Advanced Studies in Pure Mathematics 53, 2009
Advances in Discrete Dynamical Systems
pp. 169–177

# Equilibrium dynamics in an overlapping generations economy with endogenous labor supply

**Atsue Mizushima** [1]

**Abstract.**

This paper develops a two–period overlapping generation model to examine the behavior of an economy that incorporates endogenous labor–care choice. Assuming a log–linear utility function and a Cobb–Douglas production function, we show that there exists multiple equilibria, comprising a unique trajectory satisfying saddle–path stability and other equilibria, which have an infinite number of converging transition paths.

## §1. Introduction

It is wildly recognized that multiple equilibria are a common feature among several dynamic models. Typically, self–fulfilling expectations may yield many perfect foresight paths or sunspot equilibria. In an overlapping generations model, dynamic indeterminacy hinges on complicated preferences or specific restrictions on production technologies (see, for example [5] and [3]). This paper tries to complement the existing literature by studying the labor–care choice by using a two period overlapping generations model.

In the model presented later, we assume that labor supply decisions are endogenous as young agents can choose to work at household on a household–produced health or on the market place to produce market goods. When agents derive utility from the status of their aged parents'

Received December 5, 2006.
2000 *Mathematics Subject Classification.* J22, O40.
*Key words and phrases.* multiple equilibria, overlapping generations, household production.
[1] I would like to thank Koichi Futagami, Kazuo Mino, and Mitsu-yoshi Yanagihara for helpful comments and suggestions. All remaining errors are naturally my own.

health, working at household are modeled by allowing young agents to participate in the production of household health status of old agents.[1]

Using this model, we show that the equilibrium dynamics can feature multiple equilibria. More specifically, there exist multiple equilibria, comprising a unique trajectory satisfying saddle–path stability and other equilibria, which have an infinite number of converging transition paths in a simple utility function and Cobb–Douglas production technology.

The remainder of this paper is organized as follows. Section 2 sets up the basic model. Section 3 derives the equilibrium dynamics. Section 4 concludes this paper.

## §2. The model

Consider an infinite–horizon economy composed of agents and perfectly competitive firms. A new generation, referred to as generation $t$, is born in each period $t = 1, 2, 3, \cdots$. Generation $t$ is composed of a continuum of $N_t > 0$ units of agents who live for a maximum of two period; young and old age. At each date, new generations, each consisting of a continuum of agents with a unit measure, are born.

Agents have altruism and derive utility from the status of their aged parents' health. The probability that an agent dies at the beginning of old age after he or she has child is $1 - p$, and the probability that he or she lives throughout old period is $p \in (0, 1]$. If an agent alive in his or her old age, he or she also has a probability of being in poor health. The probability that an agent being in good health throughout the old age is $\psi$, and that has poor health is $1 - \psi$. Therefore, there are three different states in the two periods of life: good health, poor health, and death. A fraction $p\psi$ of young agents are of type $g$ whose parents have good health, a fraction $p(1 - \psi)$ of young agents are of type $b$ whose parents have poor health. Type $d$ agents, who constitute a fraction $1-p$ of young agents, whose parents die. We express the death–illness status of each young agents as index $i$.

Each young agent of generation $t$ allocates their unit of time between to work at household on a household–produced health $q_{i,t}^{t}$,[2] or on the market place to produce market goods $l_{i,t}^{t}$. He or she earns wage income $w_t l_{i,t}^{t}$, and saves all wage income, where $w_t$ is the real wage rate. Old

[1] Intergenerational transfers of time using household production have been studied by [4].

[2] In what follows we referred to intergenerational transfers of time synonymously with providing care to aged parents and synonymously with care provision.

agents of generation $t$ consume the proceeds of their savings, which we denote by $c_{i,t+1}^{t}$. We assume the existence of actuarially fair insurance in this paper (see [6] and [2]). Thus the rate of return on the annuities is $R_{t+1}/p$ if they are alive and 0 if they die at the end of period $t$, where $R_{t+1}$ is the real rental rate. Let us assume that the level of health of old agents whose health status is good ($g$) or poor ($b$) at time $t$ is produced at household using the following household health production function:

$$h_{g,t} = dq_{g,t}^{t}, \quad (1)$$

$$h_{b,t} = q_{b,t}^{t}, \quad (2)$$

where $d > 1$ is a productivity parameter. Thus, the marginal productivity of care provision of type $g$ young agents is higher than that of type $b$ young agents. Each young agent of generation $t$ solves the following optimization problem for a given level of $R_{t+1}$ and $w_t$:

$$U_i = \beta \ln h_{i,t} + pc_{i,t+1}^{t} \quad i = g, b, d$$

s.t.

$$\begin{aligned} c_{i,t+1}^{t} &= R_{t+1} w_t l_{i,t}^{t}, \\ q_{i,t}^{t} + l_{i,t}^{t} &= 1, \\ 0 \le q_{i,t}^{t} \le 1,\ 0 \le l_{i,t}^{t} &\le 1, \\ &(1), (2), \end{aligned} \quad (3)$$

where $\beta \in (0, 1)$ measures the degree of altruism towards parents.

The optimal care provision is derived as follows:

$$q_{g,t}^{t} = q_{b,t}^{t} = \begin{cases} \dfrac{\beta}{R_{t+1} w_t} & \text{if } \beta \le R_{t+1} w_t, \\ 1 & \text{if } R_{t+1} w_t \le \beta. \end{cases} \quad (4)$$

Due to the quasi-linear utility function, when $R_{t+1} w_t$ is sufficiently large, the opportunity cost of care provision on a household–produced health status is high, then each agent decreases his or her care provision. If parents die, young agents do not derive any utility from the level of parents' health status; thus we have:

$$q_{d,t}^{t} = 0. \quad (5)$$

Using (3), (4), and (5), we have the aggregate labor supply as follows:[3]

$$L_t \equiv l_t N_t = \begin{cases} \left(1 - \dfrac{p\beta}{R_{t+1}w_t}\right)N_t & \text{if } \beta \leq R_{t+1}w_t, \\ (1-p)N_t & \text{if } R_{t+1}w_t \leq \beta. \end{cases} \tag{6}$$

Firms are perfectly competitive profit maximizers that produce output using a production function of the Cobb–Douglas form $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$, where $Y_t$ is aggregate output, $A > 0$ is a productivity parameter and $K_t$ is the aggregate capital stock. We assume that capital depreciates fully in the process of production. Thus, profit maximization in the competitive market equates the marginal products of private labor and capital to the real wage and the real rental rate, respectively:

$$w_t = (1-\alpha)Ak_t^{\alpha}, \; R_t = \alpha A k_t^{\alpha-1}, \tag{7}$$

where $k_t \equiv K_t/L_t$.

## §3. Equilibrium dynamics and indeterminacy

The equilibrium condition for the capital market is given by $K_{t+1} = s_t N_t = w_t l_t N_t$, which implies that the savings of young agents in generation $t$ forms the aggregate capital stock in period $t+1$. Dividing both sides by $N_t$ and substituting in (7) yields the following:

$$k_{t+1} = \frac{(1-\alpha)Ak_t^{\alpha}l_t}{l_{t+1}}. \tag{8}$$

Using (6) through (8), we obtain the following complete dynamic systems:
<Regime I: $\beta \leq R_{t+1}w_t$ >

$$l_{t+1}^{1-\alpha} = \frac{p\beta l_t^{1-\alpha}}{\alpha A(1-l_t)((1-\alpha)Ak_t^{\alpha})^{\alpha}}, \tag{9}$$

$$k_{t+1}^{1-\alpha} = \frac{(1-\alpha)Ak_t^{\alpha}\alpha A(1-l_t)}{p\beta}. \tag{10}$$

<Regime II: $R_{t+1}w_t \leq \beta$ >

$$l_{t+1} = 1-p, \tag{11}$$

$$k_{t+1} = (1-\alpha)Ak_t^{\alpha}. \tag{12}$$

[3]Aggregate labor supply is derived as $L_t \equiv l_t N_t = p\psi(1-q_{g,t}^{\,t}) + p(1-\psi)(1-q_{b,t}^{\,t}) + (1-p)(1-q_{d,t}^{\,t})$.

Before stating the equilibrium, we consider the borderline between the regimes. Substituting (7), (8), and (9) into the borderline, $\beta = R_{t+1}w_t$, we can express the borderline as follows:

$$l_t = 1 - p. \tag{13}$$

Thus, Regime I and Regime II is respectively feasible on the area $1-p \leq l_t \leq 1$ and $l_t \leq 1-p$.

The equations (9) through (12) characterize the economic equilibria that are represented the sequences of $\{k_t, l_t\}_{t=1}^{\infty}$ with an initial condition $(k_1, l_1) \geq 0$. Now let us draw the phase diagram on the $(k_t, l_t)$ plane. We refer to the loci representing $k_{t+1} = k_t$ as $KK$ and that representing $l_{t+1} = l_t$ as $LL$. We have $KK$ and $LL$ loci from (9) through (12):
<Regime I: $1 - p \leq l_t$ >

$$LL_1 \ : \ l_t = 1 - \frac{p\beta}{\alpha A((1-\alpha)Ak_t^{\alpha})^{\alpha}}, \tag{14}$$

$$KK_1 \ : \ k_t = \left(\frac{(1-\alpha)A\alpha A(1-l_t)}{p\beta}\right)^{\frac{1}{1-2\alpha}} \tag{15}$$

$LL_1$ and $KK_1$ loci respectively intersect the borderline $l_t = 1-p$ at the points $A^{ll}$ and $A^{kk}$, where $A^{ll} \equiv (k_t, l_t) = (((\frac{\beta}{\alpha A})^{\frac{1}{\alpha}} \frac{1}{(1-\alpha)A})^{\frac{1}{\alpha}},\ 1-p)$ and $A^{kk} \equiv (k_t, l_t) = ((\frac{(1-\alpha)A\alpha A}{\beta})^{\frac{1}{1-2\alpha}},\ 1-p)$.
<Regime II : $l_t \leq 1-p$ >

$$LL_2 \ : \ l_t = 1 - p, \tag{16}$$

$$KK_2 \ : \ k_t = ((1-\alpha)A)^{\frac{1}{1-\alpha}}. \tag{17}$$

For simplicity of analysis, we assume the following:

**Assumption 1.** *(i)* $\alpha < \frac{1}{2}$, *and (ii)* $\left(\frac{\beta}{\alpha}\right)^{1-\alpha}\left(\frac{1}{1-\alpha}\right)^{\alpha} \leq A.$

Assumption 1-(i) implies that $KK$ locus is decreasing in $l_t$. Assumption 1-(ii) implies that $KK$ and $LL$ loci intersect in Regime I.[4]

The initial point at which the economy starts can be derived from:

$$l_1 = \frac{K_1}{k_1 N_1}. \tag{18}$$

[4]Assumption 1(i): $\frac{\partial k}{\partial l} = -\frac{1}{1-2\alpha}(\frac{(1-\alpha)A\alpha A(1-l_t)}{p\beta})^{\frac{2\alpha}{1-2\alpha}} < 0$, if $\alpha < \frac{1}{2}$.
Assumption 1(ii): We find the value of $k$ becomes $A^{ll} < ((1-\alpha)A)^{\frac{1}{1-\alpha}} < A^{kk}$, if $(\frac{\beta}{\alpha})^{1-\alpha}(\frac{1}{1-\alpha})^{\alpha} < A$.

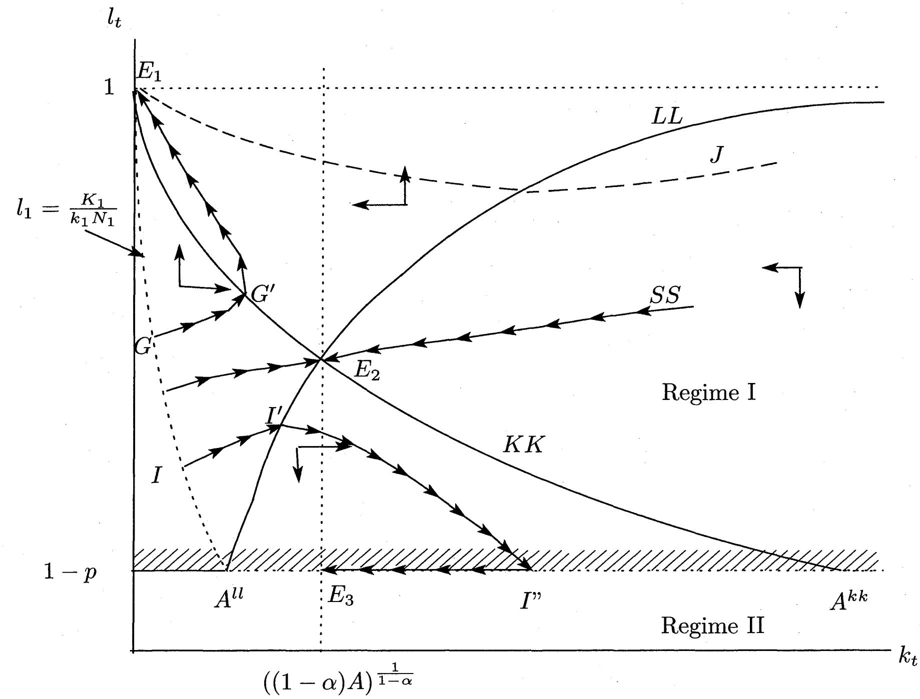


Fig. 1. Phase diagram analysis

The surface representing (18) can be drawn in $(k_t, l_t)$ space as the initial per capita capital stock $K_1/N_1$ is given exogenously. Therefore, the economy must be initially on the line (18). As can be verified immediately from (18), the contour of (18) when drawn on the $(k_t, l_t)$ plane is downward sloping. The phase diagram of this economy is depicted in Figure 1.

For later reference, we first consider the case in which the initial level of capital is sufficiently large. In this case, the initial contour is drawn in the upper-right corner in Figure 1, and the trajectory is drawn like $J$. Since any trajectory that is above $J$ does not satisfy the time constraint (see equation (3)), we can exclude these trajectories from the equilibrium. Thus, contour $J$ shows the boundary trajectory in this economy. Next, let us consider the case for example $l_1 = K_1/k_1N_1$ in Figure 1. If the economy initially happens to be on the line $SS$, it converges to

$E_2$.[5] If the economy initially happens to be above (below) the line $SS$, it converges to $E_1$ ($E_3$). In these cases, there exists an infinite number of converging transition paths towards $E_1$ ($E_3$). Therefore, for a given level of the initial capital stock, the economy converges to one of the equilibria $E_1$, $E_2$, or $E_3$ in the long run.

In order to obtain intuitive implications for these results, let us firstly consider the trajectory which converges to $E_1$ in Figure 1. On the path towards $E_1$, any trajectory that starts above the line "$SS$" for example at point $G$ in Figure 1, initial level of capital is sufficiently high and, thus, so is the opportunity cost of care provision. In the economy both labor supply and the capital–labor ratio initially increase. However, when the $KK$ locus is crossed at point $G'$ in Figure 1, the increased labor supply leads to a decrease in the capital–labor ratio $k_t$. Because the boundary trajectory of this economy is $J$, it follows that the trajectory for this regime converges to $E_1$.

Next, let us consider the equilibrium $E_3$ in Figure 1. On the equilibrium path towards $E_3$, any trajectory that starts below the line "$SS$" for example at point $I$ in Figure 1, the initial level of capital is sufficiently low and, thus, so is the opportunity cost of care provision. In the economy, both labor supply and the capital–labor ratio initially increase. However, when the $LL$ locus is crossed at point $I'$ in Figure 1, the increase in the capital–labor ratio leads to a decrease in labor supply $l_t$. When the trajectory reaches point $I''$ in Figure 1, where labor supply is as its lowest, the capital–labor ratio falls and approaches $E_3$.

## §4. Conclusion

We have investigated the steady-state equilibrium dynamics in a model that incorporates endogenous labor–care choice. Care choice is modeled by allowing young agents to participate in the household–produced health. Using a simple utility function and Cobb–Douglas production function, we have shown that the equilibrium dynamics obtained from the model can feature multiple equilibria. More specifically, there exists a unique trajectory satisfying saddle–path stability and other equilibria, which have an infinite number of converging transition paths.

---

[5]See Appendix A for the conditions for the stability of the steady state.

## § Appendix

### Appendix A

We examine the stability of the equilibrium $E_2$ in this Appendix. To examine the local dynamics of equilibrium $E_2$, we take a first-order Taylor expansion of the system around the steady state $(k^*, l^*)$. With $\check{l}_t \equiv l_t - l^*$ and $\check{k}_t \equiv k_t - k^*$, the linearization is expressed as:

$$\begin{pmatrix} \check{k}_{t+1} \\ \check{l}_{t+1} \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha}{(1-\alpha)} & -\dfrac{\alpha A}{p\beta} \cdot \dfrac{((1-\alpha)A)^{\frac{1+\alpha}{1-\alpha}}}{1-\alpha} \\ -\dfrac{l^* \alpha^2}{(1-\alpha)((1-\alpha)A)^{\frac{1}{1-\alpha}}} & 1 + \dfrac{((1-\alpha)A)^{\frac{\alpha}{1-\alpha}} l^* \alpha A}{(1-\alpha)p\beta} \end{pmatrix} \begin{pmatrix} \check{k}_t \\ \check{l}_t \end{pmatrix},$$

where $l^* \equiv 1 - \frac{p\beta}{\alpha A((1-\alpha)A)^{\frac{\alpha}{1-\alpha}}}$. The characteristic polynomial becomes:

$$P(\kappa) = \kappa^2 - T\kappa + D,$$

$$\begin{aligned} T &= \frac{\alpha}{1-\alpha} + 1 + \frac{((1-\alpha)A)^{\frac{\alpha}{1-\alpha}} l^* \alpha A}{(1-\alpha)p\beta}, \\ D &= \frac{\alpha}{1-\alpha} + \frac{\alpha}{(1-\alpha)} \cdot \frac{((1-\alpha)A)^{\frac{\alpha}{1-\alpha}} l^* \alpha A}{p\beta}. \end{aligned}$$

[1] checks that the steady state is a saddle point, when $1 - T + D < 0$ holds. It is clear that:

$$1 - T + D = -\frac{((1-\alpha)A)^{\frac{\alpha}{1-\alpha}} l^* \alpha A}{p\beta} < 0.$$

Therefore, $E_2$ is a saddle point.

## References


[1] C. Azariadis, Intertemporal Macroeconomics, Basil Blackwell, Oxford, 1993.

[2] O. J. Blanchard, Debt, deficits, and finite horizons, J. Polit. Econ., **93** (1985), 223–247.

[3] M. Boldrin, Dynamic externalities, multiple equilibria, and growth, J. Econom. Theory, **58** (1992), 198–218.

[4] E. Cardia and P. Michel, Altruism, intergenerational transfers of time and bequests, J. Econom. Dynam. Control, **28** (2004), 1681–1701.

[5] P. Reichlin, Equilibrium cycles in an overlapping generations model with production, J. Econom. Theory, **40** (1986), 89–102.

[6] M. E. Yaari, Uncertain lifetime, life insurance, and the theory of the consumer, Rev. Econom. Stud., **32** (1965), 137–150.

*Institute of Economic Research*
*Kyoto University*
*Yoshida-honmachi, Sakyo-ku*
*Kyoto, 606-8501*
*Japan*

*E-mail address*: mizushima@kier.kyoto-u.ac.jp