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On periodic points of 2-periodic dynamical systems

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§1. Introduction and statement of the result

Motivated by a recent extension of Sharkovsky's theorem to periodic difference equations [1] (see also [4]), here we show that kneading theory can be useful in the study of the periodic structure of a 2-periodic nonautonomous dynamical system.

Since the notions of zeta function and kneading determinant will play a central role in this discussion, we start by recalling them.

Let X be a set and $f : X \to X$ a map. For each $n \in \mathbb{Z}^+$, denote by f^n the nth iterate of f, defined inductively by

$$f^1 = f \text{ and } f^{n+1} = f \circ f^n, \text{ for all } n \in \mathbb{Z}^+.$$

In what follows we assume that each iterate of f has finitely many fixed points. The Artin-Mazur zeta function of f is defined in [3] as the invertible formal power series

$$\zeta_f(z) = \exp \sum_{n \geq 1} \frac{\#\mathrm{Fix}(f^n)}{n} z^n,$$

where

$$\mathrm{Fix}(f^n) = \{x \in X : f^n(x) = x\}.$$

Naturally, this definition is a particular case of a more general definition, necessary for our purposes.

Let $f : Y \to X$ be a map, with $Y \subset X$. In this case the nth iterate of f is the map $f^n : Y_n \to X$ defined inductively by:

$$Y_1 = Y, f^1 = f$$

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and

$$Y_{n+1} = f^{-n}(Y), f^{n+1} = f \circ f^n, \text{ for all } n \in \mathbb{Z}^+.$$

We define

$$\zeta_f(z) = \exp \sum_{n \geq 1} \frac{\#\mathrm{Fix}(f^n)}{n} z^n,$$

where

$$\mathrm{Fix}(f^n) = \{x \in Y_n : f^n(x) = x\}$$

Problems concerning rationality and analytic continuation of ζ_f are often considered. In some interesting cases ζ_f is a rational function of z. Notice that in such case, there exist $a_1, ..., a_k, b_1, ..., b_k \in \mathbb{C}$ such that

$$\zeta_f(z) = \prod_{i=1}^{k} \frac{1 - b_i z}{1 - a_i z},$$

and consequently

$$\#\mathrm{Fix}(f^n) = \sum_{i=1}^{k} a_i^n - b_i^n, \text{ for all } n \geq 1. \tag{1}$$

Milnor and Thurston in [5] studied the Artin-Mazur zeta function of a continuous piecewise monotone map $f : [a, b] \to [a, b]$ introducing a so called kneading determinant of f, $\mathbf{D}_f(z)$, the determinant of a finite matrix, $\mathbf{N}_f(z)$, called kneading matrix, with entries in $\mathbb{Z}[[z]]$ and depending upon the orbits of the turning points of f; they established a fundamental relation between $\mathbf{D}_f(z)$ and $\zeta_f(z)$. We illustrate this relation in the two following examples, without going into full details.

Example 1. *Let $I = [a, b] \subset \mathbb{R}$ be a compact interval. A continuous map $f : I \to I$ is called piecewise monotone if there exist points (called* turning points *of f) $a = c_0 < c_1 < \cdots < c_{k-1} < c_k = b$ such that: f is strictly monotone in $[c_i, c_{i+1}]$, and f has a local extrema at c_i.*

As an example let $s \in]1, 2]$, and $f : [-1, 1] \to [-1, 1]$ be the continuos map defined by $f(x) = s - 1 - s\,|x|$. The simplest case occurs when $s = 2$, in this case we do not need kneading theory to conclude that

$$\zeta_f(z) = \frac{1}{1 - 2z} \text{ and } \#\mathrm{Fix}(f^n) = 2^n, \text{ for all } n \geq 1.$$

The situation is much more complex when $s \in]1, 2[$. Following [2] we consider a modified kneading determinant of f given by

$$D_f(z) = (1 - z) \sum_{n \geq 0} k_n z^n,$$

where the sequence $k_n \in \{-1, 0, 1\}$ *is defined by*

$$k_0 = 1 \textit{and } k_n = -\mathrm{sign}(f^n(0))k_{n-1}, \textit{for } n \geq 1.$$

Thus, $D_f(z)$ *depends upon the orbit of the turning point* 0*, and, as a consequence of the Milnor-Thurston's identity*

$$\zeta_f(z) = \mathbf{D}_f^{-1}(z) = \frac{1}{(1-z)\sum_{n\geq 0} k_n z^n},$$

we may conclude that $\zeta_f(z)$ *is rational if and only if the sequence* k_n *is eventually periodic. For example let* $s = \frac{1+\sqrt{5}}{2}$*. Since* $f(0) > 0$, $f^2(0) < 0$ *and* $f^3(0) = 0$*, we have* $\mathbf{D}_f(z) = 1 - 2z + z^3$ *and*

$$\zeta_f(z) = \frac{1}{z^3 - 2z + 1} = \frac{1}{(1-z)(1 - \frac{1-\sqrt{5}}{2}z)(1 - \frac{1+\sqrt{5}}{2}z)},$$

and by (1)

$$\#\mathrm{Fix}(f^n) = 1 + \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n, \textit{ for all } n \geq 1.$$

Example 2. *It is possible to generalize the notion of kneading determinant for a continuous piecewise monotone map*

$$f : [a_1, b_1] \cup \cdots \cup [a_k, b_k] \to \mathbb{R}.$$

As in the previous situation, this determinant depends upon the orbits of the turning points of f *and there exists a fundamental relation between* $\mathbf{D}_f(z)$ *and* $\zeta_f(z)$*.*

As an example, let $a \in]0, 1[$ *and* $f : [-1, -a] \cup [a, 1] \to \mathbb{R}$ *be the continuos map defined by* $f(x) = 1 - 2|x|$*. A modified kneading determinant of* f *is given by*

$$D_f(z) = (1-z)\sum_{n\geq 0} k_n z^n,$$

where the sequence $k_n \in \{-1, 0, 1\}$ *is defined by*

$$k_0 = 1 \textit{ and } k_{n+1} = \epsilon(f^{n+1}(a))k_n, \textit{ for } n \geq 0,$$

and $\epsilon : \mathbb{R} \to \{-1, 0, 1\}$ *is the step function defined by*

$$\epsilon(x) = \begin{cases} 1 \textit{ if } x \in]-1, -a[\\ -1 \textit{ if } x \in]a, 1[\\ 0 \textit{ otherwise} \end{cases}.$$

As a consequence of Milnor Thurston's main identity we have

$$\zeta_f(z) = \mathbf{D}_f^{-1}(z) = \frac{1}{(1-z)\sum_{n\geq 0} k_n z^n}.$$

Consider the particular case $a = \frac{1}{8}$. *We have* $f(a) = \frac{3}{4}$, $f^2(a) = -\frac{1}{2} < 0$, $f^3(a) = 0$, *thus* $\mathbf{D}_f(z) = 1 - 2z + z^3$ *and*

$$\zeta_f(z) = \frac{1}{z^3 - 2z + 1} = \frac{1}{(1-z)(1-\frac{1-\sqrt{5}}{2}z)(1-\frac{1+\sqrt{5}}{2}z)},$$

and by (1)

$$\#\mathrm{Fix}(f^n) = 1 + \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n.$$

As mentioned above, our goal is to show that kneading theory can be useful to study the periodic structure of a periodic nonautonomous dynamical system. In this paper we shall restrict the discussion to 2-periodic dynamical systems. As in the autonomous case, we shall need a preparation theorem, which is actually a generalization of (1). First, we need to introduce some notation.

In what follows, by a dynamical system on a set X we mean a pair

$$F = \{f_0, f_1\}$$

of self mappings in X. Given $x \in X$, the orbit of x is the sequence $\{x_n\}_{n=0}^{\infty}$ on X defined by

$$x_0 = x, x_1 = f_0(x), x_2 = f_1(f_0(x)), \ldots$$

or more precisely

$$x_0 = x \text{ and } x_{n+1} = \begin{cases} f_0(x_n) \text{ if } n \text{ is even} \\ f_1(x_n) \text{ if } n \text{ is odd} \end{cases}. \tag{2}$$

The point x is called periodic, with period $\mathrm{p}(x) \in \mathbb{Z}^+$, if the orbit of x is a periodic sequence with period $\mathrm{p}(x)$. The set whose elements are the periodic points of F is denoted by Per_F. For each positive integer, n, we also define

$$\mathrm{Per}_F(n) = \{x \in \mathrm{Per}_F : \mathrm{p}(x) \text{ divides } n\}.$$

We will assume that $\mathrm{Per}_F(n)$ is a finite set for all positive integer n.

Observe that, even in the simplest cases, there exists a relevant difference between the numbers

$$\#\mathrm{Per}_F(n) \text{ and } \#\mathrm{Fix}(f^n).$$

Indeed, as the following example shows, even when the set X is finite, we can not guarantee the existence of complex numbers $a_1, ..., a_k, b_1, ..., b_k$ such that

$$\#\mathrm{Per}_F(n) = \sum_{i=1}^{k} a_i^n - b_i^n, \text{ for all } n \geq 1. \tag{3}$$

Example 3. *Let $X = \{0,1\}$ and define the maps $f_0 : \{0,1\} \to \{0,1\}$ and $f_1 : \{0,1\} \to \{0,1\}$ by $f_0(0) = f_0(1) = 0$ and $f_1(0) = f_1(1) = 1$. We have*

$$\#\mathrm{Per}_F(n) = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ 1 & \textit{if } n \textit{ is even} \end{cases}.$$

Consequently, the formal power series

$$\exp \sum_{n\geq 1} \frac{\#\mathrm{Per}_F(n)}{n} z^n = \frac{1}{\sqrt{1-z^2}}$$

is not rational, and therefore do not exist complex numbers satisfying (3).

Nevertheless, it can be shown that, if the set X is finite, then there are complex numbers $a_1, ..., a_k, b_1, ..., b_k, c_1, ..., c_k, d_1, ..., d_k$ such that

$$\#\mathrm{Per}_F(n) = \begin{cases} \sum_{i=1}^{k} a_i^n - b_i^n & \text{if } n \text{ is odd} \\ \sum_{i=1}^{k} c_i^n - d_i^n & \text{if } n \text{ is even} \end{cases}. \tag{4}$$

This fact rises the following problem. If $F = \{f_0, f_1\}$ is a dynamical system on an infinite set X, under which conditions can we guarantee the existence of complex numbers verifying (4)? Our main theorem concerns this problem. For that purpose, we need to introduce the maps

$$\begin{array}{rccc} g_0 : & X_0 \subset X & \to & X \\ & x & \to & f_0(x) \end{array},$$

where

$$X_0 = \{x \in X : f_0(x) = f_1(x)\},$$

and

$$g_1 = f_1 \circ f_0.$$

Theorem 4. *Let $F = \{f_0, f_1\}$ be a dynamical system on X. If $\zeta_{g_0}(z)$ and $\zeta_{g_1}(z)$ are rational, and $a_1, ..., a_k, b_1, ..., b_k, c_1, ..., c_k, d_1, ..., d_k$ are complex numbers such that*

$$\zeta_{g_0}(z) = \prod_{i=1}^{k} \frac{1 - b_i z}{1 - a_i z} \text{ and } \zeta_{g_1}(z) = \prod_{i=1}^{k} \frac{1 - d_i z}{1 - c_i z},$$

then we have

$$\#\mathrm{Per}_F(n) = \begin{cases} \sum_{i=1}^{k} a_i^n - b_i^n & \text{if } n \text{ is odd} \\ \sum_{i=1}^{k} c_i^{\frac{n}{2}} - d_i^{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}.$$

This general result has a relevant consequence in the context of interval maps. Let f_0 and f_1 be continuous piecewise monotone self-maps of a compact interval $I \subset \mathbb{R}$. Furthermore, assume that the set

$$X_0 = \{x \in I : f_0(x) = f_1(x)\}$$

has finitely many connected components. Notice that under these conditions both maps g_0 and g_1 are continuous piecewise monotone, and thus we can use the old results on kneading theory to study the zeta functions of g_0 and g_1.

Example 5. *Let $f_0 : [-1, 1] \to [-1, 1]$ be defined by $f_0(x) = 1 - 2|x|$, and let $f_1 : [-1, 1] \to [-1, 1]$ be any continuous expanding map such that: f_1 is increasing on $[-1, 0]$ and decreasing on $[0, 1]$; $f_1(x) = f_0(x)$, for $x \in \{-1, 0, 1\}$. Under these conditions, it is easy to see that $\zeta_{g_1}(z)$ do not depend upon f_1. As a matter of fact*

$$\zeta_{g_1}(z) = \zeta_{f_1 \circ f_0}(z) = \zeta_{f_0^2}(z) = \frac{1}{1 - 4z}.$$

The study of $\zeta_{g_0}(z)$ is much more interesting because it depends on X_0. The simplest case occurs when $X_0 = \{-1, 0, 1\}$. In this case we have

$$\zeta_{g_0}(z) = \frac{1}{1 - z},$$

and from Theorem 4

$$\#\mathrm{Per}_F(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even} \end{cases}.$$

Of course the situation is more complex if the set X_0 is infinite. As an example, assume that $X_0 = \left[-1, -\frac{1}{8}\right] \cup \{0\} \cup \left[\frac{1}{8}, 1\right]$. We have then (see Example 2)

$$\zeta_{g_0}(z) = \frac{1}{(1-z)(1-\frac{1-\sqrt{5}}{2}z)(1-\frac{1+\sqrt{5}}{2}z)},$$

and from Theorem 4

$$\#\mathrm{Per}_F(n) = \begin{cases} 1 + \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even} \end{cases}.$$

§2. Proof of Theorem 4

Let us begin by recalling some facts on generating functions. For any sequence $\{s_n\}_{n=1}^{\infty} \subset \mathbb{C}$ let us define the formal power series

$$S(z) = \exp \sum_{n \geq 1} \frac{s_n}{n} z^n.$$

It is well-known that:
i) The generating function $S(z)$ is a rational function of z if and only if there exist $a_1, ..., a_k; b_1, ..., b_k \in \mathbb{C}$ such that

$$S(z) = \prod_{i=1}^{k} \frac{1 - b_i z}{1 - a_i z}. \tag{5}$$

ii) For any $a_1, ..., a_k; b_1, ..., b_k \in \mathbb{C}$ the identity (5) holds if and only if

$$s_n = \sum_{i=1}^{k} a_i^n - b_i^n, \text{ for } n \geq 1.$$

So, for any map $f : X \to X$, we may write: $\zeta_f(z)$ is rational if and only if there exist $a_1, ..., a_k; b_1, ..., b_k \in \mathbb{C}$ such that

$$\zeta_f(z) = \prod_{i=1}^{k} \frac{1 - b_i z}{1 - a_i z}, \tag{6}$$

which is equivalent to

$$\#\mathrm{Fix}(f^n) = \sum_{i=1}^{k} a_i^n - b_i^n, \text{ for } n \geq 1. \tag{7}$$

We can now prove Theorem 4. Let $F = \{f_0, f_1\}$ be a dynamical system on X. If the zeta functions $\zeta_{g_0}(z)$ and $\zeta_{g_1}(z)$ are both rational, then by (6) there are complex numbers $a_1,...,a_k,b_1,...,b_k,c_1,...,c_k,d_1,...,d_k$ such that

$$\zeta_{g_0}(z)=\prod_{i=1}^{k}\frac{1-b_iz}{1-a_iz}\text{ and }\zeta_{g_1}(z)=\prod_{i=1}^{k}\frac{1-d_iz}{1-c_iz},$$

and by (7)

$$\#\mathrm{Fix}(g_0^n)=\sum_{i=1}^{k}a_i^n-b_i^n\text{ and }\#\mathrm{Fix}(g_1^n)=\sum_{i=1}^{k}c_i^n-d_i^n,\ n\geq 1.$$

So, the theorem will follow from the identity

$$\mathrm{Per}_F(n)=\begin{cases}\mathrm{Fix}(g_0^n) & \text{if } n \text{ is odd}\\ \mathrm{Fix}(g_1^{n/2}) & \text{if } n \text{ is even}\end{cases}. \tag{8}$$

In order to prove (8), let $x\in X$ and $\{x_i\}_{i=0}^{\infty}$ be the orbit of x. Assume first that n is even. In this case we have by (2)

$$x_0=x\text{ and }x_{kn}=(f_1\circ f_0)^{kn/2}(x)=g_1^{kn/2}(x),\text{ for }k\geq 1.$$

So, we can write

$$x_{kn}=x_0,\text{ for all }k\geq 1\text{ if and only if }g_1^{kn/2}(x)=x,\text{ for all }k\geq 1,$$

and therefore $\mathrm{Per}_F(n)=\mathrm{Fix}(g_1^{n/2})$.

For n odd, if $x\in\mathrm{Per}_F(n)$, then the period $p(x)$ is odd, and:

$$f_0(x_j)=x_{j+1}=x_{1+j+p(x)}=f_1(x_{j+p(x)})=f_1(x_j),\text{ if } j \text{ is even};$$

$$f_1(x_j)=x_{j+1}=x_{1+j+p(x)}=f_0(x_{j+p(x)})=f_0(x_j),\text{ if } j \text{ is odd},$$

which shows that

$$\{x_i\}_{i=0}^{\infty}\subset X_0=\{x\in X: f_0(x)=f_1(x)\}.$$

Therefore

$$x_i=f_0^i(x)=g_0^i(x),\text{ for }i\geq 1$$

and consequently

$$g_0^{p(x)}(x)=x_{p(x)}=x_0=x.$$

This proves the inclusion $\mathrm{Per}_F(n)\subset\mathrm{Fix}(g_0^n)$. Since the inclusion $\mathrm{Fix}(g_0^n)\subset\mathrm{Per}_F(n)$ is immediate it follows $\mathrm{Fix}(g_0^n)=\mathrm{Per}_F(n)$, as requested.

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