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Intermediate solutions for nonlinear difference equations with p-Laplacian

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Abstract.

The paper deals with the existence of the so-called intermediate solutions for the nonlinear difference equation with deviating argument. The roles of the nonlinearity and deviating argument are discussed and illustrated by examples.

§1. Introduction

Consider the difference equation

$$\Delta(a_n|\Delta x_n|^{\alpha}\text{sgn } \Delta x_n) + b_n F(x_{n+p}) = 0 \tag{1}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $\alpha > 0$ is a real number, $a = \{a_n\}$, $b = \{b_n\}$ are positive real sequences, $p \geq 0$ is a fixed integer number and F is a real positive continuous function on $(0, \infty)$.

Equation (1) is the discrete analogue of a nonlinear differential equation with p-Laplacian operator, that appears in studying spherically symmetric solutions for certain nonlinear elliptic systems.

An important special case of (1) is the discrete Emden-Fowler equation

$$\Delta(a_n|\Delta x_n|^{\alpha}\text{sgn } \Delta x_n) + b_n|x_{n+p}|^{\beta} \text{ sgn } x_{n+p} = 0, \tag{2}$$

where $\beta > 0$ is a real number. Both equations (1), (2) are widely considered in the literature, see, e.g., [2, 3, 6, 8, 9], the monograph [1] and references therein. In particular, in [2, 3] equation (1) has been

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investigated when $b_n \le 0$ for large n. In this paper we continue such a study, by assuming the positiveness of b.

Throughout the paper, for brevity, by solution of (1) we mean a nontrivial sequence satisfying (1) for $n \ge p$. As usual, a solution $x = \{x_n\}$ of (1) is said to be *nonoscillatory* if there exists $n_x \ge 1$ such that $x_n x_{n+1} > 0$ for $n \ge n_x$.

For the sake of simplicity, we restrict our study to nonoscillatory solutions x for which $x_n > 0$ for large n and we denote by $x^{[1]} = \{x_n^{[1]}\}$ its quasidifference, where

$$x_n^{[1]} = a_n |\Delta x_n|^\alpha \operatorname{sgn} \Delta x_n. \tag{3}$$

Clearly, $x^{[1]}$ is decreasing for large n. Then for any eventually positive solution x of (1) we say $x \in \mathbb{M}^+$ or $x \in \mathbb{M}^-$, according to $x_n > 0, \Delta x_n > 0$ for $n \ge n_x \ge 1$ or $x_n > 0, \Delta x_n < 0$ for $n \ge n_x \ge 1$.

We deal with the existence of a particular type of nonoscillatory solutions, namely the so called *intermediate solutions.* Solution x of (1) is said to be *intermediate* if either

$$x \in \mathbb{M}^-, \quad \lim_n x_n = 0, \quad \lim_n x_n^{[1]} = -\infty, \tag{4}$$

or

$$x \in \mathbb{M}^+, \quad \lim_n x_n = \infty, \quad \lim_n x_n^{[1]} = 0. \tag{5}$$

This terminology originates from the corresponding continuous case and very few is known in the literature (see, e.g., [7, Remark 1.1.]). A comparison with some partial results in [8, 9] is also given. A complete study concerning the nonoscillation of (1) will be given in a forthcoming paper [5].

§2. Main results

Put

$$S_a := \sum_{k=1}^{\infty} \frac{1}{(a_k)^{1/\alpha}}, \quad S_b := \sum_{k=1}^{\infty} b_k.$$

Lemma 1. *(i_1) If $S_a < \infty$, then any solution $x \in \mathbb{M}^+$ is bounded.*

(i_2) Assume that F is nondecreasing. If $S_b < \infty$, then for any solution $x \in \mathbb{M}^-$ the quasidifference $x^{[1]}$ is bounded.

Proof. Claim (i_1). Since $x^{[1]}$ is positive decreasing for large n, say $n \ge n_0$, from (3) we have

$$\Delta x_n \le \frac{1}{(a_n)^{1/\alpha}} (\Delta x_{n_0}^{[1]})^{1/\alpha}.$$

The assertion follows by summing this inequality.

Claim (i_2). Since x is positive decreasing for large n, say $n \geq n_0$, from (1) we have

$$\Delta x_n^{[1]} \geq -b_n F(x_{n_0+p})$$

and again the assertion follows by summing this inequality. Q.E.D.

As follows from Lemma 1, if F is nondecreasing, equation (1) does not admit intermediate solutions when $S_a + S_b < \infty$. Thus, two cases (i) $S_a < \infty, S_b = \infty$, (ii) $S_a = \infty, S_b < \infty$ are considered here.

Theorem 1. *Assume that F is nondecreasing on $(0,1]$ and*

$$J_p := \sum_{j=1}^{\infty} b_j F\left(\sum_{i=j+p}^{\infty} \frac{1}{(a_i)^{1/\alpha}}\right) = \infty, \tag{6}$$

$$S_1 := \sum_{j=2}^{\infty} \left(\frac{1}{a_j}\sum_{i=1}^{j-1} b_i\right)^{1/\alpha} < \infty. \tag{7}$$

Then (1) has solutions satisfying (4).

Proof. Clearly, (6) and (7) yield $S_a < \infty, S_b = \infty$. Let n_0 large so that $n_0 \geq 2$, and

$$\sum_{j=n_0}^{\infty} \frac{1}{(a_j)^{1/\alpha}} < 1, \quad F(1)\sum_{i=1}^{n_0-1} b_i \geq 1, \tag{8}$$

$$\sum_{j=n_0+1}^{\infty} \left(\frac{1}{a_j}\sum_{i=1}^{j-1} b_i\right)^{1/\alpha} \leq \frac{1}{(F(1))^{1/\alpha}}. \tag{9}$$

Let $\mathbb{N}_{n_0} = \{n \in \mathbb{N}, n \geq n_0\}$ and denote by $\mathbb{X}$ the Fréchet space of the real sequences defined for $n \in \mathbb{N}_{n_0}$, endowed with the topology of convergence on finite subsets of $\mathbb{N}_{n_0}$. Consider the set $\Omega \subset \mathbb{X}$ defined by

$$\Omega = \left\{ v = \{v_n\} \in \mathbb{X} : \sum_{j=n}^{\infty} \frac{1}{(a_j)^{1/\alpha}} \leq v_n \leq 1 \right\}.$$

Let $\mathcal{T} : \Omega \to \mathbb{X}$ be the map given by $\mathcal{T}(v) = z = \{z_n\}$, where $z_{n_0} = 1$ and

$$z_n = \sum_{j=n}^{\infty} \frac{1}{(a_j)^{1/\alpha}} \left(1 + \sum_{i=n_0}^{j-1} b_i F\left(v_{i+p}\right)\right)^{1/\alpha} \quad \text{for } n > n_0.$$

Clearly $z_n \geq \sum_{j=n}^{\infty} \frac{1}{(a_j)^{1/\alpha}}$. In view of (8) and (9), we have for $n > n_0$

$$z_n \leq \sum_{j=n}^{\infty} \frac{1}{(a_j)^{1/\alpha}} \left(1 + F(1) \sum_{i=n_0}^{j-1} b_i\right)^{1/\alpha} \leq$$
$$\leq (F(1))^{1/\alpha} \sum_{j=n_0+1}^{\infty} \frac{1}{(a_j)^{1/\alpha}} \left(\sum_{i=1}^{j-1} b_j\right)^{1/\alpha} \leq 1,$$

therefore $\mathcal{T}(\Omega) \subset \Omega$. Let us show that $\mathcal{T}(\Omega)$ is relatively compact and $\mathcal{T}$ is continuous on Ω. In virtue of the Ascoli theorem, any bounded set in $\mathbb{X}$ is relatively compact (see, e.g., [1, Theorem 5.6.1]) and so, because $\mathcal{T}(\Omega)$ is bounded on $\mathbb{X}$, the compactness follows.

Let us prove the continuity of $\mathcal{T}$ on Ω. Let $v^{(k)} = \{v_j^{(k)}\}$ be a sequence in Ω, converging on finite subsets of $\mathbb{N}_{n_0}$ to $v^{(\infty)} = \{v_j^{(\infty)}\} \in \Omega$. From (8) we have

$$A_j^{(k)} := \frac{1}{(a_j)^{1/\alpha}} \left(1 + \sum_{i=n_0}^{j-1} b_i F\left(v_{i+p}^{(k)}\right)\right)^{1/\alpha} \leq F^{1/\alpha}(1) \left(\frac{1}{a_j} \sum_{i=1}^{j-1} b_i\right)^{1/\alpha}.$$

Since F is continuous, the sequence $A^{(k)} = \{A_j^{(k)}\}$ converges, on finite subsets of $\mathbb{N}_{n_0}$, to $A^{(\infty)}$ (with clear meaning of $A^{(\infty)}$) and so $\lim_k A_j^{(k)} = A_j^{(\infty)}$ for any fixed $j \in \mathbb{N}_{n_0}$. Hence, in view of (7), the series $\sum_j A_j^{(k)}$ totally converges. Using the discrete analogue of the Lebesgue dominated convergence theorem, the sequence $\mathcal{T}(v^{(k)})$ converges on finite subsets of $\mathbb{N}_{n_0}$ to $\mathcal{T}(v^{(\infty)})$ and so the continuity of $\mathcal{T}$ is proved.

Applying the Tychonov fixed point theorem, there exists a sequence x such that for $n > n_0$

$$x_n = \sum_{j=n}^{\infty} \frac{1}{(a_j)^{1/\alpha}} \left(1 + \sum_{i=n_0}^{j-1} b_i F(x_{i+p})\right)^{1/\alpha}. \tag{10}$$

Clearly, x is solution of (1) and $x \in \Omega$. From (8) we have

$$x_n \leq (F(1))^{1/\alpha} \sum_{j=n}^{\infty} \left(\frac{1}{a_j} \sum_{i=1}^{j-1} b_j\right)^{1/\alpha}$$

and so, in view of (7), x converges to zero. Since

$$F\left(x_{i+p}\right) \geq F\left(\sum_{m=i+p}^{\infty} \frac{1}{(a_m)^{1/\alpha}}\right),$$

from (10) we obtain

$$x_n^{[1]} \leq -1 - \sum_{i=n_0}^{n-1} b_i F\left(\sum_{m=i+p}^{\infty} \frac{1}{(a_m)^{1/\alpha}}\right)$$

and so, by (6), $\lim_n x_n^{[1]} = -\infty$. Q.E.D.

Theorem 2. *Assume that F is nondecreasing on $[1,\infty)$, and*

$$I_p := \sum_{i=3}^{\infty} b_i F\left(\sum_{j=2}^{i+p-1} \frac{1}{(a_j)^{1/\alpha}}\right) < \infty, \tag{11}$$

$$\sum_{i=1}^{\infty}\left(\frac{1}{a_i}\sum_{j=i}^{\infty} b_j\right)^{1/\alpha} = \infty. \tag{12}$$

Then equation (1) has solutions satisfying (5).

Proof (outline). Clearly, $S_a = \infty$ and $S_b < \infty$. Let n_0 large so that $n_0 \geq 3$ and

$$\sum_{i=n_0}^{\infty} b_i F\left(\sum_{m=2}^{i+p-1} \frac{1}{(a_m)^{1/\alpha}}\right) < 1, \qquad \sum_{m=2}^{n_0-1} \frac{1}{(a_m)^{1/\alpha}} \geq 1. \tag{13}$$

Let $\mathbb{X}$ be defined as in Theorem 1. The assertion follows by applying the fixed point Tychonov fixed point theorem in the set

$$\Omega = \left\{u = \{u_n\} \in \mathbb{X} : 1 \leq u_n \leq \sum_{j=2}^{n-1} \frac{1}{(a_j)^{1/\alpha}}\right\}$$

to the map $\mathcal{T} : \Omega \to \mathbb{X}$ given by $\mathcal{T}(u) = y = \{y_n\}$, where $y_{n_0} = 1$,

$$y_n = 1 + \sum_{j=n_0}^{n-1} \frac{1}{(a_j)^{1/\alpha}} \left(\sum_{i=j}^{\infty} b_i F\left(u_{i+p}\right)\right)^{1/\alpha} \quad \text{for } n > n_0. \tag{14}$$

Finally, the fixed point of $\mathcal{T}$, say x, satisfies $\lim_n x_n^{[1]} = 0$ and it is unbounded in virtue of (12). Q.E.D.

If F is bounded or is bounded away zero near zero, the assumption on monotonicity of F in Theorems 1, 2 can be relaxed, as the following result shows.

Theorem 3. *(i_1) Assume* $0 < \inf_{0<u\le 1} F(u) \le \sup_{0<u\le 1} F(u) < \infty$. *If* $S_a < \infty$ *and (7) holds, then (1) has solutions satisfying (4).*

(i_2) Assume $0 < \inf_{u\ge 1} F(u) \le \sup_{u\ge 1} F(u) < \infty$. *If* $S_b < \infty$ *and (12) holds, then (1) has solutions satisfying (5).*

The proof is similar to the ones of the above theorems with minor changes.

Remark 1. In [9], a more general equation is considered, but the existence results concern with unbounded and zero-convergent solutions which are not intermediate ones; moreover these results require strong assumptions on nonlinearity which are not satisfied for any $\beta > 0$.

In [8], a second order difference system, including (1) with $p = 0$, is considered. Comparing [8, Th.4] with our Theorem 2, both summation conditions are equivalent, but [8, Th.4] is not applicable to (1), due to different assumptions on nonlinearities. In addition, the proof of [8, Th.4] seems not to be correct because a previous result [8, Th.2], with a different assumption, is applied. Comparing [8, Th.11] with our Theorem 1, one can check that assumptions of [8, Th.11] can hold for (1) only if $\limsup_{u\to\infty} F(u) < \infty$, so this result is not applicable to (2).

§3. The role of F and p

When assumptions of Theorem 3 are satisfied, the existence of intermediate solutions does not depend on p. When F is unbounded, the situation can be different, as the following two examples show.

Example 1. Consider equation (2) with $b_n = 1$, $a_n = n(n+1)$, $\alpha = 2/3$, $\beta = 1/2$. It is easy to verify that $S_1 < \infty$, $J_p = \infty$ for any $p \ge 0$ and so, from Theorem 1, equation (2) has intermediate solutions in the class $\mathbb{M}^-$.

Example 2. Consider the equations

$$\Delta(a_n|\Delta x_n|^4 \text{sgn } \Delta x_n) + b_n|x_n|^2 \text{ sgn } x_n = 0, \tag{15}$$

$$\Delta(a_n|\Delta x_n|^4 \text{sgn } \Delta x_n) + b_n|x_{n+1}|^2 \text{ sgn } x_{n+1} = 0, \tag{16}$$

where $b_n = e^{n^2}$, $a_n = \left(e^{-n^2/2} - e^{-(n+1)^2/2}\right)^{-4}$. We will show that Theorem 1 is applicable if $p = 0$ and not for $p = 1$. This means that equation (15) has solutions satisfying (5), while the existence of such solutions for (16) is an open problem. We have $\sum_{k=n}^{\infty}(a_k)^{-1/4} = e^{-n^2/2}$ and so $S_a < \infty$. Furthermore,

$$S_1 \le \sum_{i=2}^{\infty} \left(e^{-i^2/2} - e^{-(i+1)^2/2}\right) e^{i^2/4} i^{1/4} \le \sum_{i=2}^{\infty} e^{-i^2/4} i^{1/4} < \infty.$$

Concerning (6), if $p = 0$, we have

$$J_0 = \sum_{i=1}^{\infty} e^{i^2} \left(\sum_{j=i}^{\infty} \frac{1}{(a_k)^{1/4}}\right)^2 = \sum_{i=1}^{\infty} e^{i^2} e^{-i^2} = \infty,$$

and if $p = 1$, we have

$$J_1 = \sum_{i=1}^{\infty} e^{i^2} \left(\sum_{j=i+1}^{\infty} \frac{1}{(a_k)^{1/4}}\right)^2 = \sum_{i=1}^{\infty} e^{i^2} e^{-(i+1)^2} = e^{-1} \sum_{i=1}^{\infty} e^{-2i} < \infty.$$

Concerning intermediate solutions in the class $\mathbb{M}^+$, it is easy to produce an example of equation (2) for which Theorem 2 holds for any $p \ge 0$ and, similarly, an example such that $I_0 < \infty$ and $I_1 = \infty$.

Remark 2. It is possible to show, by means of some recent summation inequalities, see [4, Lemma 2], that the conditions (6) and (7) [similarly, (11) and (12)] are not compatible for $F(u) = u^\beta$ if $\beta > \alpha$ and $p \ge 1$. Thus Theorems 1, 2 can be applied to (2) only when $\beta \le \alpha$. A detailed discussion on this problem is given in [5].

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