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Local stability of a discrete competition model derived from a nonstandard numerical method

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Abstract.

The continuous two-dimensional Lotka-Volterra competition model is converted to a discrete version using a noncanonical symplectic numerical method. The local stability of the differential equations and difference equations are analyzed and compared. We found that the numerical method preserves the local dynamics of the continuous model. The local stability criteria are the same between the continuous model and the discrete model. The discrete-time model is dynamically consistent with its continuous counterpart.

§1. Introduction

Mathematical models are used to represent phenomena in the biological, ecological, and physical sciences, to name a few. Differential equations are used when the model represents continuous variables. However, when working with discrete variables, difference equations are most appropriate. For example, in ecology, predator-prey models can be formulated as discrete-time mappings. Difference equations are appropriate when organisms have discrete, nonoverlapping generations [1]. Different numerical schemes can be used to convert differential equations into difference equations. If the corresponding difference equations possess the same dynamic behavior as the continuous equations, such as local stability, bifurcations, and/or chaos, then they are said to be *dynamically consistent* [2].

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The following is the Lotka-Volterra competition model:

$$\begin{aligned} \frac{dx}{dt} &= x(r_1 - a_{11}x - a_{12}y), \\ \frac{dy}{dt} &= y(r_2 - a_{21}x - a_{22}y), \end{aligned} \tag{1}$$

where

$$r_i > 0 \text{ and } a_{ij} > 0. \tag{2}$$

The parameters r_i are the intrinsic growth rates for species x and y; a_{12} and a_{21} are the interspecific acting coefficients; and $x(t)$ and $y(t)$ represent the number of individuals or population density in species x and y at time t. The dynamics of the model is well-known. We will briefly mention the results of the Lotka-Volterra model.

The system (1) has at most four equilibria. They are the extinct equilibrium $E_0 = (0,0)$; the exclusive equilibria $E_1 = (\frac{r_1}{a_{11}}, 0)$ and $E_2 = (0, \frac{r_2}{a_{22}})$; and the possible coexistence equilibrium $E_3 = (\frac{a_{22}r_1 - a_{12}r_2}{a_{11}a_{22} - a_{12}a_{21}}, \frac{a_{11}r_2 - a_{21}r_1}{a_{11}a_{22} - a_{12}a_{21}})$. Their stability conditions are summarized as follows

(i) E_0 is a repeller, always unstable.

(ii) E_1 is locally asymptotically stable if $\frac{a_{11}}{a_{21}} < \frac{r_1}{r_2}$.

(iii) E_2 is locally asymptotically stable if $\frac{r_1}{r_2} < \frac{a_{12}}{a_{22}}$.

(iv) E_3 is locally asymptotically stable if

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}$$

and unstable if

$$\frac{a_{11}}{a_{21}} < \frac{r_1}{r_2} < \frac{a_{12}}{a_{22}}.$$

We will introduce a numerical method that converts the continuous model (1) into a discrete-time model. We are inspired by a method that was applied to a continuous predator-prey model which will be explained. The following is a simple normalized Lotka-Volterra predator-prey model:

$$\begin{aligned} \frac{dx}{dt} &= x - xy, \\ \frac{dy}{dt} &= -y + xy. \end{aligned} \tag{3}$$

All solutions of the predator-prey model are mutually stable periodic solutions if the initial conditions are positive, $x(t_0) > 0$ and $y(t_0) > 0$. Standard numerical methods will not produce periodic solutions unless the step size is small enough. Roeger [5] has shown that there are a class of nonstandard numerical methods that will produce periodic solutions for the predator-prey system. However, the methods in [5] do not apply to the Lotka-Volterra models with x^2 or y^2 terms, as it happens in the competition model (1). The simplest numerical method for the predator-prey model (3) that produces periodic solutions and also preserves positivity of the solution is the following method:

$$
\begin{aligned}
\frac{x(t+h)-x(t)}{h} &= x(t) - x(t+h)y(t), \\
\frac{y(t+h)-y(t)}{h} &= -y(t+h) + x(t+h)y(t).
\end{aligned}
\tag{4}
$$

An example solution is given in Figure 1 when $h = 0.1$.

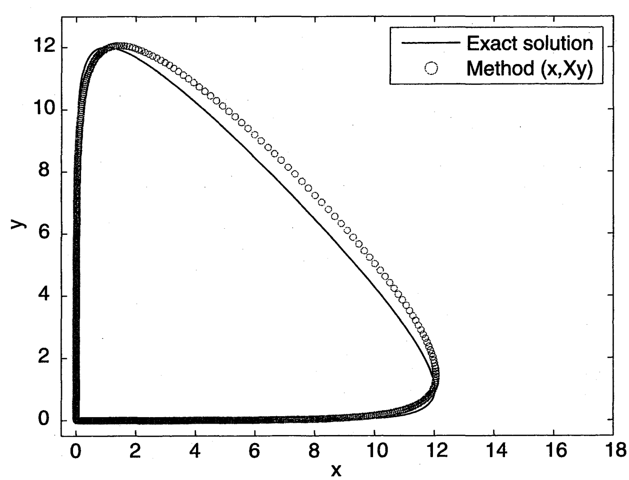

Fig. 1. The method (4) produces periodic solutions for the predator-prey model (3).

Inspired by the method (4), we decide to try the following numerical method on the competition model (1):

$$
(5)\qquad
\begin{aligned}
\frac{x(t+h)-x(t)}{h} &= r_1x(t) - a_{11}x(t)x(t+h) - a_{12}x(t+h)y(t),\\
\frac{y(t+h)-y(t)}{h} &= r_2y(t+h) - a_{21}x(t+h)y(t) - a_{22}y(t)y(t+h),
\end{aligned}
$$

where the pattern of nonlocal terms on the right hand side equations follows the numerical method (4), except the x^2 and y^2 terms which are replaced by $x(x)x(t+h)$ and $y(t)y(t+h)$ respectively. Since the system has an extra parameter h, further assumptions are needed which will be stated in Section 2. In Section 2, we will discuss the local stability results of the discrete time model (5).

Other different schemes have been used to transform the differential equation (1), into its discrete counterpart [2, 4]. For example, Liu and Elaydi [2] use the following numerical scheme to convert the competition model into its discrete version:

$$
\begin{aligned}
\frac{x(t)(t+h)-x(t)}{\varphi(h)} &= r_1x(t) - a_{11}x(t)x(t)(t+h) - a_{12}x(t)(t+h)y(t),\\
\frac{y(t)(t+h)-y(t)}{\varphi(h)} &= r_2y(t) - a_{21}x(t)y(t)(t+h) - a_{22}y(t)y(t)(t+h),
\end{aligned}
$$

where $\varphi(h) = h + O(h^2)$. Let $x_n = x(t)$, $y_n = y(t)$, $x_{n+1} = x(t+h)$, and $y_{n+1} = y(t+h)$. Its discrete version can be expressed explicitly in terms of x_n and y_n:

$$
(6)\qquad
\begin{cases}
x_{n+1} = \dfrac{[1+r_1\varphi(h)]x_n}{1+\varphi(h)[a_{11}x_n + a_{12}y_n]}\\[2ex]
y_{n+1} = \dfrac{[1+r_2\varphi(h)]y_n}{1+\varphi(h)[a_{21}x_n + a_{22}y_n]}
\end{cases}
$$

Cushing [3] has shown that the dynamics of the discrete system (6) are the same as the continuous model (1).

§2. The main results

We will analyze the local stability of the discrete time system (5). Let $x_n = x(t)$, $y_n = y(t)$, $x_{n+1} = x(t+h)$, and $y_{n+1} = y(t+h)$. The discrete-time competition model (5) can be expressed explicitly in terms of x_n and y_n.

$$
\left\{
\begin{array}{rcl}
x_{n+1} & = & \dfrac{x_n(1+hr_1)}{1+ha_{11}x_n+ha_{12}y_n}, \\[2ex]
y_{n+1} & = & \dfrac{y_n(1-ha_{21}x_{n+1})}{1-hr_2+ha_{22}y_n}.
\end{array}
\right. \tag{7}
$$

Since this is a discrete-time competition model, we would like to have the solutions to be nonnegative. Therefore, we will assume that

$$h > 0 \text{ and } 1 - hr_2 > 0. \tag{8}$$

We will show that the local stability conditions for each equilibrium of the discrete-time competitive model (7) are exactly the same as those of the continuous model (1). First, we show that under certain conditions, the solutions for the model (7) are bounded.

Theorem 2.1. *Consider the discrete-time model (7) under the assumptions (2) and (8). If $1 + hr_1 < a_{11}/a_{21}$, then for fixed $h > 0$ the solutions of the system (7) are nonnegative and bounded.*

Proof. It is not difficult to see that x_{n+1} is nonnegative and bounded since

$$0 \le x_{n+1} = \frac{x_n(1+hr_1)}{1+ha_{11}x_n+ha_{12}y_n} \le \frac{x_n(1+hr_1)}{a_{11}hx_n} = \frac{1+hr_1}{a_{11}h}.$$

If we can show that $1 - a_{21}hx_{n+1} > 0$ and since $1 - hr_2 > 0$, then y_{n+1} is nonnegative. We know that $x_{n+1} < \dfrac{1+hr_1}{a_{11}h}$, so $1 - a_{21}hx_{n+1} > 1 - \dfrac{a_{21}}{a_{11}}(1+hr_1) > 0$ if $1 + hr_1 < \dfrac{a_{11}}{a_{21}}$. Then $y_{n+1} \le \dfrac{y_n}{a_{22}hy_n} = \dfrac{1}{a_{22}h}$. y_{n+1} is bounded. Q.E.D.

The system (7) has the same equilibria as the continuous Lotka-Volterra model (1). First, we will show the stability conditions for the equilibria E_0, E_1, and E_2.

Theorem 2.2. *Consider the discrete system (7) under the assumptions (2) and (8). We have the following results.*
(i) E_0 is unstable and a repeller.
(ii) E_1 is locally asymptotically stable if $\dfrac{a_{11}}{a_{21}} < \dfrac{r_1}{r_2}$.
(iii) E_2 is locally asymptotically stable if $\dfrac{r_1}{r_2} < \dfrac{a_{12}}{a_{22}}$.

Proof. We only need to consider the eigenvalues of the Jacobian matrix evaluated at each equilibrium. The Jacobian matrices evaluated at E_0, E_1, and E_2 are

$$J(E_0) = \begin{pmatrix} 1+hr_1 & 0 \\ 0 & \dfrac{1}{1-hr_2} \end{pmatrix}, J(E_1) = \begin{pmatrix} \dfrac{1}{1+hr_1} & \dfrac{r_1 h a_{12}}{a_{11}(1+hr_1)} \\ 0 & \dfrac{a_{11}-hr_1a_{21}}{a_{11}(1-hr_2)} \end{pmatrix},$$

and

$$J(E_2) = \begin{pmatrix} \dfrac{a_{22}(1+hr_1)}{a_{22}+hr_2a_{12}} & 0 \\ \dfrac{hr_2a_{21}(1+hr_1)}{a_{22}+hr_2a_{12}} & 1-hr_2 \end{pmatrix}$$

respectively. Their eigenvalues appear along the diagonals.

For E_0, since $hr_1 > 0$, $hr_2 > 0$, and $1 - hr_2 > 0$, both eigenvalues of $J(E_0)$ are greater than one, $1 + hr_1 > 1$ and $1/(1 - hr_2) > 1$. E_0 is a repeller and unstable.

For E_1, the first eigenvalue of $J(E_1)$, $1/(1 + hr_1)$, is less than one. Its second eigenvalue is $\lambda_2 = \dfrac{a_{11}-hr_1a_{21}}{a_{11}(1-hr_2)}$. We can check that

$$\lambda_2 < 1 \Leftrightarrow \frac{a_{11}}{a_{21}} < \frac{r_1}{r_2}.$$

Therefore, E_1 is stable if $\frac{a_{11}}{a_{21}} < \frac{r_1}{r_2}$. The stability conditions for E_2 can be proved similarly as the case of E_1. Q.E.D.

The positive coexistence equilibrium $E_3 = (x^*, y^*)$ for the system (7) satisfies

$$\begin{aligned} 1+hr_1 &= 1+ha_{11}x^*+ha_{12}y^*, \\ 1-ha_{21}x^* &= 1-hr_2+ha_{22}y^*, \end{aligned} \tag{9}$$

which is the same as

$$\begin{aligned} a_{11}x^*+a_{12}y^* &= r_1, \\ a_{21}x^*+a_{22}y^* &= r_2. \end{aligned}$$

Theorem 2.3. *Consider the discrete system (7) under the assumptions (2) and (8). The coexistence equilibrium E_3 is locally asymptotically stable if*

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}$$

and unstable if

$$\frac{a_{11}}{a_{21}} < \frac{r_1}{r_2} < \frac{a_{12}}{a_{22}}.$$

Proof. The coexistence equilibrium $E_3 = (x^*, y^*) = (\frac{a_{22}r_1 - a_{12}r_2}{a_{11}a_{22} - a_{12}a_{21}}, \frac{a_{11}r_2 - a_{21}r_1}{a_{11}a_{22} - a_{12}a_{21}})$. E_3 exists if $\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}$ or $\frac{a_{11}}{a_{21}} < \frac{r_1}{r_2} < \frac{a_{12}}{a_{22}}$.

By using the identities in (9), the Jacobian matrix evaluated at E_3 can be simplified to

$$J(E_3) = \begin{pmatrix} \dfrac{1 + ha_{12}y^*}{1 + hr_1} & \dfrac{-ha_{12}x^*}{1 + hr_1} \\ \\ \dfrac{-ha_{21}y^*(1 + ha_{12}y^*)}{(1 + hr_1)(1 - ha_{21}x^*)} & \dfrac{(1 - hr_2)(1 + hr_1) + h^2 a_{12}a_{21}x^*y^*}{(1 + hr_1)(1 - ha_{21}x^*)} \end{pmatrix}.$$

From (9), we know $1 - ha_{21}x^* = 1 - hr_2 + ha_{22}y^* > 0$. Therefore, the entries of the matrix $J(E_3)$ satisfy $J_{11} > 0$, $J_{12} < 0$, $J_{21} < 0$, and $J_{22} > 0$.

To determine the stability of E_3, we conduct the Jury test: E_3 is stable if and only if $|\mathrm{tr}J| < 1 + \det J < 2$. The trace and determinant of $J(E_3)$ are

$$\mathrm{tr}J = J_{11} + J_{22} > 0$$

and

$$\det J = \frac{(1 - hr_2)(1 + ha_{12}y^*)}{(1 + hr_1)(1 - ha_{21}x^*)} > 0.$$

Since $\mathrm{tr}J > 0$, we only need to check two conditions: $1 - \det J > 0$ and $1 + \det J - \mathrm{tr}J > 0$. Using the equations in (9), we can show that

$$\begin{aligned} 1 - \det J &= 1 - \frac{(1 - hr_2)(1 + ha_{12}y^*)}{(1 + hr_1)(1 - ha_{21}x^*)} \\ &= \frac{(1 + hr_1)(1 - hr_2 + ha_{22}y^*) - (1 - hr_2)(1 + hr_1 - ha_{11}x^*)}{(1 + hr_1)(1 - ha_{21}x^*)} \\ &= \frac{ha_{22}y^*(1 + hr_1) + ha_{11}x^*(1 - hr_2)}{(1 + hr_1)(1 - ha_{21}x^*)}, \end{aligned}$$

which is always positive because $1 - hr_2 > 0$ and $1 - ha_{21}x^* > 0$.

For the condition $1 + \det J - \mathrm{tr}J$, using $a_{21}x^* + a_{22}y^* = r_2$, we have

$$\begin{aligned} 1 + \det J - \mathrm{tr}J &= \frac{h^2 y^*(a_{22}r_1 - a_{12}a_{21}x^* - a_{12}a_{22}y^*)}{(1 + hr_1)(1 - ha_{21}x^*)} \\ &= \frac{h^2 y^*(a_{22}r_1 - a_{12}r_2)}{(1 + hr_1)(1 - ha_{21}x^*)}. \end{aligned}$$

Since $1 - ha_{21}x^* > 0$, then $1 + \det J - \operatorname{tr} J$ is positive if $\frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}$ and is negative if $\frac{r_1}{r_2} < \frac{a_{12}}{a_{22}}$. Q.E.D.

§3. Conclusion

We have shown that the local stability conditions between the continuous Lotka-Volterra competition model (1) and the discrete competition model (7) are the same. In other words, the two systems are dynamically consistent.

Numerical methods can be applied to differential equation systems to obtain difference equation systems or discrete-time systems. Comparisons between the two systems help us understand the relationship between them. It will helps us to construct a more appropriate discrete model from continuous system.

The numerical method we used is only one of the methods discussed by Roeger [5]. We will look into other different methods to see if the resulting discrete model is still dynamically consistent with the continuous model.

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