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On the stable cohomology algebra of extended mapping class groups for surfaces

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Abstract.

Let $\Sigma_{g,1}$ be an oriented compact surface of genus g with 1 boundary component, and $\Gamma_{g,1}$ the mapping class group of $\Sigma_{g,1}$. We determine the stable cohomology group of $\Gamma_{g,1}$ with coefficients in $H^1(\Sigma_{g,1};\mathbb{Z})^{\otimes n}$, $n \geq 1$, explicitly modulo the stable cohomology group with trivial coefficients. As a corollary the rational stable cohomology algebra of the semi-direct product $\Gamma_{g,1} \ltimes H_1(\Sigma_{g,1};\mathbb{Z})$ (which we call the *extended mapping class group*) is proved to be freely generated by the generalized Morita-Mumford classes $\widetilde{m_{i,j}}$'s ($i \geq 0$, $j \geq 1$, $i+j \geq 2$) [11] over the rational stable cohomology algebra of the group $\Gamma_{g,1}$.

Introduction

Let $g \geq 2$ be an integer, $\Sigma_{g,1}$ an oriented compact surface of genus g with 1 boundary component, and $\Gamma_{g,1}$ the mapping class group of $\Sigma_{g,1}$. Similarly let Σ_g be an oriented closed surface of genus g, and Γ_g the mapping class group of Σ_g. We denote by H the first integral homology group of the surface $\Sigma_{g,1}$, $H_1(\Sigma_{g,1};\mathbb{Z})$, which can be regarded as that of Σ_g. The mapping class groups Γ_g and $\Gamma_{g,1}$ act on it in an obvious symplectic way.

One of the earliest works in the study of twisted (co)homology of the mapping class group with symplectic coefficients is Morita's paper [18] computing the first homology group with coefficients in H. Here he proved $H_1(\Gamma_{g,1};H) = \mathbb{Z}$ and $H_1(\Gamma_g;H) = \mathbb{Z}/(2-2g)$. In particular, the integral homology $H_1(\Gamma_g;H)$ depends on the genus g. This suggests to us the situation in the symplectic twisted (co)homology for $\Gamma_{g,1}$, which

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we would like to call the 'bordered' case, is different from that in the 'closed' case Γ_g.

The Harer stability theorem [3] and the theorem of Madsen and Weiss [16] are the most important facts on (co)homology of the mapping class groups. The former states the cohomology group of the mapping class group with trivial coefficients is independent of the genus g and the number of boundary components of the surface, provided that the degree is smaller than $g/3$ [3] or $g/2$ [8]. Moreover Ivanov [7] has generalized this theorem to those with twisted coefficients in the 'bordered' case. These theorems enable us to consider the stable cohomology group of the mapping class groups for surfaces. When we consider trivial coefficients $\mathbb{Z}$ (resp. $\mathbb{Q}$), we denote it by $H^*(\Gamma_\infty; \mathbb{Z})$ (resp. $H^*(\Gamma_\infty; \mathbb{Q})$). The latter theorem established in 2002 [16] gave a loop-space description for $H^*(\Gamma_\infty; \mathbb{Z})$. As a corollary it is proved the rational stable cohomology algebra $H^*(\Gamma_\infty; \mathbb{Q})$ is generated by the Morita-Mumford classes $e_i = (-1)^{i+1}\kappa_i$ [17] [21].

Looijenga [15] proved that the rational stable cohomology group of the mapping class group Γ_g, the 'closed' one, with coefficients in any irreducible representation of the rational symplectic group was a free module over $H^*(\Gamma_\infty; \mathbb{Q})$, and described its free basis. His computation is involved with geometric considerations on the moduli orbifold of complex algebraic curves including a theorem on Hodge theory [2]. Here it is remarkable that his results are based on the Harer stability theorem with trivial coefficients.

In [11] we independently constructed a bigraded series of cohomology classes of the mapping class group $\Gamma_{g,1}$, the 'bordered' one, with coefficients in the exterior algebra $\bigwedge^* H$. These series are twisted generalizations of the Morita-Mumford classes [17] [21], and are easily modified to those with coefficients in the n-fold tensor product $H^{\otimes n}$, $n \geq 1$.

The present paper is a revised version of the author's preprint [12]. In §§1-3 we will consider only the 'bordered' case, i.e., the case when the surface has boundaries. Our purpose is to prove the stable cohomology group of the mapping class group with coefficients in the n-fold tensor product $H^{\otimes n}$, $n \geq 1$, is a free module over the algebra $H^*(\Gamma_\infty; \mathbb{Z})$ and some combinations of the (modified) twisted Morita-Mumford classes give its free basis (Theorems 1.A and 1.B). This implies the Ivanov stability theorem with coefficients in any finite dimensional rational symplectic coefficients. Following Looijenga [15] we deduce them from the Harer stability theorem with trivial coefficients, but use the Lyndon-Hochschild-Serre spectral sequence for a pair of groups introduced in [11] instead of geometric considerations including Hodge theory. As a corollary the rational cohomology algebra of the semi-direct product

$H \rtimes \Gamma_{g,1}$ (which we call *the extended mapping class group*) is proved to be stabilized and to be freely generated by the generalized Morita-Mumford classes over the rational stable cohomology algebra of the mapping class group in the stable range (Theorem 1.C). After the original version was completed, the twisted Morita-Mumford classes turned out to lift to the mapping class group with punctures [13]. This will be discussed in §4, where we will also discuss briefly what the theorem of Madsen and Weiss [16] brought to our results.

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CONTENTS

§1. Results

Let $g \geq 2$, $r, s \geq 0$ be integers, and $\Sigma^s_{g,r}$ a 2-dimensional oriented connected C^∞ manifold (i.e., oriented surface) of genus g with r ordered boundary components and s ordered punctures. The group of path-components $\pi_0(\mathrm{Diff}^+(\Sigma^s_{g,r}))$ is denoted by $\Gamma^s_{g,r}$ (or $\mathcal{M}^s_{g,r}$) and called the mapping class group of genus g with r ordered boundary components and s ordered punctures. Here $\mathrm{Diff}^+(\Sigma^s_{g,r})$ denotes the topological group (endowed with C^∞ topology) consisting of all orientation preserving diffeomorphisms of $\Sigma^s_{g,r}$ which fix all the boundary points and the punctures pointwise. When $s = 0$, we drop the indices: $\Sigma_{g,r} = \Sigma^0_{g,r}$, $\Gamma_{g,r} = \Gamma^0_{g,r}$ and similarly $\Sigma_g = \Sigma^0_{g,0}$, $\Gamma_g = \Gamma^0_{g,0}$. Throughout this paper we often denote by $H^1(\Sigma^s_{g,r})$ the first integral singular cohomology group of the space $\Sigma^s_{g,r}$. The group $\Gamma^t_{g,q}$ acts on it in an obvious way provided that $q \geq r$ and $t \geq s$. When $s = 0$ and $r = 1$, we often write simply

$$H := H_1(\Sigma_{g,1};\mathbb{Z}) = H_1(\Sigma_g;\mathbb{Z}) = H^1(\Sigma_g;\mathbb{Z}) = H^1(\Sigma_{g,1};\mathbb{Z}). \tag{1.1}$$

The isomorphism $H_1(\Sigma_g;\mathbb{Z}) = H^1(\Sigma_g;\mathbb{Z})$ is the Poincaré duality map, which is invariant under the action of the mapping class group Γ_g.

In view of the Harer stability theorem [3] there exists an integer $N(g)$ depending only on the genus g such that the forgetful map $\Gamma^s_{g,r+1} \to \Gamma^s_{g,r}$ given by forgetting the $(r+1)$-st boundary component induces an isomorphism

$$H^*(\Gamma^s_{g,r+1};\mathbb{Z}) = H^*(\Gamma^s_{g,r};\mathbb{Z})$$

for any $* \leq N(g)$ and $s, r \geq 0$. Harer [3] proved $N(g) \geq g/3$, and later Ivanov [8] improved the inequality; $N(g) \geq g/2$. Now consider a natural central extension

$$0 \to \mathbb{Z} \to \Gamma^s_{g,r+1} \to \Gamma^{s+1}_{g,r} \to 1 \tag{1.2}$$

given by collapsing the $(r+1)$-st boundary component to the $(s+1)$-st puncture. Let $e \in H^2(\Gamma^{s+1}_{g,r};\mathbb{Z})$ be the Euler class of the central extension. Then, substituting Harer's isomorphism into the Gysin sequence of the extension (1.2), we obtain a natural decomposition

$$H^*(\Gamma^{s+1}_{g,r};\mathbb{Z}) = H^*(\Gamma^s_{g,r};\mathbb{Z}) \oplus eH^{*-2}(\Gamma^{s+1}_{g,r};\mathbb{Z}) = H^*(\Gamma^s_{g,r};\mathbb{Z})[e] \tag{1.3}$$

for $* \leq N(g)$ (cf. [17] [4] [15]).

Our first theorem in the present paper is

Theorem 1.A. *If $s \geq 0$, $r \geq 1$ and $n \geq 0$, we have*

$$H^*(\Gamma^s_{g,r}; H^1(\Sigma^s_{g,r};\mathbb{Z})^{\otimes n}) = H^*(\Gamma_{g,1}; H^{\otimes n}) \otimes_{H^*(\Gamma_{g,1};\mathbb{Z})} H^*(\Gamma^s_{g,r};\mathbb{Z})$$

for degrees $\leq N(g) - n$. Here the RHS is a tensor product over the graded algebra $H^*(\Gamma_{g,1};\mathbb{Z})$.

If we denote by $e_{(a)}$ the Euler class corresponding to the a-th puncture, $1 \leq a \leq s$, then $H^*(\Gamma^s_{g,r};\mathbb{Z}) = H^*(\Gamma_{g,1};\mathbb{Z})[e_{(1)},\dots,e_{(s)}]$ for $* \leq N(g)$ from (1.3). Hence Theorem 1.A means

$$H^*(\Gamma^s_{g,r}; H^1(\Sigma^s_{g,r};\mathbb{Z})^{\otimes n}) = H^*(\Gamma_{g,1}; H^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{Z}[e_{(1)},\dots,e_{(s)}]$$

for degrees $\leq N(g) - n$.

As a consequence one deduces the Ivanov stability theorem [7] for the $\Gamma^s_{g,r}$-module $H^1(\Sigma^s_{g,r};\mathbb{Z})^{\otimes n}$ and any finite dimensional rational Sp-modules.

To describe the cohomology group $H^*(\Gamma_{g,1}; H^{\otimes n})$ we need to introduce some notions related to the mapping class groups. Observe the surface $\Sigma^1_{g,1}$ is obtained by glueing the surfaces $\Sigma_{g,1}$ and $\Sigma^1_{0,2}$ along boundary components. The infinite cyclic group $\mathbb{Z}$ acts on the surface

$\Sigma_{0,2}^1$ by rotating the puncture and fixing all the boundary pointwise. So the group $\Gamma_{g,1} \times \mathbb{Z}$ is embedded into the group $\Gamma_{g,1}^1$ (cf. e.g.,[6]). The Lyndon-Hochschild-Serre spectral sequence of the pair of groups $(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z})$ introduced in [11]§1 induces the fiber integral

$$\pi_! : H^q(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M) \to H^{q-2}(\Gamma_{g,1}; M) \tag{1.4}$$

for any $\Gamma_{g,1}$-module M. Here we mean by $H^q(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M)$ the q-th cohomology group of the kernel of the restriction map

$$C^*(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M) := \mathrm{Ker}(C^*(\Gamma_{g,1}^1; M) \to C^*(\Gamma_{g,1} \times \mathbb{Z}; M))$$

of the normalized standard cochain complexes $C^*(\cdot;\cdot)$ [5].

The cohomology class ω defined below plays an important role throughout this paper. Regard the surface $\Sigma_{g,1}^1$ as a subsurface obtained by deleting one interior point from the surface $\Sigma_{g,1}$. The cohomology exact sequence of the pair of spaces $(\Sigma_{g,1}, \Sigma_{g,1}^1)$ gives a $\Gamma_{g,1}^1$-exact sequence

$$0 \to H^1(\Sigma_{g,1}) = H \to H^1(\Sigma_{g,1}^1) \to H^2(\Sigma_{g,1}, \Sigma_{g,1}^1) = \mathbb{Z} \to 0. \tag{1.5}$$

We denote by ω the image of $1 \in \mathbb{Z} = H^0(\Gamma_{g,1}^1; \mathbb{Z})$ under the connecting homomorphism δ^* induced by (1.5):

$$\omega := \delta^*(1) \in H^1(\Gamma_{g,1}^1; H). \tag{1.6}$$

The restriction of ω to the subgroup $\Gamma_{g,1} \times \mathbb{Z}\,(\subset \Gamma_{g,1}^1)$ is null cohomologous. In fact, choose a simple curve l inside the subsurface $\Sigma_{0,2}^1(\subset \Sigma_{g,1}^1)$ connecting the puncture to a point on the boundary of $\Sigma_{g,1}^1$. The 1-cocycle $\omega_l \in Z^1(\Gamma_{g,1}^1; H)$ given by

$$\omega_l(\gamma) = \gamma l - l \in H, \quad \gamma \in \Gamma_{g,1}^1, \tag{1.7}$$

represents the cohomology class $\omega \in H^1(\Gamma_{g,1}^1; H)$. Clearly we have $\omega_l(\gamma) = 0$ for any $\gamma \in \Gamma_{g,1} \times \mathbb{Z}$.

Thus, in view of the cohomology exact sequence

$$0 \to H^1(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; H) \to H^1(\Gamma_{g,1}^1; H) \to H^1(\Gamma_{g,1} \times \mathbb{Z}; H),$$

there exists a unique element of $H^1(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; H)$ mapping to ω. We also denote it by

$$\omega \in H^1(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; H).$$

For a finite subset S of the natural numbers $\mathbb{N}$ we form the power of ω

$$\omega^S \in H^{\sharp S}(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; H^{\otimes S}),$$

which we multiply in numerical order. Let $i \geq 0$ be an integer satisfying the condition

$$i + \sharp S \geq 2 \tag{1.8}$$

Then we define *the twisted*[1] *Morita-Mumford class* $m_{i,S}$ by

$$m_{i,S} := \pi_!(e^i \omega^S) \in H^{2i+\sharp S-2}(\Gamma_{g,1}; H^{\otimes S}), \tag{1.9}$$

where $\pi_!$ is the fiber integral given in (1.4) and $e \in H^2(\Gamma_{g,1};\mathbb{Z})$ is the Euler class of the central extension (1.2) for $r = 1$, $s = 0$.

Definition 1.1. A set $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \ldots, (S_\nu, i_\nu)\}$ is a *weighted partition* [2] *of the index set* $\{1, 2, \ldots, n\}$ if

(1) The set $\{S_1, S_2, \ldots, S_\nu\}$ is a partition of the set $\{1, 2, \ldots, n\}$

$$\{1, 2, \ldots, n\} = \coprod_{a=1}^{\nu} S_a, \quad S_a \neq \emptyset \quad (1 \leq \forall a \leq \nu).$$

(2) $i_1, i_2, \ldots, i_\nu \geq 0$ are non-negative integers.
(3) Each (S_a, i_a) satisfies the condition (1.8): $i_a + \sharp S_a \geq 2$.

We denote by $\mathcal{P}_n$ the set consisting of all weighted partitions of the index set $\{1, 2, \ldots, n\}$. For each weighted partition $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \ldots, (S_\nu, i_\nu)\} \in \mathcal{P}_n$ we define *the twisted Morita-Mumford class*

$$m_{\widehat{P}} := m_{i_1,S_1} m_{i_2,S_2} \cdots m_{i_\nu,S_\nu} \in H^{2(\sum i_a)+n-2\nu}(\Gamma_{g,1}; H^{\otimes n}). \tag{1.10}$$

Theorem 1.B. *For degrees* $\leq N(g) - n$

$$H^*(\Gamma_{g,1}; H^{\otimes n}) = \bigoplus_{\widehat{P} \in \mathcal{P}_n} H^*(\Gamma_{g,1};\mathbb{Z}) m_{\widehat{P}} = \bigoplus_{\widehat{P} \in \mathcal{P}_n} H^{*-\deg m_{\widehat{P}}}(\Gamma_{g,1};\mathbb{Z}).$$

By the *extended mapping class group* we mean the semi-direct product

$$\widetilde{\Gamma^s_{g,r}} := H \rtimes \Gamma^s_{g,r} = H_1(\Sigma_{g,1};\mathbb{Z}) \rtimes \Gamma^s_{g,r}.$$

This group was studied in [1]. The generalized Morita-Mumford classes $\widetilde{m_{i,j}} \in H^*(\widetilde{\Gamma_{g,1}};\mathbb{Z})$ are constructed as follows [11]. In a similar way to $\Gamma_{g,1} \times \mathbb{Z} \subset \Gamma^1_{g,1}$ the group $\widetilde{\Gamma_{g,1}} \times \mathbb{Z}$ is embedded into the group $\widetilde{\Gamma^1_{g,1}}$. Using

[1]In [11] and [12] the author called it the *generalized* Morita-Mumford class.
[2]We use the term 'partition of a set' following Stanley [22] p.33.

the simple curve l in (1.7), we define a 2-cocycle $\widetilde{\omega_l} \in Z^2(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z})$ by

$$\text{(1.11)} \quad \widetilde{\omega_l}(u_1\gamma_1, u_2\gamma_2) := \gamma_1(\gamma_2 l - l) \cdot u_1, \quad u_1, u_2 \in H,\ \gamma_1, \gamma_2 \in \Gamma^1_{g,1},$$

where $\cdot$ denotes the intersection product on $H = H_1(\Sigma_g; \mathbb{Z})$. Its image $\widetilde{\omega}$ in $H^2(\widetilde{\Gamma^1_{g,1}}; \mathbb{Z})$ is equal to the Euler class of the central extension

$$0 \to \mathbb{Z} \to H_1(\Sigma^1_{g,1}; \mathbb{Z}) \rtimes \Gamma^1_{g,1} \to H_1(\Sigma_{g,1}; \mathbb{Z}) \rtimes \Gamma^1_{g,1} = \widetilde{\Gamma^1_{g,1}} \to 1.$$

The forgetful map $\widetilde{\pi} : \widetilde{\Gamma^1_{g,1}} \to \widetilde{\Gamma_{g,1}}$ induces the fiber integral

$$\text{(1.12)} \qquad \widetilde{\pi}_! : H^q(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z}) \to H^{q-2}(\widetilde{\Gamma_{g,1}}; \mathbb{Z}).$$

Thus we can define *the generalized Morita-Mumford class*

$$\widetilde{m_{i,j}} := \widetilde{\pi}_!(e^i \widetilde{\omega}^j) \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}}; \mathbb{Z})$$

for $i \geq 0$, $j \geq 0$ with $i + j \geq 2$. Clearly $\widetilde{m_{i+1,0}}$ is equal to (the image of) the i-th Morita-Mumford class e_i $(= (-1)^{i+1}\kappa_i) \in H^{2i}(\Gamma_g; \mathbb{Z})$ [17] [21]:

$$\widetilde{m_{i+1,0}} = e_i \in H^{2i}(\widetilde{\Gamma_{g,1}}; \mathbb{Z}).$$

Theorem 1.C.

$$H^*(\widetilde{\Gamma_{g,1}}; \mathbb{Q}) = H^*(\Gamma_{g,1} \ltimes H_1(\Sigma_{g,1}; \mathbb{Z}); \mathbb{Q}) = H^*(\Gamma_{g,1}; \mathbb{Q}) \otimes \mathbb{Q}[\widetilde{m_{i,j}}]$$

for degrees $\leq N(g)$, where integers i and j run over the domain

$$\{(i,j) \in \mathbb{Z} \times \mathbb{Z}; \quad i \geq 0,\ j \geq 1 \text{ and } i + j \geq 2\}.$$

§2. Stable Cohomology with Coefficients in $H^1(\Sigma_{g,1}; \mathbb{Z})^{\otimes n}$

This section is devoted to the proof of Theorems 1.A and 1.B.

Suppose $r, s \geq 1$. We have a natural commutative diagram of forgetful maps

$$\text{(2.1)} \qquad \begin{array}{ccc} \Gamma^s_{g,r} & \xrightarrow{\ \pi\ } & \Gamma^{s-1}_{g,r} \\ {\scriptstyle \varpi}\big\downarrow & & {\scriptstyle \varpi}\big\downarrow \\ \Gamma^1_{g,1} & \xrightarrow{\ \pi\ } & \Gamma_{g,1}. \end{array}$$

Here the upper and the lower π's are given by forgetting the s-th and the first punctures, respectively, and the left and the right ϖ's by forgetting the punctures from the first to the $(s-1)$-st and the boundary components except the first.

We regard the surface $\Sigma^s_{g,r}$ as a subsurface obtained by deleting one interior point from the surface $\Sigma^{s-1}_{g,r}$ and numbering the resulting puncture the s-th. The inclusion homomorphism $H^1(\Sigma^{s-1}_{g,r}) \to H^1(\Sigma^s_{g,r})$ is equivariant under the forgetful map $\pi : \Gamma^s_{g,r} \to \Gamma^{s-1}_{g,r}$, and so induces a $\Gamma^s_{g,r}$-exact sequence

$$0 \to H^1(\Sigma^{s-1}_{g,r}) \to H^1(\Sigma^s_{g,r}) \to H^2(\Sigma^{s-1}_{g,r}, \Sigma^s_{g,r}) = \mathbb{Z} \to 0. \tag{2.2}$$

We denote by $\omega = \omega_{(s-1)}$ the image of $1 \in \mathbb{Z} = H^0(\Gamma^s_{g,r}; \mathbb{Z})$ under the connecting homomorphism δ^* induced by (2.2):

$$\omega = \omega_{(s-1)} := \delta^*(1) \in H^1(\Gamma^s_{g,r}; H^1(\Sigma^{s-1}_{g,r})). \tag{2.3}$$

From the commutative diagram (2.1), the homomorphism induced by the forgetful map ϖ

$$H^1(\Gamma^1_{g,1}; H) \to H^1(\Gamma^s_{g,r}; H) \to H^1(\Gamma^s_{g,r}; H^1(\Sigma^{s-1}_{g,r}))$$

maps ω defined in (1.6) to $\omega = \omega_{(s-1)}$ defined in (2.3).

The kernel of the forgetful map $\pi : \Gamma^{s+1}_{g,r} \to \Gamma^s_{g,r}$ forgetting the $(s+1)$-st puncture is naturally isomorphic to $\pi_1(\Sigma^s_{g,r})$ $(s \geq 0)$, and so we have a Gysin exact sequence

$$\begin{aligned} &\cdots \to H^q(\Gamma^s_{g,r}; M) \xrightarrow{\pi^*} H^q(\Gamma^{s+1}_{g,r}; M) \\ &\xrightarrow{\pi_\sharp} H^{q-1}(\Gamma^s_{g,r}; H^1(\Sigma^s_{g,r}) \otimes M) \to H^{q+1}(\Gamma^s_{g,r}; M) \to \cdots \end{aligned} \tag{2.4}$$

for any $\Gamma^s_{g,r}$-module M. Here we denote the Gysin map (the fiber integral) by $\pi_\sharp$ to distinguish it from the fiber integral $\pi_!$ introduced in (1.4).

Theorem 2.1. *Let I and $J \subset \mathbb{N}$ be mutually disjoint finite index sets. Suppose $s \geq 0$ and $r \geq 1$. Moreover if $I \neq \emptyset$, assume $s \geq 1$. Then the forgetful map ϖ induces an isomorphism*

$$\begin{aligned} &H^*(\Gamma^s_{g,r}; H^1(\Sigma^{s-1}_{g,r})^{\otimes I} \otimes H^1(\Sigma^s_{g,r})^{\otimes J}) \\ &= (\bigoplus_{S \subset I} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes (J \cup I - S)})) \otimes_{H^*(\Gamma_{g,1};\mathbb{Z})} H^*(\Gamma^s_{g,r}; \mathbb{Z}) \end{aligned} \tag{2.5}$$

for degrees $\leq N(g) - \sharp(I \cup J)$, where $\omega^S \in H^{\sharp S}(\Gamma^1_{g,1}; H^{\otimes S})$ is the power of the class ω defined in (1.6). *The RHS is a tensor product over the graded algebra $H^*(\Gamma_{g,1}; \mathbb{Z})$.*

In particular, if $I = \emptyset$ and $J = \{1, 2, \ldots, n\}$, we obtain Theorem 1.A stated in §1.

Proof. We write simply $H_{(s)} := H^1(\Sigma^s_{g,r}) = H^1(\Sigma^s_{g,r};\mathbb{Z})$. The $\Gamma^s_{g,r}$-exact sequence (2.2) is rewritten to

$$0 \to H_{(s-1)} \to H_{(s)} \to \mathbb{Z} \to 0. \tag{2.6}$$

We prove the theorem by double induction on $\sharp(I \cup J)$ and $\sharp I$. When $I \cup J = \emptyset$, the theorem is trivial. Suppose $\sharp(I \cup J) \geq 1$.

(A). The case $I = \emptyset$: Let j_0 be the minimum of J and set $J_- := J - \{j_0\}$. From the inductive assumption applied to J_- the forgetful homomorphism

$$\pi^* : H^*(\Gamma^s_{g,r}; H_{(s)}{}^{\otimes J_-}) \to H^*(\Gamma^{s+1}_{g,r}; H_{(s)}{}^{\otimes J_-})$$

can be identified with $\pi^* : H^*(\Gamma^s_{g,r};\mathbb{Z}) \to H^*(\Gamma^{s+1}_{g,r};\mathbb{Z})$ tensored by $H^*(\Gamma_{g,1}; H^{\otimes J_-}))\otimes_{H^*(\Gamma_{g,1};\mathbb{Z})}$, and so has a left inverse over $H^*(\Gamma^s_{g,r})$. Hence the Gysin sequence (2.4) splits into the $H^*(\Gamma^s_{g,r})$-split exact sequences

$$\begin{aligned} &0 \to H^*(\Gamma^s_{g,r}; H_{(s)}{}^{\otimes J_-}) \to \\ &H^*(\Gamma^{s+1}_{g,r}; H_{(s)}{}^{\otimes J_-}) \xrightarrow{\pi_\sharp} H^{*-1}(\Gamma^s_{g,r}; H_{(s)} \otimes H_{(s)}{}^{\otimes J_-}) \to 0 \end{aligned} \tag{2.7}$$

for $* \leq N(g) - \sharp J_-$. When $s = 0$ and $r = 1$, we have

$$\begin{aligned} &0 \to H^*(\Gamma_{g,1}; H^{\otimes J_-}) \to \\ &H^*(\Gamma^1_{g,1}; H^{\otimes J_-}) \xrightarrow{\pi_\sharp} H^{*-1}(\Gamma_{g,1}; H \otimes H^{\otimes J_-}) \to 0 \end{aligned} \tag{2.8}$$

for $* \leq N(g) - \sharp J_-$. Compare the exact sequence (2.7) with the sequence (2.8) tensored by $\otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r})$. Then the forgetful map ϖ induces an isomorphism

$$\varpi^* : H^*(\Gamma^s_{g,r}; H_{(s)} \otimes H_{(s)}{}^{\otimes J_-}) \xrightarrow{\cong} H^*(\Gamma_{g,1}; H \otimes H^{\otimes J_-}) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r})$$

for $* \leq N(g) - \sharp J$. Here we use the fact the map ϖ induces an isomorphism

$$\varpi^* : H^*(\Gamma^1_{g,1}) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}) \cong H^*(\Gamma^{s+1}_{g,r})$$

deduced from (1.3) and (2.1). Finally label the first H and $H_{(s)}$ the index j_0. Thus the induction proceeds.

(B). The case $I \neq \emptyset$: Then we suppose $s \geq 1$. Choose an index $i_0 \in I$. Set $I_- := I - \{i_0\}$ and $J_0 := J \cup \{i_0\}$. The sequence (2.6) induces a $\Gamma^s_{g,r}$-exact sequence

$$\begin{aligned} &0 \to H_{(s-1)}{}^{\otimes I} \otimes H_{(s)}{}^{\otimes J} \to \\ &H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0} \to H_{(s-1)}{}^{\otimes I_-} \otimes \mathbb{Z}^{\otimes\{i_0\}} \otimes H_{(s)}{}^{\otimes J} \to 0. \end{aligned} \tag{2.9}$$

By the inductive assumption applied to the index sets I_- and J_0

$$\begin{aligned} & H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0}) \\ = \; & (\bigoplus_{S \subset I_-} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes(J_0 \cup I_- - S)})) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}). \end{aligned}$$

Each element of the RHS lifts to a (uniquely determined) element of $H^*(\Gamma^s_{g,r}; H^{\otimes(I_- \cup J_0)})$. Hence the map

$$H^*(\Gamma^s_{g,r}; H^{\otimes(I_- \cup J_0)}) \to H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0})$$

has a right inverse for $* \leq N(g) - \sharp(I \cup J)$. Therefore the cohomology exact sequence induced by the sequence (2.9) splits into a split exact sequence

$$\begin{aligned} 0 \to H^{*-1}(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J}) \xrightarrow{\omega^{\{i_0\}}\otimes} \\ H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I} \otimes H_{(s)}{}^{\otimes J}) \to H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0}) \to 0 \end{aligned}$$

for $* \leq N(g) - \sharp(I \cup J)$. Thus we have

$$\begin{aligned} & H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I} \otimes H_{(s)}{}^{\otimes J}) \\ = \; & H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0}) \\ & \oplus(\omega^{\{i_0\}} \otimes H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J})) \\ = \; & (\bigoplus_{S \subset I_-} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes(J_0 \cup I_- - S)})) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}) \\ & \oplus(\bigoplus_{S \subset I_-} \omega^{S \cup \{i_0\}} \otimes H^*(\Gamma_{g,1}; H^{\otimes(J \cup I_- - S)})) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}) \\ = \; & (\bigoplus_{S \subset I} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes(J \cup I - S)})) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}) \end{aligned}$$

for $* \leq N(g) - \sharp(I \cup J)$. This completes the induction. Q.E.D.

We have introduced two sorts of fiber integrals or Gysin maps induced by the forgetful map $\pi : \Gamma^1_{g,1} \to \Gamma_{g,1}$ in (1.4) and (2.4). These two Gysin maps are related to each other in the following way.

Lemma 2.2. *For any $\Gamma_{g,1}$-module M we have*

$$\begin{array}{ccc} H^p(\Gamma^1_{g,1}; M) & = & H^p(\Gamma^1_{g,1}; M) \\ {\scriptstyle \omega\cup}\downarrow & & \downarrow{\scriptstyle \pi_\sharp} \\ H^{p+1}(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H \otimes M) & \xrightarrow{\pi_!} & H^{p-1}(\Gamma_{g,1}; H \otimes M). \end{array}$$

Here we write simply $H = H^1(\Sigma_{g,1}; \mathbb{Z})$ *as in* (1.1).

Proof. Let G be a group, K a subgroup of G and M a G-module. We define the cohomology group $H^*(G, K; M)$ by that of the kernel of the restriction map

$$C^*(G, K; M) := \mathrm{Ker}(C^*(G; M) \to C^*(K; M))$$

of the normalized standard cochain complexes $C^*(\cdot; \cdot)$ [11]§1. Consider a normal subgroup N of G satisfying the condition: $KN = G$. In [5] p.118, l.27ff and p.119, l.6ff two mutually equivalent filtrations (A_j) and (A_j^*) are introduced on the normalized standard cochain complex, and induce the (ordinary) Lyndon-Hochschild-Serre spectral sequence. The filtration (A_j^*) (or equivalently (A_j)) restricted to $C^*(G, K; M)$ induces the Lyndon-Hochschild-Serre spectral sequence of pairs of groups [11]:

$$E_2^{p,q} = H^p(G/N; H^q(N, N \cap K; M)) \Rightarrow H^{p+q}(G, K; H).$$

In our situation we consider the case $G = \Gamma^1_{g,1}$, $K = \Gamma_{g,1} \times \mathbb{Z}$ and $N = \pi_1(\Sigma_{g,1}) \subset \Gamma^1_{g,1}$. Since $H^{p-i}(\Gamma_{g,1}; H^i(\pi_1(\Sigma_{g,1}); M)) = 0$ for $i \geq 2$, any $u \in H^p(\Gamma^1_{g,1}; M)$ is represented by a cocycle z whose value $z(\gamma_1, \gamma_2, \ldots, \gamma_p)$, $\gamma_1, \gamma_2, \ldots, \gamma_p \in \Gamma^1_{g,1}$, depends only on γ_1 and the cosets $\gamma_2\pi_1(\Sigma_{g,1}), \ldots, \gamma_p\pi_1(\Sigma_{g,1})$. We denote by $r_{p-1}z$ the cocycle given by restricting γ_1 into $\pi_1(\Sigma_{g,1})$ and regarding $\gamma_2, \ldots, \gamma_p$ as elements of $\Gamma_{g,1} = \Gamma^1_{g,1}/\pi_1(\Sigma_{g,1})$. By definition we have $\pi_\sharp u = [r_{p-1}z] \in H^{p-1}(\Gamma_{g,1}; H \otimes M)$.

On the other hand the cocycle $\omega \cup z$ defined by

$$(\omega \cup z)(\gamma_0, \gamma_1, \ldots, \gamma_p) = \omega(\gamma_0) \otimes \gamma_0(z(\gamma_1, \gamma_2, \ldots, \gamma_p)),$$

for $\gamma_0, \gamma_1, \ldots, \gamma_p \in \Gamma^1_{g,1}$, represents the cup product $\omega \cup u \in H^{p+1}(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H \otimes M)$. The value $(\omega \cup z)(\gamma_0, \gamma_1, \ldots, \gamma_p)$ depends only on γ_0, γ_1 and the cosets $\gamma_2\pi_1(\Sigma_{g,1}), \ldots, \gamma_p\pi_1(\Sigma_{g,1})$.

Hence, from a computation involved with [11] Lemma 2.3,

$$\pi_!(\omega \cup z)(\gamma_2, \ldots, \gamma_p) = -\sum_{i=1}^{g} a_i \otimes z(b_i, \gamma_2 \ldots, \gamma_p) + b_i \otimes z({a_i}^{-1}, \gamma_2 \ldots, \gamma_p),$$

where $\{a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g\}$ is a usual symplectic generating system of the fundamental group $\pi_1(\Sigma_{g,1})$. This implies $\pi_!(\omega \cup z) = r_{p-1}z$ and so $\pi_!(\omega \cup u) = [\pi_!(\omega \cup z)] = [r_{p-1}z] = \pi_\sharp u$, as was to be shown. Q.E.D.

Proof of Theorem 1.B. We prove the theorem by induction on n. When $n = 0$, the theorem is trivial, so we assume $n \geq 1$. Set $J = \{1, 2, \ldots, n\}$, $j_0 = 1$ and $J_- = \{2, \ldots, n\}$. Recall the exact sequence (2.8) in the proof of Theorem 2.1. From Theorem 2.1 and (1.3) the Gysin map $\pi_\sharp$ restricted to

$$(H^*(\Gamma_{g,1}; H^{\otimes J_-}) \otimes e\mathbb{Z}[e]) \oplus \bigoplus_{\emptyset \neq S \subset J_-} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes (J_- - S)}) \otimes \mathbb{Z}[e]$$

is an isomorphism onto $H^{*-1}(\Gamma_{g,1}; H^{\otimes J})$ for $* \leq N(g) - n + 1$. We denote $S_+ := S \cup \{1\}$ for $S \subset J_-$. Lemma 2.2 implies

$$\pi_\sharp(e^i \omega^S) = \pi_!(e^i \omega^{S_+}) = m_{i,S_+} \in H^*(\Gamma_{g,1}; H^{\otimes S_+}).$$

Therefore from the inductive assumption we obtain

$$\begin{aligned}
(2.10) \qquad & H^*(\Gamma_{g,1}; H^{\otimes n}) = H^*(\Gamma_{g,1}; H^{\otimes J}) \\
= & \bigoplus_{\emptyset \neq S \subset J_-} \bigoplus_{i=0}^{\infty} m_{i,S_+} H^*(\Gamma_{g,1}; H^{\otimes (J_- - S)}) \\
& \qquad \oplus \bigoplus_{i=1}^{\infty} m_{i,\{1\}} H^*(\Gamma_{g,1}; H^{\otimes J_-}) \\
= & \bigoplus_{\widehat{P} \in \mathcal{P}_n} m_{\widehat{P}} H^*(\Gamma_{g,1}; \mathbb{Z})
\end{aligned}$$

for $* \leq N(g) - n$, which completes the induction. Q.E.D.

§3. Stable Cohomology Algebra of Extended Mapping Class Groups

Let i and j be integers with $i \geq 0$, $j \geq 1$ and $i + j \geq 2$. As in [11], we define

$$m_{i,j} := \pi_!(e^i \omega^j) \in H^{2i+j-2}(\Gamma_{g,1}; \textstyle\bigwedge^j H),$$

where $\pi_!$ is the fiber integral (1.4), and $\omega^j \in H^j(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; \bigwedge^j H)$ is the j-th power of ω.

The n-th symmetric group $\mathfrak{S}_n$ acts on the set $\mathcal{P}_n$ of the weighted partitions of $\{1, 2, \ldots, n\}$ by

$$\sigma \widehat{P} := \{(\sigma(S_1), i_1), (\sigma(S_2), i_2), \ldots, (\sigma(S_\nu), i_\nu)\},$$

where $\sigma \in \mathfrak{S}_n$ and $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \ldots, (S_\nu, i_\nu)\} \in \mathcal{P}_n$. The $\mathfrak{S}_n$-orbits in $\mathcal{P}_n$ are parametrized by the set $\mathcal{Q}_n$ defined as follows.

Definition 3.1. A sequence $\widehat{Q} = ((j_1, i_1), (j_2, i_2), \ldots, (j_\nu, i_\nu))$ is a *weighted partition of the number* n if

(1) The sequence $(j_1, j_2, \ldots, j_\nu)$ is a partition of the number n

$$j_1 + j_2 + \cdots + j_\nu = n, \quad j_1 \geq j_2 \geq \cdots \geq j_\nu \geq 1.$$

(2) $i_1, i_2, \ldots, i_\nu \geq 0$ are non-negative integers.
(3) $i_a \geq i_{a+1}$ if $j_a = j_{a+1}$.
(4) Each (j_a, i_a) satisfies the condition: $i_a + j_a \geq 2$.

We denote by $\mathcal{Q}_n$ the set consisting of all weighted partitions of the number n. Define

$$\lambda\widehat{P} := ((\sharp S_1, i_1), (\sharp S_2, i_2), \ldots, (\sharp S_\nu, i_\nu)) \in \mathcal{Q}_n$$

for $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \ldots, (S_\nu, i_\nu)\} \in \mathcal{P}_n$ provided that $\sharp S_1 \geq \sharp S_2 \geq \cdots \geq \sharp S_\nu$ and $\sharp S_a = \sharp S_{a+1} \Rightarrow i_a \geq i_{a+1}$, $(1 \leq a < \nu)$. Then the map $\lambda : \mathcal{P}_n \to \mathcal{Q}_n$, $\widehat{P} \mapsto \lambda\widehat{P}$ induces a bijection $\lambda : \mathcal{P}_n/\mathfrak{S}_n = \mathcal{Q}_n$. Define

$$m_{\widehat{Q}} := m_{i_1,j_1} m_{i_2,j_2} \cdots m_{i_\nu,j_\nu} \in H^{\sum(2i_a+j_a-1)}(\Gamma_{g,1}; \bigwedge^n H)$$

for $\widehat{Q} = ((j_1, i_1), (j_2, i_2), \ldots, (j_\nu, i_\nu)) \in \mathcal{Q}_n$. The canonical projection $\lambda : H^{\otimes n} \to \bigwedge^n H$ maps $m_{\widehat{P}}$ to $\pm m_{\lambda\widehat{P}}$ for any $\widehat{P} \in \mathcal{P}_n$

$$\lambda_*(m_{\widehat{P}}) = \pm m_{\lambda\widehat{P}} \in H^*(\Gamma_{g,1}; \bigwedge^n H). \tag{3.1}$$

Theorem 3.2. *Let* $\mathbf{k}$ *be a field with* $ch\,\mathbf{k} > n$ *or* $= 0$. *Then we have*

$$\begin{aligned} H^*(\Gamma_{g,1}; \bigwedge^n H^1(\Sigma_{g,1}; \mathbf{k})) &= \bigoplus_{\widehat{Q} \in \mathcal{Q}_n} H^*(\Gamma_{g,1}; \mathbf{k}) m_{\widehat{Q}} \\ &= \bigoplus_{\widehat{Q} \in \mathcal{Q}_n} H^{*-\deg m_{\widehat{Q}}}(\Gamma_{g,1}; \mathbf{k}) \end{aligned}$$

for degrees $\leq N(g) - n$.

As a corollary we obtain Theorem 1.C. In fact, let $h : H \rtimes \Gamma_{g,1} = \widetilde{\Gamma_{g,1}} \to H$ denote the twisted 1-cocycle $u\gamma \mapsto u$, $u \in H$, $\gamma \in \Gamma_{g,1}$. Then we have

$$\widetilde{\omega_l} = -\mu(h \cup \omega_l) \in Z^2(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}),$$

where $\mu : H^{\otimes 2} \to \mathbb{Z}$ is the intersection pairing. If we define $M_j : H^{\otimes 2j} \to \mathbb{Z}$ by $M_j(u_1 \otimes \cdots \otimes u_{2j}) := \prod_{k=1}^{j} \mu(u_k \otimes u_{j+k})$, then

$$e^i \widetilde{\omega_l}^j = \pm M_j(h^j e^i \omega_l^j) \in H^{2i+2j}(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}).$$

Since h^j comes from the group $\widetilde{\Gamma_{g,1}}$, we obtain

$$\widetilde{m_{i,j}} = \pm M_j(h^j m_{i,j}) \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}}). \tag{3.2}$$

The Lyndon-Hochschild-Serre spectral sequence of the group extension $H \to \widetilde{\Gamma_{g,1}} = H \rtimes \Gamma_{g,1} \to \Gamma_{g,1}$ is given by

$$E_2^{p,q} = H^p(\Gamma_{g,1}; \bigwedge^q H) \Rightarrow H^{p+q}(\widetilde{\Gamma_{g,1}}).$$

From (3.2) the class $m_{i,j} \in E_2^{2i+j-2,j}$ lifts to the class $\pm\widetilde{m_{i,j}} \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}})$. This proves Theorem 1.C.

Proof of Theorem 3.2. We define the order in each $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \dots, (S_\nu, i_\nu)\} \in \mathcal{P}_n$ by

(1) $\sharp S_1 \geq \sharp S_2 \geq \cdots \geq \sharp S_\nu$.
(2) If $\sharp S_a = \sharp S_{a+1}$, $i_a \geq i_{a+1}$.
(3) If $\sharp S_a = \sharp S_{a+1}$ and $i_a = i_{a+1}$, then the minimum of S_a is smaller than that of S_{a+1}.

Furthermore we denote

$$\tau_{\widehat{P}} = \begin{pmatrix} 1\ 2 & \cdots & n \\ S_1 & \cdots & S_\nu \end{pmatrix} \in \mathfrak{S}_n,$$

where the indices are set in numerical order inside each subset S_a. In other words, if $\psi_a : \{1, 2, \dots, \sharp S_a\} \to S_a$ is the unique order-preserving bijection, then we have $\tau_{\widehat{P}}(i) = \psi_a(i - \sum_{b=1}^{a-1} \sharp S_b)$ for $\sum_{b=1}^{a-1} \sharp S_b < i \leq \sum_{b=1}^{a} \sharp S_b$.

Let the n-th symmetric group $\mathfrak{S}_n$ act on $H^{\otimes n}$ by

$$\sigma(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = (\operatorname{sign} \sigma) u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)}$$

for $\sigma \in \mathfrak{S}_n$, $u_i \in H$ $(1 \leq i \leq n)$. Then we have

$$\tau_* m_{\widehat{P}} = m_{\widehat{P}} \in H^*(\Gamma_{g,1}; H^{\otimes n}). \tag{3.3}$$

for $\tau \in \mathfrak{S}_n$ and $\widehat{P} \in \mathcal{P}_n$ with $\tau(\widehat{P}) = \widehat{P}$. In fact, from the (anti-)commutativity of cup products we have $\tau_* \omega^S = \omega^S$ if $S \subset \{1, 2, \dots, n\}$, $\tau \in \mathfrak{S}_n$ and $\tau(S) = S$. Hence $\tau_* m_{i,S} = m_{i,S}$. Furthermore, since $\deg m_{i,S} \equiv \sharp S \mod 2$, we have

$$\begin{pmatrix} S & T \\ T & S \end{pmatrix}_* m_{i,S} m_{i,T} = m_{i,S} m_{i,T}$$

for $S, T \subset \{1, 2, \dots, n\}$ with $\sharp S = \sharp T$ and $S \cap T = \emptyset$. Here if $\varphi : S \to T$ is the unique order-preserving bijection, then $\begin{pmatrix} S & T \\ T & S \end{pmatrix}$ maps $s \in S$ to $\varphi(s)$, and $t \in T$ to $\varphi^{-1}(t)$.

Now, for any $\sigma \in \mathfrak{S}_n$, the permutation $\sigma^{-1}\tau_{\sigma(\widehat{P})}\tau_{\widehat{P}}{}^{-1}$ fixes $\widehat{P}$. From (3.3)

$$\sigma_* m_{\widehat{P}} = \tau_{\sigma(\widehat{P})}{}_* \tau_{\widehat{P}}{}^{-1}{}_* m_{\widehat{P}} = (\operatorname{sign} \tau_{\sigma(\widehat{P})}\tau_{\widehat{P}}{}^{-1}) m_{\sigma(\widehat{P})}.$$

Therefore for any $\widehat{P_0} \in \mathcal{P}_n$ the sum $\sum_{\widehat{P} \in \lambda^{-1}\lambda(\widehat{P_0})} (\operatorname{sign} \tau_{\widehat{P}}) m_{\widehat{P}}$ is invariant under the $\mathfrak{S}_n$-action. This means $m_{\lambda \widehat{P_0}} \neq 0$ in $H^*(\Gamma_{g,1}; \bigwedge^n H \otimes \mathbf{k})$.

The group $H^*(\Gamma_{g,1}; H^{\otimes n})$ is decomposed into a direct sum of $\mathfrak{S}_n$-submodules parametrized by $\mathcal{Q}_n = \mathcal{P}_n/\mathfrak{S}_n$, which implies $m_{\widehat{Q}}$'s, $\widehat{Q} \in \mathcal{Q}_n$, are linearly independent over $H^*(\Gamma_{g,1}; \mathbf{k})$.

From the assumption on the characteristic of the field $\mathbf{k}$ the map $\lambda_* : H^*(\Gamma_{g,1}; (H \otimes \mathbf{k})^{\otimes n}) \to H^*(\Gamma_{g,1}; \bigwedge^n H \otimes \mathbf{k})$ is surjective. By Theorem 1.B the $H^*(\Gamma_{g,1}; \mathbf{k})$-module $H^*(\Gamma_{g,1}; (H \otimes \mathbf{k})^{\otimes n})$ is generated by $m_{\widehat{P}}$, $\widehat{P} \in \mathcal{P}_n$ for $* \leq N(g) - n$. Hence, from (3.1), the $H^*(\Gamma_{g,1}; \mathbf{k})$-module $H^*(\Gamma_{g,1}; \bigwedge^n H \otimes \mathbf{k})$ is generated by $m_{\widehat{Q}}$, $\widehat{Q} \in \mathcal{Q}_n$. Q.E.D.

§4. Concluding Remarks

After the original version was completed in 1995, the twisted Morita-Mumford classes turned out to lift to the mapping class group $\mathcal{M}_{g,*} := \Gamma^1_{g,0}$ [13] [14]. The cohomology class $\omega \in H^1(\Gamma^1_{g,1}; H)$ in the previous sections has already appeared in Morita's work [19]. To explain it following [14], we introduce the fiber product $\overline{\mathcal{M}_{g,*}} := \mathcal{M}_{g,*} \times_{\Gamma_g} \mathcal{M}_{g,*}$ of the group $\mathcal{M}_{g,*}$ with respect to the forgetful map $\pi : \mathcal{M}_{g,*} = \Gamma^1_{g,0} \to \Gamma_g$. Since the kernel of π is naturally isomorphic to the fundamental group $\pi_1\Sigma_g$, we have an isomorphism

$$\overline{\mathcal{M}_{g,*}} \cong \pi_1\Sigma_g \rtimes \mathcal{M}_{g,*}, \quad (\varphi, \psi) \mapsto (\psi\varphi^{-1}, \varphi).$$

In [19] Morita introduced a twisted 1-cocycle

$$k_0 : \overline{\mathcal{M}_{g,*}} \to H, \quad (\varphi, \psi) \mapsto [\psi\varphi^{-1}]$$

taking the homology class of $\psi\varphi^{-1} \in \pi_1\Sigma_g$. As in §1 we choose a simple curve ℓ on $\Sigma^1_{g,1}$ connecting the puncture p_1 to a point p_0 on the boundary. Take a diffeomorphism $\psi_\ell : (\Sigma_g, p_1) \to (\Sigma_g, p_0)$ sliding the point p_1 along the curve ℓ. Define a homomorphism $\varpi' : \Gamma^1_{g,1} \to \mathcal{M}_{g,*}$ by forgetting the puncture p_1 and collapsing the boundary to a new puncture p_0, and $\varpi'' : \Gamma^1_{g,1} \to \mathcal{M}_{g,*}$ by forgetting the boundary. A homomorphism

$\alpha_\ell : \Gamma^1_{g,1} \to \mathcal{M}_{g,*}$ is defined by $\varphi \mapsto (\varpi'(\varphi), \psi_\ell \varpi''(\varphi){\psi_\ell}^{-1})$. Then we obtain

$$(4.1) \qquad {\alpha_\ell}^*(k_0) = -\omega_\ell \in Z^1(\Gamma^1_{g,1}; H)$$

by a straight forward computation ([14] Lemma 4.1).

Now let e be the Euler class of the forgetful central extension $\mathbb{Z} \to \Gamma_{g,1} \to \mathcal{M}_{g,*}$, and $\overline{e}$ the pull-back of e by the second projection $\overline{\pi} : \overline{\mathcal{M}_{g,*}} \to \mathcal{M}_{g,*}$, $(\varphi, \psi) \mapsto \psi$. Then the twisted Morita-Mumford class

$$(4.2) \qquad m_{i,S} := \pi_!(\overline{e}^i(-k_0)^{\otimes S}) \in H^{2i+\sharp S-2}(\mathcal{M}_{g,*}; H^{\otimes S})$$

is defined for any i and $S \subset \mathbb{N}$ with $i + \sharp S \geq 2$. Here $\pi_!$ is the Gysin map of the first projection $\pi : \overline{\mathcal{M}_{g,*}} \to \mathcal{M}_{g,*}$, $(\varphi, \psi) \mapsto \varphi$. As was proved in [13] [14] by (4.1), this is exactly a lift of $m_{i,S} \in H^*(\Gamma_{g,1}; H^{\otimes S})$ defined in §1. We define $m_{\widehat{P}} \in H^*(\mathcal{M}_{g,*}; H^{\otimes n})$ for $\widehat{P} \in \mathcal{P}_n$ in a similar way to §1. From Theorem 1.B we have

Theorem 4.1. *For degrees* $\leq N(g) - n$

$$H^*(\mathcal{M}_{g,*}; H^{\otimes n}) = \bigoplus_{\widehat{P} \in \mathcal{P}_n} H^*(\mathcal{M}_{g,*}; \mathbb{Z}) m_{\widehat{P}}.$$

Consider the semi-direct products $\widetilde{\mathcal{M}_{g,*}} := H \rtimes \mathcal{M}_{g,*}$ and $H \rtimes \overline{\mathcal{M}_{g,*}}$. Making use of k_0 instead of ω, we can define a 2-cocycle $\tilde{k}_0 \in Z^2(H \rtimes \overline{\mathcal{M}_{g,*}}; \mathbb{Z})$ and the generalized Morita-Mumford class $\widetilde{m_{i,j}}$ for $i, j \geq 0$ with $i + j \geq 2$ in a similar way to (1.11) and (1.12). Then from Theorem 1.C we obtain

Theorem 4.2. *For degrees* $\leq N(g)$

$$H^*(\widetilde{\mathcal{M}_{g,*}}; \mathbb{Q}) = H^*(\mathcal{M}_{g,*}; \mathbb{Q}) \otimes \mathbb{Q}[\widetilde{m_{i,j}};\ i \geq 0,\ j \geq 1 \textit{ and } i + j \geq 2].$$

In 2002 Madsen and Weiss [16] gave a loop-space description of $H^*(\Gamma_\infty; \mathbb{Z})$ and proved the algebra $H^*(\Gamma_\infty; \mathbb{Q})$ is generated by the Morita-Mumford classes. This result improves ours in the present paper in a surprising way. Theorem 1.C and Theorem 1.B are improved to the following.

Theorem 4.3. *For degrees* $\leq N(g)$

$$H^*(\widetilde{\Gamma_{g,1}}; \mathbb{Q}) = \mathbb{Q}[\widetilde{m_{i,j}};\ i, j \geq 0,\ \textit{and } i + j \geq 2].$$

Theorem 4.4. *For degrees* $\leq N(g) - n$

$$H^*(\Gamma_{g,1}; H^{\otimes n}) = \bigoplus_{\widehat{P} \in \mathcal{P}_n} \mathbb{Q}[e_j;\ j \geq 1] m_{\widehat{P}}.$$

Theorems 4.1 and 4.2 also have appropriate improvements.

Finally we give a brief remark on a relation to the Johnson homomorphism [9]. Let $\mathcal{I}_{g,1}$ be the Torelli group, i.e., the kernel of the action of the mapping class group $\Gamma_{g,1}$ on the integral homology group H. Johnson [10] proved the first Johnson homomorphism $\tau_1 : \mathcal{I}_{g,1} \to \bigwedge^3 H$ induces the isomorphism $H_1(\mathcal{I}_{g,1}; \mathbb{Q}) \cong \bigwedge^3 H \otimes \mathbb{Q}$. The twisted Morita-Mumford classes are related to the extended Johnson homomorphism $\tilde{k} : \Gamma_{g,1} \to \frac{1}{2} \bigwedge^3 H$ introduced by Morita [20]. It extends the homomorphism τ_1 and is equal to $\frac{1}{6} m_{0,3}$. Let $Sp(H)$ denote the symplectic group of H. The extended homomorphism $\tilde{k}$ induces a natural homomorphism

$$\tilde{k}_M : ((\bigwedge^* H^1(\mathcal{I}_{g,1}; \mathbb{Q})) \otimes M)^{Sp(H)} \to H^*(\Gamma_{g,1}; M) \tag{4.3}$$

for any finite dimensional $\mathbb{Q}[Sp(H)]$-module M. It was proved in [13] [14] the image of the map is exactly the submodule generated by the Morita-Mumford classes e_i's and the twisted ones. The Madsen-Weiss theorem implies the map $\tilde{k}_M$ is stably surjective. In fact, Theorem 4.4 is applicable, because M is regarded as a $Sp(H)$-submodule of some $H^{\otimes n} \otimes \mathbb{Q}$.

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