Advanced Studies in Pure Mathematics 52, 2008
Groups of Diffeomorphisms
pp. 469–477

Mapping configuration spaces to moduli spaces

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§1. Introduction

Let $\mathcal{C}_n$ be the space of unordered n-tuples of distinct points in the interior of the unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$, and let $\mathcal{M}_{g,2}$ denote the moduli space of connected Riemann surfaces of genus g with two ordered and parametrised boundary components. There is a natural map

$$\Phi : \mathcal{C}_{2g+2} \longrightarrow \mathcal{M}_{g,2};$$

it takes a subset $\underline{a} = \{a_1, \ldots, a_{2g+2}\} \subset D$ to the part $\Sigma_{\underline{a}}$ of the Riemann surface associated to the function

$$f_{\underline{a}}(z) = ((z - a_1) \ldots (z - a_{2g+2}))^{1/2}$$

which lies over the disc D.

The purpose of this paper is to describe this map in topological terms. On passing to the fundamental groups it gives a geometric definition of a well-known algebraic homomorphism $\phi : \mathfrak{Br}_{2g+2} \to \Gamma_{g,2}$ from the braid group on $2g + 2$ strings to the mapping class group which is described below. The main result is that Φ is compatible with naturally defined actions of the framed little 2-discs operad on configuration spaces and moduli spaces. As immediate consequences we deduce that Φ is trivial in homology with field coefficients, and trivial on stable homology with any constant coefficient system. In particular, this proves an unstable version of a conjecture by Harer, and simplifies the proof of the triviality of the map in stable homology given in [ST], to which we refer for more background on this problem.

Received July 29, 2007.
Revised October 23, 2007.

In an appendix we discuss the map in homology given by the inclusion of the hyper-elliptic involution into the mapping class group.

§2. The induced map on fundamental groups

Let $F_{g,2}$ be a surface of genus g with two boundary circles. The moduli space $\mathcal{M}_{g,2}$ is homotopic to the classifying space of the group of homeomorphisms of $F_{g,2}$ which leave the boundary circles pointwise fixed:

$$\mathcal{M}_{g,2} \simeq B\mathrm{Homeo}^+(F_{g,2};\partial) \simeq B\Gamma_{g,2};$$

here $\Gamma_{g,2}$ is the associated mapping class group. Similarly, the configuration space $\mathcal{C}_{2g+2}$ is homotopic to the classifying space of the group of homeomorphisms of $D \setminus \underline{a}$ — the disc with $2g+2$ interior points removed — which fix the boundary of the disc pointwise but are allowed to permute the points of $\underline{a}$:

$$\mathcal{C}_{2g+2} \simeq B\mathrm{Homeo}^+(D \setminus \underline{a}) \simeq B\mathfrak{Br}_{2g+2};$$

the associated mapping class group is the braid group $\mathfrak{Br}_{2g+2}$ on $2g+2$ strings. Thought of as the fundamental group of $\mathcal{C}_{2g+2}$ based at $\underline{a}$, the group $\mathfrak{Br}_{2g+2}$ is generated by the braids σ_i, $i = 1, \ldots, 2g+1$, where σ_i interchanges the points a_i and a_{i+1}.

We shall now determine the map that Φ induces on fundamental groups. Write $\pi : \Sigma_{\underline{a}} \to D$ for the double covering map branched at $a_1, \ldots, a_{2g+2}$. Let D_i be a subdisc in the interior of D containing the two points a_i and a_{i+1}. Then $\pi^{-1}(D_i)$ is an annulus contained in $\Sigma_{\underline{a}}$. To see this, note that $\pi^{-1}(D_i)$ has two boundary components and that its Euler characteristic is given by

$$\chi(\pi^{-1}(D_i)) = 2\chi(D_i) - 2 = 0.$$

Proposition 2.1. *On fundamental groups, the map Φ takes σ_i to the Dehn twist corresponding to the annulus $\pi^{-1}(D_i)$ in $\Sigma_{\underline{a}}$.*

Proof. Homeomorphisms of the base of a cyclic branched covering of a surface that permute the branch points can be lifted, and this lift is unique up to covering transformations, cf. Lemma 5.1 of [BH]. The braid σ_i corresponds to a homeomorphism h_i of D supported in the subdisc

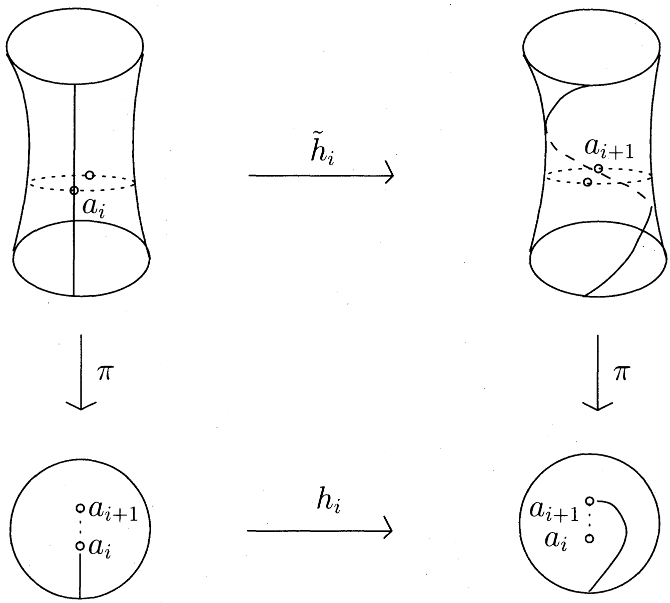

Fig. 1. Lifting h_i

D_i, and hence lifts to a homeomorphism $\tilde{h}_i$ of $\Sigma_{\underline{a}}$ supported in the annulus $\pi^{-1}(D_i)$. There is only one non-trivial covering transformation for π, given by the hyper-elliptic involution J, which interchanges the boundary components of the annulus $\pi^{-1}(D_i)$. Thus the lift $\tilde{h}_i$ is unique if we ask for the boundary circles to be fixed.

The mapping class group of an annulus is generated by the Dehn twist around its waist. By analysing the effect of $\tilde{h}_i$ on a path connecting the two boundary components of $\pi^{-1}(D_i)$, one checks the claim. See Figure 1. Q.E.D.

Thus, on fundamental groups, Φ induces the group homomorphism

$$\phi : \mathfrak{Br}_{2g+2} \to \Gamma_{g,2} \to \Gamma_g$$

where the generator σ_i is mapped to the Dehn twist around the curve α_i in Figure 2. This is the group homomorphism studied in [ST].

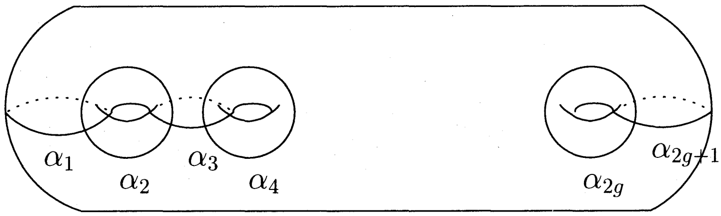

Fig. 2. The image of ϕ

§3. Action of the framed little 2-discs operad

Let $\mathcal{D}_k$ be the space of smooth embeddings of k disjoint, ordered copies of the disc D into its interior which restrict on each disc to the composition of a translation and a multiplication by an element of $\mathbb{C}^\times$. The spaces $\{\mathcal{D}_k\}$ form an operad $\mathcal{D}$, the framed little 2-discs operad, with structure maps

$$\gamma : \mathcal{D}_k \times (\mathcal{D}_{m_1} \times \cdots \times \mathcal{D}_{m_k}) \longrightarrow \mathcal{D}_{\Sigma m_i}$$

given by composition of embeddings. The operad $\mathcal{D}$ acts naturally on the configuration spaces $\mathcal{X} = \coprod_{m \geq 0} \mathcal{X}_m$ with $\mathcal{X}_m = \mathcal{C}_{2m}$. This action is defined by maps

$$\gamma_{\mathcal{X}} : \mathcal{D}_k \times (\mathcal{X}_{m_1} \times \cdots \times \mathcal{X}_{m_k}) \longrightarrow \mathcal{X}_{\Sigma m_i}$$

which take a point $(f; \underline{a}_1, \dots, \underline{a}_k)$ to the image of the union $\underline{a}_1 \cup \cdots \cup \underline{a}_k$ under the embedding f.

Now put $\mathcal{Y}_m = \mathcal{M}_{m-1,2}$ for $m \geq 1$, and $\mathcal{Y}_0 = \mathcal{M}_{0,1} \sqcup \mathcal{M}_{0,1}$. Each surface $\Sigma \in \mathcal{Y}_m$ has two ordered parametrised boundary circles, $\partial_1 \Sigma$ and $\partial_2 \Sigma$. For $f \in \mathcal{D}_k$, let $P_f = D \setminus f(D \cup \cdots \cup D)$. We think of P_f as a Riemann surface with $k+1$ parametrised boundary components. There is an operad action

$$\gamma_{\mathcal{Y}} : \mathcal{D}_k \times (\mathcal{Y}_{m_1} \times \cdots \times \mathcal{Y}_{m_k}) \longrightarrow \mathcal{Y}_{\Sigma m_i}$$

defined by

$$(f; \Sigma_1 \cup \cdots \cup \Sigma_k) \mapsto (P_f \cup P_f \cup \Sigma_1 \cup \cdots \cup \Sigma_k)/\equiv$$

where the union of the first boundary components $\cup_i \partial_1 \Sigma_i$ is identified with the interior boundary circles of the first P_f, and $\cup_i \partial_2 \Sigma_i$ is identified with the interior boundary circles of the second P_f using the parametrisations, cf. Figure 3. The resulting surface has a well-defined complex structure that restricts to the given complex structures on the subsurfaces, cf. [S].

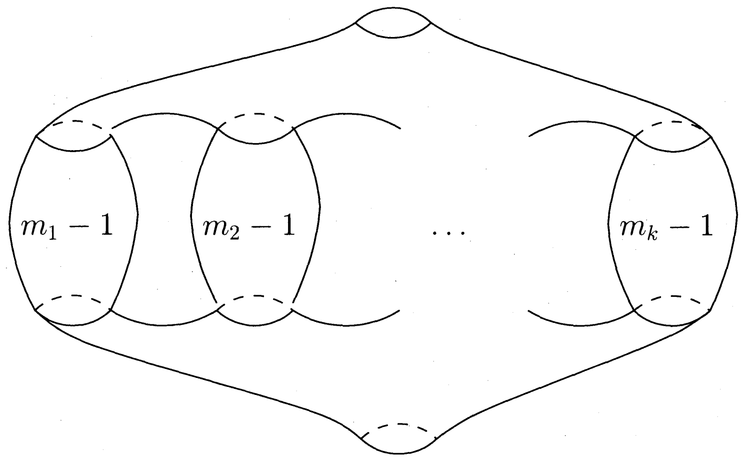

Fig. 3. $\mathcal{D}$ acting on moduli spaces

Proposition 3.1. *The map* Φ *defines a map* $\mathcal{X} \to \mathcal{Y}$ *of algebras over the framed little* 2*-discs operad.*

Proof. The index m was chosen so that Φ takes $\mathcal{X}_m$ to $\mathcal{Y}_m$. Both $\gamma_\mathcal{X}$ and $\gamma_\mathcal{Y}$ are defined by gluing of surfaces. The composed surface has a unique conformal structure that restricts to the given conformal structures on the subsurfaces. Furthermore, the covering map $\pi : \Sigma_{\underline{a}} \to D$ is conformal away from the set $\underline{a}$. It therefore follows that the assignment $\underline{a} \mapsto \Sigma_{\underline{a}}$ commutes with the gluing operations of $\gamma_\mathcal{X}$ and $\gamma_\mathcal{Y}$. Q.E.D.

Remark 3.2. We can also work on the level of mapping class groups and their classifying spaces. For this, replace $\mathcal{D}_k$ by the homotopy equivalent space $B\Gamma_{0,k+1}$, the classifying space of the mapping class group of D with k interior discs removed. Similarly, we also replace $\mathcal{X}_m$ by $B\mathfrak{Br}_{2m}$ and $\mathcal{Y}_m$ by $B\Gamma_{m-1,2}$, for $m > 0$. The actions of $\gamma_\mathcal{X}$ and $\gamma_\mathcal{Y}$, reinterpreted in these terms, are now induced by maps of the underlying mapping class groups. By definition they commute with Φ, or more precisely with $B\phi$. This corrects the statement and proof of 2.6 in [C].

§4. The induced map in homology

Proposition 3.1 has immediate consequences for the map that Φ induces on homology. Let p be a prime. The idea for the following is due to F. Cohen [C]. The case $p = 2$ is the (stronger) unstable version of Harer's conjecture, cf. [ST].

Corollary 4.1. *For $g > 2$ and $* > 0$,*

$$\Phi_* : H_*(\mathcal{C}_{2g+2}; \mathbb{Z}/p\mathbb{Z}) \to H_*(\mathcal{M}_{g,2}; \mathbb{Z}/p\mathbb{Z}) \text{ is zero .}$$

Proof. Recall that the first homology group of the mapping class group $\Gamma_{g,2}$, and hence that of $\mathcal{M}_{g,2}$ is zero for $g > 2$. On the other hand, $H_*(\mathcal{C}_{2g+2}, \mathbb{Z}/p\mathbb{Z})$ is generated by the first homology group under products and the operations induced from the action of $\mathcal{D}$, cf. [CLM]. The result now follows by Proposition 3.1. Q.E.D.

This implies that also the map in homology with rational coefficients (and hence with any field coefficients) is trivial as can be seen as follows. If the image of a class in rational homology is non-trivial then there must be a class in integer homology which is mapped by Φ_* to a class of infinite order. As all homology groups of the moduli spaces are finitely generated, this image class is divisible by at most finitely many primes and would therefore be non-zero in homology with coefficients in $\mathbb{Z}/p\mathbb{Z}$ for any prime p coprime to the maximal divisor. But this cannot happen by Corollary 4.1.

We now deduce the main result of [ST]. Maps of algebras over the little framed 2-discs operad group-complete to maps of double loop spaces. In particular, Φ induces a map of double loop spaces from the group-completion of $\mathcal{X}$ to that of $\mathcal{Y}$,

$$\Phi^+ : 2\mathbb{Z} \times B\mathfrak{Br}_\infty^+ \longrightarrow \mathbb{Z} \times B\Gamma_\infty^+;$$

here X^+ denotes Quillen's plus construction on X, while $\mathfrak{Br}_\infty := \lim_{k\to\infty} \mathfrak{Br}_k$ and $\Gamma_\infty := \lim_{g\to\infty} \Gamma_{g,2}$ are the infinite braid and stable mapping class groups. Note that here the zero component in $\mathbb{Z} \times B\Gamma_\infty^+$ corresponds to surfaces of Euler characteristic 2.

Corollary 4.2. *$\phi_* : H_*(\mathfrak{Br}_\infty; k) \to H_*(\Gamma_\infty; k)$ is zero for $* > 0$, and any constant coefficient system k.*

Proof. As the plus construction does not alter the homology, this follows because on connected components the map Φ^+ is homotopic to the constant map. Indeed, any double loop space map from $B\mathfrak{Br}_\infty^+ \simeq \Omega^2 S^3$ is determined by its restriction to $S^1 \subset \Omega^2\Sigma^2(S^1)$. For Φ^+ this restriction has to be homotopic to the constant map as $B\Gamma_\infty^+$ is simply connected. Q.E.D.

Recall that by the Harer-Ivanov homology stability theorem, the homology of the mapping class group is independent of the genus g and the number of boundary components in dimensions $* < g/2$. Thus Corollary 4.2 can be rephrased as follows.

Corollary 4.3. *For* $g > 2* > 0$,

$$\Phi_* : H_*(\mathcal{C}_{2g+2}; k) \to H_*(\mathcal{M}_{g,2}; k) \text{ is zero }.$$

§5. Appendix: Relation to hyper-elliptic mapping class groups

By definition Φ factors through the subspace of hyper-elliptic curves in $\mathcal{M}_{g,2}$. Note that the hyper-elliptic involution J interchanges the boundary components of $F_{g,2}$ and is only an element of the extension $\Gamma_{g,(2)}$ of $\Gamma_{g,2}$, the mapping class group associated to homeomorphisms that may exchange the boundary components as long as the parametrisations are preserved. The hyper-elliptic mapping class groups $\triangle_{g,(2)}$ and $\triangle_g$ are the commutants of J in $\Gamma_{g,(2)}$ and Γ_g. They are extensions of the braid groups of the disc D and the sphere S^2 respectively:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & <J> & \longrightarrow & \triangle_{g,(2)} & \longrightarrow & \mathfrak{Br}_{2g+2} & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & <J> & \longrightarrow & \triangle_g & \longrightarrow & \mathfrak{Br}_{2g+2}(S^2) & \longrightarrow & 0. \end{array}$$

The kernel of the middle and right vertical maps is an extension of the free group on $2g+1$ generators by $\mathbb{Z}$ for $g > 0$. The homomorphism ϕ provides a splitting of the top row. As J maps the curves α_i, $i = 1, \ldots, 2g+1$, in Figure 2 to themselves, J commutes with the corresponding Dehn twists, cf. Lemma 4.6.7 of [B]. Hence

$$\triangle_{g,(2)} \simeq \mathfrak{Br}_{2g+2} \times <J> .$$

At the conference, R. Hain raised the question whether Corollary 4.2 can be extended to the hyper-elliptic mapping class group. This is not the case, as we shall explain now.

Consider the universal surface bundle E over $B\Gamma_g \simeq B\text{Homeo}^+(F_g)$ and the associated pull-back bundle $incl^*(E)$ over $B(< J >) = BC_2$. The vertical tangent bundle on E is classified by a map $\omega : E \to \mathbb{C}P^\infty$. Consider the composition

$$\theta : BC_2 \xrightarrow{incl} B\Gamma_g \xrightarrow{tr} Q(E_+) \xrightarrow{\omega} Q(\mathbb{C}P^\infty_+) = Q(\mathbb{C}P^\infty) \times Q(S^0);$$

here tr denotes the Becker-Gottlieb transfer map of the bundle E, while X_+ stands for the union of X with a disjoint base point, and $Q = \lim_{n\to\infty} \Omega^n S^n$ is the free infinite loop space functor. A straightforward computation of transfer maps, cf. Lemma 5.2 of [GMT], gives

$$\theta \simeq (2g+2)\psi + (-2g)\hat{t},$$

where

$$\psi : BC_2 \longrightarrow \mathbb{C}P^\infty \longrightarrow Q(\mathbb{C}P^\infty)$$

is induced by the inclusion $C_2 \to S^1$, and $\hat{t}$ is the transfer associated to the universal bundle $EC_2 \to BC_2$. The map ψ is non-trivial in homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, and hence so is $incl$. We have proved

Proposition 5.1. *The inclusion $incl :< J >= C_2 \to \Gamma_g$ is non-trivial on stable homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients.*

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