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Calculating the image of the second Johnson-Morita representation

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Abstract.

Johnson has defined a surjective homomorphism from the *Torelli subgroup* of the mapping class group of the surface of genus g with one boundary component to $\wedge^3 H$, the third exterior product of the homology of the surface. Morita then extended Johnson's homomorphism to a homomorphism from the entire mapping class group to $\frac{1}{2} \wedge^3 H \rtimes \mathrm{Sp}(H)$. This *Johnson-Morita homomorphism* is not surjective, but its image is finite index in $\frac{1}{2} \wedge^3 H \rtimes \mathrm{Sp}(H)$ [11]. Here we give a description of the exact image of Morita's homomorphism. Further, we compute the image of the *handlebody subgroup* of the mapping class group under the same map.

§1. Introduction

Let S_g be a closed surface of genus g. We fix a closed disk D in S_g, and by deleting its interior, obtain $S_{g,1}$, a genus g surface with one boundary component, as illustrated in Figure 1. Let $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) denote the mapping class group of the surface S_g (resp. $S_{g,1}$). In the case of $\mathcal{M}_{g,1}$ we assume the boundary component is fixed pointwise.

We choose a base point on $\partial S_{g,1}$, and let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ denote the based loops illustrated in Figure 1(b). Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ denote the corresponding homology classes, as in Figure 1(a). It will sometimes be convenient to denote these same homology classes by $x_1, \ldots, x_{2g}$ with the understanding that $x_i = a_i$ and $x_{i+g} = b_i$ for

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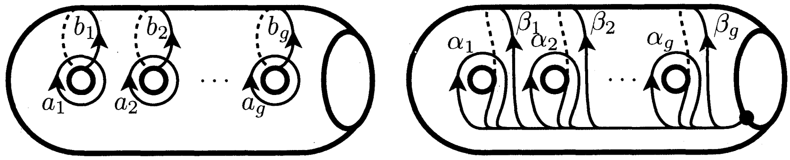

Fig. 1. (a) A basis for $H_1(S_{g,1})$ (b) Generators for $\pi_1(S_{g,1})$

$1 \leq i \leq g$. Likewise, we will sometimes refer to the based loops $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ by $\xi_1, \dots, \xi_{2g}$ with the understanding that $\xi_i = \alpha_i$ and $\xi_{i+g} = \beta_i$ for $1 \leq i \leq g$.

Now, let $H = H_1(S_{g,1})$ be the free abelian group with generating set $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ and $\pi = \pi_1(S_{g,1})$ which is a free group on the generating set $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$. The action of $\mathcal{M}_{g,1}$ on π gives an injection $\mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}(\pi)$. More generally, we can compose with the homomorphism $\mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi/\chi)$ for any characteristic subgroup $\chi \subset \pi$. The *lower central series* of the free group π is a sequence of characteristic subgroups defined inductively by setting $\pi^{(0)} = \pi$ and $\pi^{(k+1)} = [\pi, \pi^{(k)}]$. We define the k^{th} *Johnson-Morita representation* to be the map

$$\rho_k : \mathcal{M}_{g,1} \to \mathrm{Aut}(\pi/\pi^{(k)})$$

We note that these maps were first studied by Johnson in [7, 6] and subsequently developed by Morita in a series of papers [11, 12, 13, 14].

Observe that the first Johnson-Morita map is just the classical symplectic representation $\rho_1 : \mathcal{M}_{g,1} \to \mathrm{Sp}(H)$ which is surjective ([4], in particular pp. 209-212). In [11, Theorem 4.8] Morita shows that the image of ρ_2 is isomorphic to a subgroup of finite index in $\frac{1}{2} \wedge^3 H \rtimes \mathrm{Sp}(H)$. Our first main result in this paper, given in Theorem 2.4, is to identify the precise image $\rho_2(\mathcal{M}_{g,1})$ using a formulation due to Perron [16].

Let us now consider S_g as ∂X_g, where X_g is a genus g handlebody. Let $\mathcal{H}_g$ denote the *handlebody subgroup* of $\mathcal{M}_g$, that is, the subgroup consisting of maps of S_g which extend to the handlebody X_g. There is a natural surjection $\mathcal{M}_{g,1} \to \mathcal{M}_g$ obtained by extending via the identity map along D. The kernel of this surjection is generated by two kinds of elements: the Dehn twist along the boundary curve, and "push" maps along elements of $\pi_1(S_{g,1})$ [1]. Note that any map in this kernel extends to X_g. Hence, we are justified in defining the handlebody subgroup $\mathcal{H}_{g,1}$ of $\mathcal{M}_{g,1}$ as the pullback of $\mathcal{H}_g$.

The handlebody group arises naturally in a number of applications in 3-manifold topology, particularly through Heegaard splittings of 3-manifolds. Our second result in this paper is to compute $\rho_2(\mathcal{H}_{g,1})$, given in Theorem 3.5.

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§2. The second Johnson-Morita map

In this section we will describe Perron's formulation [16] of the second Johnson-Morita representation. We will give a precise characterization of the image of the mapping class group under this map. First, it will be useful to review the image of the first Johnson-Morita representation, i.e., the symplectic group.

2.1. The symplectic group

The group $H = H_1(S_{g,1})$ is free abelian with free basis $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, as in Figure 1(a), and has a symplectic intersection form given by signed intersection of curves which is preserved by every mapping class $f \in \mathcal{M}_{g,1}$. In the basis above, the intersection form is given by the the matrix J with $g \times g$ block form

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{1}$$

The intersection form got by acting by the linear transformation M on an intersection form with matrix L is given by $ML\overline{M}$ where $\overline{M}$ denotes the transpose of M. Hence for every M in the image of the mapping class group

$$MJ\overline{M} = J, \qquad \text{or equivalently} \qquad \overline{M}JM = J \tag{2}$$

In fact (2) is a sufficient condition for M to be in the image of the mapping class group under ρ_1. It is sometimes useful to write a symplectic matrix M in $g \times g$ block form as

$$M = \begin{pmatrix} S & T \\ P & Q \end{pmatrix}$$

A convenient consequence of (2) is that $M^{-1} = J\overline{M}J^{-1}$. In block form this becomes

$$\begin{pmatrix} S & T \\ P & Q \end{pmatrix}^{-1} = \begin{pmatrix} \overline{Q} & -\overline{T} \\ -\overline{P} & \overline{S} \end{pmatrix}$$

The group of such matrices form the *symplectic group.* Writing M and $\overline{M}$ in $g \times g$ block form

$$M = \begin{pmatrix} S & T \\ P & Q \end{pmatrix}, \qquad \overline{M} = \begin{pmatrix} \overline{S} & \overline{P} \\ \overline{T} & \overline{Q} \end{pmatrix}$$

we derive the *symplectic constraints*, which follow directly from the condition in (2):

(3) (i) $Q\overline{S} - P\overline{T} = I$, (ii) $S\overline{T}$ symmetric, (iii) $P\overline{Q}$ symmetric.

2.2. Perron's formulation of ρ_2

The *Torelli group* $\mathcal{I}_{g,1}$ is the kernel of the symplectic representation $\rho_1 : \mathcal{M}_{g,1} \to \mathrm{Sp}(H)$. Johnson proved, in [5], that the image of the Torelli group under ρ_2 is $\wedge^3 H$. In the next section we will go a step further, and describe, in Theorem 2.4, the image of the full mapping class group $\mathcal{M}_{g,1}$ under ρ_2 noting that Morita [11, Theorem 4.8] has already identified this image as being finite index in $\frac{1}{2} \wedge^3 H \rtimes \mathrm{Sp}(H)$. We begin by summarizing Morita's explicit description of ρ_2 as given in [11, Section 4]. Consider the 2-step nilpotent group

$$\Phi_2 = \left\{ (\eta, y) \middle| \eta \in \frac{1}{2} \wedge^2 H,\ y \in H \right\}$$

with multiplication in Φ_2 given by $(\eta, y)(\nu, z) = (\eta + \nu + \frac{1}{2} y \wedge z, y + z)$. It contains a subgroup of finite index which can be identified (see [8, Sec. 5.5]) with the second nilpotent quotient $\pi/\pi^{(2)} = \pi/[\pi, [\pi, \pi]]$ of our surface group via the homomorphism $\phi_2 : \pi \to \Phi_2$

$$\phi_2(\xi_i) = (0, x_i)$$

where $\{\xi_1, \cdots, \xi_{2g}\}$ generate $\pi = \pi_1(S_{g,1})$ and $\{x_1, \cdots, x_{2g}\}$ is our basis for $H = H_1(S_{g,1})$ (see Figure 1(a-b)). The group Φ_2 can be viewed as a subgroup of the Mal'cev completion of the nilpotent group $\pi/\pi^{(2)}$. Any automorphism of $\pi/\pi^{(2)}$ extends to the Mal'cev completion and preserves Φ_2 so we may think of $\mathcal{M}_{g,1}$ as acting on Φ_2 [11, Proposition 2.5].

In [11, Section 3] Morita describes a function $\mathcal{M}_{g,1} \to \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$. An automorphism f of Φ_2 coming from an automorphism of the Mal'cev completion of $\pi/\pi^{(2)}$ can be specified by the images

$$f(0, x_i) = (w_i, h_i) \qquad w_i \in \frac{1}{2} \wedge^2 H,\ h_i \in H$$

for each x_i. The homomorphism $\rho_1(f) : H \to H$ given by $\rho_1(f)(x_i) = h_i$ is just the image of f under the symplectic representation. Johnson looks at the homomorphism $\tilde{\tau}_2(f) : H \to \frac{1}{2} \wedge^2 H$ given by

$$\tilde{\tau}_2(f)(x_i) = w_i$$

The function $\tilde{\tau}_2 : \mathcal{M}_{g,1} \to \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$ is a homomorphism when restricted to the kernel $\mathcal{I}_{g,1}$ of the symplectic representation. Johnson [5, Theorem 1] identifies its image as $\wedge^3 H \subset \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$, where $x_i \wedge x_j \wedge x_k \in \wedge^3 H$ is understood to be the homomorphism

$$(4) \quad (x_i \wedge x_j \wedge x_k)(y) = \langle y, x_k \rangle x_i \wedge x_j + \langle y, x_i \rangle x_j \wedge x_k + \langle y, x_j \rangle x_k \wedge x_i$$

where $\langle , \rangle$ gives the symplectic pairing for vectors in H. The map $\mathcal{I}_{g,1} \to \wedge^3 H \subset \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$ is usually referred to as the *Johnson homomorphism*.

Morita [11, Section 3] begins by considering this map $\tilde{\tau}_2 : \mathcal{M}_{g,1} \to \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$ (in Morita's notation this is the map $\tilde{k}$). While not a homomorphism it is a crossed homomorphism with respect to the symplectic action of the mapping class group on $\mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$. In other words, the map $\tilde{\tau}_2$ satisfies:

$$\tilde{\tau}_2(fg) = \tilde{\tau}_2(f) + \rho_1(f)\tilde{\tau}_2(g) \qquad f, g \in \mathcal{M}_{g,1}$$

Choose $R \in \mathrm{Sp}(H)$, $y \in H$, and $m \in \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$. We note that the action of $\mathrm{Sp}(H)$ on $\mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$ in the equation above (and in the remainder of this paper) is the natural "change-of-basis" action:

$$(5) \qquad (Rm)(y) = Rm(R^{-1}y)$$

The crossed homomorphism property is exactly what is needed for the map $\tilde{\rho}_2 : \mathcal{M}_{g,1} \to \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H) \rtimes \mathrm{Sp}(H)$ given by

$$\tilde{\rho}_2(f) = (\tilde{\tau}_2(f), \rho_1(f))$$

to be a homomorphism. The homomorphism $\tilde{\rho}_2$ gives the action of $\mathcal{M}_{g,1}$ on $\phi_2(\pi) \subset \Phi_2$, via the action of $(r, R) \in \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H) \rtimes \mathrm{Sp}(H)$ on Φ_2:

$$(6) \qquad (r, R) * (\eta, y) = (r(Ry) + R\eta, Ry)$$

Morita shows that by modifying the crossed homomorphism $\tilde{\tau}_2 : \mathcal{M}_{g,1} \to \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$, one obtains a crossed homomorphism $\tilde{\tau}_2'$ (Morita denotes this map by $\tilde{k}'$ in [11, Section 4] and $\tilde{k}$ in [11, Section 5]) from $\mathcal{M}_{g,1}$ to the submodule $\frac{1}{2} \wedge^3 H$ of $\mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$ which extends the Johnson homomorphism. We will modify $\tilde{\tau}_2$ to get a different crossed homomorphism $\tau_2 : \mathcal{M}_{g,1} \to \frac{1}{2} \wedge^3 H$ extending the Johnson homomorphism. Our map τ_2 is a trivial modification of Morita's map $\tilde{\tau}_2'$ which will lend itself to later calculations.

For any $m \in \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$, the map $\sigma_m : \mathcal{M}_{g,1} \to \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$ given by

$$\sigma_m(f) = m - \rho_1(f)m$$

is a crossed homomorphism. Such a crossed homomorphism is called *principal*; two crossed homomorphisms are cohomologous if they differ by a principal crossed homomorphism [3, Chapter IV.2].

Let $\kappa \in \mathrm{Hom}(H, \frac{1}{2} \wedge^2 H)$ be the homomorphism

$$\kappa(a_i) = \frac{1}{2} a_i \wedge b_i \qquad \kappa(b_i) = -\frac{1}{2} a_i \wedge b_i$$

or equivalently

$$\kappa(x_i) = \frac{1}{2} x_i \wedge C x_i \tag{7}$$

where C is the $2g \times 2g$ matrix with $g \times g$ block form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Define

$$\tau_2(f) = \tilde{\tau}_2(f) + \kappa - \rho_1(f)\kappa \tag{8}$$

This is the crossed homomorphism that Perron [16, Remark 5.5] denotes $-\frac{1}{6}\widetilde{A_1}$. We note that by comparing the above with [11, Proposition 4.7], it is straightforward to see that Morita's crossed homomorphism $\tilde{\tau}_2'$ can be expressed as

$$\tilde{\tau}_2'(f) = \tau_2(f) + m - \rho_1(f)m$$

where $m = -\frac{1}{2}(\sum_{i=1}^{g} a_i + b_i) \wedge (\sum_{i=1}^{g} a_i \wedge b_i)$. In other words, the map τ_2 and Morita's original map $\tilde{\tau}_2'$ are cohomologous, that is, they represent the same element of $H^1(\mathcal{M}_{g,1}, \frac{1}{2} \wedge^3 H)$.

We can now define a homomorphism $\rho_2 : \mathcal{M}_{g,1} \to \frac{1}{2} \wedge^3 H \rtimes \mathrm{Sp}(H)$ as follows:

$$\rho_2(f) = (\tau_2(f), \rho_1(f))$$

Using (8), (6), (5), and (4), we obtain the correct action of $\rho_2(\mathcal{M}_{g,1})$ on Φ_2:

$$\begin{aligned}
&\Big(\sum r_{ijk} x_i \wedge x_j \wedge x_k, R\Big) * (\eta, y) \\
(9)\qquad &= (R\eta - \kappa(Ry) + R(\kappa(y)) + r(y), Ry) \\
(10)\qquad &= \left(R\eta - \kappa(Ry) + R(\kappa(y)) + \sum r_{ijk} \begin{pmatrix} \langle Ry, x_k\rangle x_i \wedge x_j \\ + \langle Ry, x_i\rangle x_j \wedge x_k \\ + \langle Ry, x_j\rangle x_k \wedge x_i \end{pmatrix}, Ry \right)
\end{aligned}$$

where $\langle , \rangle$ is the symplectic pairing on H and the sums are taken over $1 \le i < j < k \le 2g$.

2.3. Calculating the image of the mapping class group

In this section we compute $\rho_2(\mathcal{M}_{g,1})$. See Theorem 2.4 below.

Recall the map $\phi_2 : \pi \to \Phi_2$ given in the previous section. It will be helpful for us to identify $\phi_2(\pi) \subset \Phi_2$ precisely. The gist of the following lemma is that for pairs in the image of ϕ_2, the second coordinate determines the first coordinate modulo 1.

Lemma 2.1. *The image of π under the map ϕ_2 is given as follows.*

$$\phi_2(\pi) = \left\{ \left(\sum_{1<i<j<2g} \left(n_{ij} + \frac{l_i l_j}{2} \right) x_i \wedge x_j \, , \, \sum_{i=1}^{2g} l_i x_i \right) \,\middle|\, n_{ij}, \ l_i \in \mathbb{Z} \right\}$$

Proof. Let $G \subset \Phi_2$ denote the set on the right-hand side of the equation in the lemma. We claim that the set G is a subgroup of Φ_2. First, G is closed under inversion since $(\eta, y)^{-1} = (-\eta, -y)$. For closure under products consider

$$\begin{aligned}
&\left(\sum_{1<i<j<2g} \left(n_{ij} + \frac{l_i l_j}{2} \right) x_i \wedge x_j \, , \, \sum_{i=1}^{2g} l_i x_i \right) \\
&\qquad \cdot \left(\sum_{1<i<j<2g} \left(n'_{ij} + \frac{l'_i l'_j}{2} \right) x_i \wedge x_j \, , \, \sum_{i=1}^{2g} l'_i x_i \right) \\
&= \left(\sum_{1<i<j<2g} \begin{pmatrix} n_{ij} + n'_{ij} + \frac{l_i l_j}{2} \\ + \frac{l'_i l'_j}{2} + \frac{l_i l'_j}{2} - \frac{l_j l'_i}{2} \end{pmatrix} x_i \wedge x_j \, , \, \sum_{i=1}^{2g} (l_i + l'_i) x_i \right)
\end{aligned}$$

This product is in G because $l_i l_j + l'_i l'_j + l_i l'_j - l_j l'_i \equiv (l_i + l'_i)(l_j + l'_j) \bmod 2$.

Clearly, G contains each generator $\phi_2(\xi_i) = (0, x_i)$ of $\phi_2(\pi)$. For the reverse inclusion, note that any element of the form

$$(0, x_i)(0, x_j)(0, -x_i)(0, -x_j) = (x_i \wedge x_j, 0)$$

lies in $\phi_2(\pi)$. In fact such an element is in the center of G. Now, any element of G can be written as a product of $(0, x_i)$'s to get the correct second coordinate, followed by a product of $(x_i \wedge x_j, 0)$'s to get the correct first coordinate. Hence $G \subset \phi_2(\pi)$. Q.E.D.

We are almost ready to characterize the subgroup $\rho_2(\mathcal{M}_{g,1}) \subset \frac{1}{2} \wedge^3 H \rtimes \mathrm{Sp}(H)$. We begin with a simple yet fundamental observation.

Remark 2.2. Suppose R is a symplectic matrix and (r_1, R), $(r_2, R) \in \rho_2(\mathcal{M}_{g,1})$. Then $(r_1, R)^{-1} = (-R^{-1}r_1, R^{-1}) \in \rho_2(\mathcal{M}_{g,1})$ so

$$(r_2, R)(-R^{-1}r_1, R^{-1}) = (r_2 - r_1, I) \in \rho_2(\mathcal{M}_{g,1}).$$

In other words, we have that $(r_2 - r_1, I) \in \rho_2(\mathcal{I}_{g,1})$. Using Johnson's characterization of $\tau_2(\mathcal{I}_{g,1})$ [5, Theorem 1] we conclude that if two elements of $\rho_2(\mathcal{M}_{g,1})$ have identical symplectic matrices, then their $\frac{1}{2} \wedge^3 H$ coordinate must differ by an *integral* element of $\wedge^3 H$.

As a consequence of this observation, we expect that the symplectic matrix R will determine the coefficients of r_1 and r_2 modulo 1. Theorem 2.4 makes this precise and gives the characterization of $\rho_2(\mathcal{M}_{g,1})$. First we give a short definition.

Definition 2.3. Given three n-dimensional vectors $\vec{w} = (w_1, \ldots, w_n)$, $\vec{y} = (y_1, \ldots, y_n)$, $\vec{z} = (z_1, \ldots, z_n)$ in basis $\mathcal{B}$, their *$\mathcal{B}$-triple dot product* is the scalar

$$\bullet_{\mathcal{B}}(\vec{w}, \vec{y}, \vec{z}) = \sum_{i=1}^{n} w_i y_i z_i.$$

When the basis $\mathcal{B}$ is clear, we will write $\bullet(\vec{w}, \vec{y}, \vec{z})$.

Recall that J is the matrix given in (1).

Theorem 2.4. *Let $R \in \mathrm{Sp}(2g, \mathbb{Z})$ be an arbitrary symplectic matrix. Let r be any element of $\frac{1}{2} \wedge^3 H$ with $r = \sum_{1 \le i < j < k \le 2g} r_{ijk} x_i \wedge x_j \wedge x_k$. Then $(r, R) \in \rho_2(\mathcal{M}_{g,1})$ if and only if*

$$r_{ijk} \equiv \frac{E_{ijk}}{2} \bmod 1$$

where

$$\begin{aligned} E_{ijk} \quad = \quad & \bullet(\mathrm{row}_i(RJ), \mathrm{row}_j(R), \mathrm{row}_k(R)) \\ & - \bullet\, (\mathrm{row}_i(R), \mathrm{row}_j(RJ), \mathrm{row}_k(R)) \\ & + \bullet\, (\mathrm{row}_i(R), \mathrm{row}_j(R), \mathrm{row}_k(RJ)) \end{aligned}$$

for all $1 \le i < j < k \le 2g$.

Proof. Let $(r, R) \in \rho_2(\mathcal{M}_{g,1})$, and let

$$r = \sum_{1 \leq i < j < k \leq 2g} r_{ijk} x_i \wedge x_j \wedge x_k.$$

For $1 \leq i, j, k \leq 2g$ we set $r_{ijk} = 0$ unless $i < j < k$. The group $\rho_2(\mathcal{M}_{g,1})$ preserves $\phi_2(\pi)$, described in Lemma 2.1. Let x_n be an arbitrary basis element of H, and consider the action of (r, R) on $(0, x_n)$. We will use the standard notation M_{ij} to denote the entry in the i^{th} row and j^{th} column of a matrix M throughout. By (10), we get that the second coordinate of $(r, R) * (0, x_n)$ is simply Rx_n, which we can write as $\sum_{i=1}^{2g} R_{in} x_i$, with an eye on eventually applying Lemma 2.1. Using (10) and (7), we obtain the following for the first coordinate of $(r, R) * (0, x_n)$:

$$-\kappa(Rx_n) + R(\kappa(x_n)) + \sum_{1 \leq i < j < k \leq 2g} r_{ijk} \begin{pmatrix} \langle Rx_n, x_k \rangle x_i \wedge x_j \\ +\langle Rx_n, x_i \rangle x_j \wedge x_k \\ -\langle Rx_n, x_j \rangle x_i \wedge x_k \end{pmatrix}$$

Notice that under the symplectic pairing $\langle Rx_n, x_k \rangle = (JR)_{kn}$ so the above can be rewritten:

$$\begin{aligned}
& -\kappa\left(\sum_{i=1}^{2g} R_{in} x_i\right) + R\left(\frac{1}{2} x_n \wedge C x_n\right) \\
& + \sum_{1 \leq i < j < k \leq 2g} r_{ijk} \begin{pmatrix} ((JR)_{kn}) x_i \wedge x_j \\ +((JR)_{in}) x_j \wedge x_k \\ -((JR)_{jn}) x_i \wedge x_k \end{pmatrix} \\
= \; & -\left(\sum_{i=1}^{2g} \frac{R_{in}}{2} x_i \wedge C x_i\right) + \left(\sum_{1 \leq i,j \leq 2g} \frac{R_{in}(RC)_{jn}}{2} x_i \wedge x_j\right) \\
& + \sum_{1 \leq i < j < k \leq 2g} r_{ijk} \begin{pmatrix} ((JR)_{kn}) x_i \wedge x_j \\ +((JR)_{in}) x_j \wedge x_k \\ -((JR)_{jn}) x_i \wedge x_k \end{pmatrix} \\
= \; & \left(\sum_{i=1}^{g} \frac{(CR)_{in} - R_{in}}{2} x_i \wedge x_{i+g}\right) \\
& + \left(\sum_{1 \leq i < j \leq 2g} \frac{R_{in}(RC)_{jn} - R_{jn}(RC)_{in}}{2} x_i \wedge x_j\right) \\
& + \sum_{1 \leq i < j < k \leq 2g} r_{ijk} \begin{pmatrix} ((JR)_{kn}) x_i \wedge x_j \\ +((JR)_{in}) x_j \wedge x_k \\ -((JR)_{jn}) x_i \wedge x_k \end{pmatrix}
\end{aligned}$$

Now, applying Lemma 2.1 to the coefficient of $x_p \wedge x_q$, where $p < q$, gives

$$\frac{\delta_{q,p+g}((CR)_{pn} - R_{pn}) + R_{pn}(RC)_{qn} - R_{qn}(RC)_{pn}}{2} + \sum_{i=1}^{2g} (r_{ipq}(JR)_{in} - r_{piq}(JR)_{in} + r_{pqi}(JR)_{in}) \equiv \frac{R_{pn}R_{qn}}{2} \bmod 1$$

Note that for fixed i, p, q, at most one of the r-coefficients in the above summation is nonzero. For bookkeeping purposes, when $1 \leq j < r \leq 2g$ we define $\vec{r}_{jk}$ be the $2g$-dimensional column vector whose i^{th} entry is r_{ijk} if $i < j$, $-r_{jik}$ if $j < i < k$, r_{jki} if $k < i$, and 0 otherwise. If $\mathrm{col}_n(M)$ denotes the n^{th} column vector of M, we may rewrite this to obtain that $\mathrm{col}_n(JR) \cdot \vec{r}_{pq}$ is congruent $(\bmod 1)$ to

$$\frac{\delta_{q,p+g}(R_{pn} - (CR)_{pn}) + R_{pn}R_{qn} - R_{pn}(RC)_{qn} + R_{qn}(RC)_{pn}}{2}$$

In order to write this a bit more compactly, for $1 \leq j < k \leq 2g$, we define $\vec{t}_{jk}$ to be the $2g$-dimensional column vector whose i^{th} entry is $\delta_{k,j+g}(R_{ji} - (CR)_{ji}) + R_{ji}R_{ki} - R_{ji}(RC)_{ki} + R_{ki}(RC)_{ji}$. Combining the equations above for all $1 \leq n \leq 2g$ we get:

$$\overline{JR}\vec{r}_{pq} \equiv \frac{\vec{t}_{pq}}{2} \bmod 1 \qquad \forall 1 \leq p < q \leq 2g$$

Solving for $\vec{r}_{pq}$, we obtain:

$$\vec{r}_{pq} \equiv \frac{(\overline{JR})^{-1}\vec{t}_{pq}}{2} \bmod 1$$

Since R is assumed to be symplectic, we can rewrite this as:

$$\vec{r}_{pq} \equiv \frac{RJ\vec{t}_{pq}}{2} \bmod 1$$

Observe that the i^{th} entry of the vector on the right-hand side is

$$\begin{aligned}
&\frac{1}{2}\delta_{q,p+g}\mathrm{row}_i(RJ) \cdot (\mathrm{row}_p(R) - \mathrm{row}_p(CR)) \\
&+\frac{1}{2} \bullet (\mathrm{row}_i(RJ), \mathrm{row}_p(R), \mathrm{row}_q(R)) \\
&-\frac{1}{2} \bullet (\mathrm{row}_i(RJ), \mathrm{row}_p(R), \mathrm{row}_q(RC)) \\
&+\frac{1}{2} \bullet (\mathrm{row}_i(RJ), \mathrm{row}_p(RC), \mathrm{row}_q(R))
\end{aligned} \tag{11}$$

We are interested in calculating the coefficients r_{ipq} for $1 \leq i < p < q \leq 2g$. Thus we are interested in the i^{th} entry of $\vec{r}_{pq}$ when $1 \leq i < p < q \leq 2g$. If $q \neq p+g$ then $\delta_{q,p+g} = 0$. Assume that $q = p+g$. Then $1 \leq i < p \leq g$, and if we write $R = \begin{pmatrix} S & T \\ P & Q \end{pmatrix}$, we have

$$\begin{aligned}
&\mathrm{row}_i(RJ) \cdot (\mathrm{row}_p(R) - \mathrm{row}_p(CR)) \\
&= \mathrm{row}_i(T) \cdot \mathrm{row}_p(S) - \mathrm{row}_i(S) \cdot \mathrm{row}_p(T) \\
&\quad -\mathrm{row}_i(T) \cdot \mathrm{row}_p(P) + \mathrm{row}_i(S) \cdot \mathrm{row}_p(Q) \\
&= (T\overline{S})_{ip} - (S\overline{T})_{ip} - (T\overline{P})_{ip} + (S\overline{Q})_{ip} \\
&= 0 - 0
\end{aligned}$$

The last equality results from using the symplectic conditions (3 i,ii) and by our assumption that $i \neq p$. Thus we may drop the first term of (11). In other words, for $1 \leq i < p < q \leq 2g$ the i^{th} entry of $\vec{r}_{pq}$ (mod 1) is given by

$$\begin{aligned}
&\frac{1}{2} \bullet (\mathrm{row}_i(RJ), \mathrm{row}_p(R), \mathrm{row}_q(R)) \\
&-\frac{1}{2} \bullet (\mathrm{row}_i(RJ), \mathrm{row}_p(R), \mathrm{row}_q(RC)) \\
&+\frac{1}{2} \bullet (\mathrm{row}_i(RJ), \mathrm{row}_p(RC), \mathrm{row}_q(R)) \mod 1
\end{aligned}$$

For aesthetic reasons we rewrite the expression above more symmetrically to show that i^{th} entry of $\vec{r}_{pq}$ (mod 1) is:

$$\begin{aligned}
&\frac{1}{2} \bullet (\mathrm{row}_i(RJ), \mathrm{row}_p(R), \mathrm{row}_q(R)) \\
&-\frac{1}{2} \bullet (\mathrm{row}_i(R), \mathrm{row}_p(RJ), \mathrm{row}_q(R)) \\
&+\frac{1}{2} \bullet (\mathrm{row}_i(R), \mathrm{row}_p(R), \mathrm{row}_q(RJ)) \mod 1
\end{aligned}$$

We have just shown that the $\binom{2g}{3}$ equations in the statement of the lemma are necessary for (r, R) to be an element of $\rho_2(\mathcal{M}_{g,1})$. Since the symplectic representation ρ_1 is surjective, $\rho_2(\mathcal{M}_{g,1})$ contains an element of the form (r, R) for any given R. Johnson [5, Theorem 1] showed that any element of the form (w, I) with $w \in \wedge^3 H$ is in $\rho_2(\mathcal{M}_{g,1})$. Then if $(r, R) \in \rho_2(\mathcal{M}_{g,1})$, so is $(w, I)(r, R) = (w+r, R)$ for any $w \in \wedge^3 H$. Hence we can hit any other possible choice of the coefficients r_{ijk} satisfying the "mod 1" conditions imposed by R by composing our map with different choices of Torelli elements. This shows sufficiency. Q.E.D.

§3. The handlebody group

Our primary goal in this section is to compute $\rho_2(\mathcal{H}_{g,1})$ explicitly. We will begin with some known algebraic characterizations of $\mathcal{H}_{g,1}$ and of $\rho_1(\mathcal{H}_{g,1})$ which will be helpful to us, and use them to derive an analogous characterization at the second level. Thus equipped, we derive an explicit formulation of $\rho_2(\mathcal{H}_{g,1})$ in Section 3.2.

3.1. Algebraic characterizations of the handlebody subgroup

Let $\mathfrak{b}$ denote the normal closure in π of $\{\beta_1, \ldots, \beta_g\}$. Note that $\mathfrak{b}$ is also the kernel of the homomorphism $\pi \to \pi_1(X_g)$ induced by inclusion.

The following proposition was first proved by McMillan [9]. The proof given here was suggested to the authors by Saul Schleimer.

Proposition 3.1. *The handlebody subgroup $\mathcal{H}_{g,1}$ of the mapping class group $\mathcal{M}_{g,1} \subset \mathrm{Aut}(\pi_1(S_{g,1}))$ is precisely the subgroup which preserves $\mathfrak{b}$.*

Proof. One direction is immediate; in order for a mapping class in $\mathcal{M}_{g,1}$ to extend to the X_g it must preserve $\mathfrak{b}$. Now suppose f is a mapping class which preserves $\mathfrak{b}$. Then f sends each β_i to a loop that can be represented by a simple closed curve which is trivial in $\pi_1(X_g)$. Dehn's Lemma [15] shows that these curves bound disks in X_g that can be made disjoint. By matching these disks to the ones bounded by each β_i we may construct a homeomorphism from X_g to itself restricting to f on its boundary. Q.E.D.

Moving on to level one of the Johnson-Morita representations, Birman has shown that the image of the handlebody group in $\mathrm{Sp}(2g, \mathbb{Z})$ is particularly nice [2, Lemma 2.2]. All subblocks are $g \times g$ matrices.

Proposition 3.2 (Birman). *The image of the handlebody group under the symplectic representation is characterized by a $g \times g$ block of zeroes in the upper-right corner. That is,*

$$\rho_1(\mathcal{H}_{g,1}) = \left\{ M \in \mathrm{Sp}(2g;\mathbb{Z}) \,\middle|\, M \textit{ has block form } \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$$

Sufficiency is shown in [2] by exhibiting generators for $\rho_1(\mathcal{H}_{g,1})$ which are in the image of the handlebody group. The necessity of this condition for membership in $\rho_1(\mathcal{H}_{g,1})$ follows from the observation that in the handlebody X_g, the homology classes of the generators of type b_i are all 0. Any homeomorphism of S_g which extends to X_g must take trivial elements in the homology of the handlebody to trivial elements in the homology of the handlebody. In other words, $\rho_1(\mathcal{H}_{g,1})$ is characterized by the property that its elements must preserve the subgroup of H generated by the b_i's.

We will now give a second-level analogue of these characterizations by describing a subgroup of $\pi/\pi^{(2)}$ which must be preserved by $\rho_2(\mathcal{H}_{g,1})$, thus giving a restriction on the image of the handlebody group.

The second Johnson-Morita homomorphism is given by the action of $\mathcal{M}_{g,1}$ on the nilpotent quotient $\pi/\pi^{(2)}$. Let $\mathfrak{b} \subset \pi$ be as above, and recall from Section 2.2 the map $\phi_2 : \pi \longrightarrow \Phi_2$ be as above. The following lemma computes $\phi_2(\mathfrak{b})$.

Lemma 3.3.

$$\phi_2(\mathfrak{b}) = \left\{ \left(\begin{array}{c} \sum_{1\le i,j\le g} m_{ij} a_i \wedge b_j \\ +\sum_{1\le i<j\le g} \left(n_{ij} + \frac{l_i l_j}{2}\right) b_i \wedge b_j \end{array}, \sum_{i=1}^{g} l_i b_i \right) \middle| m_{ij}, n_{ij}, l_i \in \mathbb{Z} \right\}$$

Proof. In light of Lemma 2.1, the right-hand side above is clearly the kernel of the quotient homomorphism $\pi/\pi^{(2)} \longrightarrow \pi_1(X_g)/\pi_1(X_g)^{(2)}$. Q.E.D.

Now that we have identified $\phi_2(\mathfrak{b})$ we will describe $\rho_2(\mathcal{H}_{g,1})$.

3.2. Image of the handlebody subgroup under ρ_2

Theorem 2.4 above gives $\rho_2(\mathcal{M}_{g,1})$. The missing ingredient for a characterization of $\rho_2(\mathcal{H}_{g,1})$ is $\rho_2(\mathcal{I}_{g,1} \cap \mathcal{H}_{g,1})$ which was computed by Morita.

Proposition 3.4 ([10, Lemma 2.5]). *$\rho_2(\mathcal{I}_{g,1} \cap \mathcal{H}_{g,1})$ is the free abelian group with free basis:*

$$(b_i \wedge b_j \wedge b_k, I), \quad (a_i \wedge b_j \wedge b_k, I), \quad \text{and} \quad (a_i \wedge a_j \wedge b_k, I) \quad 1 \le i,j,k \le g.$$

Now we have the tools to assemble a description of $\rho_2(\mathcal{H}_{g,1})$. The following theorem gives a complete characterization of $\rho_2(\mathcal{H}_{g,1})$; it says that an element is in this image if and only if its first factor has no "triple-a" terms and its second factor has the form of Proposition 3.2.

Theorem 3.5. *Let $R \in \mathrm{Sp}(2g, \mathbb{Z})$ be an arbitrary symplectic matrix. Let r be any element of $\frac{1}{2} \wedge^3 H$ with $r = \sum_{1\le i<j<k\le 2g} r_{ijk} x_i \wedge x_j \wedge x_k$. Then $(r, R) \in \rho_2(\mathcal{H}_{g,1})$ if and only if all of the following three conditions hold:*

(1) *R has $g \times g$ block form* $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$

(2) $r_{ijk} \equiv \frac{1}{2} E_{ijk} \bmod 1$ *for all* $1 \le i < j < k \le 2g$.

(3) *$r_{ijk} = 0$ for all i, j, k with $0 \leq i < j < k \leq g$. (i.e. r contains no terms of the form $a_i \wedge a_j \wedge a_k$.)*

We refer the reader to Theorem 2.4 for the definition of E_{ijk}, which depends on the matrix R.

Proof. The necessity of condition 1 has already been established in [2, Lemma 2.2]. We claim that only elements of $\frac{1}{2}\wedge^3 H \rtimes \mathrm{Sp}(H)$ satisfying condition 3 above preserve $\phi_2(\mathfrak{b})$ under the action of (10). Suppose R is symplectic with the required block form and r contains a term of the form $ca_i \wedge a_j \wedge a_k$. Since R^{-1} must satisfy condition 1 above and using Lemma 3.3, there is an element $(\nu, R^{-1}b_i) \in \phi_2(\mathfrak{b})$ where ν has only terms of the form $\frac{1}{2}b_n \wedge b_m$. Applying (9) we get

$$\begin{aligned}
&(r, R) * (\nu, R^{-1}b_i) = \\
&= \big(R(\nu) + \kappa(RR^{-1}b_i) + R\kappa(R^{-1}b_i) + r(RR^{-1}b_i), RR^{-1}b_i\big) \\
&= \big(R(\nu) + \kappa(b_i) + R\kappa(R^{-1}b_i) + r(b_i), b_i\big)
\end{aligned}$$

Consider each of the terms in the first coordinate of the ordered pair above. Since ν only has terms of the form $\frac{1}{2}b_n \wedge b_m$ and the matrix R has the block form given in condition 1, we must have that $R(\nu)$ contains no terms of the form $a_j \wedge a_k$. The image of the homomorphism κ has no $a_j \wedge a_k$ terms so neither $\kappa(b_i)$ nor $\kappa(R^{-1}b_i)$ contains any $a_j \wedge a_k$ terms. Application of the matrix R preserves this quality; hence $R\kappa(R^{-1}b_i)$ contains no $a_j \wedge a_k$ terms. We can see using (4) that $r(b_i)$ will contain a term of the form $-ca_j \wedge a_k$ by construction. Then Lemma 3.3 implies that $c = 0$. It follows that the two conditions of the corollary are necessary.

For each R satisfying 1 there is some mapping class $f \in \mathcal{H}_{g,1}$ with $\rho_1(f) = R$ as shown in [2, Lemma 2.2]. We have shown that $\rho_2(f)$ satisfies conditions 1 and 2. Applying Proposition 3.4 we can get every other element of the form (w, R) satisfying 1 and 2 as a product $(z, I)\rho_2(f)$ where $(z, I) \in \rho_2(\mathcal{I}_{g,1} \cap \mathcal{H}_{g,1})$. This establishes sufficiency. Q.E.D.

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