$ and x^TQx had vanished for the same vector $x \in \mathbb{R}^N$. However, the former vanishes if and only if $v = t\,[1,1,\dots,1]^T$ and $t \in \mathbb{R}$. But for such a v (now considering only $t \neq 0$) we get

$$v^T Q v = t^2 \sum_{k=1}^{N} q(k) > 0,$$

since at least one $q(k)$ is positive. This implies that A_{pq} is positive definite. Q.E.D.

§4. Application of variational methods

At this stage, we are ready to show that even the simplest conlusions of the theory of variational methods can provide very strong statements if we consider the above matrix formulation of discrete nonlinear problems.

Namely, we first use the following result:

Theorem 2. [8, Theorem 6.2.8.] *Let H be a Hilbert space. Let $F : H \to \mathbb{R}$ be a weakly sequentially lower semi-continuous and weakly coercive functional. Then F is bounded from below on H and there exists $x_0 \in H$ such that $F(x_0) = \min F(x)$. Moreover, if the Fréchet derivative $F'(x_0)$ exists then*

$$F'(x_0) = o.$$

If we work with locally integrable functions $g_k(u)$ then there exists a functional $\widetilde{G} : \mathbb{R}^N \to \mathbb{R}$ such that $\nabla\widetilde{G} = G$. Thanks to the symmetry of A_{pq}, there exists also a functional $F : \mathbb{R}^N \to \mathbb{R}$ defined by ($\langle\cdot,\cdot\rangle$ denotes a scalar product in $\mathbb{R}^N$.)

$$F(x) := \frac{1}{2}\langle A_{pq}x, x\rangle - \widetilde{G}(x), \tag{6}$$

whose local extremum $x_0 \in \mathbb{R}^N$ is a solution of (4). Therefore, we are looking for extrema of the functional (6).

Theorem 3. *Let* (Q^0) *or* (Q^+) *be satisfied. Let us suppose that $g : \{1, 2, \dots, N\} \times \mathbb{R} \to \mathbb{R}$ is a function such that for each $k = 1, 2, \dots, N$:*

(I) $g_k \in L^1_{loc}(\mathbb{R})$,

(L) *there exists $M > 0$ such that*

$$\lim_{u\to\infty} g_k(u) \le -M, \qquad \textit{and}$$
$$\lim_{u\to-\infty} g_k(u) \ge M.$$

Then BVP (1) *has a solution.*

Proof. Taking into account the assumption (I) and the matrix procedure from the above sections, we can transform the problem (1) into the problem of existence of an extremum of the funcional F defined in (6). Moreover, the integrability of $g_k(u)$ ensures that the functional F is continuous (this is a consequence of Fundamental theorem of calculus for Lebesgue Integral, see e.g. [13, Theorem 23.4]). This implies, thanks to the finite dimension of $\mathbb{R}^N$, that F is also weakly sequentially semi-continuous. In order to apply Theorem 2 it suffices to show that the functional is weakly coercive.

Condition (I) implies that for each $k = 1, 2, \dots, N$, there exists an antiderivative $\widetilde{g}_k(u) = \int\limits_0^u g_k(\tau)\mathrm{d}\tau$. If we take into account the assumption (L) we obtain

$$\lim_{u\to\infty} \widetilde{g}_k(u) = -\infty,$$
$$\lim_{u\to-\infty} \widetilde{g}_k(u) = -\infty.$$

Since $\widetilde{G}(x) = \sum_{k=1}^{N} \widetilde{g}_k(x_k)$ and A_{pq} is at least a positive semidefinite matrix ($\langle A_{pq}x, x\rangle \geq 0$ for each nonzero vector $x \in \mathbb{R}^N$), the following inequality holds

$$\lim_{\|x\|\to\infty} F(x) \geq \lim_{\|x\|\to\infty} -\widetilde{G}(x) = +\infty.$$

Thus F is a weakly coercive functional.

Consequently, Theorem 2 yields that the functional (6) has a minimum $x_0 \in \mathbb{R}^N$, in other words $A_{pq}x_0 = G(x_0)$ which completes the proof. Q.E.D.

In order to obtain a uniqueness of the solution we suppose additionally that the functional in Theorem 2 is strictly coercive. In this case the minimum of the functional is unique which is summarized in the following theorem.

Theorem 4. [8, Theorem 6.2.12]*: If, in addition to assumptions of Theorem 2, the functional F is continuous and strictly convex, then x_0 is uniquely determined.*

As usual, it is the monotonicity of right-hand sides that ensures the convexity of the functional F in our case.

Theorem 5. *Let us suppose that $g : \{1, 2, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$ is a function such that for each $k = 1, 2, \ldots, N$ the assumptions* (I), (L) *and*

(M) *g_k is nonincreasing,*

are satisfied. If (Q^+) *holds, then the solution of* (1) *is unique. Moreover, if*

(M') *g_k is strictly decreasing,*

then the solution is unique also in the case (Q^0).

Proof. Similarly as in the proof of Theorem 3, one can show that F exists and is weakly coercive and continuous. In order to apply Theorem 4 it suffices to prove the strict convexity.

First, let us realize that the assumption (M) is sufficient for the concavity of $\widetilde{g}_k(u) = \int_0^u g_k(\tau)\mathrm{d}\tau$ for each $k = 1, 2, \ldots, N$ (e.g. [15, Theorem 12A]). Thus for each $x, y \in \mathbb{R}^N$:

$$
\begin{aligned}
\widetilde{G}(tx+(1-t)y) &= \sum_{k=1}^{N} \widetilde{g}_k(tx_k+(1-t)y_k) \\
&\geq t\sum_{k=1}^{N} \widetilde{g}_k(x_k) + (1-t)\sum_{k=1}^{N} \widetilde{g}_k(y_k) \\
&= t\widetilde{G}(x) + (1-t)\widetilde{G}(y),
\end{aligned}
$$

and therefore $\widetilde{G}$ is concave, i.e. $-\widetilde{G}$ is a convex functional. Since A_{pq} is at least positive semidefinite, we obtain that $\frac{1}{2}\langle A_{pq}x, x\rangle$ is convex. Hence, F is a convex functional.

Thus the functional F has a minimum. Moreover, it is unique if the assumption (Q^+) is satisfied. This is caused by the fact that in this case A_{pq} is positive definite and thus F is not only convex, but also strictly convex.

Finally, if g_k are strictly decreasing for each $k = 1, 2, \ldots, N$, then $-\widetilde{G}$ is strictly convex and, consequently, F is strictly convex also in the case (Q^0). Therefore, Theorem 4 yields a unique solution even in this case. Q.E.D.

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