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Conjugation-invariant norms on groups of geometric origin

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Abstract.

A group is said to be bounded if it has a finite diameter with respect to any bi-invariant metric. In the present paper we discuss boundedness of various groups of diffeomorphisms.

§1. Introduction and main results

1.1. The main phenomenon

A group G is said to be *bounded* if it is bounded with respect to any bi-invariant metric (that is, as a metric space, it has a finite diameter).

A *conjugation-invariant norm* $\nu : G \to [0; +\infty)$ is a function which satisfies the following axioms:

(i) $\nu(1) = 0$;
(ii) $\nu(f) = \nu(f^{-1})\ \ \forall f \in G$;
(iii) $\nu(fg) \leq \nu(f) + \nu(g)\ \ \forall f, g \in G$;
(iv) $\nu(f) = \nu(gfg^{-1})\ \ \forall f, g \in G$;
(v) $\nu(f) > 0$ for all $f \neq 1$.

Thus a group is bounded iff every conjugation-invariant norm is bounded.

Convention: In this paper we work only with conjugation-invariant norms, so by default a *norm* is a conjugation-invariant norm.

If one drops condition (v), ν is said to be a *pseudo-norm.* It can immediately be converted into a norm by adding 1 to all elements except the

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unity. Hence a group is unbounded if it admits an unbounded pseudo-norm. Observe that on a simple group every non-trivial pseudo-norm is automatically a norm: Indeed, the set of all elements with vanishing pseudo-norm forms a normal subgroup. Hence in the sequel condition (v) can be dropped everywhere when we deal with simple groups such as various groups of smooth diffeomorphisms.[d]

Two norms on a group are called *equivalent* if their ratio is bounded away from 0 and ∞. *The trivial norm*, which exists on any group, equals 1 on every element except the identity.

Given a connected manifold M, denote by $\mathrm{Diff}_0(M)$ the identity component of the group of C^∞ smooth compactly supported diffeomorphisms. This group is simple due to a theorem by Thurston [34]. The central phenomenon discussed in this paper is as follows: *in all known to us examples any norm on $\mathit{Diff}_0(M)$ is equivalent to the trivial one.* Below we confirm this phenomenon for spheres, all closed connected three-manifolds and the annulus. However we have neither a proof nor a counter-example for closed surfaces of genus ≥ 1 and the Möbius strip.

1.2. Setting the stage

1.2.1. *Conjugation-generated norms.* Many interesting norms come from the following construction: Let G be a group, and let $K \subset G$ be a symmetric subset, that is $x \in K$ whenever $x^{-1} \in K$. We say that the set K *conjugation-generates* (or, for brevity, *c-generates*) G if every element $h \in G$ can be represented as a product

$$h = \tilde{h}_1 \tilde{h}_2 \dots \tilde{h}_N \tag{1}$$

where each $\tilde{h}_i$ is conjugate to some element $h_i \in K$: $\tilde{h}_i = \alpha_i h_i \alpha_i^{-1}$, $\alpha_i \in G$. In this case define a norm $q_K(h)$ as the minimal N for which such a representation exists. We shall say that *the norm q_K is c-generated by the subset K*. If K is finite, G is said to be *finitely c-generated.* For instance, every simple group G is finitely c-generated by $K = \{x, x^{-1}\}$ with an arbitrary $x \neq 1$.

Note that the norm q_K has the following extremal property: for any norm q bounded on K there is a constant λ such that $q \leq \lambda q_K$. Hence, if K is finite, the group G is bounded if and only if q_K is bounded.

[d]A group is called simple if it has no non-trivial normal subgroups. In the 1970-ies, simplicity of various interesting groups of diffeomorphisms was established by highly non-trivial methods in works of Herman [16], Thurston [34], Mather [21, 22], Banyaga [2]. We refer to Banyaga's book [3] for a detailed discussion.

Example 1.1. Groups $SL(n,\mathbb{R})$ for $n \geq 2$ and $SL(n,\mathbb{Z})$ for $n \geq 3$ are finitely c-generated by the set K of all elementary matrices whose off-diagonal term equals ± 1. Moreover we claim that the number of terms in the decomposition (1) is bounded by a constant which does not depend on h.

In the case of $SL(n,\mathbb{R})$ the claim follows from an appropriate version of the Gauss elimination process.

As for $SL(n,\mathbb{Z})$, denote by $\mathcal{E}$ the set of all elementary matrices whose only non-zero off-diagonal element equals to 1. There exists $N = N(n) \in \mathbb{N}$ so that every element from $SL(n,\mathbb{Z})$ can be written as a product of $\leq N$ matrices of the form E^p, where $E \in \mathcal{E}$ and $p \in \mathbb{Z}$ (in other words, $SL(n,\mathbb{Z})$ possesses a bounded generation by elements from $\mathcal{E}$), see [9, 37]. The claim readily follows from the fact that each $E^p = [A, B^p]$ for some $A, B \in \mathcal{E}$. Let us prove this identity: let E_{ij} (where $i \neq j$) denotes the elementary matrix from $\mathcal{E}$ whose only non-zero off-diagonal element stands in the i-th raw and j-th column. Without loss of generality, put $i = 1, j = 3$. Then $E_{13}^p = [E_{12}, E_{23}^p]$ as required.

It follows from the claim that the "extremal" norm q_K is bounded, and hence the groups in question are bounded in view of extremality of q_K.

Example 1.2. The commutator length. Given a group G, denote by G' its commutator subgroup. The norm on G' c-generated by the set of all simple commutators $[a,b] = aba^{-1}b^{-1}$ is called the *commutator length* and is denoted by cl_G. This norm has a long history and has been intensively studied in various contexts, see e.g. [5].

1.2.2. *The role of the commutator subgroup.* The next observations suggest that the commutator subgroup plays a significant role in the study of boundedness.

Proposition 1.3. *If $H_1(G) := G/G'$ is infinite then G is unbounded.*

In particular, *an abelian group is bounded if and only if it is finite.*

Note that unbounded norms maybe non-extendable from a normal subgroups to the ambient group. Consider, for instance, the group $Aff(\mathbb{Z})$ of transformations of the real line of the form $u \mapsto \epsilon u + z$ with $\epsilon = \pm 1$ and $z \in \mathbb{Z}$. It can be considered as an extension of $\mathbb{Z}$ (the group of integer translations) by an element t of order 2 (the reflection over the origin) and with one additional relation $tz = z^{-1}t$. Thus $\mathbb{Z}$ is a normal subgroup of index 2 in $Aff(\mathbb{Z})$. Of course, $\mathbb{Z}$ has an unbounded norm, while $Aff(\mathbb{Z})$ admits no unbounded norms since t is conjugate to tz^{2n} (by z^n) for all integers n. However, the situation changes when one deals with the commutator length on the commutator subgroup:

Proposition 1.4. *Let G be any group. If the commutator length on G' is unbounded then G itself is unbounded.*

Propositions 1.3 and 1.4 are proved in Section 1.2.5 below.

1.2.3. *Stably unbounded norms.* Given a conjugation-invariant norm ν on a group G, we define its *stabilization* by

$$\nu_\infty(f) = \lim_{n\to\infty} \frac{\nu(f^n)}{n} .$$

Let us emphasize that stabilization of a norm is not in general a norm. An unbounded norm ν is called *stably unbounded* if $\nu_\infty(f) \neq 0$ for some $f \in G$.

For instance, an infinite abelian torsion group is unbounded by Proposition 1.3 but never stably unbounded.

Example 1.5. Consider a group $\mathbb{Z}_2^\infty$ of all finite words over $\{0,1\}$ with componentwise addition mod 2 (that is, a direct sum of countably many copies of $\mathbb{Z}_2$). This group admits no stably unbounded norms since the order of every element is 2. On the other hand, the length of a word is an unbounded norm. There is a natural action of $\mathbb{Z}_2^\infty$ on $\mathbb{Z}\times\mathbb{Z}_2$: the i-th generator swaps $(i,0)$ and $(i,1)$. Thus the norm in our example can be interpreted as "the size of support".

Open Problem. Does there exist a group that does not admit a stably unbounded norm and yet admits a norm unbounded on some cyclic subgroup?

1.2.4. *Stable commutator length and quasi-morphisms.* In what follows we shall focus on the stable commutator length. Let G be any group. The commutator length cl_G on G' is stably unbounded if and only if G admits *non-trivial homogeneous quasi-morphisms* [5]. Recall that a function $r : G \to R$ is called a quasi-morphism if there exists $C > 0$ so that

$$|r(ab) - r(a) - r(b)| \le C \;\; \forall a, b \in G .$$

A quasi-morphism is called *homogeneous* if $r(a^n) = nr(a)$ for all $a \in G$ and $n \in \mathbb{Z}$. A quasi-morphism is called non-trivial if it is not a morphism.

Convention: In this paper we deal with homogeneous quasi-morphisms only, so by default quasi-morphism means a homogeneous quasi-morphism.

Example 1.6. $G = SL(2,\mathbb{Z})$ carries an abundance of quasi-morphisms (cf. e.g. [4]) and hence the commutator norm on $SL(2,\mathbb{Z})$ is stably unbounded. Thus G is unbounded in view of Proposition 1.4, in contrast with $SL(n,\mathbb{Z})$ for $n \geq 3$ (see Example 1.1 above).

Introduce the class $\mathcal{G}$ of groups G with finite $H_1(G) = G/G'$ (we wish to rule out conjugation-invariant stably unbounded norms coming from the first homology, see Proposition 1.3 above). Note that various interesting groups of diffeomorphisms are simple (see footnote in Section 1.1 above) and hence belong to this class.

Open Problem. Does there exist a finitely presented group $G \in \mathcal{G}$ whose commutator length is unbounded but stably bounded?

Open Problem. Does there exist an unbounded finitely presented group which admits no unbounded quasi-morphisms?

A. Muranov informed us that he has an example of a finitely generated, but not finitely presented, group from $\mathcal{G}$ whose commutator length is unbounded but stably bounded. The existence of an infinitely generated group with this property readily follows from Muranov's work [26], who constructed a sequence of simple groups $G_i, i \in \mathbb{N}$ of finite commutator length diameter n_i, where $n_i \to \infty$. The infinite direct product $G = \prod_i G_i$ is as required.

A mystery related to the notion of stable unboundedness is as follows.

Open Problem. Does there exist a group $G \in \mathcal{G}$ whose commutator length is stably bounded, but which admits a stably unbounded norm? In other words, does the existence of a stably unbounded norm on G yields existence of non-trivial quasi-morphisms? In fact, we do not know even a single example of a group from $\mathcal{G}$ that admits no non-trivial quasi-morphisms but carries a norm that is unbounded on some cyclic subgroup.

Here is a (somewhat artificial) example of groups for which existence of a stably unbounded norm yields existence of non-trivial quasi-morphisms. Start with an arbitrary group $G \in \mathcal{G}$ and set $\bar{G}$ to be the extension of $G \times G$ by an element t so that

$$t^2 = 1, \quad \text{and} \quad t(g_1, g_2)t^{-1} = (g_2, g_1) \;\; \forall g_1, g_2 \in G.$$

Proposition 1.7. *The group $\bar{G}$ lies in $\mathcal{G}$ for every $G \in \mathcal{G}$.*

Proposition 1.8. *Suppose that for some $G \in \mathcal{G}$, the group $\bar{G}$ admits a stably unbounded norm. Then $\bar{G}$ admits a non-trivial quasi-morphism.*

1.2.5. *Quasi-norms.*

Definition 1.9. *Let G be a group. We say that a function $q : G \to [0; +\infty)$ is a a quasi-norm (for brevity, a q-norm) if:*

(i) *q is quasi-subadditive: there is a constant c such that*

$$q(ab) \leq q(a) + q(b) + c\,;$$

(ii) *q is quasi-conjugation-invariant: there is a constant c such that*

$$|q(b^{-1}ab) - q(a)| \leq c\,;$$

(iii) *q is unbounded.*

One can see that in fact the existence of a q-norm implies the existence of an unbounded norm: This norm can be constructed by (i) symmetrization: taking the maximum of the norm of a and a^{-1} for each a, (ii) redefining the norm of a to be the maximum of norms of its conjugates $b^{-1}ab$, and (iii) by adding a sufficiently large constant to the norm of all elements excluding the identity.

Hence a group is unbounded if it admits a q-norm; in other words, the existence of unbounded norms and q-norms are equivalent. However q-norms are often defined in a more natural way: A motivating example is provided by the absolute value of a non-trivial homogeneous quasi-morphism. Another advantage of q-norms is that they behave nicely under epimorphisms:

Lemma 1.10. *The pull-back of a q-norm under an epimorphism is a q-norm. In particular, if a group G admits a homomorphism onto an unbounded group, G itself is unbounded.*

This follows immediately from the definitions and discussion above. Let us apply the lemma for proving results stated in 1.2.2:

Proof of Proposition 1.3:

STEP 1: Let us show that any infinite abelian group G admits an unbounded norm.

If G is finitely generated, than by the classification theorem it has a $\mathbb{Z}$ as a direct factor, and hence it admits an epimorphism onto $\mathbb{Z}$. Thus G admits an unbounded norm by Lemma 1.10.

For a countably generated G, let us enumerate its generators g_1, $g_2, \ldots$. Define the norm of g to be the smallest k such that g lies in the subgroup generated by $g_1, g_2, \ldots, g_k$. This norm is unbounded.

In general, any infinite abelian group contains an infinite finitely or countably generated subgroup, and the above construction provides us with a norm on this subgroup H. Now choose any element g from $G \setminus H$ and consider a subgroup H' generated by the union of H and g. Combining the easily verifiable fact that the norm extends from H to H' with Zorn's lemma completes the proof.

STEP 2: Assume now that G/G' is infinite. By Step 1, it admits an unbounded norm. Look at the epimorphism $G \to G/G'$. Applying Lemma 1.10 we conclude that G is unbounded. ■

Proof of Proposition 1.4: If $[G, G]$ has infinite index, look at the epimorphism $G \to H := G/G'$. The group H is an infinite abelian group, thus by Proposition 1.3 H is unbounded, and hence G is unbounded in view of Lemma 1.10.

Otherwise, if H is finite, one can check that the commutator norm can be extended from the commutator to the whole group (even though in general q-norms cannot be extended from finite index subgroups, see an example above). Indeed, pick a (finite!) set S of representatives from cosets of G'. Then every element of G can be uniquely written as hs where $h \in G'$, $s \in S$. Define a q-norm of such an element $g = hs$ by $q(g) = cl_G(h)$. The approximate conjugation invariance of this norm follows from the fact that conjugation can be written as a multiplication by a commutator (and hence it changes the norm by at most 1). To prove the approximate triangle inequality, note that for $g_1 = h_1 s_1$ and $g_2 = h_2 s_2$

$$g_1 g_2 = h_1 h_2 [h_2^{-1}, s_1] s_1 s_2 \ .$$

Write

$$s_1 s_2 = h(s_1, s_2) t(s_1, s_2) \ ,$$

where $h(s_1, s_2) \in G'$ and $t(s_1, s_2) \in S$. Thus

$$q(g_1 g_2) = cl_G(h_1 h_2 [h_2^{-1}, s_1] h(s_1, s_2)) \ .$$

Put $C = \max_{s_1, s_2 \in S} cl_G(h(s_1, s_2))$. Applying the triangle inequality for the commutator length, we get

$$q(g_1 g_2) \leq cl_G(h_1) + cl_G(h_2) + 1 + C = q(g_1) + q(g_2) + 1 + C \ .$$

Thus q is indeed a q-norm. ■

1.2.6. *Fine norms.* A norm ν on G is called *fine* if 0 is a limit point of $\nu(G)$. Otherwise the norm is called, following a suggestion by Yehuda Shalom, *discrete.* For instance, conjugation-generated norms assume integer values only and hence are discrete. On the other hand a bi-invariant Riemannian metric on a compact Lie group gives rise to a bounded fine norm on the group.

1.2.7. *Meager groups.* A norm ν on a group is **not** equivalent to the trivial norm if it is either unbounded or fine. A group G is called *meager* if every conjugation-invariant norm on G is equivalent to the trivial one (i.e. is bounded and discrete).

1.3. Norms on diffeomorphism groups

1.3.1. *Smooth diffeomorphisms.* In this section we present the main results of the paper which deal with norms on groups $\mathrm{Diff}_0(M)$, where M is a smooth connected manifold. We start with the case of closed manifolds.

Theorem 1.11 (Main Theorem)**.**

(i) *The group $Diff_0(M)$ does not admit a fine conjugation-invariant norm for all connected manifolds M.*
(ii) *The group $Diff_0(S^n)$ is meager (where S^n is a sphere);*
(iii) *The group $Diff_0(M)$ is meager for any closed connected 3-dimensional manifold M.*

After the first draft of this paper appeared, T.Tsuboi [35] generalized this result and, remarkably, established meagerness of $Diff_0(M)$ for all odd-dimensional closed manifolds.

Let us give two important examples of conjugation-invariant norms on $\mathrm{Diff}_0(M)$.

Example 1.12. The commutator length: Since $\mathrm{Diff}_0(M)$ is a simple group [34] it coincides with its commutator subgroup and hence the commutator length (see Example 1.2) is a well-defined invariant norm on $\mathrm{Diff}_0(M)$. Introduce the *commutator length diameter* $cld(M) \in \mathbb{N} \cup \infty$ as $\max cl(f)$ over all $f \in \mathrm{Diff}_0(M)$.

Theorem 1.13.

(i) *For the sphere, $cld(S^n) \le 4$;*
(ii) *For any closed connected 3-dimensional manifold M, $cld(M) \le 10$.*

Example 1.14. The fragmentation norm: Every element $f \in \mathrm{Diff}_0(M)$ can be represented as a finite product of diffeomorphisms supported in an embedded open ball (this is the famous fragmentation lemma, see e.g. [3]). The *fragmentation norm $frag(f)$* is the minimal number of factors required to represent an element $f \in \mathrm{Diff}_0(M)$. Clearly, $frag$ is an conjugation-invariant norm on $\mathrm{Diff}_0(M)$. The next result shows that the fragmentation norm is responsible for meagerness of $\mathrm{Diff}_0(M)$.

Proposition 1.15. *The group $\mathit{Diff}_0(M)$ is meager if and only if the fragmentation norm is bounded.*

Open Problem. Is the fragmentation norm bounded for the case of closed surfaces?

Let us now turn to open manifolds.

Definition 1.16. We say that a smooth connected open manifold M is *portable*[e] if it admits a complete vector field X and a compact subset M_0 with the following properties:

- M_0 is an attractor of the flow X^t generated by X: for every compact subset $K \subset M$ there exists $\tau > 0$ so that $X^\tau(K) \subset M_0$.
- There exists a diffeomorphism $\theta \in \mathrm{Diff}_0(M)$ so that $\theta(M_0) \cap M_0 = \emptyset$.

The set M_0 is called *the core* of a portable manifold M.

For instance, any manifold M which splits as $P \times \mathbb{R}^n$, where P is a closed manifold, is portable. Indeed, the vector field $X(p,z) = -z\frac{\partial}{\partial z}$ and the compact $M_0 = P \times \{|z| \leq 1\}$ satisfy the conditions above. Furthermore, M is portable if it admits an exhausting Morse function with finite number of critical points so that all the indices are strictly less than $\frac{1}{2}\dim M$. This implies, for example, that every 3-dimensional handlebody is a portable manifold.

The next result is the main "local" block in the proof of Theorem 1.11(ii) and (iii).

Theorem 1.17. *The group $\mathit{Diff}_0(M)$ is meager provided M is portable.*

For instance, any norm on Diff_0 of an open ball is bounded. Together with Theorem 1.11(i) this immediately yields Proposition 1.15. Furthermore, Diff_0 of a 2-dimensional annulus is meager (as well as for any product $\mathbb{R} \times M$). However, it is still unknown whether the same holds for the open Möbius band!

Our next result deals with the commutator length diameter of a portable manifold.

Theorem 1.18. *For a portable manifold M, $\mathit{cld}(M) \leq 2$.*

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[e]This notion is a mock version of subcritical Liouville manifolds in symplectic topology.

1.3.2. *Volume-preserving and symplectic diffeomorphisms: examples and problems.* In contrast to groups Diff_0, the identity components of groups of compactly supported volume preserving and symplectic diffeomorphisms, as well as their commutator subgroups, are **never** meager: they admit a fine norm.

Example 1.19. The size-of-support norm: The counterpart of Example 1.5 above for diffeomorphism groups is as follows. Consider the identity component $\mathrm{Diff}_0(M, \mathrm{vol})$ of the group of compactly supported volume-preserving diffeomorphisms of a smooth manifold M of dimension > 0. Define the norm of a diffeomorphism as the volume of its support. This norm is necessarily fine, and it is unbounded whenever the volume of M is infinite. However this norm is never stably unbounded: in fact, it is bounded on all cyclic subgroups.

In some situations, stably unbounded norms on the commutator subgroup of $\mathrm{Diff}_0(M, \mathrm{vol})$ can be "induced" from the fundamental group of M even when the volume of M is finite:

Example 1.20. Suppose that M is a closed manifold equipped with a volume form. Suppose that $H := \pi_1(M)$ has trivial center. Then the commutator length on the commutator subgroup of $\mathrm{Diff}_0(M, \mathrm{vol})$ is stably unbounded provided the commutator length on H' is stably unbounded, see [15, 29].

However, in dimension ≥ 3 no unbounded norms on volume-preserving diffeomorphisms are known so far in the cases when the manifold has simple topology and finite volume.

Open Problem. Assume that $n \geq 3$. Does the identity component of the group of volume preserving diffeomorphisms of the sphere S^n admit an unbounded conjugation-invariant norm? Does the identity component of the group of compactly supported volume preserving diffeomorphisms of the ball of finite volume admit an unbounded conjugation-invariant norm?

In the symplectic category, interesting norms inhabit the group $\mathrm{Ham}(M, \omega)$ of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold (M, ω).

Example 1.21. The Hofer norm on $\mathrm{Ham}(M, \omega)$ ([17], see also [28]) is fine. Its unboundedness is a long-standing conjecture in symplectic topology. Nowadays it is confirmed for various symplectic manifolds including for instance surfaces, complex projective spaces with the Fubini-Studi symplectic form and closed manifolds with $\pi_2 = 0$. Further, the Hofer norm on groups of Hamiltonian diffeomorphisms is known to be stably unbounded for various closed symplectic manifolds. However it is unbounded, but not stably unbounded, for the standard symplectic vector space $\mathbb{R}^{2n}$ (Sikorav, [33]).

Example 1.22. The commutator length on $\mathrm{Ham}(M, \omega)$ is known to be stably unbounded for various symplectic manifolds (see [4, 12, 13, 7, 15, 30, 31]) including all surfaces and complex projective spaces of arbitrary dimension.

Example 1.23. The group $\mathrm{Ham}(\mathbb{R}^{2n})$ admits the Calabi homomorphism (the average Hamiltonian) to $\mathbb{R}$. The kernel of the Calabi homomorphism coincides with the commutator subgroup of $\mathrm{Ham}(\mathbb{R}^{2n})$, which is known to be simple [2]. This group is stably bounded with respect to the commutator length. This is proved by D. Kotschick in [18]. Alternatively, this readily follows from the algebraic packing inequality given by Theorem 2.7 below. In contrast to this, the commutator length on $[\mathrm{Ham}(B^{2n}), \mathrm{Ham}(B^{2n})]$, where B^{2n} is the standard symplectic ball, is stably unbounded, see [4].

Example 1.24. A somewhat less understood example is *the fragmentation norm* (cf. Example 1.14 above). Let (M, ω) be a closed symplectic manifold and let $U \subset M$ be an open subset. The Hamiltonian fragmentation lemma (see [2]) states that every Hamiltonian diffeomorphism f can be written as a product $h_1 \circ \ldots \circ h_N$, where each h_i is conjugate to an element from $\mathrm{Ham}(U)$. Define the fragmentation norm $frag_U(f)$ as the minimal number of factors in such a decomposition. Using methods of [14], one can show that for certain symplectic manifolds $frag_U$ is unbounded on $\mathrm{Ham}(M)$ provided the subset U is *displaceable by a Hamiltonian diffeomorphism* (e.g. U is a ball of a small diameter). Let us elaborate this statement.

First of all recall [32, 27, 25] that for elements of *the universal cover* $\widetilde{\mathrm{Ham}}(M)$ of the group of Hamiltonian diffeomorphisms one can define *spectral invariants* which come from Floer homology of the action functional. Denote by $\widetilde{\mu} : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ the asymptotic spectral invariant as defined in [14]. For various interesting symplectic manifolds $\widetilde{\mu}$ descends to a function μ on $\mathrm{Ham}(M)$. This is for instance the case for symplectic manifolds with $\pi_2(M) = 0$ (due to M. Schwarz [32]) and for standard complex projective spaces (see [13]). [f] We continue discussion on the fragmentation norm assuming that $\widetilde{\mu}$ does descend to μ.

Second, if U is displaceable, Theorem 7.1 in [14] guarantees that

$$|\mu(\phi\psi) - \mu(\phi) - \mu(\psi)| \leq \min(frag_U(\phi), frag_U(\psi)) \tag{2}$$

for all $\phi, \psi \in \mathrm{Ham}(M)$. At this point there is a dichotomy which roughly speaking depends on the algebraic structure of the quantum homology ring of (M, ω):

POSSIBILITY 1: The left hand side of (2) is a bounded function on $\mathrm{Ham}(M) \times \mathrm{Ham}(M)$, and thus μ is a quasi-morphism on $\mathrm{Ham}(M)$. For instance, this is the case for the complex projective spaces [13].

POSSIBILITY 2: The left hand side of (2) is unbounded, and thus *a fortiori* the fragmentation norm on $\mathrm{Ham}(M)$ is unbounded. For instance this is the case for the standard symplectic tori $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, dp \wedge dq)$.

Let us explain why Possibility 2 holds for the two-torus: Take a pair of disjoint meridians L and K on the torus. Let Φ, Ψ be two smooth cut off functions on the torus with disjoint supports which equal 1 near L and K respectively. Let $\{\phi_t\}$ and $\{\psi_t\}$ be the Hamiltonian flows generated by Φ and Ψ. A standard calculation in Floer homology shows that the left hand side of (2) with $\phi = \phi_t, \psi = \psi_t$ goes to infinity as $t \to \infty$. This proves unboundedness of the Hamiltonian fragmentation norm $frag_U$ for the 2-torus.

We conclude with an open problem. In spite of the fact that the complex projective spaces enjoy Possibility 1 above, the question on unboundedness of the Hamiltonian fragmentation norm in this case is widely open even for $\mathbb{C}P^1 = S^2$.

ORGANIZATION OF THE PAPER: In the next section we introduce algebraic packing and displacement technique which is used for the proof of the main results stated in the introduction. As an illustration, we deduce there Theorem 1.11(i) and Proposition 1.8. Theorems 1.17 and 1.18 are proved in Section 3.1. These theorems, combined with topological decomposition technique (which is standard in the case of spheres, and less trivial in the case of three-manifolds) are applied to the proof of Theorems 1.11(ii),1.13(i) in Section 3.2 and of Theorems 1.11(iii),1.13(ii) in Section 3.3.

[f] In general, $\widetilde{\mu}$ may descend to $\mathrm{Ham}(M)$ and may not. We refer to a recent paper [23] by D. McDuff for new results and a detailed discussion of the current state of art in this problem. We thank D. McDuff for an illuminating discussion on this topic.

§2. Algebraic tools: packing and displacement

Here we present the algebraic tools used for proving Theorems 1.11 (i), 1.17 and 1.18. We use a number of tricks which imitate displacement of supports of diffeomorphisms and decomposition of diffeomorphisms into products of commutators in a more general algebraic setting. The tricks of this nature appear in the context of transformation groups at least since the beginning of 1960-ies (see e.g. [1]). The system of notions introduced below in parts imitates and extends the one arising in the study of Hofer's geometry on the group of Hamiltonian diffeomorphisms. Note also that various interesting results on infinitely displaceable subgroups were obtained in a recent work of D. Kotschick [18].

2.1. Algebraic packing and displacement energy

Let G be any group. We say that two subgroups $H_1, H_2 \subset G$ commute if $h_1h_2 = h_2h_1$ for all $h_1 \in H_1, h_2 \in H_2$. We denote by $Conj_\phi$ the automorphism of G given by $g \mapsto \phi g \phi^{-1}$. A subgroup $H \subset G$ is called *m-displaceable* (where $m \geq 1$ is an integer) if there exist elements $\phi_0 := 1, \phi_1, ..., \phi_m \in G$ so that the subgroups $\mathrm{Conj}_{\phi_i}(H), \mathrm{Conj}_{\phi_j}(H)$ pair-wise commute for all distinct $i, j \in \{0; ...; m\}$. A subgroup H is called *strongly* m-displaceable if in the previous definition one can choose ϕ_k's to be consecutive powers of the same element $\phi \in G$: $\phi_k = \phi^k$. In this case we shall say that ϕ m-displaces H.

Note that for $m = 1$ both notions coincide, and, for brevity, we refer to a 1-displaceable subgroup as to displaceable.

Introduce two numerical invariants related to the above notions. The *algebraic packing number* $p(G, H) = m + 1$, where m is the maximal integer such that H is m-displaceable. This is a purely algebraic invariant. The second quantity involves a conjugation-invariant norm, say ν on G. Define the *order m displacement energy* [g] of H with respect to ν as $e_m(H) = \inf \nu(\phi)$ where the infimum is taken over all $\phi \in G$ which m-displace H. We put $e_m(H) = +\infty$ if H is not strongly m-displaceable.

While speaking on displaceability, we tacitly assume that the subgroup H is non-abelian. Indeed, every abelian subgroup H is m-displaceable by 1 for every $m \in \mathbb{N}$ and hence $e_m(H) = 0$.

Example 2.1. Let M be a smooth connected manifold. Put $G = \mathrm{Diff}_0(M)$. Take any open ball $B \subset M$. Let H be the subgroup of G

[g]This notion is an algebraic counterpart of the symplectic displacement energy introduced by Hofer in [17].

consisting of all diffeomorphisms supported in B. Choose any diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ which displaces B: $B \cap \phi(B) = \emptyset$. Then H commutes with $Conj_\phi(H)$, so H is displaceable.

Theorem 2.2. *Let $H \subset G$ be a strongly m-displaceable subgroup of G. Assume that G is endowed with a conjugation-invariant norm ν.*

(i) *For every element $x \in H'$ with $cl_H(x) = m$ the following inequalities hold:*

$$\nu(x) \leq 14 e_m(H) \tag{3}$$

and

$$cl_G(x) \leq 2; \tag{4}$$

(ii) *In the case $cl_H(x) = 1$, that is $x = [f, g]$ for some $f, g \in H$, we have that*

$$\nu(x) \leq 4 e_1(H)\ ; \tag{5}$$

Corollary 2.3. *Assume that an element $F \in G$ m-displaces H for every $m \geq 1$. Then $cl_G(h) \leq 2$ for all $h \in H'$.*

This follows immediately from inequality (4).

Theorem 2.2(ii) is proved in [11]. The argument is very short: indeed, assume that $\mathrm{Conj}_\phi(H)$ commutes with H. Then $[f, g] = [f \cdot \phi f^{-1} \phi^{-1}, g]$. Using bi-invariance of ν we get that

$$\nu([f, g]) \leq 2\nu([f, \phi]) \leq 4\nu(\phi)\ .$$

Taking the infimum over all ϕ displacing H we get inequality (5). The proof of Theorem 2.2(i) is more involved, see Section 2.2 below.

Let us give some sample applications of Theorem 2.2. First, we deduce from inequality (5) the fact that the group $\mathrm{Diff}_0(M)$ does not admit a fine norm.

Proof of Theorem 1.11(i): Assume on the contrary that $\mathrm{Diff}_0(M)$ admits a fine norm, say ν. Take any ball $B \subset M$ and pick two non-commuting diffeomorphisms f and g supported in B. For any $\epsilon > 0$ take $h \in \mathrm{Diff}_0(M)$ with $0 < \nu(h) < \epsilon$. Note that since $h \neq \mathbb{1}$ there exists a ball $C \subset M$ so that h displaces C. Since all balls in M are isotopic, there is a diffeomorphism $\psi \in \mathrm{Diff}_0(M)$ with $\psi(C) = B$. Therefore $\phi := \psi h \psi^{-1}$

displaces B, and hence ϕ displaces the subgroup $\mathrm{Diff}_0(B) \subset \mathrm{Diff}_0(M)$. Applying inequality (5) we get that

$$\nu([f,g]) \leq 4\nu(\phi) = 4\nu(h) < 4\epsilon \, .$$

Sending ϵ to zero, we conclude that $\nu([f,g]) = 0$, a contradiction with the non-degeneracy of a norm. ■

Next, we apply Theorem 2.2 to proving that for a class of groups introduced in Section 1.2.4 existence of stably unbounded norms yields existence of quasi-morphisms.

Proof of Propositions 1.7 and 1.8: First of all note that every element $h \in \bar{G}$ can be uniquely written in the following *normal form*: either $h = (g_1, g_2)$ or $h = (g_1, g_2)t$. This readily yields Proposition 1.7. Second, we claim that it suffices to show that G has a non-trivial homogeneous quasi-morphism, say r. Indeed, put $\bar{r}(h) = r(g_1) + r(g_2)$, where h is in the normal form as above. A straightforward analysis shows that $\bar{r}$ is a (not necessarily homogeneous!) quasi-morphism on $\bar{G}$. For instance, if $h = (h_1, h_2)t$ and $f = (f_1, f_2)$ then $hf = (h_1 f_2, h_2 f_1)t$ and hence

$$|\bar{r}(hf) - \bar{r}(h) - \bar{r}(f)| \leq |r(h_1 f_2) - r(h_1) - r(f_2)| + |r(h_2 f_1) - r(h_2) - r(f_1)|$$

and hence is uniformly bounded. The other cases are considered similarly. Finally note that the stabilization $\bar{r}_\infty(h) := \lim_{n\to\infty} \bar{r}(h^n)/n$ does not vanish on $h = (g, 1)$ provided $r(g) \neq 0$. Since $\bar{r}_\infty$ is a homogeneous quasi-morphism, the claim follows.

Let ν be a stably unbounded norm on $\bar{G}$. Assume that $\nu_\infty(w) > 0$ for some $w \in \bar{G}$.

CASE 1: $w = (g_1, g_2)$. Put $w_1 = (g_1, 1)$ and $w_2 = (1, g_2)$. We claim that either $\nu_\infty(w_1) > 0$ or $\nu_\infty(w_2) > 0$. Indeed, $w^k = w_1^k w_2^k$ and hence

$$0 < \nu_\infty(w) \leq \nu_\infty(w_1) + \nu_\infty(w_2) \, ,$$

which yields the claim.

CASE 2: $w = (g_1, g_2)t$. Put $w_1 = (g_1 g_2, 1)$ and $w_2 = (1, g_2 g_1)$. We claim that either $\nu_\infty(w_1) > 0$ or $\nu_\infty(w_2) > 0$. Indeed, $w^{2k} = w_1^k w_2^k$ and hence

$$0 < \nu_\infty(w) \leq \frac{1}{2}(\nu_\infty(w_1) + \nu_\infty(w_2)) \, ,$$

which yields the claim.

Looking at elements w_1 and tw_2t above we conclude that there exists an element $u = (g, 1)$ with $\nu_\infty(u) > 0$. Replacing, if necessary, u by its power we can assume that $g \in G'$ (here we use that $H_1(G)$ is finite). Denote by $H \subset \bar{G}$ the subgroup consisting of all elements of the form $(f, 1)$ where $f \in G$. Clearly, H is isomorphic to G and $u \in H'$. Furthermore, t displaces H. Thus inequality (5) yields that

$$\nu(z) \le 4\nu(t) \cdot cl_H(z) \;\; \forall z \in H' .$$

Substituting $z = u^k$, dividing by k and passing to the limit as $k \to \infty$ we get that

$$0 < \nu_\infty(u) \le 4\nu(t) \cdot scl_H(u) .$$

Thus $scl_G(g) = scl_H(u) > 0$. Therefore Bavard's theorem [5] yields existence of a non-trivial homogeneous quasi-morphism on G. ■

2.2. Inequalities with commutators

Here we prove Theorem 2.2(i). For an element $F \in G$, we say that $g \in G$ is an F-*commutator* if $g = \mathrm{Conj}_f[F, h]$ for some $f, h \in G$. Note that the inverse of an F-commutator is again an F-commutator.

Fix $F \in G$ such that the subgroups

$$H_0 := H, \;\; H_1 := \mathrm{Conj}_F H, \;\; ..., \;\; H_m := \mathrm{Conj}_{F^m} H$$

pair-wise commute. We shall show that every element x from the commutator subgroup H' with $cl_H(x) = m$ can be represented as a product of seven F-commutators. Note that given a conjugation-invariant norm ν on G, for every F-commutator g we have $\nu(g) \le 2\nu(F)$. Thus we shall get that $\nu(x) \le 14\nu(F)$, which yields inequality (3).

We shall consider products $\prod_0^m \mathrm{Conj}_{F^i}(g_i)$, where $g_i \in H$, $i = 0, ..., m$. Since H_i's pair-wise commute, the product of such elements $\prod_0^m \mathrm{Conj}_{F^i}(f_i)$ and $\prod_0^m \mathrm{Conj}_{F^i}(g_i)$ can be computed component-wise: it equals $\prod_0^m \mathrm{Conj}_{F^i}(f_i g_i)$.

Lemma 2.4. *Let a collection of $g_i \in H$, $i = 0, 1, \ldots, m$ be such that $\prod_0^m g_i = 1$. Then the product $g = \prod_0^m \mathrm{Conj}_{F^i}(g_i)$ is an F-commutator.*

Proof. We will show that $g = [F, \phi^{-1}]$ where $\phi = \prod_0^{m-1} \mathrm{Conj}_{F^i}(\phi_i)$, $\{\phi_i\}_{i=0}^{m-1}$ is a collection of elements of H which will be defined later. We set $\phi_m = 1$ for convenience of notation.

Note that $[F, \phi^{-1}] = \mathrm{Conj}_F(\phi^{-1})\phi$ and $\mathrm{Conj}_F(\phi^{-1})$ equals the product $\prod_0^{m-1} \mathrm{Conj}_{F^{i+1}}(\phi_i^{-1}) = \prod_1^m \mathrm{Conj}_{F^i}(\phi_{i-1}^{-1})$ whose terms lie in $H_1, ...,$ H_m. Hence

$$[F, \phi^{-1}] = \mathrm{Conj}_F(\phi^{-1})\phi = \phi_0 \cdot \prod_1^m \mathrm{Conj}_{F^i}(\phi_{i-1}^{-1}\phi_i)$$

and the equation $[F, \phi^{-1}] = g$ is equivalent to the system

$$\begin{cases} \phi_0 = g_0 \\ \phi_0^{-1}\phi_1 = g_1 \\ \phi_1^{-1}\phi_2 = g_2 \\ \dots \\ \phi_{m-1}^{-1}\phi_m = g_m \end{cases}$$

The solution of this system is $\phi_k = \prod_0^k g_i$, $k = 0, 1, \dots, m$. The equation $\phi_m = 1$ is satisfied by the assumption $\prod g_i = 1$. ■

Lemma 2.5. *Let $g_1, g_2, \dots, g_m$ be a collection of elements of H. Then $g = \prod_m^1 g_i$ equals an F-commutator times the product $\prod_1^m Conj_{F^i}(g_i)$.*

Proof. Introduce $g_0' = g$ and $g_i' = g_i^{-1}$. Note that $\prod_0^m g_i' = 1$. Then apply the previous lemma. ■

Lemma 2.6. *Any commutator from H is a product of two F-commutators.*

Proof. Consider a commutator $[f, g]$ with $f, g \in H$. Then by Lemma 2.4, the elements

$$(fg)\mathrm{Conj}_F(g^{-1})\mathrm{Conj}_{F^2}(f^{-1})$$

and

$$(f^{-1}g^{-1})\mathrm{Conj}_F(g)\mathrm{Conj}_{F^2}(f)$$

are F-commutators. Their product is $[f, g]$. ■

End of the proof of Theorem 2.2(i): Consider $h = \prod_m^1 [f_i, g_i]$ with $f_i, g_i \in H$. By Lemma 2.5, h equals an F-commutator times a product $\theta := \prod_1^m \mathrm{Conj}_{F^i}([f_i, g_i])$. The latter in its turn is equal to the commutator of two products $\phi := \prod_1^m \mathrm{Conj}_{F^i}(f_i)$ and $\psi := \prod_1^m \mathrm{Conj}_{F^i}(g_i)$ since the subgroups H_i and H_j commute for $i \neq j$. This proves inequality (4).

Applying again Lemma 2.5 we have that $\phi = fx$ and $\psi = gy$ where $f = f_m \dots f_1$ and $g = g_m \dots g_1$ and x, y are F-commutators. We write

$$\theta = [fx, gy] = [f, g] \cdot \mathrm{Conj}_g\{\mathrm{Conj}_f(g^{-1}xg \cdot y \cdot x^{-1}) \cdot y^{-1}\}\,.$$

Since $f, g \in H$, we have by Lemma 2.6 that $[f, g]$ equals a product of two F-commutators. Hence θ is a product of six F-commutators and therefore h is a product of seven F-commutators. As we explained in the beginning of this section, this completes the proof of the theorem.■

2.3. Packing and distortion of subgroups

Let G be a group and $H \subset G$ a subgroup. Consider the embedding of metric spaces $(H', cl_H) \mapsto (G', cl_G)$. Obviously $cl_G(w) \leq cl_H(w)$ for all $w \in H'$. It turns out that, after stabilization, this inequality can be refined provided H is m-displaceable in G: the larger m is, the stronger H' is distorted in G' with respect to the stable commutator lengths.

Theorem 2.7.

$$scl_G(w) \leq \frac{1}{p(G,H)} scl_H(w) \ \ \forall w \in H' .$$

Example 2.8. Let $G = \widetilde{Sp(2n,R)}$ be the universal cover of the linear symplectic group and let $H = \widetilde{Sp(2,R)} \subset G$. Here we fix the splitting $\mathbb{R}^{2n} = \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$. The monomorphism $Sp(2,\mathbb{R}) \to Sp(2n,\mathbb{R})$ which sends a matrix A to $A \oplus \mathbf{1}_{2n-2}$ induces the isomorphism of the fundamental groups $\pi_1(Sp(2,\mathbb{R})) = \pi_1(Sp(2n,\mathbb{R})) = \mathbb{Z}$, and hence H naturally embeds into G. Let $(p_1, q_1, ..., p_n, q_n)$ be the standard symplectic coordinates on $\mathbb{R}^{2n}$. Denote by I_j the symplectic transformation which permutes (p_1, q_1) and (p_j, q_j)-coordinates. Write $\widetilde{I_j}$ for a lift of I_j to G. Then the subgroups $\mathrm{Conj}_{\widetilde{I_j}}(H)$ pairwise commute, and hence $p(G,H) \geq n$. Denote by $e \in H$ the generator of the center of H. One can show (see Remark 2.11 below) that

$$scl_H(e) = n \cdot scl_G(e) . \tag{6}$$

Thus the inequality in Theorem 2.7 yields $p(G,H) \leq n$. We conclude that $p(G,H) = n$ and the inequality is sharp.

Example 2.9. Let (M,ω) be a symplectic manifold, and let $U \subset M$ be an open subset. Let $G = \mathrm{Ham}(M,\omega)$ and let $H = \mathrm{Ham}(U,\omega)$. In this case the algebraic packing number $p(G,H)$ has a simple geometric meaning: It equals to the *geometric packing number* $p_{geom}(M,U)$ which is defined as the maximal number of diffeomorphisms from G which take U to pairwise disjoint subsets of M. In the case when U is a standard symplectic ball the geometric packing number was intensively studied in the framework of the symplectic packing problem (see [6] for a survey). For instance, assume that M and U are $2n$-dimensional symplectic balls. In the case $n = 1$ the geometric packing number is simply the integer part of the ratio of the areas. In the case $n = 2$ the situation is more complicated: For instance, if the ratio of volumes of M and U lies in the interval $(8; (1 + 1/288) \cdot 8)$, the geometric packing number equals 7 (see [24]). It would be interesting to explore the sharpness of the inequality in Theorem 2.7 in these examples.

The proof of Theorem 2.7 is based on the following observation (thanks to Sasha Furman for help). For a subgroup $H \subset G$ write $Q(H)$ for the set of homogeneous quasi-morphisms on H modulo morphisms, and for $\phi \in Q(H)$ put

$$||\phi||_H = \sup_{x,y \in H} \phi([x, y]) .$$

Proposition 2.10. *Let $H_1, ..., H_N$ be subgroups of G so that H_i and H_j commute for $i \neq j$. Put $K = H_1 \cdot ... \cdot H_N$. Then for every $\phi \in Q(K)$*

$$||\phi||_K = \sum_{i=1}^{N} ||\phi||_{H_i} .$$

Proof of Proposition 2.10: Take any $x, y \in K$ and write

$$x = x_1 \cdot ... \cdot x_N,\ y = y_1 \cdot ... \cdot y_N ,$$

where $x_i, y_i \in H_i$. Then

$$[x, y] = [x_1, y_1] \cdot ... \cdot [x_N, y_N] .$$

Since the commutators in the right hand side pair-wise commute we get that for every quasi-morphism $\phi \in Q(K)$

$$\phi([x, y]) = \sum_{i=1}^{N} \phi([x_i, y_i]) .$$

Since pairs x_i, y_i can be chosen in an arbitrary way we get the desired equality. ■

Proof of Theorem 2.7: Suppose that $p(G, H) \geq N$. Then there exist elements $g_1 = 1, g_2, ..., g_N$ so that subgroups $H_i := g_i H g_i^{-1}$ pair-wise commute. For every $\phi \in Q(G)$ we have $||\phi||_{H_i} = ||\phi||_H$. Put $K = H_1 \cdot ... \cdot H_N$. Applying Proposition 2.10 we have

$$||\phi||_G \geq ||\phi||_K = N||\phi||_H . \tag{7}$$

Denote by $Q_*(H)$ the set of non-trivial quasi-morphisms from $Q(H)$, and by $Q_*(G, H)$ the set of quasi-morphisms from $Q_*(G)$ which restrict to a non-trivial quasi-morphism on H. Apply now Bavard's theorem [5]: given $w \in H'$ we have

$$scl_H(w) = \frac{1}{2} \sup_{\phi \in Q_*(H)} \frac{\phi(w)}{||\phi||_H} \geq \frac{1}{2} \sup_{\phi \in Q_*(G,H)} \frac{\phi(w)}{||\phi||_H} .$$

Using inequality (7) above and applying the same Bavard's theorem we have

$$\begin{aligned} scl_H(w) &\geq N \cdot \frac{1}{2} \sup_{\phi \in Q_*(G,H)} \frac{\phi(w)}{||\phi||_G} \\ &= N \cdot \frac{1}{2} \sup_{\phi \in Q_*(G)} \frac{\phi(w)}{||\phi||_G} = N scl_G(w). \end{aligned} \tag{8}$$

The equality in the middle follows from the fact that for $\phi \in Q_*(G) \setminus Q_*(G,H)$ and $w \in H'$ one has $\phi(w) = 0$. Using inequality (8), we readily complete the proof. ■

Remark 2.11. Denote by G_n the universal cover of the group $Sp(2n, \mathbb{R})$ and by $e_n \in G_n$ the generator of $\pi_1(Sp(2n, \mathbb{R}))$ with Maslov index 2. The group G_n carries unique homogeneous quasi-morphism μ_n with $\mu_n(e_n) = 1$ (see [4]). Put

$$I_n := \frac{||\mu_n||_{G_n}}{||\mu_1||_{G_1}} .$$

One can show that $I_n = n$. The only known to us proof of this innocently looking fact is surprisingly involved: it can be extracted from [8] (thanks to A. Iozzi and A. Wienhard for illuminating consultations). By the above-cited theorem due to Bavard

$$\frac{scl_{G_1}(e_1)}{scl_{G_n}(e_n)} = I_n ,$$

which proves equality (6) above.

§3. Topological arguments

3.1. Portable manifolds

Let M be a portable manifold. We shall use notations of Definition 1.16.

Lemma 3.1. *There exists a neighborhood U of the core M_0 of M and a diffeomorphism $\phi \in Diff_0(M)$ so that the sets $\phi^i(U)$, $i \geq 1$ are pair-wise disjoint.*

Proof. Choose a sufficiently small neighbourhood U of the core so that $\theta(U) \cap \mathrm{Closure}(U) = \emptyset$. Put $V = \theta(U)$ and consider the vector field $Y = \theta_* X$ on M. Note that V is an attractor of Y. In particular there exists $\tau > 0$ large enough so that the closure of $Y^\tau(U \cup V)$ is contained in V. Cutting off Y^τ outside a sufficiently large compact set, we get that there exists a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ so that

$$\mathrm{Closure}\ \phi(U \cup V) \subset V .$$

Observe that $\phi^i(U) \subset \phi^{i-1}(V) \setminus \phi^i(V)$. Thus the sets $\phi^i(U)$, $i \geq 1$ are pair-wise disjoint. ■

Proof of Theorem 1.17: Let ν be any conjugation-invariant norm on $\mathrm{Diff}_0(M)$. It suffices to show that ν is bounded.

We shall use notations of Definition 1.16 of a portable manifold. Look at the neighborhood U of the core and at the diffeomorphism ϕ from Lemma 3.1. Note that ϕ m-displaces the subgroup $\mathrm{Diff}_0(U)$ for any m. Take any diffeomorphism $h \in \mathrm{Diff}_0(U)$. Since the group $\mathrm{Diff}_0(U)$ is perfect, it follows from inequality (3) that $\nu(h) \leq 14\nu(\phi)$.

Further, take any diffeomorphism $f \in \mathrm{Diff}_0(M)$. The first item of the Definition 1.16 guarantees that for $\tau > 0$ large enough $X^\tau(\mathrm{support} f) \subset U$. Applying the ambient isotopy theorem, we can find a diffeomorphism $\psi \in \mathrm{Diff}_0(M)$ with $\psi(\mathrm{support} f) \subset U$. Thus $\psi f \psi^{-1}$ lies in $\mathrm{Diff}_0(U)$. We conclude that

$$\nu(f) = \nu(\psi f \psi^{-1}) \leq 14\nu(\phi)$$

which implies that ν is bounded. This completes the proof. ■

Proof of Theorem 1.18: The proof above shows that the diffeomorphism ϕ m-displaces the subgroup $H := \mathrm{Diff}_0(U)$ for any m. Corollary 2.3 above implies that $cl_G(h) \leq 2$ for all $h \in \mathrm{Diff}_0(U)$, where $G = \mathrm{Diff}_0(M)$. But every element $f \in G$ is conjugate to an element from H. Thus $cld(M) \leq 2$. ■

Remark 3.2. Theorem 1.17 admits the following straightforward generalization. Let G be any group acting by homeomorphisms on a topological space X. Assume that there exist two disjoint open subsets $U, V \subset X$ and an element $\phi \in G$ which satisfy the following two easily verifiable properties:

(i) $\mathrm{Closure}\ \phi(U \cup V) \subset V$;

(ii) For every finite collection of elements $\psi_1, ..., \psi_k \subset G$ there exists $h \in G$ so that

$$h\Big(\bigcup_{i=1}^{k} \mathrm{support}(\psi_i)\Big) \subset U .$$

Then any invariant norm on G is bounded on the commutator subgroup G'.

3.2. Spheres

Lemma 3.3. *Every diffeomorphism $f \in \mathit{Diff}_0(S^n)$ can be written as $f = gh$ where $g \in \mathit{Diff}_0(S^n \setminus \{z\})$ and $h \in \mathit{Diff}_0(S^n \setminus \{w\})$ for some points $z, w \in S^n$.*

Since $S^n \setminus \{\text{point}\} = \mathbb{R}^n$ is a portable manifold, Theorem 1.11(ii) follows from Theorem 1.17 and Theorem 1.13(i) follows from Theorem 1.18.

Proof of Lemma 3.3: This fact is standard: Let $\{f_t\}$, $t \in [0;1]$ be a path in $\mathrm{Diff}_0(S^n)$ with $f_0 = \mathbf{1}$ and $f_1 = f$. Choose a sufficiently small closed disc $D \subset S^n$ so that $X := \bigcup_t f_t(D) \neq S^n$. Pick a point $z \notin X$. Since $S^n \setminus \{z\}$ is diffeomorphic to $\mathbb{R}^n$, there exists a path $\{g_t\}$ of diffeomorphisms from $\mathrm{Diff}_0(S^n \setminus \{z\})$ such that $g_0 = \mathbf{1}, g_t|_D = f_t|_D$. Pick a point w in the interior of D. Note the path $\{g_t^{-1} f_t\}$ is compactly supported in $S^n \setminus \{w\}$. Thus the diffeomorphisms $g := g_1$ and $h := g^{-1} f$ are as required in the lemma. ■

3.3. Three-manifolds

Here we prove Theorem 1.11(iii). By a *graph* in a manifold we mean a piecewise smoothly embedded graph. By a smooth isotopy of a graph we mean an isotopy which extends to a smooth isotopy of its tubular neighborhood. We shall use without a special mentioning the following fact (see e.g. [20]): any smooth compactly supported diffeomorphism ϕ of an open handlebody U is isotopic to the identity through compactly supported diffeomorphisms, that is $f \in \mathrm{Diff}_0(U)$.

Lemma 3.4 (Fundamental Lemma)**.** *Let Γ and K be two disjoint graphs and M. Let $f_t : \Gamma \to M$, $t \in [0;1]$ be a smooth isotopy with $f_0|_\Gamma = \mathbf{1}$ and $f_1(\Gamma) \cap K = \emptyset$. Then there exist a diffeomorphism h of M supported in a ball and a diffeomorphism $\phi \in \mathit{Diff}_0(M \setminus K)$ so that*

$$f_1|_\Gamma = h \circ \phi|_\Gamma \ .$$

Let us prove the theorem assuming the lemma.

Proof of Theorem 1.11(iii): Take any norm ν on $\mathrm{Diff}_0(M)$. A graph is called *the Heegard graph* if its complement is diffeomorphic to an open handlebody. Every three-manifold contains a Heegard graph (for instance, a neighborhood of the 1-skeleton of a triangulation of M). Choose a pair of disjoint Heegard graphs L and K in M. Fix a sufficiently small tubular neighborhood U of L. Since $U, M \setminus K$ and $M \setminus L$

are open handlebodies and therefore are portable, Theorem 1.17 implies that the norm ν, when restricted to Diff_0 of these submanifolds, does not exceed some constant $C > 0$. We shall assume also that the same inequality holds for the restriction of ν to Diff_0 of any ball in M (we use here that all balls are pair-wise isotopic and portable).

We shall show that

$$\nu(f) \leq 5C \tag{9}$$

for every $f \in \mathrm{Diff}_0(M)$ with $f(U) \cap K = \emptyset$. Note that this yields the same inequality for *every* f. Indeed, perturbing K to K' by a small ambient isotopy of M and shrinking U to U' by an ambient isotopy of M we can always achieve that $f(U') \cap K' = \emptyset$. But the subgroups $\mathrm{Diff}_0(U')$ and $\mathrm{Diff}_0(M \setminus K')$ are conjugate in $\mathrm{Diff}_0(M)$ to $\mathrm{Diff}_0(U)$ and $\mathrm{Diff}_0(M \setminus K)$ respectively, and hence the restriction of the norm ν to these subgroups is bounded by the same constant C which yields inequality (9). From now on we assume that $f(U) \cap K = \emptyset$.

Let $N \subset U \setminus L$ be any embedded graph so that the induced homomorphism $\pi_1(N) \to \pi_1(U \setminus L)$ is a surjection. Put $\Gamma = L \cup N$, and apply the Fundamental Lemma. We get a diffeomorphism h supported in a ball, and a diffeomorphism $\phi \in \mathrm{Diff}_0(M \setminus K)$ so that $f|_\Gamma = h \circ \phi|_\Gamma$. Denote $\psi = (h\phi)^{-1} f$ and observe that $\psi|_\Gamma = \mathbf{1}$.

In particular, ψ fixes L. We wish to correct ψ and get a diffeomorphism fixing *a neighborhood* of L. This is the point where the graph N enters the play. More precisely, we claim that there exist diffeomorphisms $\xi, \theta \in \mathrm{Diff}_0(U)$ and $\eta \in \mathrm{Diff}_0(M \setminus L)$ so that $\psi = \xi\eta\theta$. Indeed, since ψ fixes L, there exists a sufficiently small tubular neighborhood $V \subset U$ of L and a diffeomorphism $\theta \in \mathrm{Diff}_0(U)$ so that $\psi\theta^{-1}(V) = V$. Put $\tau := \psi\theta^{-1}$. Since $U \setminus L$ retracts to ∂V and ψ fixes N we conclude that τ induces the identity isomorphism of $\pi_1(\partial V)$. It is well known (see e.g. [36, 19, 20]) that therefore $\tau|_V : V \to V$ is isotopic to the identity. Hence there exists a diffeomorphism $\xi \in \mathrm{Diff}_0(U)$ which coincides with τ on V, and so $\eta := \xi^{-1}\tau$ is supported in $M \setminus L$. The claim follows.

Finally, write

$$f = h\phi\psi = h\phi\xi\eta\theta\ .$$

Note that $h \in \mathrm{Diff}_0(B)$ where B is a ball, and hence $\nu(h) \leq C$ where the constant C was chosen in the beginning of the proof. Furthermore, $\phi \in \mathrm{Diff}_0(M \setminus K)$, $\xi, \theta \in \mathrm{Diff}_0(U)$ and $\eta \in \mathrm{Diff}_0(M \setminus L)$. Thus $\nu(f) \leq 5C$ which proves inequality (9). This completes the proof. ■

Proof of Theorem 1.13(ii): In the proof above we represented every diffeomorphism from Diff_0 of a closed connected three-manifold M as a product of 5 diffeomorphisms from Diff_0 of portable manifolds. Applying Theorem 1.18 we get the desired estimate $cld(M) \leq 10$. ■

Proof of Lemma 3.4: The proof is divided into several steps.

STEP 1: Let $\Gamma, L \subset M$ be disjoint embedded graphs, and $f_t : \Gamma \to M$ be a smooth isotopy. Put $\Gamma_t := f_t(\Gamma)$. We say that the crossing point $y = f_\tau(x) \in \Gamma_\tau \cap L$ is *generic* if the points x and y lie in smooth interior parts of Γ and L respectively and

$$f_{\tau *}(T_x\Gamma) \oplus T_y L \oplus \mathbb{R} \cdot \frac{\partial}{\partial t}\Big|_{t=\tau} f_t x = T_y M \ .$$

Introduce two modifications of the isotopy f_t at a generic crossing point.

TYPE I MODIFICATION (REMOVING THE CROSSING POINT): Here we assume that L is **a segment** with the endpoints A and B and $y = \Gamma_\tau \cap L$ is a generic crossing point. Choose $\epsilon > 0$ small enough so that y is the only crossing point on the time interval $I := [\tau - \epsilon; \tau + \epsilon]$. Choose a sufficiently small neighborhood U of L. Let $h_s, s \in I$ be a path in $\mathrm{Diff}_0(U)$ so that $h_s = \mathbf{1}$ outside a small neighborhood of $s = \tau$, $h_s(L) \subset L$ and $h_s(B) = B$ for all s, and h_τ shrinks L so that $y \notin h_\tau(L)$. Replace the piece $\{\Gamma_t\}_{t\in I}$ of the original isotopy by $\{\Gamma'_t\}_{t\in I}$ where $\Gamma'_t = h_t^{-1}\Gamma_t$. Note that $\Gamma_t \cap h_t(L) = \emptyset$, and hence $\Gamma'_t \cap L = \emptyset$, for all $t \in I$.

TYPE II MODIFICATION (DECOMPOSITION): Here Γ and L are arbitrary graphs, and $y = f_\tau(x) \in \Gamma_\tau \cap L$ is a generic crossing point. Choose $\epsilon > 0$ small enough so that y is the only crossing point on the time interval $I := [\tau - 2\epsilon; \tau + 2\epsilon]$. There exists a neighborhood E of y diffeomorphic to a Euclidean cube

$$Q = \{(u, v, w) \in \mathbb{R}^3 \ \Big| \ |u|, |v|, |w| < 2\epsilon\}$$

so that $L \cap Q$ is the vertical segment $\{u = v = 0, w \in [-2\epsilon; 2\epsilon]\}$ and $\Gamma_t \cap Q$ is the segment $c_{t-\tau} := \{u = t - \tau, v \in [-2\epsilon; 2\epsilon], w = 0\}$ for $t \in I$. Thus the isotopy Γ_t inside Q is given by the motion of the segment $c_{-2\epsilon}$ in the (u, v)-plane in the direction of the u-axis. In this picture, the crossing point y is the origin.

Let us agree on the following wording: Suppose that two curves α_0 and α_1 in the (u, v)-plane are given by the graphs $\{u = F_0(v)\}$ and $\{u = F_1(v)\}$ of smooth functions $F_0, F_1 : [-2\epsilon; 2\epsilon] \to \mathbb{R}$. The *linear isotopy* between α_0 and α_1 is formed by graphs of $(1 - s)F_0 + sF_1$, $s \in [0; 1]$.

The modification we are going to describe is local. Fix a smooth cut-off function $\rho : [-2\epsilon; 2\epsilon] \to [0; 3\epsilon/2]$ which is supported in a very small neighborhood of 0 and which satisfies $\rho(0) = 3\epsilon/2$. Denote $\beta^{\pm} = c_{\pm\epsilon}$. Consider the curve

$$\alpha = \{u = -\epsilon + \rho(v), v \in [-2\epsilon; 2\epsilon], w = 0\} .$$

Modify the original isotopy on the time interval $I' := [\tau - \epsilon; \tau + \epsilon]$ as follows: first make a linear isotopy from β^- to α, and then a linear isotopy from α to β^+. We extend the curves appearing in the process of this isotopy outside Q by appropriate Γ_t's and make an obvious change of time in order to fit into the time interval I'.

The following features of the modified isotopy are crucial for our further purposes. The isotopy from β^- to α can be realized by an isotopy of diffeomorphisms of M supported in a ball $B \subset Q$. The isotopy from α to β^+ does not hit L and hence can be extended to an ambient isotopy of M which is fixed near L.

STEP 2: After these preliminaries, we pass to the situation described in the formulation of the lemma: Let Γ, K be two disjoint graphs in M and let $f_t : \Gamma \to M$, $t \in [0; 1]$ be a smooth isotopy with $f_1(\Gamma) \cap K = \emptyset$. After a small perturbation of the isotopy with fixed end points we can assume that the following conditions hold:

(C1) The set

$$\{(x,t) \in \Gamma \times [0;1] \;\Big|\; f_t(x) \in K\}$$

consists of N pairs (x_i, t_i), $i = 1, ..., N$ so that $\{x_i\}$ are distinct points of Γ, $0 < t_1 < ... < t_N < 1$ and $y_i = f_{t_i}(x)$ are distinct generic crossing points.

(C2) The curves $\gamma_i := \{f_t(x_i)\}_{t\in[0;1]}$ are pairwise disjoint embedded segments.

(C3) For each i, the isotopy $f_t : \Gamma \setminus \{x_i\} \to M$ crosses γ_i generically.

We shall remove the latter crossings using the Type I modification (see Step 1): Note that each such crossing occurs in the subsegment of γ_i which is either of the form $[x_i; f_{t_i-\delta}x_i]$ or $[f_{t_i+\delta}x_i; f_1x_i]$, where $\delta > 0$ is small enough. We apply Type I modification to these segments keeping the end point $f_{t_i\pm\delta}x_i$ fixed (such an end point is denoted by B in the local description of a Type I modification above). Note that each such modification is localized near some γ_i and hence does not create new crossings, so the process stops after a finite number of modifications. Thus we replace assumption (C3) above by a stronger one:

(C3') For each i, the isotopy $f_t : \Gamma \setminus \{x_i\} \to M$ does not hit γ_i.

STEP 3: It would be convenient to make a change of time in our isotopy as follows. We assume that f_t is defined on the time interval $t \in [0; N+1]$ and the crossings times are consecutive integers $t_i = i$, $i = 1, ..., N$. Assumptions (C1) and (C2) of the previous step yield existence of embedded pair-wise disjoint parallelepipeds $P_i \subset M$, $i = 1, ..., N$ (each parallelepiped P_i is a neighborhood of the segment γ_i) equipped with local coordinates $u \in [-1; N+2], v \in [-1; 1], w \in [-1; 1]$ so that the following holds:

$$\gamma_i = \{(u, 0, 0) \mid u \in [0; N+1]\} ,$$

$$K \cap P_i = \{(i, 0, w) \mid w \in [-1; 1]\}, \quad \Gamma \cap P_i = \{(0, v, 0) \mid v \in [-1; 1]\},$$

and

$$f_t(0, v, 0) = (t, v, 0) \ \forall t \in [0; N+1], v \in [-1; 1] .$$

In addition, assumption (C3') of the previous step guarantees that P_i's can be chosen so thin that

$$f_t(\Gamma \setminus P_i) \cap P_i = \emptyset \ \forall t \in [0; N+1] . \tag{10}$$

STEP 4: Let $Q_i \subset P_i$ be a sufficiently small cube centered at the crossing $(i, 0, 0)$ whose edges have the length 4ϵ and are parallel to the coordinate axes. Perform a Type II modification of our isotopy inside Q_i: We keep notations $\alpha_i, \beta_i^{\pm}$ (with the extra sub-index i) for special curves appearing in the description of the modification presented in Step 1. The reader should have in mind that the current u-coordinate is shifted by i in comparison to the one of Step 1, and the crossing time τ equals i.

Thus we assume that

$$\beta_i^{\pm} = \{(i \pm \epsilon, v, 0) \mid v \in [-2\epsilon, 2\epsilon] \} .$$

Set

$$\Gamma_i^- = f_{i-\epsilon}(\Gamma) \quad \text{and} \quad \Gamma_i^+ = (f_{i-\epsilon}(\Gamma) \setminus \beta_i^-) \cup \alpha_i \ , \quad i = 1, ..., N .$$

Note that $\Gamma_i^+ = h_i(\Gamma_i^-)$, where $h_i \in \mathrm{Diff}_0(Q_i)$.

It will be convenient to put $\Gamma_0^+ = \Gamma$ and $\Gamma_{N+1}^- = f_1(\Gamma)$. Recall that we write $\Gamma_t = f_t(\Gamma)$.

STEP 5: Fix $i \in \{0; ...; N\}$. Let us focus on the following isotopy taking Γ_i^+ to Γ_{i+1}^- : we proceed according to the description of the Type II modification (see Step 1) until we reach the graph $\Gamma_{i+\epsilon}$ which extends β_i^+

(this move is empty when $i = 0$), and then move on with the original isotopy f_t until Γ_{i+1}^-. Note that this isotopy does not hit K. Furthermore, the (time-dependent) vector field $\zeta_t^{(i)}$ of this isotopy, which is defined along the image of Γ_i^+ at the time moment t, is parallel to the u-axis *in each of the parallelepipeds* P_j, $j = 1, ..., N$. Now we shall use property (10) of the original isotopy: It guarantees that one can cut off $\zeta_t^{(i)}$ near K and extend it to the whole M so that *it remains parallel to the u-axis in all P_j's.* After such an extension we get an isotopy supported in $M \setminus K$ so that its time-1-map ϕ_i sends Γ_i^+ to Γ_{i+1}^-.

The following property of maps ϕ_i, which readily follows from the above discussion on vector fields $\zeta_t^{(i)}$, is crucial for the final step of the proof:

$$\phi_N \circ ... \circ \phi_i(Q_i) \subset P_i \ \ \forall i = 1, ..., N \ . \tag{11}$$

STEP 6: We have

$$f_1|_\Gamma = \phi_N h_N \circ ... \circ \phi_1 h_1 \phi_0|_\Gamma \ , \tag{12}$$

where the diffeomorphisms $h_i \in \text{Diff}_0(Q_i)$ and the cubes Q_i appear in Step 4, and the diffeomorphisms $\phi_i \in \text{Diff}_0(M \setminus K)$ are constructed in the previous step. Put

$$g_i = (\phi_N \circ ... \circ \phi_i) h_i (\phi_N \circ ... \circ \phi_i)^{-1} \ , \ i = 1, ..., N \ .$$

Note that $g_i \in \text{Diff}_0(Q_i')$ where $Q_i' = \phi_N \circ ... \circ \phi_i(Q_i)$. By (11), the sets Q_i' are pair-wise disjoint. Since each of Q_i' is diffeomorphic to an open ball, the diffeomorphism $h := g_N \circ ... \circ g_1$ is supported in a ball. Finally, put

$$\phi = \phi_N \circ ... \circ \phi_0 \in \text{Diff}_0(M \setminus K)$$

and observe that in view of equation (12) $f_1|_\Gamma = h\phi|_\Gamma$. This finishes off the proof of the lemma. ■

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